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15 August 2014

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August 15, 2014

Keywords: Hyperbolic Discounting, naivety, bargaining

JEL Codes: C72, C78, D03, D91

Abstract

We propose a solution concept for games that are played among hyperbolic discounters that are possibly naive about their own, or about their opponent's future time inconsistency. Our perception-perfect equilibrium essentially requires each player to take an action consistent with the subgame perfect equilibrium, given her perceptions concerning future types, and under the assumption that other present and future players have the same perceptions. Applications include a common pool problem and Rubinstein bargaining. When players are naive about their own time consistency and sophisticated about their opponent's, the common pool problem is exacerbated, and Rubinstein bargaining breaks down completely.

1 Introduction

Time-inconsistent present-biased preferences are among the most prominent and persistent behavioral biases in economics. For example, most people would prefer to do an unpleasant task on May 1 rather than on May 15 when faced with that choice on April 1. But on May 1, almost everyone will be inclined to postpone it to May

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15. This type of time inconsistency (often also referred to as hyperbolic discounting) has been put forward as an explanation of savings behavior and, more generally, to explain why economic agents would choose to use commitment devices to restrict their future selves.[1] O'Donoghue and Rabin (1999) provide a model for behavior with such present-biased preferences. In their model, an individual decision-maker can be time-consistent, or she can have present-biased preferences. Importantly, she can either be sophisticated concerning her time inconsistency, or she can be naive. A sophisticated individual knows that she will have present-based preferences in the future, and hence will have an incentive today to restrict the choices of that future self. If she is naive, then she believes that although her current self has present-biased preferences, her future self will behave in a time-consistent manner.

Many situations that are of interest to economists, however, concern the interaction between economic agents. Consider for example the case in which two individuals, A and B, bargain over the distribution of some future payoff. As in the simple one-person model. Player A's behavior will depend on whether she is time inconsistent and, if so, whether she is naive or sophisticated about that. However, her behavior will also depend on whether she perceives player B to be time-consistent or not, and whether she believes player B is naive or sophisticated. It may even depend on her perceptions concerning player B's perceptions about player A. Where the one-person model implies a game played between two players (the current and future self), a two-person model effectively implies a game played between four players (both A and B's current and future self).

In this paper, we study such games. We introduce an equilibrium concept for games played between possibly time inconsistent players. As a starting point, we take O'Donoghue and Rabin (1999). They consider a one-player game played by a current self against her future self. In their model, players have to decide whether to do a task now, or to do it later. The authors introduce the concept of a perception-perfect strategy, which essentially is a course of action that maximizes the current player's utility given her perception about the type of her future self, and the behavior she rationally expects from such a type. Possible types then refer to whether the future self will be time-consistent or time-inconsistent. We first extend the analysis to one-person games with a richer strategy space, both in the two-period case as well as in

[1]For a survey, see e.g. Frederick et al., 2002

a set-up with more periods. We introduce a perception-perfect equilibrium, which is an extension of O'Donoghue and Rabin (1999)'s perception perfect strategy, that can also be applied to a multi-player set-up. We then analyze games with two players. We apply our equilibrium concept to a common pool problem, and to a model of Rubinstein bargaining.

When we allow players to have perceptions concerning the type of their competitors, higher-order beliefs are going to play a crucial role. Behavior will depend not only on player A's perception about player B, but also on player A's perception about player B's perception about player A, etcetera. To deal with this complication we impose, first, that players assume that future incarnations of themselves have the same perceptions as their current self has. We coin this *intraplayer perception naivety.* Thus, if at time $t = 1$ player A perceives that she will be time consistent at time $t = 3$, then she will also perceive that her future self at $t = 2$ perceives her incarnation at $t = 3$ to be time consistent. We believe this to be innocuous. In fact, the same assumption is also implicitly made in O'Donoghue and Rabin (1999). Second, we impose that players assume that *other* players have the same perceptions as they themselves have. We coin this *interplayer perception naivety.* Thus, if at time $t = 1$ player A perceives that she will be time consistent at time $t = 3$, then she will also perceive that player B perceives that A will be time consistent at time $t = 3$. Admittedly, this assumption is somewhat stronger than intraplayer perception consistency. Essentially, it requires that players are unable to imagine, or to think through the implications of the possibility that the other players has different perceptions about herself than she herself has. Yet, in our world of naive players, we feel it makes sense. Moreover, it greatly simplifies the analysis.

