Matching with Couples: Stability and Algorithm

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Abstract

This paper defines a notion of semi-stability for matching problem with couples, which is a natural generalization of, and further identical to, the conventional stability for matching without couples. It is shown that there always exists a semi-stable matching for couples markets with strict preferences, and the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice. We further prove that a semi-stable matching is stable when couples play reservation strategies. This result perfectly explains the puzzle of NRMP even for finite markets. Moreover, we define a notion of asymptotic stability and present sufficient conditions for a sequential couples market to be asymptotically stable. Another remarkable contribution is that we develop a new algorithm, called Persistent Improvement Algorithm, for finding semi-stable matchings, which is also more efficient than the Gale-Shapley algorithm for finding stable matchings for singles markets. Lastly, this paper investigates the welfare property and incentive issues of semi-stable mechanisms.

Keywords: Matching with couples; stability; semi-stability; asymptotic stability; algorithm.

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1 Introduction

This paper studies the matching problem with couples. One typical feature of the problem is that stable matchings may not exist in the presence of couples. To avoid this defect and find sufficient conditions for the existence of stable matchings, we define the notion of semi-stability as a generalized solution for matching problem with couples, which is a relaxation and natural generalization of the conventional stability, and is further identical to the conventional stability for matching without couples.

Matching is one of the most important natures of market. Many problems, such as trade problem in consumption goods markets, employment problem in labor markets and auction problem of indivisible/public goods, etc., can be regarded as matching problems. Gale and Shapley (1962) were the first to introduce the notion of stable matching and regarded it as a solution of matching problem. The *deferred acceptance algorithm* proposed by them reveals that a stable matching always exists in one-to-one matching markets with strict preferences. Since then, a lot of important theoretical results and their practice on matching have been developed in key areas including education, health care, and army program such as the National Resident Matching Program (NRMP) for medical students; student assignment mechanisms in major school districts; kidney exchange programs; army programs, etc.

The NRMP has a long history. The original algorithm for NRMP proposed by Mullen and Stalnaker (1952) was an unstable mechanism, and later, it was revised repeatedly as discussed in detail in Roth and Peranson (1999). One of the reasons is that, since the 1970s, more and more female medical students had entered the job market, which made NRMP's algorithm run into difficulties in finding stable outcomes. For instance, couples would often decline the job offers assigned by the clearinghouse and search positions themselves in order to stay together, say, they would prefer to have jobs in the same city, although the choice may not be the best for their professional development. This implies that couple students' preferences are complements. In order to make the NRMP's algorithm also work for couple medical students, Roth and Peranson (1999) designed the NRMP's present algorithm, but it may result in an empty set of stable matchings since stable matching may not exist at all.

Sun and Yang (2006, 2009) studied the auction problem in economies where agents of the same type are substitutes for one another, but agents of different types are complements. They showed that equilibrium always exists in economies with quasi-linear preferences. Ostrovsky (2008) studied a more generalized problem of supply chain networks in which there are similar restrictions—same-side substitutability and cross-side complementarity, and showed that the

set of chain-stable networks is non-empty. The problem of matching with couples, however, is different from that of cross-side complementarity in that agents of same-couple are complements. Ronn (1990) demonstrated that when couple factors in preferences are taken into account, it is an NP-complete problem to show whether there exist stable matchings by computational methods.[1]

As Kojima *et al.* (2013) pointed out, although there may not exist any stable matching in couples markets, market practice in the past two decades indicated that the NRMP's present algorithm rarely failed for the clearinghouse to find stable matchings.[2] Why can the clearinghouse find stable outcomes while the theories indicate that there may not exist any stable outcomes for couples markets? Kojima *et al.* (2013) regarded it as a puzzle and the reason may be that there are relatively few couples and preference lists are sufficiently short relative to market size. They showed that under some regularity conditions, as the size of market tends to infinity whereas the number of couples relative to the size of market does not grow rapidly, the probability that there exists a stable matching tends to 1. However, this is just an asymptotic result and a one based on probability bounds. Indeed, any particular market in practice has only a finite number of market participants and the probability that a stable matching exists is not guaranteed to reach 1 in any finite market. As such, this asymptotic prediction is not directly applicable to any finite economy.

In their attempt to find a generalized solution for the matching problem with couples, Klijn and Masso (2003) introduced the notion of weakly stable matching[3] in order to extend the existence result to a larger class of preferences. For singles markets, they showed that the set of weakly stable and weakly efficient matchings is identical to Zhou's (1994) bargaining set. However, as Klaus and Klijn (2005) indicated, the set of weakly stable matchings may still be empty in matching markets with couples. A question is then under what conditions there exist stable matchings in a matching market with couples. Klaus and Klijn (2005) showed that a stable matching exists if all couples' preferences are (weakly) responsive, which means that the unilateral improvement of one partner's acceptable job is considered beneficial for the couple as well, and thus it reflects situations where couples search for jobs in the same metropolitan area.

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[1]The abbreviation NP refers to nondeterministic polynomial time, which is a common term in computational complexity theory.

[2]The NRMP's present algorithm proposed by Roth and Peranson (1999) is an adjustment of the instability-chaining algorithm for singles markets, which was proposed by Roth and Vande Vate (1990), so that the revised matching system can accept couples' preferences.

[3]A matching is weakly stable if it is individually rational and all blocking coalitions are dominated. The detailed definition can be seen in Klijn and Masso (2003), Klaus and Klijn (2005).

As a result, (weak) responsiveness essentially excludes complementarities in couples' preferences. When couples' preferences are not (weakly) responsive, Klaus and Klijn[4] showed that stable outcomes may not exist even in the system containing only one couple. Thus, their result is limited in application.

Moreover, Aldershof and Carducci (1996) showed that, for couples markets, even when the set of stable matchings is non-empty, there may not be a lattice structure, the set of unmatched objects may not be the same at every stable matching, and further there may not be any optimal stable matchings. Klaus and Klijn (2005) demonstrated that there are not any ready parallels to any of the standard results in marriage matching markets, even if preferences are responsive.

All in all, there has been no satisfactory result so far to the problem of matching with couples. There is neither any concept of outcome that is generally applicable, nor any generalized algorithm applicable to couples markets for general settings.

In this paper we define a notion of semi-stability that can be seen as a generalized solution of the matching problem with couples, and show that there always exists a semi-stable matching for couples markets with strict preferences. Semi-stable matching means that it is individually rational and there does not exist any blocking coalition of the matching that contains singleton, that is, a blocking coalition of a semi-stable matching, if any, contains a real couple. As such, the set of stable matchings is clearly a subset of semi-stable matchings. For a special singles market, a semi-stable matching is identical to a stable matching. Thus, the notion of semi-stable matching is a natural generalization of the conventional stable matching without couples. The Persistent Improvement Algorithm (PI-Algorithm in short) proposed in the paper reveals that semi-stable matching always exists in matching markets with couples and strict preferences.

We then provide sufficient conditions for the existence of stable matchings for couples markets even with complementary preferences. It is shown that there exists a stable matching with couples provided every real couple plays reservation strategies, i.e., some reservation preference of couples, which can secure a pair of jobs if they want, are placed on top of their rank lists of preferences. The notion of reservation preference is similar to the conventional reservation utility of an individual: a utility a person can surely obtain. When a couple plays reservation strategies, like individuals' preferences in game theory, mechanism design, auction theory, and market design, their preferences depend not only on their own choice but also on the choice of jobs of hospitals. The reason why couples play reservation strategies is that their preferences have couple-complementarity, that is, although popular jobs are personally desirable, the pair

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[4]The counterexample is seen in Roth (2008), but it belongs to Klaus and Klijn.

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of popular jobs may not be the most preferred choice for couples, as they may not be at the same hospital or in the same city. In order to stay together, the most preferred pair of jobs may not be popular jobs, which is consistent with the practice of NRMP. As a consequence of the sufficiency results, we provide a new explanation for the puzzle of NRMP raised in Kojima *et al.* (2013). Moreover, these results perfectly explain the puzzle even for finite market because they are not based on probability bounds and can be directly applicable to any finite economy.

This paper also defines a notion of asymptotic stability. In a large couples market, if the number of couples is sufficiently small relative to that of singles, a semi-stable matching can be deemed as an approaching stable matching. The number of blocking coalitions of any semi-stable matching must be very small relative to the size of the market, if the length of rank list of couples' preferences does not quickly increase as the market goes very large. We define the notion of the degree of unstable matching to indicate the unorderly degree of a matching. The degree of instability is 0 for a stable matching, and 1 for the null matching in which all players are unmatched. It is shown that under some simple regularity conditions, couples markets are asymptotically stable, i.e., there exists a matching sequence whose instability degree sequence tends to zero when the size of market tends to infinity. This conclusion is similar to the result in Kojima *et al.* (2013), who demonstrated that the probability that a stable matching exists converges to 1 as the market size approaches infinity under some regularity conditions. The simple regularity conditions defined in this paper, however, are weaker than their regularity conditions. As such, their result can be regarded as a special case of our result.

Another important contribution of the paper is to provide an algorithm called Persistent Improvement Algorithm (PI-Algorithm in short) for finding a semi-stable matching, which is also more efficient than the Gale-Shapley algorithm in the sense that it is quicker to find a stable marching for singles markets. Crawford and Knoer (1981) and Kelso and Crawford (1982) studied the employment problem in labor markets, and generalized the Gale-Shapley algorithm by introducing the salary adjustment process. Hatfield and Milgrom (2005) extended the Gale-Shapley algorithm into a generalized algorithm for matching with contracts, which is in turn a generalization of the salary adjustment process of Kelso and Crawford (1982). Ostrovsky (2008) studied a more generalized problem about supply chain networks in which there are restrictions—same-side substitutability and cross-side complementarity, and presented the T-Algorithm which generalizes the algorithms in Kelso and Crawford (1982), Hatfield and Milgrom (2005), as well as the Gale-Shapley algorithm for marriage matching. However, the problem of matching with couples is different from that of cross-side complementarity in that

agents of same-couple are complements. Those algorithms mentioned above cannot fit into the couples markets. Roth and Vande Vate (1990) presented the instability-chaining algorithm for one-to-one matching markets. The NRMP's present algorithm proposed in Roth and Peranson (1999) is an improvement of the instability-chaining algorithm for singles markets such that the clearinghouse can accept the preferences of couples, but the algorithm may not converge. This paper presents the PI-Algorithm which fits the couples markets. Moreover, it improves the Gale-Shapley algorithm, through which we can find a stable matching quickly and which must end in finite rounds for singles markets.

We also show that the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice, and in each of which there exist optimal semi-stable matchings for couples side, truth-telling is a dominant strategy for hospitals side, and the set of unmatched objects is the same at every semi-stable matching. We study welfare property and incentive issues of semi-stable matching mechanisms from the perspective of market design, and generalize respective results in marriage matching markets.

The remainder of this paper is organized as follows. Section II describes the setup and introduces the notion of semi-stability in couples matchings. Section III provides the main results on the existence of semi-stable matchings, stable matchings, asymptotically stable matching sequence, and a generalized lattice theorem. We also provide a new explanation for the NRMP puzzle. Section IV presents the PI-Algorithm and its properties. Section V discusses the welfare and incentive properties of semi-stable matching mechanisms from the perspective of market design. Section VI concludes and the appendix gives proofs of theorems.

2   A General Setting of Matching with Couples

To study matching with couples, without loss of generality, we restrict ourselves to a matching market that consists of jobs of hospitals, job-seeking medical students and their preferences. Although a hospital may provide many jobs, yet as Gale and Shapley (1962) and Roth and Sotomayor (1990) pointed out, when medical students' preferences are on specific jobs, it is equivalent to the one-to-one marriage matching market. In fact, a hospital may provide some jobs of special profession, such as physician jobs, surgeon jobs or gynecologist jobs, etc., and the requirements for the jobs are generally different. As such, in this paper, matching objects of medical students are jobs rather than hospitals.

Let H denote the set of jobs of hospitals, S the set of medical students and C the set of student couples. Their elements are written as h, s, and $c = (s, s')$ respectively. For convenience

of discussion, we assume that null outcome is an element of sets H and S and a couple with null partner is an element of C. To save notation, here we abuse ϕ to have denoted the null elements/partner of sets H, S, and C. Specifically, null ϕ in H (resp. S) denotes the outside option for doctors (resp. jobs), and $c = (s, \phi)$ in C denotes a special couple—single student. As such, the model in this paper can actually be applied to two markets, i.e., the market containing singles only, in short, singles market; the market containing couples (may containing singles or not), in short, couples market.

We assume that all preferences of jobs and couples are strict. Let $\succ_h$ and $\succ_c$ denote h's and c's preference relation, and P^h and P^c denote h's and c's preference list over students and pairs of jobs, respectively. It is said that a student $s \in S$ is acceptable (resp. unacceptable) to h if $s \succ_h \phi$ (resp. $\phi \succ_h s$), and $(h, h') \in H^2$ is acceptable (resp. unacceptable) pair of jobs to c if $(h, h') \succ_c (\phi, \phi)$ (resp. $(\phi, \phi) \succ_c (h, h')$). For convenience of discussion, we assume that ϕ and (ϕ, ϕ) are at the last in P^h and P^c respectively. Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ denote a couples market.

A matching μ is a one-to-one idempotent function from the set $H \cup S$ onto itself (i.e., $\mu^2(x) = x$ for all x) such that $\mu(s) \in H$ and $\mu(h) \in S$, where $\mu(s)$ and $\mu(h)$ are the matched objects of s and h in μ. When a medical student or a job is not matched in μ, ϕ is regarded as its matched object. For convenience, we assume $\mu(\phi) = \phi$. Let $\mu(c) = (\mu(s), \mu(s'))$ with $\mu(s) \in H$ and $\mu(s') \in H$. For any μ, $\mu(s) = h$ if and only if $\mu(h) = s$; $\mu(c) = (h, h')$ if and only if $\mu(h) = s$ and $\mu(h') = s'$.

If a job or a couple cannot be improved upon by voluntarily abandoning its matched object, the matching is individually rational. Formally,

Definition 2.1 A matching μ is *individually rational* if (i) for all $h \in H$ with $\mu(h) \neq \phi$, $\mu(h) \succ_h \phi$; and (ii) for all $c = (s, s') \in C$, $\mu(c) \succ_c (\phi, \mu(s'))$ when $\mu(s) \neq \phi$, $\mu(c) \succ_c (\mu(s), \phi)$ when $\mu(s') \neq \phi$, and $\mu(c) \succ_c (\phi, \phi)$ when $\mu(c) \neq (\phi, \phi)$.

A couple and a pair of hospital jobs constitute a coalition. We then have the following definitions.

Definition 2.2 $\{(s, s'), (h, h')\} \in C \times H^2$ is called a *blocking coalition of matching μ* if (i) $(h, h') \succ_c \mu(c)$; and (ii) $[h \neq \phi$ and $\mu(h) \neq s$ imply $s \succ_h \mu(h)]$ and $[h' \neq \phi$ and $\mu(h') \neq s'$ imply $s' \succ_{h'} \mu(h')]$.

Thus, a blocking coalition means that agents can be improved upon by matching with each other.

Definition 2.3 A matching is said to be *stable* if it is individually rational and there exist no blocking coalitions.

Definition 2.4 A matching is said to be *semi-stable* if it is individually rational and there are no blocking coalitions containing a single.

It is obvious that a stable matching is semi-stable, but the reverse in general is not true. However, a semi-stable matching is a stable matching for any singles market. Indeed, when all couples are (s, ϕ), it is identical to the definition of stable matching for singles markets. Thus, the notion of semi-stability for couples markets is a natural generalization of the conventional stability for singles markets. As Gale and Shapley (1962) showed, a stable matching always exists for singles markets with strict preferences. However, for couples markets, Roth (1984) showed that there may not exist any stable matching. In the next section, departing from Roth's example, we show that a semi-stable matching always exists for any couples market with strict preferences and also provide sufficient conditions for the existence of stable matchings.

3 Main Existence Results and NRMP Puzzle Revisited

In this section, we first investigate the existence of semi-stable matching for couples markets with strict preferences. We then show that the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice so that the couple-optimal matching exists on each of partition sets. We also provide sufficient conditions for the existence of stability. The results enable us to provide a new explanation for the puzzle of NRMP raised in Kojima *et al.* (2013). Moreover, we define a notion of asymptotic stability and provide sufficient conditions for a matching sequence to be asymptotically stable.

3.1 Existence of Semi-Stable Matching and Distributive Lattice

The example in Klaus and Klijn (2005) shows that, even if there is only one couple in a matching market, there may not exist any stable matching. As such, if one focuses only on stable matchings, the set of outcomes may be empty. This makes us introduce the notion of semi-stable matching, which means that there does not exist any blocking coalition of the matching that contains a single. A question is then whether there exists a semi-stable matching for a couples market. The following theorem gives an affirmative answer.

Theorem 3.1 (Existence of Semi-Stable Matching) *For any couples market* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ *with strict preferences, there exists a semi-stable matching* μ.

The theorem indicates that a semi-stable matching always exists for strict preferences. Since the theorem is proved by a constructive way, we actually obtain an algorithm to find a semi-stable outcome. In addition, the algorithm also provides an approach to find a stable matching, if any, in couples markets. Indeed, we first find a semi-stable matching, and then see if the semi-stable matching is stable by verifying that there is no blocking coalition containing a real couple. Of course, this is only a sufficient condition for stable matching, that is, if the semi-stable matching is not stable, we cannot assert that there do not exist any stable matchings.

The Conway lattice theorem in the literature shows that the set of all stable matchings forms a distributive lattice for a singles market with strict preferences so that there is a polarization of interests between the two sides of the market along the set of stable matchings.[5] This implies that there exists a unique best stable matching μ_S favored by the students and a unique best stable matching μ_H favored by the hospitals. Despite this nice property, Aldershof and Carducci (1996) and Klaus and Klijin (2005) showed that, for couples markets, even when the set of stable matchings is non-empty, there may be no optimal matching for either side of the market even for the responsive preferences.

Can we have a similar result of this nice property for semi-stable matchings with couples? The answer is affirmative in some sense. In any couples market with strict preferences, there is a partition of the set of semi-stable matchings, each section of which forms a distributive lattice. To show this, define two partial ordering relations $\geq_C$ and $\geq_H$ on matchings as follows. For any two matchings μ_1 and μ_2, $\mu_1 \geq_C \mu_2$ (resp. $\mu_1 \geq_H \mu_2$) if and only if $\mu_1(c) \succ_c \mu_2(c)$ or $\mu_1(c) = \mu_2(c)$ (resp. $\mu_1(h) \succ_h \mu_2(h)$ or $\mu_1(h) = \mu_2(h)$) for every $c \in C$ (resp. $h \in H$). It is easily seen that both $\geq_C$ and $\geq_H$ are partial ordering relations, i.e., they are irreflexive, anti-symmetric and transitive.

Consider a couples market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$. Let F be the set of all semi-stable matchings, and μ_1 and μ_2 be two semi-stable matchings in F. We define two functions $\vee_C$ and $\wedge_C$ that assign to each student his/her more preferred and less preferred match from μ_1 and μ_2, respectively. Formally, define two operators $\vee_C$ and $\wedge_C$ as follows: for any $c \in C$ and $\mu_1, \mu_2 \in F$, let $\lambda = \mu_1 \vee_C \mu_2$ and $\nu = \mu_1 \wedge_C \mu_2$ where $\lambda(c) = \max_{\succ_c} \{\mu_1(c), \mu_2(c)\}$, $\nu(c) = \min_{\succ_c} \{\mu_1(c), \mu_2(c)\}$. Similarly, we can define functions $\vee_H$ and $\wedge_H$. We then have the following generalized lattice theorem.

Theorem 3.2 (Generalized Lattice Theorem) *Let* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ *be a*

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[5]The theorem is seen in Knuth (1976) and Roth and Sotomayor (1990), but it belongs to John Conway. A lattice is a partially ordered set in which every two elements have a supremum and an infimum. A lattice is called a distributive lattice if it also satisfies the law of distributivity.

couples market with strict preferences. Then the set of all semi-stable matchings F can be partitioned into subsets F_i $(i = 1, \ldots, m)$ with $\cup_{i=1}^{m} F_i = F$ and $F_i \cap F_j = \emptyset$ $(j \neq i)$ such that for any two semi-stable matchings μ_1 and μ_2 in F_i, $\mu_1 \vee_C \mu_2 = \mu_1 \wedge_H \mu_2$ and $\mu_1 \wedge_C \mu_2 = \mu_1 \vee_H \mu_2$ are semi-stable matchings. Furthermore, each of them forms a distributive lattice for operations $\vee_C$ and $\wedge_C$ (resp. $\vee_H$ and $\wedge_H$).

The above Generalized Lattice Theorem implies that in each subset of the partition, there exists a unique best semi-stable matching μ_C (called the student-optimal semi-stable matching) favored by the couples, which is the worst semi-stable matching for the hospital jobs, and there exists a unique worst semi-stable matching for the couples μ_H (called the hospital-optimal semi-stable matching) favored by the hospital jobs, which is the best semi-stable matching for the hospital jobs.[6]

For a singles market, the set of all semi-stable matchings is identical to the set of all stable matchings. As a corollary, the above theorem covers the Conway lattice theorem for marriage matching markets as a special case.

Corollary 3.1 (Conway Lattice Theorem) *If all preferences are strict, then the set of all stable matchings in marriage matching markets forms a lattice for partial ordering relation.*

3.2 Sufficient Condition for the Existence of Stable Matching

For a couples market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, under what conditions does there exist a stable matching? Klaus and Klijn (2005) provided an answer by introducing the notion of (weakly) responsive preference[7], and showed that relative personal preferences can be induced from the preference of couple when it is (weakly) responsive, and these personal preferences induced must be unique. In such situations, the stable matchings found by the Gale-Shapley algorithm are also stable in couple context.

However, (weakly) responsive preference implies that there is no complementarity for preferences of couple, but in real world, they are generally complementary. For example, although

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[6]In fact, supposing that there are m semi-stable matchings in the subset, then we can easily see that $\mu_C \equiv \mu_1 \vee_C \mu_2 \vee_C \cdots \vee_C \mu_m$ and $\mu_H \equiv \mu_1 \wedge_C \mu_2 \wedge_C \cdots \wedge_C \mu_m$ are respectively the best and worst semi-stable matching for couples and hospital jobs in the subset. The argument can be seen in the remark after the proof of the Generalized Lattice Theorem in Appendix.

[7]The preference of couple $c = (s, s') \in C$ is (weakly) responsive if there exist single preferences $\succ_s$ and $\succ_{s'}$ such that: 1) for all $h \in H$, $(h, \phi) \succ_c (\phi, \phi)$ if and only if $h \succ_s \phi$; $(\phi, h) \succ_c (\phi, \phi)$ if and only if $h \succ_{s'} \phi$; and 2) for all $h_p, h_q, h_r \in H$, if $h_q \succ_{s'} \phi$, $h_p \succ_s h_r \succ_s \phi$, then $(h_p, h_q) \succ_c (h_r, h_q)$; if $h_q \succ_s \phi$, $h_p \succ_{s'} h_r \succ_{s'} \phi$, then $(h_q, h_p) \succ_c (h_q, h_r)$.

for an individual s, $h_p \succ_s h_r$, yet for couple $c = (s, s') \in C$, $(h_r, h_q) \succ_c (h_p, h_q)$, as h_q and h_r are in Boston whereas h_p is in New York. Thus, the preference of the couple c is not (weakly) responsive. If so, there may not exist any stable outcomes even in markets containing only one couple.

In this subsection, we provide a sufficient condition for the existence of stable matching even in the presence of complementary preferences of couples. To do so, we first introduce the following notion.

Definition 3.1 (Reservation Preference) A pair of jobs $(h, h') \in H^2$ is said to be a *reservation preference job pair* of a couple $c = (s, s') \in C$ if (i) $(h, h') \succ_c (\phi, \phi)$, (ii) whenever $h \neq \phi$, $s \succ_h \widetilde{s}$ for all $\widetilde{s} \in P^h \backslash \{s\}$, and (iii) whenever $h' \neq \phi$, $s' \succ_{h'} \widetilde{s'}$ for all $\widetilde{s'} \in P^{h'} \backslash \{s'\}$.

A reservation preference of a couple means that the couple can get a pair of jobs if they want, as the members of the couple are respectively the most preferred medical students for the relevant jobs of hospitals. It may be remarked that, when a couple plays reservation strategies, like individuals' preferences in game theory, mechanism design, auction theory, and market design, their preferences depend not only on their own choice but also on the choice of jobs of hospitals. The notion of reservation preference is also similar to the conventional reservation utility of an individual. While reservation utility is a utility that a person can surely obtain by outside opportunities and in turn may depend on preferences of other individuals, reservation preference means that the couple can surely get the jobs since the couple is most preferred students to the jobs.

We then have the following theorem which shows that there must exist a stable matching if the first preference of each couple is one of its reservation preferences.

Theorem 3.3 (Sufficient Condition I for the Existence of Stable Matching) *Let* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ *be a couples market with strict preferences. Suppose that for all* $c \in C$ *with* $s, s' \neq \phi$, *the first preference job pair* (h, h') *in* P^c *is a reservation preference job pair of* c. *Then, there exists a stable matching* μ.

The reason why a reservation preference job pair may be the first priority of a real couple is that its preferences have couple-complementarity, that is, although one wants some popular jobs, the pair of popular jobs may not be the most preferred choice for couples. Since the pair of popular jobs may not be at the same hospital or in the same city, the most preferred job pair for a couple may not be popular jobs, but may be its reservation preference job pair. In

the later subsection, we will give a generalized version of the theorem with more slack condition that reservation preference job pair of real couples may not be their first preference job pair.

3.3 Asymptotic Stability

If we regard stable matching as orderly matching whereas unstable matching as unorderly matching, then the degree of an unstable matching may be used to measure the unorderly degree of a matching. For a couples market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, the most unorderly matching is the null matching μ^0, i.e., for any $c \in C$, $\mu^0(c) = (\phi, \phi)$, whose instability degree is denoted by 1. The most orderly matchings are stable matchings. For any stable matching μ^1, its instability degree is denoted by 0. As such, for a matching μ, its instability degree is a real number between 0 and 1. The intuition is that the higher the instability degree of a matching is, the more unorderly the matching is.

Definition 3.2 (Effective Preference) A pair of jobs $(h, h') \in H^2$ is said to be an *effective preference job pair* of a couple c if (i) $(h, h') \succ_c (\phi, \phi)$, (ii) $s \succ_h \phi$ whenever $h \neq \phi$, and (iii) $s' \succ_{h'} \phi$ whenever $h' \neq \phi$. Student s is said to be an *effective preference student of h* if (i) $s \succ_h \phi$ and (ii) there exists $\overline{h} \in H$ such that $(h, \overline{h}) \succ_c (\phi, \phi)$. Student s' as h''s effective preference student can be similarly defined.

If jobs in a couple c's preference list can accept its corresponding members, then the couple's preference job pair is an effective preference job pair to the couple. If (h, h') is not an effective preference job pair of c, then there does not exist any individually rational matching μ such that $\mu(c) = (h, h')$. Similarly, if s is not an effective preference student of h, there does not exist any individually rational matching μ such that $\mu(h) = s$.

Definition 3.3 If the set C contains n elements, the preference list P^{c_i} contains l_i effective preference job pairs and the number of blocking coalitions of matching μ is m, then the instability degree of μ is denoted by $\theta(\mu) = m/N$, where $N = \sum_{i=1}^{n} l_i$.

Definition 3.3 implies that instability degree is a function from the set of all the matchings in matching market Γ onto the unit interval. The null matching μ^0 has $N = \sum_{i=1}^{n} l_i$ blocking coalitions[8] and each stable matching μ^1 has no blocking coalition, so $\theta(\mu^0) = 1$ and $\theta(\mu^1) = 0$.

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[8]Each blocking coalition $\{c, (h, h')\}$ of the null matching μ^0 contains a couple and one of its effective preference job pairs; conversely, for any $c \in C$ and any job pair (h, h') of its effective preference job pairs, $\{c, (h, h')\}$ must be a blocking coalition of the null matching μ^0. Thus, the null matching μ^0 has exactly $N = \sum_{i=1}^{n} l_i$ blocking coalitions.

Thus, for any matching μ, if it has m blocking coalitions, obviously $0 \leq m \leq N$, and thus the degree of instability $\theta(\mu) \in [0, 1]$. Intuitively, the more blocking coalitions a matching has, the more unorderly it is. Thus, the lower the instability degree of a matching is, the more orderly and stable it is.

Definition 3.4 Let $\{\Gamma^k\}_{k=1}^{\infty}$ be a sequence of couples markets with $\Gamma^k = (H^k, S^k, C^k, (\succ_h)_{h \in H^k}, (\succ_c)_{c \in C^k})$, and let μ^k be a matching of Γ^k, $k = 1, 2, \cdots$. The matching sequence $\{\mu^k\}_{k=1}^{\infty}$ is said to be *asymptotically stable* if $\lim_{k \to \infty}\{\theta(\mu^k)\}_{k=1}^{\infty} = 0$. $\{\Gamma^k\}_{k=1}^{\infty}$ is said to be asymptotically stable if there exists *a matching sequence* $\{\mu^k\}_{k=1}^{\infty}$ that is asymptotically stable.

Based on some common features of large matching markets in reality, Kojima and Pathak (2009) first presented the notion of regular markets. Kojima *et al.* (2013) then defined the regular markets for couples matchings, and demonstrated that for a regular sequence of couples markets, the probability that a stable matching exists converges to 1 as the market size approaches infinity whereas the number of couples relative to the market size does not grow rapidly.

Here we define the notion of simple regular couples markets. Consider a sequence of markets of different sizes. For a sequence of couples markets, $\{\Gamma^k\}_{k=1}^{\infty}$ with $\Gamma^k = (H^k, S^k, C^k, (\succ_h)_{h \in H^k}, (\succ_c)_{c \in C^k})$, there are $|C^k| = n_k$, m_k real couples, and l_c effective preference job pairs in P^c.

Definition 3.5 A sequence of markets $\{\Gamma^k\}_{k=1}^{\infty}$ is said to be *simple regular* if it satisfies the following conditions:

(1) $m_k \leq n_k^{1-\epsilon}$ for all k and a small positive ϵ;

(2) For any $c \in C$ with $s' \neq \phi$, $l_c \leq \gamma(\ln n_k)^{\lambda}$, where γ and λ are constants;

(3) (Participation Constraint) For any $c \in C$, $l_c > 0$.

Condition (1) implies the fact that the number of real couples is small relative to the number of singles. Condition (2) requires that the number of effective preference job pairs of real couples be very small relative to the number of possible pairs of jobs. Condition (3) is actually a participation constraint. For any couple c, if $l_c = 0$, then for any individually rational matching μ, $\mu(c) = (\phi, \phi)$, so it will not participate in the matching market. We then have the following theorem.

Theorem 3.4 (Asymptotic Stability) *Suppose that $\{\Gamma^k\}_{k=1}^{\infty}$ is a sequence of simple regular couples markets with strict preferences where $\Gamma^k = (H^k, S^k, C^k, (\succ_h)_{h \in H^k}, (\succ_c)_{c \in C^k})$. Then*

there exists a matching sequence $\{\mu^k\}_{k=1}^{\infty}$ that is asymptotically stable when n_k tends to infinity, that is, $\{\Gamma^k\}_{k=1}^{\infty}$ is asymptotically stable.

The theorem indicates that there almost always exists a stable matching when the size of a simple regular market tends to infinity. As the simple regularity conditions are weaker than the regularity conditions proposed in Kojima *et al.* (2013), their result that the probability of the existence of a stable matching converges to 1 as the market size approaches infinity can be regarded as a special case of the above asymptotic stability theorem.

3.4 NRMP Puzzle Revisited

In the past two decades, NRMP's practice has shown that the clearinghouse seldom fails to find a stable matching. Kojima *et al.* (2013) pointed out that it is a puzzle. In fact, the reason is that the NRMP market has many special features, which are described as eight stylized facts by them. Here are the first four stylized facts.

Fact 1: Applicants who participate as couples constitute a small fraction of all participating applicants.

Fact 2: The length of the rank order lists of applicants who are single or couples is small relative to the number of possible programs.

Fact 3: The most popular programs are ranked as a top choice by a small number of applicants.

Fact 4: A pair of internship programs ranked by doctors who participate as a couple tend to be in the same region.

Kojima *et al.* (2013) pointed out that, in the data of NRMP during 1992-2009, applicants who participated as couples are on average 4.4% of all applicants, the length of single applicants' preference lists is on average about 7-9 programs, which is about 0.3% of the number of all possible programs, and the length of couple applicants' rank lists is about 81 on average, which is less than 3% of the number of all possible programs. Thus, both are small relative to the number of possible programs.