Our concept of perception perfect equilibrium then entails the following. Consider player A. She has certain perceptions about her own future type, and about the future type of the other player. Given those perceptions, and under the assumption that all other present and future players have the same perceptions, we can derive the subgame perfect equilibrium that player A perceives to be played. We call this the equilibrium as perceived by player A. Similarly, we can derive the equilibrium as perceived by player B. The perception perfect equilibrium in period $t = 1$ then consists of an action taken by player A that is consistent with an equilibrium as perceived by A, and an action taken by player B that is consistent with an equilibrium as perceived by B. In all later periods the same is true, but given the actions that were played in the past.

From our two main applications, the common pool problem and Rubinstein bargaining, we derive the following insights. First, suppose that players are naive about their own future selves, but are sophisticated about the future self of others. This is consistent with a lot of psychological evidence, as e.g. Kahneman (2011) argues. We then find that the common pool problem becomes much worse than in a standard world with rational actors. This can be seen as follows. Suppose player A perceives B to be time inconsistent in the future. That implies that B will be impatient then, and hence claim a large share of the common pool. Given that that is the case, A has an incentive to preempt B and claim a large share today. But the same holds for B. As a result, both players claim a large share of the pool today, completely exhausting it. We show that this effect is even stronger than in a case where both players know their future selves to also be time inconsistent.

In the case of Rubinstein bargaining, we show that the assumption that players are naive about their own future selves, but are sophisticated about the future self of others, implies a breakdown in bargaining. Suppose that it is player A's turn to make an offer. She will base that offer on the assumption that B will be impatient in the future. Yet B perceives herself to be patient in the future, and hence turns down A's offer. This process will continue indefinitely.

We are neither the first to develop approaches to solve games with possibly naive hyperbolic discounters, nor are we the only scholars to solve Rubinstein bargaining with such players. In an unpublished working paper, Sarafidis (2006) proposes "naive backward induction" with possibly naive hyperbolic discounters. He applies the concept to Rubinstein bargaining and shows that it yields similar results provided that both players are sophisticated. His naive players are similar ours, but his sophisticated players know everything, including the perceptions of the naive players. Hence players are either naive about both players, or sophisticated about both. Akin (2007) shows that the bargaining process breaks down if naive bargainers meet, but in his definition of naivety players are sophisticated about the time inconsistency of their opponents. Compared to the existing literature, our approach thus yields a framework that is more consistent and more flexible concerning the types and perceptions of players. Other related literature includes Akin (2009), in which a naive player plays against a sophisticated player but learns about her naivety in the course of play, and Chade et al. (2008) who analyze repeated games between sophisticated hyperbolic discounters.

The remainder of this paper is structured as follows. Section 2 looks at the case of one person. We first look at the case of a two-period model, and generalize the equilibrium concept introduced by O'Donoghue and Rabin (1999). Section 4 further generalizes to a multi-period model, and gives examples in the context of intertemporal consumption decisions. We then extend the analysis to a two-person game, and introduce the concept of a perception-perfect equilibrium. We do so for the two-period case in section 6, and apply our equilibrium concept to a common pool problem in section 7. Section 8 looks at a multi-period model, and section 9 applies our analysis to Rubinstein bargaining. Section 10 concludes.

2 The one-player case: two periods

In this section, we consider the simplest set-up. Suppose that one player has to make decisions at times $t = 1$ and $t = 2$. Yet, the player may have intertemporal preferences that are time-inconsistent. Moreover, she may not be aware that her future self (i.e. the one that makes the decision at $t = 2$) may also be time-inconsistent. The problem of the current self then is what action to take at $t = 1$, taking into account her perceptions concerning the preferences of the future self.

Throughout this paper we consider the following preferences. Let u_t be a person's *instantaneous utility* or *felicity* in period t. In a model with T periods, we let $U_t\left(u_t, u_{t+1}, \ldots, u_T; \beta^i\right)$ represent a person's intertemporal preferences, where β^i is a parameter. We assume

$$U_t\left(u_t, u_{t+1}, \ldots, u_T; \beta^i\right) \equiv u_t + \beta^i \sum_{\tau=t+1}^{T} \delta^\tau u_\tau \tag{1}$$

with $0 < \beta^i, \delta \leq 1$. Note that with $\beta^i = 1$, equation (1) collapses into the standard exponential discounting function with discount factor δ. With $\beta^i < 1$, we have the canonical model of hyperbolic discounting introduced by Phelps and Pollak (1968). In that case, the person has present-biased preferences, where β^i represents the bias for the present. In other words, she is time inconsistent.

In this context, consider a one-player game with 3 periods, $t = 1, 2, 3$, in which individual A makes two sequential decisions at $t = 1$ and $t = 2$. Therefore, we refer to this as a two-period model. In $t = 1$, she chooses action $a_1 \in \mathcal{A}_1$, with $\mathcal{A}_1$ the set of feasible actions that the current self has. In $t = 2$, she chooses action $a_2 \in \mathcal{A}_2(a_1)$, with $\mathcal{A}_2(a_1)$ the set of feasible actions available at $t = 2$, that may depend on a_1. Her felicity in period 1 will depend on her action in period 1; that in periods 2 and 3 will depend on all actions. Thus $u_1^A = u_1^A(a_1)$, while $u_2^A = u_2^A(a_1, a_2)$ and $u_3^A = u_3^A(a_1, a_2)$.

The present-bias of the current self (that at $t = 1$) is denoted β^A. Following O'Donoghue and Rabin (1999), we allow for two possibilities: she is either a hyperbolic discounter that has $\beta^A = \beta$, where $\beta < 1$ is some exogenously given fixed value, or she is time-consistent and has $\beta^A = 1$. For ease of discussion, we denote the true present-bias of the *future* self (i.e. that at $t = 2$) as γ^A, where we also assume that $\gamma^A \in \{\beta, 1\}$. Using (1) A's lifetime utility at both dates is thus given by

$$U_1^A\left(a_1, a_2; \beta^A\right) = u_1^A(a_1) + \beta^A \delta u_2^A\left(a_1, a_2\right) + \beta^A \delta^2 u_3^A\left(a_1, a_2\right) \tag{2}$$

$$U_2^A\left(a_1, a_2; \gamma^A\right) = u_2^A(a_1, a_2) + \gamma^A \delta u_3^A\left(a_1, a_2\right) \tag{3}$$

where we have now written utilities as functions of actions.

Following Strotz (1956) and Pollak (1968), we allow A either to be sophisticated (knowing exactly what her future preferences will be), or to be naive (believing that her future biases will be identical to her current ones).[2] First, suppose that $\beta^A = 1$. In that case, she must believe that $\gamma^A = 1$ as well. It makes no sense for the current self to believe that she will be a hyperbolic discounter in the future if that is not the case today. Second, suppose that $\beta^A = \beta$. In that case, the current self is time-inconsistent. By construction, a player that is a hyperbolic discounter will not only be so today, but also at any point in the future. Yet, she may not be aware of that. Naive time-inconsistent players know that they have a present-bias today, but do not realize that they also have such a bias in the future. Such a naive player will assume that $\gamma^A = 1$. Sophisticated time-inconsistent players know that they will also have a present-bias in the future, and will assume that $\gamma^A = \beta$.

[2] Arguably, in reality, people are likely to be partly naive and partly sophisticated. Our set-up is flexible enough to allow for this.

We denote by $\mu^A(\gamma)$ the individual's belief that she has $\gamma^A = \gamma$ in the future, with $\gamma \in \{\beta, 1\}$ and $\mu^A(\beta) + \mu^A(1) = 1$. Hence, in her perception,[3] $\mu^A(\gamma) = \Pr\left(\gamma^A = \gamma | \beta^A = \beta\right)$. Thus, a naive player has $\mu^A(1) = 1$, a sophisticated player $\mu^A(\beta) = 1$. As noted, a player that has no present-bias today will also not have such a bias in the future. Thus $\beta^A = 1$ must imply $\mu^A(1) = 1$.