Since a matching of which the instability degree is zero must be stable, the asymptotic stability theorem indicates that there almost always exists a stable matching when the size of simple regular markets approaches infinity. Facts 1 and 2 imply that the NRMP's practice satisfies the simple regularity conditions, so the asymptotic stability theorem, Theorem 3.4, is a good interpretation for the puzzle of NRMP.

Considering the stylized Fact 3 of NRMP market, most of popular jobs are placed at the top

of their preference rank lists by only a small number of medical students. This indicates that the first preference jobs of most students are not their most preferred choices. Why is this true? It is because couple are complementary, and they want to stay in the same location. As such, the pair of popular jobs may not be the most preferred choice for couples as they may not be at the same hospital or in the same city. Fact 4 exactly describes this, which indicates that a pair of internship programs ranked by doctors who participate as a couple tend to be in the same region. Combining Facts 3 and 4, one can assess that real couples have more incentives to play reservation strategies than singles, i.e., they place one of their reservation preference job pairs, rather than popular jobs, at the top of their rank order list (ROL) of preferences. As a result, by Theorem 3.3, there exists a stable matching. Thus, this theoretical result explains why the clearinghouse seldom fails to find a stable matching in NRMP's practice even if the market may not be large.

Table 1: Summary Statistics of Psychology Labor Market

	Total	Mean length for rank order list (ROL)	Geographic similarity for preference
Single doctors	3010	7.6	
♯ Regions ranked			2.5
Couples	19	81.2	
♯ Regions ranked			3.9
Fraction of ROL where members rank same region			73.4%

Notes: The data are from Kojima et al. (2013). This table reports descriptive information from the Association of Psychology Postdoctoral and Internship Centers match, averaged over 1999-2007. Single doctors' rank order lists consist of a ranking over hospital jobs, while couples' indicate rankings over pairs of hospital jobs.

From the statistical data in Tables 1 and 2, we can see these facts are true. Table 1 indicates that the fraction of rank order list where both members rank the same region, i.e., the preference having couple-complementarity, is 73.4%, which coincides with Fact 4. It shows that the pair of jobs having couple-complementarity surely provides the couple with extra welfare, so it gives couples more incentives than singles to play reservation strategies. Real couples play reservation strategies, which coincides with Fact 3 that the most popular programs are ranked as a top choice by a small number of applicants. This actually shows that not only most real couples but

also most singles play reservation strategies.

Intuitively, if the first preference of a couple or single is successfully matched, then the preference may be seen as one of its reservation preferences. The data in Table 2 show that the fractions of the first preferences of singles that are successfully matched are respectively 36.8% and 36.0% in markets without couples and with couples. If the second preference of singles that is successfully matched is also seen as their reservation preference, then the fractions of singles that play reservation strategies are respectively 53.7% and 52.6%. Although the fraction of the first and second preferences[9] of couples that are successfully matched is only 27.9% and 41.2% plus the third and fourth preferences, as couples have more incentives than singles to play reservation strategies, we can conclude that there is a larger fraction of couples than 53.7% to place their reservation preferences at the top of their rank order lists. It can be partly explained by the fractions of couples and singles unassigned. On average, about 19.5% of all singles are unassigned but only 5.1% for couples, that is, about 80.5% of all singles and 94.9% of all couples are assigned. It may be interpreted as a larger fraction of couples than singles play reservation strategies.[10]

Table 2: Summary Statistics of Psychology Labor Market

Matching Markets	Doctor Type	Doctor's Choice Received					
		1st	2nd	3rd	4th	5th	unassigned
without couples	single	36.8%	16.9%	10.1%	6.1%	11.2%	18.9%
with couples	single	36.0%	16.6%	10.1%	6.2%	11.6%	19.5%
	couple	18.0%	9.9%	8.2%	5.1%	53.7%	5.1%

Notes: The data are from Kojima et al. (2013). This table reports the fractions of preferences of singles that are successfully matched in the doctor-optimal stable matching in a singles market versus a couples market in the Association of Psychology Postdoctoral and Internship Centers match, averaged over 1999-2007. A doctor is counted as unassigned even if being unassigned is among her top five choices.

All in all, Facts 3 and 4 imply that the assumption that real couples play reservation strategies is a very reasonable assumption, and thus Theorem 3.3 perfectly interprets the puzzle of NRMP

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[9]Since the preference of couples is a pair of jobs, two jobs can constitute two preferences, such as (h_1, h_2) and (h_2, h_1).

[10]Of course, it also has another interpretation that, originally, jobs h_1 and h_2 are not to be accepted by either member of a couple, yet both jobs are at the same hospital or in the same city, so as a couple they may accept the pairs of jobs (h_1, h_2) or (h_2, h_1). Thus, the probability that the couple is assigned is increased.

because we need not impose the unrealistic assumption that market goes to infinity and it is not based on probability bounds. Theorem 3.3 holds even for finite market. In contrast, the result of Kojima *et al.* (2013) is an asymptotic result that is based on probability bounds. The probability that a stable matching exists is not guaranteed to reach 1 in any finite market. Since any particular market in practice has only a finite number of market participants, their asymptotic prediction is not directly applicable to any finite economy.

In fact, the condition of Theorem 3.3 can be further weakened. Provided the preferences in front of real couples' first reservation preferences are not pairs of popular jobs, then there exists a stable matching. Since the number of couples is very small relative to the number of singles, we may consider that all popular jobs are assigned to singles. The data in Table 1 show that the number of singles is 3010 whereas the number of couples is 19, and the fraction of singles whose first preferences are successfully assigned is over 36% (which means more than 1000 jobs, and we may consider that almost all popular jobs are among the 1000 jobs). The following theorem shows that there exists a stable matching in such markets, which is another strong interpretation for the puzzle of NRMP.

Theorem 3.5 (Sufficient Condition II for the Existence of Stable Matching) *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market with strict preferences, and for any $c \in C$ with $s, s' \neq \phi$, let $(h, h') \in H^2$ be the first reservation preference in P^c. If for any job pair $(\overline{h}, \overline{h'}) \in H^2$ before (h, h') in the couple's preference list P^c, $\overline{h} \neq \phi$ or $\overline{h'} \neq \phi$ is not only the first preference but also reservation preference job of a single, then there exists a stable matching.*

It can be easily seen that, when the condition of Theorem 3.3 is met, the condition of Theorem 3.5 must also be satisfied. Thus, Theorem 3.5 is a generalization of Theorem 3.3.

4 Persistent Improvement Algorithm (PI-Algorithm)

Hatfield and Milgrom (2005) presented the generalized Gale-Shapley algorithm for matching with contracts. Ostrovsky (2008) studied the more generalized problem about supply chain networks with same-side substitutability and cross-side complementarity. He presented the T-Algorithm that generalizes the result of Hatfield and Milgrom (2005) and also the Gale-Shapley algorithm for one-to-one matching. However, the problem of matching with couples is different from that of cross-side complementarity in that agents of same-couple are complements. As such, these algorithms cannot be applied to couples markets. This section provides a new algorithm, called PI-Algorithm, which can be not only used to find a semi-stable matching according to

the steps described in the proof of Theorem 3.1, but also more efficient than the Gale-Shapley algorithm for finding stable matchings in singles markets.

4.1 The PI-Algorithm

Given a couples market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, let P^h and P^c be h's and c's preference list. Similar to Hatfield and Milgrom (2005), we denote the space of contract $X = (S \times H) \cup \{\Phi\}$ and the space of contract pairs $Z = X \times X$, where Φ denotes null contract. For convenience, we say that (s, ϕ) and (ϕ, h) are both null contracts. A contract $x = (s, h) \in X$ denotes a matching pair between a medical student s and a hospital job h, and a medical student can sign only a contract with any given job. If a medical student (resp. a hospital job) does not sign any contract with any hospital job (resp. any medical student), we say it signs a null contract Φ. That is, in any case, a medical student (resp. a hospital job) can sign a null contract if he/she (resp. it) does not want to sign a contract with any hospital job (resp. any medical student). A contract pair $z = ((s, h), (s', h')) \in Z$ denotes a group of matchings between a couple $c = (s, s')$ and a pair of jobs (h, h'). The running steps of the PI-Algorithm are a sequential process in which contracts are chosen by hospital jobs and couples.

Given that a set of contracts $X' \subsetneq X$, a job h and a couple c will make their optimal choices by their preference lists. Let $Ch_h : 2^X \to X$ and $Ch_c : 2^X \to Z$ denote their best-response mappings so that every subset X' of X corresponds to $Ch_h(X')$ and $Ch_c(X')$, which are respectively a contract and a contract pair. If there is no better choice than vacancy for h, h's choice is a null contract; otherwise, it is a contract between h and its most preferred student. For couples' choices, it is a little more complicated. The couple c chooses an optimal contract pair by its preference list which may contain two null contracts, a null contract or no null contract. Formally, the best-response mappings $Ch_h(\cdot)$ and $Ch_c(\cdot)$ of hospital jobs and couples are defined as follows:

$$Ch_h(X') = \begin{cases} \Phi & \text{if } A_h = \emptyset \\ (s, h) & \text{otherwise} \end{cases},$$

where $A_h = \{\bar{s} : \bar{s} \succ_h \phi \text{ and } (\bar{s}, h) \in X'\}$ is the set of students acceptable to the job h and $\emptyset$ denotes the empty set, and $s = \max \succ_h \{\bar{s} : (\bar{s}, h) \in X'\}$ is the maximal element in A_h.

For $c = (s, s')$, the best-response mapping $Ch_c(\cdot)$ of couples is given by

$$Ch_c(X') = \begin{cases} (\Phi, \Phi) & \text{if } A_c = \emptyset \\ ((s, h), (s', h')) & \text{otherwise} \end{cases},$$

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where $A_c = \{(\bar{h}, \bar{h}') : (\bar{h}, \bar{h}') \succ_c (\phi, \phi) \text{ and } ((s, \bar{h}), (s', \bar{h}')) \in X' \times X'\}$ is the set of job pairs acceptable to the couple c, and $(h, h') = \max \succ_c \{(\bar{h}, \bar{h}') : ((s, \bar{h}), (s', \bar{h}')) \in X' \times X'\}$ is the maximal element in A_c. Specifically, $Ch_c(X')$ can be expressed as

$$Ch_c(X') = \begin{cases} (\Phi, \Phi) & \text{if } h = \phi \text{ and } h' = \phi \\ ((s, h), \Phi) & \text{if } h \neq \phi \text{ and } h' = \phi \\ (\Phi, (s', h')) & \text{if } h = \phi \text{ and } h' \neq \phi \\ ((s, h), (s', h')) & \text{otherwise.} \end{cases}$$

Denote by $Ch_H(X') = \{x \in X : x = Ch_h(X'), h \in H\}$ the best-response set for all hospital jobs and $Ch_C(X') = \{x \in X : \exists \, \bar{x} \in X \text{ s.t. } (x, \bar{x}) = Ch_c(X') \text{ or } (\bar{x}, x) = Ch_c(X'), c \in C\}$ the best-response set for all couples.

The PI-Algorithm starts from the initial matching μ_0 at which matched objects of all couples are (ϕ, ϕ). After running each round, a new matching μ_t is created, which is a Pareto improvement on μ_{t-1} for all couples, i.e., all couples c weakly prefer $\mu_t(c)$ to $\mu_{t-1}(c)$ with at least one couple strictly preferring $\mu_t(c)$. The PI-Algorithm ends if there is no further Pareto improvement for all couples.

Let $Q = \{P_1^c, P_2^c, ..., P_m^c, P_1^h, P_2^h, ..., P_n^h\}$ be the profile of stated preference lists, one for each couple and hospital job, where each P^c and P^h are couple's and job's preference lists. After hospital jobs and couples have submitted their preference lists, all calculation of the PI-Algorithm is executed by the clearinghouse. In round 0 of the PI-Algorithm, it produces preference list $P^c(0)$ of each couple c so that all such items (h, h') of P^c will be removed whenever s or s' is unacceptable to job h or job h' respectively. The PI-Algorithm consists of repeated rounds of calculation. There are four steps in each round except round 0. Step 1 determines preference list $P^c(t)$ of each couple c where all the items of $P^c(t-1)$ behind $\mu_{t-1}(c)$ will be removed. Step 2 determines the set of contracts $X(t)$ for hospital jobs to make a choice. Step 3 determines the set of contracts $Y(t) = Ch_H(X(t))$, which is the result chosen by all hospital jobs, for couples to make a choice. Step 4 determines the set of contracts $Z(t) = Ch_C(Y(t))$ which is the result chosen by all couples. All the contracts in $Z(t)$ form a matching μ_t in round t. Running round-by-round calculation, when $Z(t) = Z(t-1)$, the PI-Algorithm ends and all the contracts in $Z(t)$ form the last matching μ_E. Formally, we have

Round 0, for all $c \in C$, $P^c(0) = P^c \setminus \{(h, h') \in P^c : s \notin P^h \text{ or } s' \notin P^{h'}\}$, $t = 1$.

Round t, for all $c \in C$,

Step 1: $P^c(t) = P^c(t-1) \setminus \{(h, h') \in P^c(t-1) : \mu_{t-1}(c) \succ_c (h, h')\}$;

Step 2: $X(t) = X_1(t) \cup X_2(t)$ where

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$$X_1(t) = \cup_{c \in C}\{(s,h) \in X : there\ exists\ h' \in H\ such\ that\ (h,h') \in P^c(t) \setminus \{(\phi,\phi)\}\},$$

$$X_2(t) = \cup_{c \in C}\{(s',h') \in X : there\ exists\ h \in H\ such\ that\ (h,h') \in P^c(t) \setminus \{(\phi,\phi)\}\};$$

Step 3: $Y(t) = Ch_H(X(t))$;

Step 4: $Z(t) = Ch_C(Y(t))$, we obtain the matching μ_t. If $Z(t) = Z(t-1)$, then the PI-Algorithm ends; else $t = t+1$ and it goes to Round t.

We will illustrate these steps of the PI-Algorithm by the following example.

4.2 The PI-Algorithm: An Example

Example 4.1 $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$ are couples, $c_3 = (s_5, \phi)$ and $c_4 = (s_6, \phi)$ are singles. There are five hospital jobs h_1, h_2, h_3, h_4 and h_5. Their preference lists are as follows:

$c_1 : \{(h_1, h_2), (h_3, h_4), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_4), (\phi, \phi)\}$;

$c_2 : \{(h_1, h_2), (h_3, h_5), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_5), (\phi, \phi)\}$;

$c_3 : \{(h_1, \phi), (h_2, \phi), (h_3, \phi), (h_5, \phi), (\phi, \phi)\}$;

$c_4 : \{(h_1, \phi), (h_2, \phi), (h_3, \phi), (h_4, \phi), (\phi, \phi)\}$;

$h_1 : \{s_1, s_3, s_5, \phi\}$; $h_2 : \{s_2, s_4, s_6, \phi\}$; $h_3 : \{s_1, s_3, s_5, \phi\}$;

$h_4 : \{s_2, s_4, s_6, \phi\}$; $h_5 : \{s_2, s_4, s_5, \phi\}$.

The running procedures of the PI-Algorithm for the example are specified as in Table 3. In round 0, it removes all the items of preference lists of all couples that are not acceptable to hospital jobs. After round 1 and round 2, it has actually produced the last matching. Round 3 repeats round 2 and thus the PI-Algorithm ends. We obtain the following matching.

$\mu_E(c_1) = (h_1, h_2)$, $\mu_E(c_2) = (h_3, h_5)$, $\mu_E(c_3) = (\phi, \phi)$, $\mu_E(c_4) = (h_4, \phi)$.

We can easily verify the matching obtained is stable. The matched object of c_1 is the most preferred choice and therefore there does not exist any blocking coalition containing c_1. As h_1 and h_2 also obtain their most preferred objects, there does not exist any blocking coalition containing h_1 or h_2. Thus, possible blocking coalitions must contain $\{s_1, h_3\}$, $\{s_2, h_4\}$, $\{s_4, h_4\}$ or $\{s_2, h_5\}$. However, such blocking coalition does not exist because there is no participation incentive for c_1 and c_2. Hence there does not exist any blocking coalition.

4.3 Properties of the PI-Algorithm

Suppose the PI-Algorithm ends in round T. Denote the matching produced in round t by μ_t and the last matching by μ_E. Then the PI-Algorithm implies the following two lemmas.

Lemma 4.1 $\mu_{T-1} = \mu_T = \mu_E$; $P^c(t+1) \subseteq P^c(t)$, $X(t+1) \subseteq X(t)$ for any $c \in C$, $0 < t < T$.

Table 3: Running Procedures of the PI-Algorithm

Round 0		$P^{c_1}(0) = \{(h_1, h_2), (h_3, h_4), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_4), (\phi, \phi)\};$ $P^{c_2}(0) = \{(h_1, h_2), (h_3, h_5), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_5), (\phi, \phi)\};$ $P^{c_3}(0) = \{(h_1, \phi), (h_3, \phi), (h_5, \phi), (\phi, \phi)\};$ $P^{c_4}(0) = \{(h_2, \phi), (h_4, \phi), (\phi, \phi)\}.$ $\mu_0(c_1) = \mu_0(c_2) = \mu_0(c_3) = \mu_0(c_4) = (\phi, \phi).$
Round 1	step 1	$P^{c_1}(1) = P^{c_1}(0), P^{c_2}(1) = P^{c_2}(0), P^{c_3}(1) = P^{c_3}(0), P^{c_4}(1) = P^{c_4}(0).$
	step 2	$X(1) = \{(s_1, h_1), (s_1, h_3), (s_2, h_2), (s_2, h_4), (s_3, h_1), (s_3, h_3),$ $(s_4, h_2), (s_4, h_5), (s_5, h_1), (s_5, h_3), (s_5, h_5), (s_6, h_2), (s_6, h_4)\}.$
	step 3	$Ch_H(X(1)) = \{(s_1, h_1), (s_2, h_2), (s_1, h_3), (s_2, h_4), (s_4, h_5)\}.$
	step 4	$Ch_C(Ch_H(X(1))) = \{(s_1, h_1), (s_2, h_2), (s_4, h_5)\}.$ $\mu_1(c_1) = (h_1, h_2), \mu_1(c_2) = (\phi, h_5), \mu_1(c_3) = \mu_1(c_4) = (\phi, \phi).$
Round 2	step 1	$P^{c_1}(2) = \{(h_1, h_2)\}, P^{c_3}(2) = P^{c_3}(1), P^{c_4}(2) = P^{c_4}(1),$ $P^{c_2}(2) = \{(h_1, h_2), (h_3, h_5), (h_1, \phi), (h_3, \phi), (\phi, h_2), (\phi, h_5)\}.$
	step 2	$X(2) = \{(s_1, h_1), (s_2, h_2), (s_3, h_1), (s_3, h_3), (s_4, h_2), (s_4, h_5), (s_5, h_1),$ $(s_5, h_3), (s_5, h_5), (s_6, h_2), (s_6, h_4)\}.$
	step 3	$Ch_H(X(2)) = \{(s_1, h_1), (s_2, h_2), (s_3, h_3), (s_6, h_4), (s_4, h_5)\}.$
	step 4	$Ch_C(Ch_H(X(2))) = \{(s_1, h_1), (s_2, h_2), (s_3, h_3), (s_6, h_4), (s_4, h_5)\}.$ $\mu_2(c_1) = (h_1, h_2), \mu_2(c_2) = (h_3, h_5), \mu_2(c_3) = (\phi, \phi), \mu_2(c_4) = (h_4, \phi).$
Round 3	step 1	$P^{c_1}(3) = \{(h_1, h_2)\},\ P^{c_2}(3) = \{(h_1, h_2), (h_3, h_5)\},$ $P^{c_3}(3) = P^{c_3}(2),\ P^{c_4}(3) = \{(h_2, \phi), (h_4, \phi)\}.$
	step 2	$X(3) = \{(s_1, h_1), (s_2, h_2), (s_3, h_1), (s_3, h_3), (s_4, h_2), (s_4, h_5),$ $(s_5, h_1), (s_5, h_3), (s_5, h_5), (s_6, h_2), (s_6, h_4)\}.$
	step 3	$Ch_H(X(3)) = \{(s_1, h_1), (s_2, h_2), (s_3, h_3), (s_6, h_4), (s_4, h_5)\}.$
	step 4	$Ch_C(Ch_H(X(3))) = \{(s_1, h_1), (s_2, h_2), (s_3, h_3), (s_6, h_4), (s_4, h_5)\}.$ Since $Ch_C(Ch_H(X(3))) = Ch_C(Ch_H(X(2)))$, END. $\mu_E = \mu_3 = \mu_2.$

Lemma 4.2 $\mu_t(c) \succ_c \mu_{t-1}(c)$ or $\mu_t(c) = \mu_{t-1}(c)$ for all $c \in C$, and $\mu_t(c) \succ_c \mu_{t-1}(c)$ for some $c \in C$, $0 < t < T$.

Lemma 4.2 implies that the PI-Algorithm brings a Pareto improvement in each round except round 0 and the last round. In addition, Lemma 4.2 implies that the PI-Algorithm must end in finite rounds, unlike the NRMP's present algorithm that may encounter an infinite loop. Suppose there exist n couples and $l_i + 1$ preferences in couple c_i's preference list P^{c_i}. Since at least one couple gets strict improvement in each round except round 0 and the last round, the PI-Algorithm ends after at most $T = \sum_{i=1}^{n} l_i + 2$ rounds. We then have the following theorem.

Theorem 4.1 *For any couples market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ with strict preferences, the matching μ_E obtained by running the PI-Algorithm is a stable matching on the sub-market $(C, \overline{H})$ where $\overline{H} = \{h \in H : \mu_E(h) \neq \phi\}$.*

In fact, the matching μ_N found by the NRMP's present algorithm[11] is also a stable matching on the sub-market $(\overline{C}, H)$ where $\overline{C} = \{c \in C : \mu_N(\overline{c}) \neq (\phi, \phi)\}$.[12] However, the NRMP's present algorithm does not necessarily converge, which will encounter an infinite loop when no stable matching exists. Even so, we cannot assert that there does not exist any stable matching by this argument because infinite loop may also occur when there exist some stable matchings. In contrast to the NRMP's present algorithm, our PI-Algorithm must end with finite rounds. The following example shows that the NRMP's present algorithm may not converge even when there exist some stable matchings.

Example 4.2 There is a single medical student s_1, a couple $c_1 = (s_2, s_3)$ and two hospital jobs h_1 and h_2, and their preference lists are as follows:

$P^{s_1} : \{h_2, h_1, \phi\}$;

$P^{c_1} : \{(h_1, h_2), (h_2, h_1), (h_1, \phi), (h_2, \phi), (\phi, h_2), (\phi, h_1), (\phi, \phi)\}$;

$P^{h_1} : \{s_1, s_2, s_3, \phi\}$;

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[11]See Roth and Peranson (1999) for a detailed description of the algorithm. In order to avoid the infinite loop that may occur, Kojima *et al.* (2013) presented the sequential couples algorithm similar to the Roth-Peranson algorithm, which are slightly different in two aspects. Firstly, where the sequential couples algorithm fails, the Roth-Peranson algorithm proceeds and tries to find a stable matching. Secondly, in the Roth-Peranson algorithm, when a couple is added to the market with single doctors, any single doctor who is displaced by the couple is placed before another couple is added. By contrast, the sequential couples algorithm holds any displaced single doctor without letting her apply until it processes all applications by couples.

[12]When the NRMP's present algorithm stops at an infinite loop, or when the sequential couples algorithm ends, the matching obtained is stable if not considering the unmatched medical students. See Roth and Peranson (1999) and Kojima *et al.* (2013) for details.

$P^{h_2} : \{s_3, s_1, s_2, \phi\}$.

By using the PI-Algorithm, one can quickly find a stable matching μ, where $\mu(s_1) = h_1$, $\mu(c_1) = (\phi, h_2)$. However, the NRMP's present algorithm will encounter an infinite loop, no matter the processing sequence starts from s_1 or c_1. Indeed, consider two cases.

Case 1. Suppose that s_1 is first processed. The tentative matching is μ_1 with $\mu_1(s_1) = h_2$. At step 2, when c_1 is then processed, the algorithm results in another tentative matching μ_2 with $\mu_2(c_1) = (h_1, h_2)$ and s_1 is added to an "applicant stack". At step 3, when s_1 is added again, the algorithm gets the third tentative matching μ_3 with $\mu_3(s_1) = h_1$, and c_1 and h_2 are added to the "applicant stack" and "program stack", respectively. At step 4, when c_1 is added again, the algorithm gets the fourth tentative matching μ_4 with $\mu_4(c_1) = (h_2, \phi)$, and s_1 is added again to the "applicant stack" as s_1 and h_2 form a blocking coalition of the matching μ_4. At step 5, when s_1 is added again, the algorithm gets again the tentative matching μ_1. Continuing step by step, it will result in an infinite loop.

Case 2. Now suppose that c_1 is first processed. The algorithm will first get the tentative matching μ_2, and the later steps will be the same as those in Case 1[13].

Theorem 4.2 *Suppose that the market contains no real couples. Then, matching μ_E obtained by running the PI-Algorithm is a stable matching.*

The theorem indicates that the PI-Algorithm also finds a stable matching when a matching market only contains single medical students. It simplifies the matching process of the Gale-Shapley algorithm by accelerating the matching process in the way of Pareto improvement, and thus is more efficient. To compare the two algorithms, we first briefly state the running process of the Gale-Shapley algorithm in Gale and Shapley (1962).

The running procedures of the Gale-Shapley algorithm that jobs first propose to medical students are as follows: each job h proposes to medical students of its preference list starting from its first choice (if it has some acceptable choices); each medical student "holds" the most preferred job offer and rejects all others; any job rejected at some step makes a new proposal by sequential order to the next preferred medical student who has not rejected it; when no further proposals are made, the job finally accepted by medical students (if any) forms the last matching.

As for the PI-Algorithm, in round 1, each medical student proposes to all of his or her acceptable choices, and each job chooses its most preferred contract and sends back to the student. The result is identical to that each job selects its most preferred student from its

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[13]For the detailed steps of the NRMP's present algorithm, see Roth and Peranson (1999).

preference list because its most preferred student must have proposed to it.[14] As such, each medical student's choices by the two algorithms in round 1 are identical. After round 1, the PI-Algorithm is varied from the Gale-Shapley algorithm, as medical students do not propose to those jobs to which they prefer their current matched objects. The procedure is that each job proposes to medical students from more preferred to less preferred ones, but not in strict sequential order. In other words, it will skip those medical students who will reject its proposals. Compared with the Gale-Shapley algorithm, the PI-Algorithm obviously accelerates the matching process and improves the efficiency of algorithm. To see this, consider the following example.

Example 4.3 There are n single medical students and $n+1$ hospital jobs, and their preference lists are as follows:

$P^{s_1} : \{h_{n+1}, h_1, h_2, \cdots, h_{n-1}, h_n, \phi\}$;

$P^{s_2} : \{h_1, h_2, h_3, \cdots, h_n, h_{n+1}, \phi\}$;

$P^{s_3} : \{h_2, h_3, h_4, \cdots, h_{n+1}, h_1, \phi\}$;

$\cdots$;

$P^{s_n} : \{h_{n-1}, h_n, h_{n+1}, \cdots, h_{n-3}, h_{n-2}, \phi\}$;

$P^{h_1} : \{s_1, s_3, s_4, \cdots, s_{n-1}, s_n, s_2, \phi\}$;

$P^{h_2} : \{s_2, s_4, s_5, \cdots, , s_n, s_1, s_3, \phi\}$;

$P^{h_3} : \{s_3, s_5, s_6, \cdots, s_1, s_2, s_4, \phi\}$;

$\cdots$;

$P^{h_n} : \{s_n, s_2, s_3, \cdots, s_{n-2}, s_{n-1}, s_1, \phi\}$;

$P^{h_{n+1}} : \{s_1, \phi\}$.

In round 1, both the PI-Algorithm and Gale-Shapley algorithm by which the jobs first propose to medical students get the matching μ_1, where $\mu_1(s_1) = h_{n+1}$, $\mu_1(s_k) = h_k$, for all $2 \leq k \leq n$. In round 2, the PI-Algorithm gets a matching μ_2 with $\mu_2(s_1) = h_{n+1}$, $\mu_2(s_2) = h_1$, $\mu_2(s_k) = h_k$, for all $3 \leq k \leq n$. By the Gale-Shapley algorithm, however, h_1 rejected by s_1 in round 1 first proposes to s_3, and sequentially to s_4, $\cdots$, s_n. After rejected by those medical students, h_1 proposes to s_2. s_2 accepts h_1's proposal and rejects h_2. The Gale-Shapley algorithm also gets the matching μ_2 after h_1 sends $n-1$ proposals. Likewise, in round 3, the PI-Algorithm gets a matching μ_3 with $\mu_3(s_1) = h_{n+1}$, $\mu_3(s_2) = h_1$, $\mu_3(s_3) = h_2$, $\mu_3(s_k) = h_k$, for all $4 \leq k \leq n$. The Gale-Shapley algorithm also gets the matching μ_3 after h_2 sends $n-1$ proposals. By round n, the PI-Algorithm gets the last matching $\mu_E = \mu_n$ with $\mu_n(s_1) = h_{n+1}$, $\mu_n(s_k) = h_{k-1}$, for all

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[14]Assume that each preference of hospital jobs is acceptable to couples; otherwise, this preference cannot be considered effective, which does not affect any individually rational matching and can be deleted from the preference lists of hospital jobs.

$2 \leq k \leq n$, and thus the process ends. However, the Gale-Shapley algorithm will not get the last matching μ_E till $n(n-1)+1$ rounds. Therefore, this example shows that the PI-Algorithm significantly accelerates the matching process and thus is more efficient than the Gale-Shapley algorithm.

For a singles market, similar to the Gale-Shapley algorithm, the PI-Algorithm may begin with either proposals by medical students or proposals by jobs. That is, the PI-Algorithm can begin from the set S to the set H, or similarly from the set H to the set S, and obtain stable matchings μ_S^E and μ_H^E. The following theorem shows that the PI-Algorithm results in the same matching outcomes μ_H and μ_S that are obtained by H-optimal and S-optimal[15] Gale-Shapley algorithm. Thus, the PI-Algorithm can be seen as an improvement of the Gale-Shapley algorithm which only fits the singles market.

Theorem 4.3 *For any singles market $\Gamma = (H, S, (\succ_h)_{h \in H}, (\succ_s)_{s \in S})$ with strict preferences, the matchings μ_S^E and μ_H^E obtained by running the PI-Algorithm are the same as μ_H and μ_S obtained by H-optimal and S-optimal Gale-Shapley algorithm, respectively.*

5 Market Design

In the practice of matching markets, stable matching mechanisms play an important role. However, in some special markets, theoretically there may not exist any stable matching mechanism, such as roommate allocation problem (Gale and Shapley, 1962) and matching with couples, etc. For matching markets with couples, if we employ semi-stable matching mechanisms, Theorem 3.1 guarantees the existence of semi-stable matching mechanisms, and the PI-Algorithm also ensures that semi-stable matching mechanisms are computationally feasible.

Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market, and $Q = \{P_1^c, P_2^c, ..., P_m^c, P_1^h, P_2^h, ..., P_n^h\}$ be the set of stated preference lists, one for each couple and hospital job, where each P^c and P^h are couple's and job's preference lists.

Definition 5.1 A matching mechanism induced by the matching market Γ is a function g whose range is the set of all possible inputs (C, H, Q) and whose output $g(Q)$ is a matching between C and H. If $g(Q)$ is always stable with respect to Q, it can be called a stable matching mechanism; if $g(Q)$ is always semi-stable with respect to Q, it can be called a semi-stable

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[15]H-optimal and S-optimal Gale-Shapley algorithms are respectively the algorithm that jobs first propose to medical students and the algorithm that students first propose to jobs.

matching mechanism.[16]

For any singles market, stable mechanisms have the following properties: 1) At every stable matching, the set of unassigned agents is the same (cf. McVitie and Wilson (1970)); 2) There exist weakly Pareto efficient stable matchings for one side of agents (cf. Roth (1982a)); 3) Stable mechanisms in general are not strategy-proof (cf. Dubins and Freedman (1981), Roth (1982, 1982a, 1985), Sönmez (1997), Martinez *et al.* (2004), Abdulkadiroğlu (2005), Hatfield and Milgrom (2005), Klaus and Klijn (2005)). For couples markets, in general, there does not exist a stable matching mechanism, but Theorem 3.3 guarantees the existence of a stable matching mechanism when real couple plays reservation strategies. Also, Theorem 3.1 guarantees the existence of a semi-stable matching mechanism.