We now introduce a formal solution concept for this game. Note that the model we have is a generalization of O'Donoghue and Rabin (1999).[4] They define a *perception-perfect strategy* as one in which in all periods a person chooses the optimal action given her current preferences and her perceptions of future behavior. Define $\mu^{\mathbf{A}}$ as the vector of perceptions: $\mu^{\mathbf{A}} \equiv \left(\mu^A(\beta), \mu^A(1)\right)$. In our set-up, we then have the following:

Definition 1 *In the two-period one-person game, a perception-perfect strategy at $t = 1$ for a time-inconsistent player, given her perceptions $\mu^{\mathbf{A}}$, is a strategy profile (a_1^*, a_2^*) such that*

$$a_2^*(a_1; \mu^{\mathbf{A}}) \equiv \arg \max_{a_2 \in \mathcal{A}_2(a_1)} \sum_{\gamma \in \{\beta, 1\}} \mu^A(\gamma) U_2^A(a_1, a_2; \gamma), \forall a_1 \in \mathcal{A}_1; \tag{4}$$

$$a_1^*(\beta; \mu^{\mathbf{A}}) = \arg \max_{a_1 \in \mathcal{A}_1} U_1^A\left(a_1, a_2^*\left(a_1; \mu^{\mathbf{A}}\right); \beta\right) \tag{5}$$

Trivially, a perception-perfect strategy for a time-consistent player is a strategy profile (a_1^*, a_2^*) such that

$$a_2^*(a_1; (0,1)) = \arg \max_{a_2 \in \mathcal{A}_2(a_1)} U_2^A(a_1, a_2; 1)$$
$$a_1^*(1; (0,1)) = \arg \max_{a_1 \in \mathcal{A}_1} U_1^A(a_1, a_2^*(a_1; (0,1)); 1)$$

The perception-perfect strategy for the time-inconsistent player can be understood as follows. First, given a_1, the current self assumes that the future self is going to take

[3]In what follows, we use "perception" rather than "belief" to clearly differentiate from most of the literature where beliefs are rationally formed using Bayes' rule. That is clearly not the case here.

[4]In that paper, a possibly time-inconsistent player has to perform an action once, and has to choose some date in the future at which to perform that action. Yet, she has the possibility to renege on her plan in the future. Hence, if today she plans to do it tomorrow, when tomorrow comes she may decide to postpone the action for another day. A sophisticated player will foresee this future tendency, but a naive player will not.

the action that maximizes the future self's utility. In the current self's perception, with probability $\mu^A(\beta)$, the future self's utility is given by $U_2^A(a_1, a_2; \beta)$, while with probability $\mu^A(1)$, it is given by $U_2^A(a_1, a_2; 1)$. The maximizer is thus given by equation (4) and denoted $a_2^*(a_1; \mu^{\mathbf{A}})$. In period 1, given her perceptions, the current self's lifetime utility if she takes action a_1 today is given by $U_1^A\left(a_1, a_2^*\left(a_1; \mu^{\mathbf{A}}\right); \beta\right)$. The current self thus chooses a_1 to maximize this expression, hence (5). The perception-perfect strategy for the time-inconsistent player follows directly from backward induction.

Definition 2 *In the two-period one-person game, a perception-perfect equilibrium is a strategy profile (a_1^*, a_2^*) such that a_1^* is part of a perception-perfect strategy at period 1, while a_2^* maximizes the future self's utility at $t = 2$, given the action a_1^* that was taken in period 1.*

Note therefore that there is a crucial difference between a perception-perfect strategy and a perception-perfect equilibrium; a perception-perfect strategy is a strategy profile that a player *perceives* to be played, while a perception-perfect equilibrium is the strategy profile that actually *will* be played. There may be a difference between the two if the player is time-inconsistent and naive. This distinction will become even more important in the T-period case.

3 Example: intertemporal consumption, 2 periods

Consider a person that lives for 3 periods, and starts out with wealth 1 in period 1. Instantaneous utility in each period is given by $u_t^A(a_t) = \sqrt{a_t}$, with a_t consumption in period t. For simplicity, the discount factor δ equals 1. The standard model, with time-consistent preferences, would have the person maximizing

$$U_1^A(a_1, a_2) = \sqrt{a_1} + \sqrt{a_2} + \sqrt{1 - a_1 - a_2}$$

which would obviously result in $a_1^* = a_2^* = 1/3$. Note that this simple decision problem satisfies our set-up. This person has to make two decisions; the consumption decision a_1 and the consumption decision a_2, with $\mathcal{A}_1 = [0, 1]$ and $\mathcal{A}_2(a_1) = [0, 1 - a_1]$. In period 3, she consumes whatever is left of her initial wealth. Obviously, both the perception-perfect strategy and the perception-perfect equilibrium of a time-consistent player would be to have $a_1^* = a_2^* = 1/3$ as well.