Let μ^E denote a semi-stable matching obtained by the PI-Algorithm following the steps described in the proof of Theorem 3.1, then when $g(Q) = \mu^E$, Theorem 3.1 ensures that the mechanism is a semi-stable matching revelation mechanism, which is called PI-Algorithm mechanism. For a matching market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, let F be the set of all semi-stable matchings. Define a correspondence $K : F \to 2^F$ by $K(\mu) = \{\nu : \nu(c) = \mu(c)$ for all $c = (s, s') \in C$ with $s' \neq \phi\}$, i.e., the matched objects to real couples are the same at every semi-stable matching of $K(\mu)$.

For marriage matching markets with strict preferences, McVitie and Wilson (1970) showed that the set of unmatched men and women is the same at every stable matching. The following theorem generalizes the result of McVitie and Wilson (1970), which shows that for any couples market, the subset of semi-stable matchings $K(\mu^E)$ coincides with the set of all stable matchings.

Theorem 5.1 *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market with strict preferences. Suppose that μ is a semi-stable matching of Γ. Then, the set of unmatched medical students and hospital jobs is the same at every semi-stable matching of $K(\mu)$.*

The next theorem shows that there exist weakly Pareto efficient semi-stable matchings for the side of hospital jobs.

Theorem 5.2 *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market with strict preferences. Then, the semi-stable matching μ^E obtained by the PI-Algorithm mechanism is weakly Pareto efficient on $K(\mu^E)$ for the side of hospital jobs.*

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[16]This definition follows Roth and Sotomayor (1990). A mechanism in which players must state their preferences is called revelation mechanism in the literature.

Theorem 5.2 implies that we have an optimal result for the side of hospital jobs in couples markets. It then can be regarded as a generalization of the optimal theorem on marriage matching markets in Roth (1982a). Also, by Theorem 4.3, for a singles market, $K(\mu^E)$ coincides with the set of stable matchings, and thus we have the following corollary.

Corollary 5.1 *Let $\Gamma = (H, S, (\succ_h)_{h \in H}, (\succ_s)_{s \in S})$ be a singles matching market with strict preferences. Then the stable matching μ^E obtained by running the PI-Algorithm is weakly Pareto efficient for the side of hospital jobs.*

While Theorem 5.2 shows that the PI-Algorithm mechanism results in weak Pareto optimality for hospital jobs, the following theorem, however, shows that it is not strategy-proof.

Theorem 5.3 *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples matching market with strict preferences. Then the PI-Algorithm mechanism is not strategy-proof on Q.*

For any matching market with strict preferences $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, Theorem 3.2 shows that the set of semi-stable matchings forms a partition[17], that is, $F = \cup_{r=1}^{m} F_r$, where F is the set of all semi-stable matchings, and when $m > 1$, $F_i \cap F_j = \emptyset$ for all $1 \le i < j \le m$. Indeed, here F_is denote all different $K(\mu)$s for all $\mu \in F$. Theorem 5.2 implies that there exists, for hospital jobs, weak Pareto optimal semi-stable matching in $K(\mu^E)$. Let μ_r denote the optimal semi-stable matching for the side of hospital jobs, then $F_r = K(\mu_r)$. Although Theorem 5.3 is a negative result on strategy-proofness of the PI-Algorithm mechanism in the whole domain, the following theorem is relatively positive.

Theorem 5.4 *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market with strict preferences, and μ^E be the semi-stable matching obtained by the PI-Algorithm mechanism. Suppose that the PI-Algorithm mechanism is restricted in $K(\mu^E)$. Then it is a dominant strategy for every hospital job to state its true preferences.*

Since the singles market is a special case of the couples market, Theorem 5.4 generalizes the results in Dubins and Freedman (1981) and Roth (1982a).

In the PI-Algorithm mechanism, the outcome will be a random one among $\mu_1, \mu_2, \cdots, \mu_m$. Example 5.1 below is an instance that shows the outcome is completely random. For hospital jobs, though their welfare may be different among $\mu_1, \mu_2, \cdots, \mu_m$, they cannot anticipate which μ_r will be obtained by the algorithm. As such, every hospital job still has incentive to tell the truth in the PI-Algorithm mechanism. The reason is that every μ_r is the optimal semi-stable

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[17]For details, see the proof of Theorem 3.2 in Appendix.

matching in F_r for hospital jobs, and thus, when the outcomes of mechanism are restricted in F_r, truth-telling is a dominant strategy for every hospital job.

Example 5.1 Consider a matching market with two couples $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$, one single $c_3 = (s_5, \phi)$, and four jobs of hospitals. Their rank lists of preferences are as follows:

$P^{c_1} : \{(h_1, h_2), (h_3, h_1), (h_3, \phi), (\phi, \phi)\}$; $P^{c_2} : \{(h_1, h_2), (\phi, h_4), (\phi, \phi)\}$;

$P^{c_3} : \{(h_1, \phi), (h_2, \phi), (\phi, \phi)\}$;

$P^{h_1} : \{s_3, s_2, s_5, \phi\}$; $P^{h_2} : \{s_2, s_4, s_5, \phi\}$; $P^{h_3} : \{s_1, \phi\}$; $P^{h_4} : \{s_4, \phi\}$.

By the PI-Algorithm mechanism, after processing PI-A1,[18] we get a matching μ as follows: $\mu(c_1) = (h_3, \phi), \mu(c_2) = (\phi, h_4), \mu(c_3) = (\phi, \phi)$.

In the process of PI-A2, if the blocking coalition $\{c_3, (h_1, \phi)\}$ is stochastically obtained first, we can get the semi-stable matching μ_1 with $\mu_1(c_1) = (h_3, \phi), \mu_1(c_2) = (\phi, h_4), \mu_1(c_3) = (h_1, \phi)$.

In the process of PI-A2, if the blocking coalition $\{c_3, (h_2, \phi)\}$ is stochastically obtained first, we can get the semi-stable matching μ_2 with $\mu_2(c_1) = (h_3, h_1), \mu_2(c_2) = (\phi, h_4), \mu_2(c_3) = (h_2, \phi)$.

In the two semi-stable matchings μ_1 and μ_2, for couple c_1, single c_3, job h_1 and job h_2, their welfare is changed.

6   Conclusion

This paper studies the problem of matching with couples, which can be seen as an instance of problems with same-side complementarity. One of the typical characteristics of such problems is that stable outcome may not exist. To overcome this defect and provide sufficient conditions for the existence of stable matchings, we introduce the notion of semi-stable matching and consider it as a generalized solution, which is a natural generalization of, and identical to, the conventional stability for singles markets. It is shown that there always exists a semi-stable matching for couples markets with strict preferences, and further the set of semi-stable matchings can be partitioned into subsets, each of which forms a distributive lattice. When the couples market is specialized as the singles market, semi-stable matchings of the market become stable matchings. We also provide sufficient conditions for a semi-stable matching to be stable. If the first preference of all real couples is one of their reservation preferences, then there exist some stable matchings.

This result provides a perfect explanation on the puzzle of NRMP introduced by Kojima *et al.* (2013). For a couples matching market, if all couples play reservation strategies, i.e., they

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[18]For details, see the proof of Theorem 3.1 in Appendix.

place their reservation preferences at the top of their rank order list of preferences, which is consistent with the stylized Facts 3 and 4 of NRMP market described by Kojima *et al.* (2013), then the semi-stable matching obtained by the PI-Algorithm is a stable matching. We define the notion of simple regular market, which simplifies the regular market presented by Kojima *et al.* (2013). The stylized Facts 1 and 2 imply that NRMP market satisfies the conditions of simple regular market. For a sequence of simple regular couples markets, when the size of market tends to infinity, the sequence of semi-stable matchings found by the PI-Algorithm is asymptotically stable. This also provides an interpretation for the puzzle of NRMP, which is similar to Kojima *et al.* (2013).

Another remarkable contribution of the present paper is that we provide a uniform algorithm, called the PI-Algorithm for matching with couples, which ensures finding a stable matching on a subset of the domain that exists. Moreover, if a matching is not a semi-stable matching during the process, the PI-Algorithm goes on processing by canceling some items from some couples' rank lists of preferences until it converges to a semi-stable matching. In contrast to the existing algorithms, this approach ensures finding a semi-stable matching in couples markets with strict preferences.

Moreover, this paper studies the welfare property and incentive issues of the PI-Algorithm mechanism from the perspective of market design. For a singles market, the results obtained by the PI-Algorithm mechanism are the same with those of the Gale-Shapley algorithm mechanism, while the PI-Algorithm is more efficient. For a couples market, the semi-stable matching μ^E obtained by the PI-Algorithm mechanism is the optimal semi-stable matching in a subset $K(\mu^E)$ of the set of all semi-stable matchings for the side of hospital jobs.

This paper also motivates further topics for research. For instance, what are the necessary and sufficient conditions for the semi-stable matching obtained by the PI-Algorithm mechanism to be stable? In what matching mechanisms will players have incentives to play reservation strategies? Another important future research is to study similar issues for couples markets with weak preferences.

Appendix: Proofs

Since the proofs of Theorem 3.1 and some other results need to apply Lemmas 4.1 and 4.2, we prove them first.

Proof of Lemma 4.1: The PI-Algorithm ends when the matching of the current round repeats the previous round. Obviously, $\mu_{T-1} = \mu_T = \mu_E$. The process of the PI-Algorithm indicates that the preference list of each round is derived from the previous round by deleting a part of elements. Deleting the items after $\mu_t(c)$ from $P^c(t+1)$ results in $P^c(t+1) \subseteq P^c(t)$. $X(t)$ and $X(t+1)$ are respectively derived from $P^c(t)$ and $P^c(t+1)$. As such, $X(t+1) \subseteq X(t)$. Q.E.D.

Proof of Lemma 4.2: Given $X(t) \subseteq X(t-1)$ by Lemma 4.1, for any $h \in H$, if $(s, h) \in X(t)$, we have $(s, h) \in X(t-1)$. Thus, if $Ch_h(X(t-1)) = (s, h)$ and $(s, h) \in X(t)$, then we must have $Ch_h(X(t)) = (s, h)$.

Consider two cases: (1) $\mu_{t-1}(c) = (\phi, \phi)$. Then $\mu_t(c) \succ_c \mu_{t-1}(c)$ or $\mu_t(c) = \mu_{t-1}(c)$. (2) $\mu_{t-1}(c) = (h, h') \neq (\phi, \phi)$. Since $(h, h') \in P^c(t)$, in the round t, we have $(s, h) \in X(t)$ and $(s, h) \in Ch_H(X(t))$ if $h \neq \phi$, also $(s', h') \in X(t)$ and $(s', h') \in Ch_H(X(t))$ if $h' \neq \phi$. Hence, (h, h') is selectable for c. Thus, we also have $\mu_t(c) \succ_c \mu_{t-1}(c)$ or $\mu_t(c) = \mu_{t-1}(c)$. When $0 < t < T$, there exists some $c \in C$ such that $\mu_t(c) \succ_c \mu_{t-1}(c)$; otherwise, the PI-Algorithm ends before round T, which contradicts the fact that it ends in round T. Q.E.D.

Proof of Theorem 3.1: We prove the theorem by finding a semi-stable matching through the PI-Algorithm. To do so, we first prove the following two lemmas.

Lemma 6.1 *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market with strict preferences, and the matching finally obtained by the PI-Algorithm be μ_E. Suppose that the PI-Algorithm ends in round T and $\{(s, s'), (h, h')\}$ is a blocking coalition of μ_E. Then, we have (1) whenever $h \neq \phi$ and $\mu_E(h) \neq s$, $\mu_E(h) = \phi$; (2) whenever $h' \neq \phi$ and $\mu_E(h') \neq s'$, $\mu_E(h') = \phi$.*

Proof: We only show statement (1), and the proof of (2) is similar. Since the PI-Algorithm ends in round T, we have $\mu_{T-1} = \mu_T = \mu_E$ by Lemma 4.1. Suppose, by way of contradiction, that $\mu_E(h) \neq \phi$ for $h \neq \phi$ and $\mu_E(h) \neq s$. There are two cases to be considered.

Case 1: $h' = \phi$. Since $\{(s, s'), (h, \phi)\}$ is a blocking coalition of μ_E, we have $s \succ_h \mu_E(h)$ and $(h, \phi) \succ_c \mu_E(c)$. Also, since $\mu_{T-1} = \mu_T = \mu_E$, we have $s \succ_h \mu_{T-1}(h)$ and $(h, \phi) \succ_c \mu_{T-1}(c)$. Then $(h, \phi) \in P^c(T)$, and thus $(s, h) \in X(T)$. We then must have $(s, h) \notin Ch_H(X(T))$

(Otherwise, we have $\mu_T(c) \succ_c (h, \phi)$ or $\mu_T(c) = (h, \phi)$, which contradicts the fact that $(h, \phi) \succ_c$ $\mu_E(c) = \mu_T(c)$.). Therefore, there exists $\bar{s} \neq \phi$ such that $(\bar{s}, h) \in X(T)$ and $(\bar{s}, h) \in Ch_H(X(T))$. As such, $Ch_h(X(T)) = (\bar{s}, h)$. Now, if $(\bar{s}, h) \notin Ch_C(Ch_H(X(T)))$, then $\mu_T(h) = \phi$, contradicting $\mu_T(h) = \mu_E(h) \neq \phi$. Thus, we must have $(\bar{s}, h) \in Ch_C(Ch_H(X(T)))$, and therefore $\mu_T(h) = \bar{s}$. However, $(s, h) \notin Ch_H(X(T))$, and $(\bar{s}, h) \in Ch_H(X(T))$ implies $\mu_E(h) = \mu_T(h) = \bar{s} \succ_h s$, which contradicts $s \succ_h \mu_E(h)$. Hence, we must have $\mu_E(h) = \phi$.

Case 2: $h' \neq \phi$. Obviously, it implies $s' \neq \phi$. Since $\{(s, s'), (h, h')\}$ is a blocking coalition of μ_E, $(h, h') \succ_c \mu_E(c)$, $s \succ_h \mu_E(h)$ or $s = \mu_E(h)$, $s' \succ_{h'} \mu_E(h')$ or $s' = \mu_E(h')$. As $(h, h') \succ_c$ $\mu_E(c) = \mu_{T-1}(c)$, we have $(h, h') \in P^c(T)$. Thus, $(s, h) \in X(T)$ and $(s', h') \in X(T)$. If $(s, h) \in Ch_H(X(T))$ and $(s', h') \in Ch_H(X(T))$, then $\mu_T(c) \succ_c (h, h')$ or $\mu_T(c) = (h, h')$, which contradicts $(h, h') \succ_c \mu_E(c) = \mu_T(c)$. Thus, either $(s, h) \notin Ch_H(X(T))$ or $(s', h') \notin Ch_H(X(T))$. Without loss of generality, suppose $(s, h) \notin Ch_H(X(T))$. Then, there exists $\bar{s} \neq \phi$ such that $(\bar{s}, h) \in X(T)$ and $(\bar{s}, h) \in Ch_H(X(T))$. As $\mu_T(h) = \mu_E(h) \neq \phi$, $\mu_T(h) = \bar{s}$. $(s, h) \notin Ch_H(X(T))$ and $(\bar{s}, h) \in Ch_H(X(T))$ imply $\mu_E(h) = \mu_T(h) = \bar{s} \succ_h s$, which contradicts $s \succ_h \mu_E(h)$ or $s = \mu_E(h)$. Hence, we must also have $\mu_E(h) = \phi$.

Thus, in either case, we have proved that whenever $h \neq \phi$ and $\mu_E(h) \neq s$, we must have $\mu_E(h) = \phi$. Q.E.D.

Lemma 6.2 *Let $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ be a couples market with strict preferences, and the matching finally obtained by the PI-Algorithm be μ_E. Suppose that for $c = (s, \phi) \in C$, $h \in H$, $\{c, (h, \phi)\}$ is a blocking coalition of the matching μ_E. Then there exists $\bar{c} = (\bar{s}, \bar{s}')$ with $\bar{s}' \neq \phi$ and $\bar{h} \in H$ (or $\tilde{c} = (\tilde{s}, \tilde{s}')$ with $\tilde{s} \neq \phi$ and $\tilde{h} \in H$) such that $(h, \bar{h}) \succ_{\bar{c}} \mu_E(\bar{c})$ (or $(\tilde{h}, h) \succ_{\tilde{c}} \mu_E(\tilde{c})$).*

Proof: Since the PI-Algorithm ends in round T, we have $\mu_{T-1} = \mu_T = \mu_E$ by Lemma 4.1. Also, since $\{c, (h, \phi)\}$ is a blocking coalition of μ_E, we have $s \succ_h \mu_E(h)$ and $(h, \phi) \succ_c \mu_E(c)$. Then $(h, \phi) \succ_c \mu_{T-1}(c)$ and $(h, \phi) \in P^c(T)$. Thus, $(s, h) \in X(T)$. We then must have $(s, h) \notin$ $Ch_H(X(T))$ (If so, $\mu_T(c) \succ_c (h, \phi)$ or $\mu_T(c) = (h, \phi)$, contradicting $(h, \phi) \succ_c \mu_E(c) = \mu_T(c)$.). Thus, there is $\bar{s} \neq \phi$ such that $(\bar{s}, h) \in X(T)$ with $(\bar{s}, h) \in Ch_H(X(T))$.

We show that $\bar{s}$ must be a member of real couple. Suppose not. $\bar{s}$ is then a single. Let $c_1 = (\bar{s}, \phi)$. $(\bar{s}, h) \in Ch_H(X(T))$ implies either $\mu_T(c_1) = (h, \phi)$ or $\mu_T(c_1) \succ_{c_1} (h, \phi)$. However, it is impossible to have $\mu_T(c_1) = (h, \phi)$ (Otherwise, we have $\mu_T(h) = \bar{s}$, which contradicts $\mu_T(h) = \phi$ by Lemma 6.1.). As such, we have $\mu_{T-1}(c_1) = \mu_T(c_1) \succ_{c_1} (h, \phi)$, which implies $(h, \phi) \notin P^{c_1}(T)$. Then, $(\bar{s}, h) \notin X(T)$, contradicting $(\bar{s}, h) \in X(T)$. Hence, we must have $\bar{s}$ is a member of real couple.

If $\bar{s}$ is the first member of a real couple, denote $\bar{c} = (\bar{s}, \bar{s}')$ with $\bar{s}' \neq \phi$. We first show that $(h, \phi) \notin P^{\bar{c}}(T)$. Note that $(\bar{s}, h) \in Ch_H(X(T))$ implies $\mu_T(\bar{c}) \succ_{\bar{c}} (h, \phi)$ or $\mu_T(\bar{c}) = (h, \phi)$. We then only have $\mu_{T-1}(\bar{c}) = \mu_T(\bar{c}) \succ_{\bar{c}} (h, \phi)$, which implies $(h, \phi) \notin P^{\bar{c}}(T)$. To see this, suppose $\mu_T(\bar{c}) = (h, \phi)$. Then $\mu_T(h) = \bar{s}$, which is impossible by noting that $\mu_T(h) = \phi$. Thus $(\bar{s}, h) \in X(T)$ implies the existence of $\overline{h} \in H$ such that $(h, \overline{h}) \in P^{\bar{c}}(T)$, which in turn implies $(h, \overline{h}) \succ_{\bar{c}} \mu_{T-1}(\bar{c}) = \mu_T(\bar{c})$ or $\mu_T(\bar{c}) = \mu_{T-1}(\bar{c}) = (h, \overline{h})$. If $\mu_T(\bar{c}) = (h, \overline{h})$, $\mu_T(h) = \bar{s}$, which contradicts $\mu_T(h) = \phi$. Therefore, $(h, \overline{h}) \succ_{\bar{c}} \mu_T(\bar{c}) = \mu_E(\bar{c})$.

If $\bar{s}$ is the second member of a real couple, denote $\tilde{c} = (\tilde{s}, \tilde{s}')$ with $\tilde{s}' = \bar{s}$ and $\tilde{s} \neq \phi$. Likewise, we can show that there exists $\tilde{h} \in H$ such that $(\tilde{h}, h) \succ_{\tilde{c}} \mu_E(\tilde{c})$. Q.E.D.

With these two lemmas, we are now ready to prove the theorem. The process of finding a semi-stable matching by the algorithm can be divided into two stages as follows:

Process PI-A1: For any couples market with strict preferences $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, running the procedures of the PI-Algorithm described in Section IV, and supposing the PI-Algorithm ends at round T, the matching finally obtained is μ_E. If μ_E is a semi-stable matching, we complete the searching; otherwise we go on to the second stage.

Process PI-A2: There is a blocking coalition $\{c_1, (h, \phi)\}$ of μ_E, where $c_1 = (s_1, \phi)$ is a single. By Lemma 6.2, there exists $\bar{c} = (\bar{s}, \bar{s}')$ with $\bar{s}' \neq \phi$ and $\overline{h} \in H$ (or $\tilde{c} = (\tilde{s}, \tilde{s}')$ with $\tilde{s} \neq \phi$ and $\tilde{h} \in H$) such that $(h, \overline{h}) \succ_{\bar{c}} \mu_E(\bar{c})$ (or $(\tilde{h}, h) \succ_{\tilde{c}} \mu_E(\tilde{c})$). We only consider the case of $\bar{c}$; the case for $\tilde{c}$ is similar. For $\bar{c}$, deleting all of preference job pairs before $\mu_E(\bar{c})$ containing h from the preference list of $\bar{c}$, and letting $P^{\bar{c}}(T+1) = P^{\bar{c}}(T) \setminus \{(h_1, h_2) \in H \times H : h_1 = h, or, h_2 = h\}$, the PI-Algorithm continues on the basis of the round T. Letting the new process start from round $T+1$ to round T_1, the matching obtained at the new ending is μ_E^1.

We first prove for any $c \in C$, $\mu_{T+1}(c) \succ_c \mu_T(c)$ or $\mu_{T+1}(c) = \mu_T(c)$. In round $T+1$, obviously $X(T+1) \subset X(T)$. Then, for any $\widetilde{h} \in H$, if $\mu_T(\widetilde{h}) = \widetilde{s} \neq \phi$, we have $(\widetilde{s}, \widetilde{h}) \in X(T+1)$, and thus $Ch_{\widetilde{h}}(X(T+1)) = Ch_{\widetilde{h}}(X(T))$ by the same argument as in the proof of Lemma 4.2. This implies for any $\widetilde{c} \in C$, if $\mu_T(\widetilde{c}) \neq (\phi, \phi)$, $\mu_T(\widetilde{c})$ can be chosen by $\widetilde{c}$ at round $T+1$. Therefore, $\mu_{T+1}(\widetilde{c}) \succ_{\widetilde{c}} \mu_T(\widetilde{c})$ or $\mu_{T+1}(\widetilde{c}) = \mu_T(\widetilde{c})$.

We then prove there is at least one $c \in C$ such that $\mu_{T+1}(c) \succ_c \mu_T(c)$. Since $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_E, $(h, \phi) \succ_{c_1} \mu_E(c_1) = \mu_T(c_1)$, $(h, \phi) \in P^{c_1}(T) = P^{c_1}(T+1)$, and $(s_1, h) \in X(T+1)$. Suppose $Ch_h(X(T+1)) = (s, h)$. Then $s \succ_h s_1$ or $s = s_1$. If $s = s_1$, then $\mu_{T+1}(c_1) \succ_{c_1} (h, \phi)$ or $\mu_{T+1}(c_1) = (h, \phi)$, and thus $\mu_{T+1}(c_1) \succ_{c_1} \mu_T(c_1)$; if $s \succ_h s_1$, $\mu_T(h) = \mu_E(h) \neq s_1$ because $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_E. Thus we have $\mu_T(h) = \mu_E(h) = \phi$

by Lemma 6.1, which implies $\mu_T(s) \neq h$. $Ch_h(X(T+1)) = (s,h)$ implies $(s,h) \in X(T+1)$. For any real couple $\bar{c}$, we have already deleted all the preference job pairs before $\mu_E(\bar{c})$ containing h from the preference list of $\bar{c}$, so s must be a single. Let $c = (s,\phi)$, then $(s,h) \in X(T+1)$, which means $(h,\phi) \in P^c(T+1)$. $\mu_T(s) \neq h$ implies $\mu_T(c) \neq (h,\phi)$, and $(h,\phi) \succ_c \mu_T(c)$. As $Ch_h(X(T+1)) = (s,h)$, $\mu_{T+1}(c) \succ_c (h,\phi)$ or $\mu_{T+1}(c) = (h,\phi)$, and $\mu_{T+1}(c) \succ_c \mu_T(c)$. Thus we show that there exists a $c \in C$ such that $\mu_{T+1}(c) \succ_c \mu_T(c)$.

By Lemma 4.2, for any $c \in C$, $\mu_E^1(c) \succ_c \mu_{T+1}(c)$ or $\mu_E^1(c) = \mu_{T+1}(c)$. Thus $\mu_E^1(c) \succ_c \mu_T(c)$ or $\mu_E^1(c) = \mu_T(c)$, and there exists at least one $c \in C$ such that $\mu_E^1(c) \succ_c \mu_T(c)$. If μ_E^1 is a semi-stable matching, the process ends; otherwise it goes on with PI-A2. Repeating this process, we can get matchings μ_E^1, μ_E^2, $\cdots$, μ_E^m until there is no blocking coalition of μ_E^m, $\{(s,\phi),(h,\phi)\}$. The repetition will terminate because the terms of the preference lists of all $c \in C$ are finite, and for all $c \in C$, the new matching is a Pareto improvement after each repetition. As this repetition is always in process, each single medical student will obtain his/her most preferred job, i.e., the first item in his/her preference list. In this case, there is no blocking coalition $\{(s,\phi),(h,\phi)\}$ of μ_E^m.

Obviously, the PI-Algorithm indicates that the matchings μ_E and μ_E^1, μ_E^2, $\cdots$, μ_E^m are also individually rational. Thus we show that the matching μ_E^m finally obtained is a semi-stable matching of the matching market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$. Q.E.D.

Proof of Theorem 3.2: Define a correspondence $K : F \to 2^F$ by

$$K(\mu) = \{\nu : \nu(c) = \mu(c) \ \forall \ c = (s,s') \in C \ with \ s' \neq \phi\},$$

so that all the matchings in $K(\mu)$ respectively match the same objects to real couples. Obviously, $F = \cup_{\mu \in F} K(\mu)$ and further $K(\mu_1) \cap K(\mu_2) = \emptyset$ for $K(\mu_1) \neq K(\mu_2)$. Since such subsets are finite, we can label them by F_i, $i = 1, 2, \cdots, m$. Then $F = \cup_{i=1}^m F_i$, and when $m > 1$, $F_i \cap F_j = \emptyset$ for all $1 \leq i < j \leq m$. Thus, these subsets $\{F_i\}$ constitute a partition of F. If we can show that operations $\vee_C$ and $\wedge_C$ are closed in $K(\mu)$, it is easy to verify that they meet the requirements of idempotent law, commutative law, associative law, absorption law and distributive law, and then, by the definition of distributive lattice,[19] $K(\mu)$ constitutes a distributive lattice for operations $\vee_C$ and $\wedge_C$. Consequently, the theorem is proved.

To show that operations $\vee_C$ and $\wedge_C$ are indeed closed in $K(\mu)$, i.e., $\lambda = \mu_1 \vee_C \mu_2 \in K(\mu)$

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[19]From Birkhoff and Mac Lane (2007), a lattice is a set L of elements with two binary operations $\wedge$ and $\vee$ which are idempotent, commutative, and associative and which satisfy the absorption law. If in addition the distributive law holds, L is called a distributive lattice.

and $v = \mu_1 \wedge_C \mu_2 \in K(\mu)$, consider any $c = (s, s') \in C$. If $s' \neq \phi$, by the definition of $K(\mu)$, $\mu_1(c) = \mu_2(c)$, thus $\lambda(c) = v(c) = \mu_1(c) = \mu_2(c)$; if $s' = \phi$, when $\mu_1(c) = \mu_2(c)$, $\lambda(c) = v(c) = \mu_1(c) = \mu_2(c)$; when $\mu_1(c) \succ_c \mu_2(c)$, $\lambda(c) = \mu_1(c)$ and $v(c) = \mu_2(c)$; when $\mu_2(c) \succ_c \mu_1(c)$, $\lambda(c) = \mu_2(c)$ and $v(c) = \mu_1(c)$.

We now show that $\lambda \in K(\mu)$. We first prove λ is a matching. To do so, we need to show that for any $c_1 \in C$, $c_2 \in C$, $c_1 \neq c_2$, we have (1) $\lambda(s_1) \neq \lambda(s_2)$ for $\lambda(s_1) \neq \phi$ and $\lambda(s_2) \neq \phi$, (2) $\lambda(s_1') \neq \lambda(s_2')$ for $\lambda(s_1') \neq \phi$ and $\lambda(s_2') \neq \phi$, (3) $\lambda(s_1') \neq \lambda(s_2)$ for $\lambda(s_1') \neq \phi$ and $\lambda(s_2) \neq \phi$, and (4) $\lambda(s_1) \neq \lambda(s_2')$ for $\lambda(s_1) \neq \phi$ and $\lambda(s_2') \neq \phi$.

(1) $\lambda(s_1) \neq \lambda(s_2)$. Suppose not. There are two cases to be considered.

Case 1: $s_1' \neq \phi$ or $s_2' \neq \phi$. Without loss of generality, suppose $s_1' \neq \phi$.

Since $\lambda(c_1) = \mu_1(c_1) = \mu_2(c_1)$, $\lambda(s_1) = \mu_1(s_1) = \mu_2(s_1)$. If $\lambda(c_2) = \mu_1(c_2)$, $\lambda(s_2) = \mu_1(s_2)$, then $\mu_1(s_1) = \lambda(s_1) = \lambda(s_2) = \mu_1(s_2)$, which contradicts that μ_1 is a matching; if $\lambda(c_2) = \mu_2(c_2)$, $\lambda(s_2) = \mu_2(s_2)$, then $\mu_2(s_1) = \mu_2(s_2)$, which contradicts that μ_2 is a matching.

Case 2: $s_1' = \phi$ and $s_2' = \phi$.

Case A: $\lambda(c_1) = \mu_1(c_1)$ and $\lambda(c_2) = \mu_1(c_2)$. It implies $\lambda(s_1) = \mu_1(s_1)$ and $\lambda(s_2) = \mu_1(s_2)$. Then, $\mu_1(s_1) = \mu_1(s_2)$, which contradicts that μ_1 is a matching.

Case B: $\lambda(c_1) = \mu_2(c_1)$ and $\lambda(c_2) = \mu_2(c_2)$. It implies $\lambda(s_1) = \mu_2(s_1)$ and $\lambda(s_2) = \mu_2(s_2)$. Then, $\mu_2(s_1) = \mu_2(s_2)$, which contradicts that μ_2 is a matching.

Case C: $\lambda(c_1) = \mu_1(c_1)$ and $\lambda(c_2) = \mu_2(c_2)$. It implies $\lambda(s_1) = \mu_1(s_1)$ and $\lambda(s_2) = \mu_2(s_2)$. Then, $\mu_1(s_1) = \mu_2(s_2)$. Let $h = \mu_1(s_1) = \mu_2(s_2)$, $\mu_1(h) = s_1$ and $\mu_2(h) = s_2$. Since $s_1' = \phi$ and $s_2' = \phi$, $\mu_1(c_1) = (h, \phi)$ and $\mu_2(c_2) = (h, \phi)$. $\lambda(c_1) = \mu_1(c_1)$ and $\lambda(c_2) = \mu_2(c_2)$ respectively imply that $(h, \phi) = \mu_1(c_1) \succ_{c_1} \mu_2(c_1)$ and $(h, \phi) = \mu_2(c_2) \succ_{c_2} \mu_1(c_2)$. $c_1 \neq c_2$ implies $s_1 \neq s_2$. Therefore, if $s_1 \succ_h s_2$, $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_2, which contradicts that μ_2 is a semi-stable matching; if $s_2 \succ_h s_1$, $\{c_2, (h, \phi)\}$ is a blocking coalition of μ_1, which contradicts that μ_1 is a semi-stable matching.

Case D: $\lambda(c_1) = \mu_2(c_1)$ and $\lambda(c_2) = \mu_1(c_2)$. It can also be proved by the same argument of Case C.

Thus, by Case 1 and Case 2, if $c_1 \neq c_2$, we must have $\lambda(s_1) \neq \lambda(s_2)$ for $\lambda(s_1) \neq \phi$ and $\lambda(s_2) \neq \phi$.

(2) $\lambda(s_1') \neq \lambda(s_2')$. As $\lambda(s_1') \neq \phi$ and $\lambda(s_2') \neq \phi$, we have $s_1' \neq \phi$ and $s_2' \neq \phi$, and thus $\lambda(c_1) = \mu_1(c_1) = \mu_2(c_1)$ and $\lambda(c_2) = \mu_1(c_2) = \mu_2(c_2)$. Since $c_1 \neq c_2$ implies $\mu_1(s_1') \neq \mu_1(s_2')$, $\lambda(s_1') \neq \lambda(s_2')$.

(3) $\lambda(s_1') \neq \lambda(s_2)$. Since $\lambda(s_1') \neq \phi$ implies $s_1' \neq \phi$, $\lambda(c_1) = \mu_1(c_1) = \mu_2(c_1)$. Also, $c_1 \neq c_2$

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implies $\mu_1(s_1') \neq \mu_1(s_2)$ and $\mu_2(s_1') \neq \mu_2(s_2)$, but $\lambda(s_1') = \mu_1(s_1') = \mu_2(s_1')$ and $\lambda(s_2) = \mu_1(s_2)$ or $\mu_2(s_2)$. Thus $\lambda(s_1') \neq \lambda(s_2)$.

(4) $\lambda(s_1) \neq \lambda(s_2')$. The proof is similar to that of (3).

Therefore, for all $c_1, c_2 \in C$ with $c_1 \neq c_2$, we have $\lambda(s_1) \neq \lambda(s_2)$ for $\lambda(s_1) \neq \phi$ and $\lambda(s_2) \neq \phi$; $\lambda(s_1') \neq \lambda(s_2')$ for $\lambda(s_1') \neq \phi$ and $\lambda(s_2') \neq \phi$; $\lambda(s_1) \neq \lambda(s_2')$ for $\lambda(s_1) \neq \phi$ and $\lambda(s_2') \neq \phi$; $\lambda(s_1') \neq \lambda(s_2)$ for $\lambda(s_1') \neq \phi$ and $\lambda(s_2) \neq \phi$. Hence $\lambda = \mu_1 \vee_C \mu_2$ is a matching.

Next, we prove $\lambda = \mu_1 \vee_C \mu_2$ is a semi-stable matching.

Let $C = C_1 \cup C_2$, where $s_1' = \phi$ for all $c_1 \in C_1$ and $s_2' \neq \phi$ for all $c_2 \in C_2$. Obviously, $C_1 \cap C_2 = \emptyset$. Let $H_C(\mu) = \{h \in H : \text{there exists } c_2 \in C_2 \text{ such that } \mu(h) = s_2 \text{ or } \mu(h) = s_2'\}$. For any $\alpha \in K(\mu)$, let $\widetilde{\alpha}$ denote a restricted matching on $(C_1, [H \setminus H_C(\mu)])$, where $\widetilde{\alpha}(c_1) = \alpha(c_1)$ for all $c_1 \in C_1$. Since α is a semi-stable matching, $\widetilde{\alpha}$ is a stable matching. Otherwise, there exists a blocking coalition $\{c_1, (h_1, \phi)\}$ of $\widetilde{\alpha}$, and obviously $\{c_1, (h_1, \phi)\}$ is also a blocking coalition of α, which contradicts that α is a semi-stable matching.

For any $\mu_1 \in K(\mu)$ and $\mu_2 \in K(\mu)$, $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are stable matchings on $(C_1, [H \setminus H_C(\mu)])$. By Conway's lattice theorem for singles markets, $\widetilde{\lambda} = \widetilde{\mu}_1 \vee_C \widetilde{\mu}_2$ is also a stable matching on $(C_1, [H \setminus H_C(\mu)])$. Therefore, $\lambda = \mu_1 \vee_C \mu_2$ must be a semi-stable matching. Otherwise, there exists a blocking coalition $\{c_1, (h, \phi)\}$ of λ, where $c_1 \in C_1$. If $h \in [H \setminus H_C(\mu)]$, $\{c_1, (h, \phi)\}$ must be a blocking coalition of $\widetilde{\lambda}$, which contradicts that $\widetilde{\lambda}$ is a stable matching; if $h \in H_C(\mu)$, $\lambda(h) = \mu_1(h) = \mu_2(h)$. We have $\lambda(c_1) = \mu_1(c_1)$ or $\lambda(c_1) = \mu_2(c_1)$, and $\{c_1, (h, \phi)\}$ is a blocking coalition of λ implies $s_1 \succ_h \lambda(h)$ and $(h, \phi) \succ_{c_1} \lambda(c_1)$. Therefore, $\{c_1, (h, \phi)\}$ is a blocking coalition of μ_1 or μ_2, which contradicts that μ_1 and μ_2 are semi-stable matchings. Therefore, $\lambda = \mu_1 \vee_C \mu_2$ must be a semi-stable matching. Thus $\lambda = \mu_1 \vee_C \mu_2 \in K(\mu)$. Similarly, we can prove that $\upsilon = \mu_1 \wedge_C \mu_2 \in K(\mu)$. Thus, the theorem is proved. Q.E.D.

Remark: Suppose that there are m semi-stable matchings in $K(\mu)$. Then we can easily see that $\mu_C \equiv \mu_1 \vee_C \mu_2 \vee_C \cdots \vee_C \mu_m$ and $\mu_H \equiv \mu_1 \wedge_C \mu_2 \wedge_C \cdots \wedge_C \mu_m$ are respectively the best and worst semi-stable matching for the couples. Indeed, if $c \in C_2$ or $h \in H_C(\mu)$, then $\mu(c) = \mu_C(c) = \mu_H(c)$ and $\mu(h) = \mu_C(h) = \mu_H(h)$ for all $\mu \in K(\mu)$. For any $\alpha \in K(\mu)$, let $\widetilde{\alpha}$ denote a restricted matching on $(C_1, [H \setminus H_C(\mu)])$, where $\widetilde{\alpha}(c_1) = \alpha(c_1)$ for all $c_1 \in C_1$. Since α is a semi-stable matching, $\widetilde{\alpha}$ is a stable matching. By the conclusion of the marriage matching market (Knuth, 1976; Roth and Sotomayor, 1990), we have $\widetilde{\alpha} \geq_H \widetilde{\mu_C}$ and $\widetilde{\alpha} \leq_H \widetilde{\mu_H}$. Thus, $\alpha \geq_H \mu_C$ and $\alpha \leq_H \mu_H$ for all $\alpha \in K(\mu)$. Hence, μ_C and μ_H are respectively the worst and best semi-stable matching for the hospital jobs in $K(\mu)$.

Proof of Theorem 3.3: Let the matching obtained by the PI-Algorithm be μ_E. We show it must be a stable matching if every real couple c's first preference job pair (h, h') in P^c is a reservation preference job pair of c. Obviously, the PI-Algorithm implies that μ_E must be an individually rational matching. We first show $\mu_E(c) = (h, h')$ for any $c \in C$ with $s' \neq \phi$. Indeed, at round 1 of the PI-Algorithm, the first job pair (h, h') in P^c is c's reservation preference job pair. If $h \neq \phi$, then $(s, h) \in X(T)$ and $(s, h) \in Ch_H(X(T))$; if $h' \neq \phi$, then $(s', h') \in X(T)$ and $(s', h') \in Ch_H(X(T))$. Thus, (h, h') is selectable for c, and it is the first item of P^c. Therefore, $\mu_1(c) = (h, h')$. By Lemma 4.2, $\mu_E(c) \succ_c \mu_1(c) = (h, h')$ or $\mu_E(c) = \mu_1(c)$. As (h, h') is the first item of P^c, $(h, h') \succ_c \mu_E(c)$ or $(h, h') = \mu_E(c)$. As such, we must have $\mu_E(c) = (h, h')$.

We then show that μ_E is stable. If not, there exists at least one blocking coalition $\{c_1, (h_1, h_1')\}$ with $c_1 = (s_1, s_1')$. If $s_1' \neq \phi$, $(h_1, h_1') \succ_c \mu_E(c_1)$, which contradicts that $\mu_E(c_1)$ equals to the first preference of c_1. If $s_1' = \phi$, then by Lemma 6.2, there exists $\bar{c} = (\bar{s}, \bar{s}')$ with $\bar{s}' \neq \phi$ and $\bar{h} \in H$ (or $\tilde{c} = (\tilde{s}, \tilde{s}')$ with $\tilde{s} \neq \phi$ and $\tilde{h} \in H$) such that $(h_1, \bar{h}) \succ_{\bar{c}} \mu_E(\bar{c})$ (or $(\tilde{h}, h_1) \succ_{\tilde{c}} \mu_E(\tilde{c})$), which also contradicts that $\mu_E(\bar{c})$ (or $\mu_E(\tilde{c})$) equals to the first preference of $\bar{c}$ (or $\tilde{c}$). Thus, μ_E must be stable. Q.E.D.

Proof of Theorem 3.4: By Theorem 3.1, for any $k = 1, 2, \ldots$, there is a semi-stable matching μ_E^k in market Γ^k. By the definition of semi-stable matching, any blocking coalition $\{c, (h, h')\}$ of μ_E^k contains a real couple $c = (s, s')$, and (h, h') must be its effective preference. By Condition 2 for simple regularity, the number of blocking coalitions of μ_E^k must be less than $m_k \cdot \gamma (\ln n_k)^\lambda$. For the null matching μ^0, the number of its blocking coalitions is $N = \sum_{c \in C^k} l_c$. By Condition 3 for simple regularity, we have $N \geq n_k$. Thus, the instability degree is $\theta(\mu_E^k) = m/N \leq m_k \cdot \gamma (\ln n_k)^\lambda / n_k$. By Condition 1 for simple regularity, $m_k \leq n_k^{1-\epsilon}$, and we have $\theta(\mu_E^k) \leq \gamma (\ln n_k)^\lambda / n_k^\epsilon$. Thus $\lim_{n_k \to \infty} \theta(\mu_E^k) = 0$, that is, as n_k tends to infinity, the sequence of instability degree $\{\theta(\mu_E^k)\}_{k=1}^\infty$ tends to zero. Therefore, the matching sequence $\{\mu_E^k\}_{k=1}^\infty$ is asymptotically stable, that is, $\{\Gamma^k\}_{k=1}^\infty$ is asymptotically stable. Q.E.D.

Proof of Theorem 3.5: Let the semi-stable matching obtained by the algorithm described in Section IV be μ_E. We need to show that μ_E is a stable matching. It is clear that the PI-Algorithm indicates that μ_E must be individually rational.

We first show that for any $c \in C$ with $s' \neq \phi$, $\mu_E(c) = (h, h')$. Indeed, at round 1 of the PI-Algorithm, (h, h') is c's reservation preference job pair, which implies that if $h \neq \phi$, then

$(s, h) \in X(T)$ and $(s, h) \in Ch_H(X(T))$; if $h' \neq \phi$, then $(s', h') \in X(T)$ and $(s', h') \in Ch_H(X(T))$. Thus, (h, h') is selectable for c at round 1, and $\mu_1(c) \succ_c (h, h')$ or $\mu_1(c) = (h, h')$.

We now show $\mu_1(c) = (h, h')$. Suppose, by way of contradiction, that $\mu_1(c) \neq (h, h')$. Then we must have $\mu_1(c) = (\bar{h}, \bar{h}') \succ_c (h, h')$. Thus, $(\bar{h}, \phi) \neq (\phi, \phi)$ or $(\bar{h}', \phi) \neq (\phi, \phi)$ implies that it is not only a reservation preference term but also the first preference term of a single. Therefore, if $\bar{h} \neq \phi$, there exists $\bar{c} = (\bar{s}, \phi)$ such that $(\bar{s}, \bar{h}) \in X(T)$ and $(\bar{s}, \bar{h}) \in Ch_H(X(T))$, and $\mu_1(\bar{c}) = (\bar{h}, \phi)$ because $(\bar{h}, \phi)$ is the first preference term of $\bar{c}$; if $\bar{h}' \neq \phi$, there exists $\tilde{c} = (\tilde{s}, \phi)$ such that $(\tilde{s}, \bar{h}') \in X(T)$ and $(\tilde{s}, \bar{h}') \in Ch_H(X(T))$, and $\mu_1(\tilde{c}) = (\bar{h}', \phi)$ because $(\bar{h}', \phi)$ is the first preference term of $\tilde{c}$. This contradicts $\mu_1(c) = (\bar{h}, \bar{h}')$. Hence $\mu_1(c) = (h, h')$.

For any preference $(\bar{h}, \bar{h}')$ of c before (h, h'), if $\bar{h} \neq \phi$, there exists $\bar{c} = (\bar{s}, \phi)$ such that $\mu_1(\bar{c}) = (\bar{h}, \phi)$. The process of the PI-Algorithm for finding semi-stable matching μ_E implies that $\mu_E(c) \succ_c \mu_1(c)$ for any $c \in C$. However, since $(\bar{h}, \phi)$ is the first preference term of $\bar{c}$, $\mu_E(\bar{c}) = (\bar{h}, \phi)$. Likewise, if $\bar{h}' \neq \phi$, there exists $\tilde{c} = (\tilde{s}, \phi)$ such that $\mu_E(\tilde{c}) = (\bar{h}', \phi)$. Thus $\mu_E(c) = \mu_1(c) = (h, h')$ by noting that it cannot be matched to an object which is better than (h, h') for c.

Now, if the semi-stable matching μ_E is not a stable matching, there exists at least one blocking coalition $\{c_1, (h_1, h_1')\}$ with $s_1' \neq \phi$. As such, we have $(h_1, h_1') \succ_c \mu_E(c_1)$. Moreover, $s_1 \succ_{h_1} \mu_E(h_1)$ if $h_1 \neq \phi$, and $s_1' \succ_{h_1'} \mu_E(h_1')$ if $h_1' \neq \phi$. However, as shown above, if $h_1 \neq \phi$, then there exists $\bar{c}_1 = (\bar{s}_1, \phi)$ such that $\mu_E(\bar{c}_1) = (h_1, \phi)$, which contradicts $s_1 \succ_{h_1} \mu_E(h_1) = \bar{s}_1$ by noting that (h_1, ϕ) is a reservation preference term of $\bar{c}_1$ implies that $\bar{s}_1 \succ_{h_1} s_1$; if $h_1' \neq \phi$, then there exists $\tilde{c}_1 = (\tilde{s}_1, \phi)$ such that $\mu_1(\tilde{c}_1) = (h_1', \phi)$, which contradicts $s_1' \succ_{h_1'} \mu_E(h_1') = \tilde{s}_1$ by noting that (h_1', ϕ) is a reservation preference term of $\tilde{c}_1$ implies that $\tilde{s}_1 \succ_{h_1'} s_1'$. Thus μ_E must be a stable matching. Q.E.D.

Proof of Theorem 4.1: It is clear that the PI-Algorithm implies the matching μ_E is an individually rational matching. Suppose, by way of contradiction, that μ_E is not a stable matching. Then there exists a blocking coalition $\{(s, s'), (h, h')\}$ with $\mu_E(h) \neq \phi$ and $\mu_E(h') \neq \phi$. However, by Lemma 6.1, $\mu_E(h) = \phi = \mu_E(h')$ when $h \neq \phi$, $\mu_E(h) \neq s$, $h' \neq \phi$, $\mu_E(h') \neq s'$. As such, $\mu_E(h) \neq \phi$ and $\mu_E(h') \neq \phi$ imply $\mu_E(h) = s$ and $\mu_E(h') = s'$, which contradicts that $\{(s, s'), (h, h')\}$ is a blocking coalition of μ_E. Q.E.D.

Proof of Theorem 4.2: Again, the PI-Algorithm clearly implies that the matching μ_E must be an individually rational matching. Suppose that μ_E is not a stable matching, there

exists at least one blocking coalition $\{(s, s'), (h, \phi)\}$ of μ_E. By Lemma 6.2, there exists $\bar{c} = (\bar{s}, \bar{s}')$ with $\bar{s}' \neq \phi$ and $\bar{h} \in H$ (or $\tilde{c} = (\tilde{s}, \tilde{s}')$ with $\tilde{s} \neq \phi$ and $\tilde{h} \in H$) such that $(h, \bar{h}) \succ_{\bar{c}} \mu_E(\bar{c})$ (or $(\tilde{h}, h) \succ_{\tilde{c}} \mu_E(\tilde{c})$), which contradicts that Γ has no real couples. Q.E.D.

Proof of Theorem 4.3: For any singles market $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$, the subset $K(\mu_S^E)$ of the set of semi-stable matchings is identical to the set of stable matchings. By Theorem 5.2, the matching μ_S^E is the optimal stable matching for the side of hospital jobs. The optimal theorem of marriage matching markets (Roth, 1982a) implies that μ_H is the optimal stable matching for the side of hospital jobs. Since the optimal matching is unique, we have $\mu_S^E = \mu_H$. For any singles market, μ_H^E and μ_S^E are logically symmetrical. Likewise, we have $\mu_H^E = \mu_S$. Q.E.D.

Proof of Theorem 5.1: Since matched object of each real couple is the same for all semi-stable matchings in $K(\mu)$, the set of unmatched real couples is the same at every semi-stable matching in $K(\mu)$. After excluding all real couples and their matched objects, the semi-stable matching is actually a stable matching for singles and all remaining jobs of hospitals. By McVitie and Wilson theorem, the set of unmatched single medical students and remaining jobs of hospitals is the same for every stable matching. Q.E.D.

Proof of Theorem 5.2: To prove the theorem, we need the following lemma:

Lemma 6.3 *Let* $\Gamma = (H, S, C, (\succ_h)_{h \in H}, (\succ_c)_{c \in C})$ *be a couples market with strict preferences, and the outcome obtained by the PI-Algorithm mechanism be* μ^E. *Suppose that for every* $\mu \in K(\mu^E)$, *there exists* $h \in H$ *such that* $s = \mu(h) \succ_h \mu^E(h)$. *Then s must be a single,* $h_1 = \mu^E(s) \neq \phi$, *and* $\mu(h_1) \succ_{h_1} \mu^E(h_1)$.

Proof: Since μ and μ^E are semi-stable matchings, they are both individually rational, and $\mu^E(h) \succ_h \phi$ or $\mu^E(h) = \phi$. Since $s = \mu(h) \succ_h \mu^E(h)$, $s = \mu(h) \succ_h \phi$, which implies $\mu(s) = h \neq \phi$. By Theorem 5.1, the set of unmatched medical students is the same for every semi-stable matching in $K(\mu^E)$, which implies that the set of matched medical students is also the same for every semi-stable matching in $K(\mu^E)$. Thus, $h_1 = \mu^E(\mu(h)) = \mu^E(s) \neq \phi$, $s = \mu^E(h_1)$. Let $s_1 = \mu(h_1)$, then $\mu(h) \succ_h \mu^E(h)$ implies $s = \mu(h) \neq \mu^E(h)$. Thus $h_1 = \mu^E(s) \neq h$, and $s_1 = \mu(h_1) \neq \mu(h) = s$. We show s must be a single. If not, s is a member of a real couple c, then $\mu(c) = \mu^E(c)$ because $\mu \in K(\mu^E)$. Thus $h = \mu(s) = \mu^E(s) = h_1$, contradicting $h_1 \neq h$. Next, we show that $h_1 \succ_s h = \mu(s)$.

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Since μ and μ^E are individually rational and $h_1 \neq \phi \neq h$, we have $h_1 = \mu^E(s) \succ_s \phi$ and $h = \mu(s) \succ_s \phi$. Thus, at step 1 of round 1 in the process of the PI-Algorithm, we have $(s, h) \in X(1)$ and $(s, h_1) \in X(1)$. Noting that the PI-Algorithm ends at round T and letting $\bar{s} = \mu^E(h)$, we have $(\bar{s}, h) \in X(T)$ and $(\bar{s}, h) \in Ch_H(X(T))$. Since $s = \mu(h) \succ_h \mu^E(h) = \bar{s}$, we have $(s, h) \notin X(T)$; otherwise, it contradicts $(\bar{s}, h) \in Ch_H(X(T))$. Since $(s, h) \notin X(T)$, by Lemma 4.1 ,we have $h_1 = \mu^E(s) = \mu_{T-1}(s) \succ_s h$, where μ_{T-1} is the temporary matching obtained at round T-1.

If $s \succ_{h_1} s_1 = \mu(h_1)$, as $h_1 \succ_s h = \mu(s)$, $\{(s, \phi), (h_1, \phi)\}$ constitutes a blocking coalition of matching μ, which contradicts that μ is a semi-stable matching. Hence $\mu(h_1) = s_1 \succ_{h_1} s = \mu^E(h_1)$. Q.E.D.

Now we begin to prove the theorem. Suppose, by way of contradiction, that μ^E is not weakly Pareto efficient on $K(\mu^E)$ for the side of hospital jobs, then there exists a matching $\mu \in K(\mu^E)$ such that for all $h \in H$, $\mu(h) \succ_h \mu^E(h)$ or $\mu(h) = \mu^E(h)$, and there exists an $h \in H$ such that $\mu(h) \succ_h \mu^E(h)$. Let $s_1 = \mu(h)$ and $h_1 = \mu^E(s_1)$. By Lemma 6.3, s_1 must be a single, $h_1 \neq \phi$ and $\mu(h_1) \succ_{h_1} \mu^E(h_1)$. Repeatedly applying Lemma 6.3, we can obtain two sequences, one is the sequence of medical students $\{s_1, s_2, \cdots\}$, and the other is the sequence of hospital jobs $\{h_1, h_2, \ldots, \}$, where $s_{k+1} = \mu(h_k)$ and $h_k = \mu^E(s_k)$ for any $k > 0$. Due to the limited number of medical students, there exists the least k and r such that $s_{k+r} = s_k$. Thus $k = 1$; otherwise $h_{k+r-1} = \mu(s_k) = h_{k-1}$ and $s_{k+r-1} = \mu^E(h_{k-1}) = s_{k-1}$, which contradicts that k is the least. Therefore, $k = 1$, that is, $s_1 = s_{r+1}$ and $h_0 = h_r$. It is easy to deduce that $s_l = s_{l+r}$ and $h_l = h_{l+r}$ for any $l > 0$. Thus, we obtain twin circulating sequences: the sequence of medical students $\{s_1, s_2, \cdots, s_r\}$ and the sequence of hospital jobs $\{h_1, h_2, \cdots, h_r\}$, such that $s_{k+1} = \mu(h_k)$, $h_k = \mu^E(s_k)$ and $s_{k+1} \succ_{h_k} s_k$ for any $k > 0$. We now show that there are no such twin circulating sequences. As a result, the theorem is proved.

Denote by $\overline{H} \subseteq H$ the set of all jobs satisfying the condition $\mu(h) \succ_h \mu^E(h)$. We consider two cases.

Case 1: $\overline{H} = \{h_1, h_2, \cdots, h_r\}$. For any $2 \leq k \leq r+1$, μ and μ^E are both semi-stable matchings, which implies that they are both individually rational, and thus $h_k = \mu^E(s_k) \succ_{s_k} \phi$ and $h_{k-1} = \mu^E(s_{k-1}) \succ_{s_{k-1}} \phi$. At the step 1 of round 1 in the process of the PI-Algorithm, $(s_k, h_k) \in X(1)$ and $(s_{k-1}, h_{k-1}) \in X(1)$. Suppose the PI-Algorithm ends at round T and gives the temporary matching μ_t at round t, since $h_{k-1} = \mu^E(s_{k-1}) = \mu_T(s_{k-1})$, we have $(s_{k-1}, h_{k-1}) \in X(T)$ and $(s_{k-1}, h_{k-1}) \in Ch_H(X(T))$. Also, since $s_k = \mu(h_{k-1}) \succ_{h_{k-1}} \mu^E(h_{k-1}) = s_{k-1}$,

$(s_k, h_{k-1}) \notin X(T)$ (Otherwise, it contradicts $(s_{k-1}, h_{k-1}) \in Ch_H(X(T))$.). Thus, $(s_k, h_{k-1}) \in X(1)$ and $(s_k, h_{k-1}) \notin X(T)$ imply that there exists $1 < t_k < T$ such that $(s_k, h_{k-1}) \in X(t_k)$ and $(s_k, h_{k-1}) \notin X(t_k + 1)$. Without loss of generality, suppose t_k is the least one. Then $\overline{h} \equiv \mu_{t_k}(s_k) \succ_{s_k} h_{k-1} = \mu(s_k)$. Thus we must have $\overline{h} \notin \overline{H}$. Indeed, suppose not. Then $\mu_{t_k}(s_k) = \overline{h} = h_j = \mu^E(s_j)$. As such, $(s_j, h_j) \in Ch_H(X(t_k))$ and $(s_{j+1}, h_j) \notin X(t_k)$ by noting that $s_{j+1} = \mu(h_j) \succ_{h_j} \mu^E(h_j) = s_j$. Also, $(s_{j+1}, h_j) \in X(1)$ by noting that $s_{j+1} \succ_{h_j} s_j \succ_{h_j} \phi$. Thus, there exists t_j such that $(s_{j+1}, h_j) \in X(t_j)$ and $(s_{j+1}, h_j) \notin X(t_j + 1)$. We then have $t_j < t_k$ because $(s_{j+1}, h_j) \in X(t_j)$ and $(s_{j+1}, h_j) \notin X(t_k)$, which contradicts the hypothesis that t_k is the least. So $\overline{h} \notin \overline{H}$.

Let $\overline{s} = \mu(\overline{h})$. Then $\overline{s} \neq \phi$. Suppose not. Since $\overline{h} = \mu_{t_k}(s_k) \succ_{s_k} h_{k-1} = \mu(s_k)$, $s_k \succ_{\overline{h}} \phi$ and s_k is a single, $\{(s_k, \phi), (\overline{h}, \phi)\}$ forms a blocking coalition of matching μ, which contradicts the fact that μ is a semi-stable matching. So we must have $\overline{s} \neq \phi$. Then, by Theorem 5.1, we have $\widetilde{s} \equiv \mu^E(\overline{h}) \neq \phi$. Therefore, $\overline{s}$, $\widetilde{s}$ and s_k are all acceptable medical students for $\overline{h}$. Since $\overline{h}$ has chosen s_k before the final matching $\widetilde{s}$, we have $s_k \succ_{\overline{h}} \widetilde{s}$. If $\widetilde{s} \succ_{\overline{h}} \overline{s}$ or $\widetilde{s} = \overline{s}$, then $s_k \succ_{\overline{h}} \overline{s} = \mu(\overline{h})$. Consequently, $\{(s_k, \phi), (\overline{h}, \phi)\}$ is a blocking coalition of the matching μ, which contradicts that μ is a semi-stable matching. Thus $\overline{s} \succ_{\overline{h}} \widetilde{s}$, that is, $\mu(\overline{h}) \succ_{\overline{h}} \mu^E(\overline{h})$, and $\overline{h} \in \overline{H}$. However, this contradicts $\overline{h} \notin \overline{H}$. Hence μ^E must be an optimal semi-stable matching in $K(\mu^E)$ for the side of hospital jobs when $\overline{H} = \{h_1, h_2, \cdots, h_r\}$.

Case 2: $\overline{H} \neq \{h_1, h_2, \cdots, h_r\}$. By repeating the above proof, we can obtain the circulating sequence of medical students $\{s_1, s_2, \cdots, s_r\}$ and the circulating sequence of hospital jobs $\{h_1, h_2, \cdots, h_r\}$ such that $s_{k+1} = \mu(h_k)$, $h_k = \mu^E(s_k)$ and $s_{k+1} \succ_{h_k} s_k$ for any $k > 0$. The prerequisite of twin circulating sequences is that there exists at least one $\overline{h} \notin \{h_1, h_2, \cdots, h_r\}$ such that $\mu(\overline{h}) \succ_{\overline{h}} \mu^E(\overline{h})$. Applying Lemma 6.3 again, we obtain another twin circulating sequences satisfying the same conditions. By repeating the proving process in Case 1, the prerequisite of this new twin circulating sequences is another new twin circulating sequences, and it similarly shows that these different twin circulating sequences do not intersect with each other. The same argument can be repeated endlessly, thus we can obtain infinite twin circulating sequences. Since H is a finite set, all these twin circulating sequences are bound to constitute a causal circle, that is, the prerequisite of the first twin circulating sequences is the existence of the second twin circulating sequences, and the prerequisite of the second twin circulating sequences is the existence of the third twin circulating sequences, and so on, which leads to a contradiction. Thus, μ^E must be a weakly Pareto efficient semi-stable matching in $K(\mu^E)$ for the side of hospital jobs when $\overline{H} \neq \{h_1, h_2, \cdots, h_r\}$. Q.E.D.

Proof of Theorem 5.3: To prove the theorem, it is sufficient to demonstrate some matching markets in which truth-telling is not the best response for some agent even though all others state their preferences truthfully.

Consider a matching market with one couple $c_1 = (s_1, s_2)$ and one single $c_2 = (s_3, \phi)$, and four jobs of hospitals. Their rank lists of preferences are as follows:

$P^{c_1} : \{(h_1, h_2), (h_2, h_1), (h_3, \phi), (\phi, \phi)\}$; $P^{c_2} : \{(h_1, \phi), (h_4, \phi), (h_2, \phi), (\phi, \phi)\}$;

$P^{h_1} : \{s_2, s_1, s_3, \phi\}$; $P^{h_2} : \{s_3, s_1, s_2, \phi\}$; $P^{h_3} : \{s_1, \phi\}$; $P^{h_4} : \{s_3, \phi\}$.

After processing PI-A1 in the PI-Algorithm mechanism, we get a matching μ with $\mu(c_1) = (h_3, \phi)$ and $\mu(c_2) = (h_4, \phi)$. $\{c_2, (h_1, \phi)\}$ is the only blocking coalition containing a single. Thus, after deleting the preference containing h_1 in P^{c_1}, in the process of PI-A2, we get a semi-stable matching μ_1 with $\mu_1(c_1) = (h_3, \phi)$ and $\mu_1(c_2) = (h_1, \phi)$ by continuing the PI-Algorithm.

Suppose that all others state their true preferences. Then, c_1 can be better off by manipulating their own preferences through reporting $P^{'c_1} : \{(h_1, h_2), (\phi, \phi)\}$. As such, we obtain another semi-stable matching μ_2 with $\mu_2(c_1) = (h_1, h_2)$ and $\mu_2(c_2) = (h_4, \phi)$ by the PI-Algorithm mechanism, and thus $\mu_2(c_1) = (h_1, h_2) \succ_{c_1} (h_3, \phi) = \mu_1(c_1)$. Therefore, the PI-Algorithm mechanism is not strategy-proof.

We can also provide an example of market in which a job has incentive not to state its true preferences. To see this, consider another matching market with two couples $c_1 = (s_1, s_2)$ and $c_2 = (s_3, s_4)$, and four jobs of hospitals. Their rank lists of preferences are as follows:

$P^{c_1} : \{(h_1, h_2), (h_3, \phi), (\phi, \phi)\}$; $P^{c_2} : \{(h_1, h_2), (h_4, \phi), (\phi, \phi)\}$;

$P^{h_1} : \{s_1, s_3, \phi\}$; $P^{h_2} : \{s_4, s_2, \phi\}$; $P^{h_3} : \{s_1, \phi\}$; $P^{h_4} : \{s_3, \phi\}$.

After processing PI-A1 in the PI-Algorithm mechanism, we get a semi-stable matching μ_1 with $\mu_1(c_1) = (h_3, \phi)$ and $\mu_1(c_2) = (h_4, \phi)$.

Suppose that all others state their true preferences. h_1 then can be better off by manipulating its own preferences through reporting $P^{'h_1} : \{s_3, \phi\}$. After processing PI-A1 in the PI-Algorithm mechanism, we get another semi-stable matching μ_2 with $\mu_2(c_1) = (h_3, \phi)$ and $\mu_2(c_2) = (h_1, h_2)$. $\mu_2(h_1) = s_3 \succ_{h_1} \phi = \mu_1(h_1)$. Again, this shows the PI-Algorithm mechanism is not strategy-proof. Q.E.D.

Proof of Theorem 5.4: Since matched objects of real couples are the same for every semi-stable matching in $K(\mu^E)$, the welfare is unchanged for all real couples and their matched jobs. Thus, truth-telling is a dominant strategy for all real couples and their matched jobs. After

excluding all real couples and their matched objects, the semi-stable matching is actually a stable matching for all singles and remaining hospital jobs. By Theorem 4.3, for any matching market containing only singles with strict preferences, the matching obtained by the PI-Algorithm is identical to μ_H which is obtained by jobs optimal Gale-Shapley algorithm. By the dominant strategy theorem on marriage matching markets in Dubins and Freedman (1981) and Roth (1982a), telling the truth is a dominant strategy for each remaining job of hospitals. Q.E.D.

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