We now solve for the perception-perfect strategy of the time-inconsistent player. Using (4), at $t = 2$, and given first-period consumption a_1 and future time inconsistency γ, the player will choose a_2 as to maximize

$$U_2^A\left(a_1, a_2; \gamma^A\right) = \sqrt{a_2} + \gamma^A \sqrt{1 - a_1 - a_2}.$$

This yields

$$a_2^*\left(a_1; \mu^{\mathbf{A}}\right) = \frac{1 - a_1}{1 + \left[\beta\mu^A(\beta) + \mu^A(1)\right]^2} = \frac{1 - a_1}{1 + \tilde{\beta}^2},$$

where, for ease of exposition, we write

$$\tilde{\beta} \equiv \beta\mu^A(\beta) + \mu^A(1). \tag{6}$$

Perceived consumption in the last period is then given by

$$a_3^*\left(a_1; \mu^{\mathbf{A}}\right) = \frac{\tilde{\beta}^2(1 - a_1)}{1 + \tilde{\beta}^2}.$$

Plugging this back into the lifetime utility of the current self yields

$$\begin{aligned} U_1^A\left(a_1, a_2^*\left(a_1; \mu^{\mathbf{A}}\right); \beta\right) &= \sqrt{a_1} + \beta\sqrt{\frac{1 - a_1}{1 + \tilde{\beta}^2}} + \beta\sqrt{\frac{\tilde{\beta}^2(1 - a_1)}{1 + \tilde{\beta}^2}} \\ &= \sqrt{a_1} + \beta\frac{1 + \tilde{\beta}}{\sqrt{1 + \tilde{\beta}^2}}\sqrt{1 - a_1} \end{aligned}$$

The current self thus sets

$$a_1^*\left(\beta; \mu^{\mathbf{A}}\right) = \frac{1 + \tilde{\beta}^2}{\beta^2\left(1 + \tilde{\beta}\right)^2 + 1 + \tilde{\beta}^2}.$$

A sophisticated time-inconsistent player has $\mu^A(\beta) = 1$ and $\mu^A(1) = 0$, so $\tilde{\beta} = \beta$.

She would thus choose

$$a_1^*(\beta;(1,0)) = \frac{1+\beta^2}{\beta^2(1+\beta)^2+1+\beta^2}.$$

and plan to have

$$a_2^*(a_1;(1,0)) = \frac{1-a_1^*}{1+\beta^2} = \frac{\beta^2(1+\beta)^2}{(1+\beta^2)(2\beta^2+2\beta^3+\beta^4+1)}.$$

As the future self indeed has $\gamma^A = \beta$, the strategy profile $(a_1^*(\beta;(1,0)), a_2^*(a_1;(1,0)))$ is both the perception-perfect strategy in period 1, and the perception-perfect equilibrium of the game.

Now consider a naive time-inconsistent player. She has $\mu^A(\beta) = 0$ and $\mu^A(1) = 1$, so $\tilde{\beta} = \beta$. Hence

$$a_1^*(\beta;(0,1)) = \frac{1}{1+2\beta^2}$$

and she plans to have

$$a_2^*(a_1;(0,1)) = \frac{1-a_1^*}{2} = \frac{\beta^2}{1+2\beta^2}.$$

In period 2, however, she will find herself with $\gamma^A = \beta$ rather than $\gamma^A = 1$ as she expected. Hence, true second-period consumption will be

$$a_2^*(a_1,\beta) = \frac{1-a_1}{1+\beta^2} = \frac{1}{1+2\beta^2}.$$

Thus, in this case, a perception-perfect strategy in period 1 is to choose $(a_1^*, a_2^*) = \left(\frac{1}{1+2\beta^2}, \frac{\beta^2}{1+2\beta^2}\right)$, while the perception-perfect equilibrium will turn out to be $(a_1^*, a_2^*) = \left(\frac{1}{1+2\beta^2}, \frac{1}{1+2\beta^2}\right)$. It is interesting to note that $a_1^*(\beta;(0,1)) < a_1^*(\beta;(1,0))$. Hence, a naive player will choose a lower first-period consumption than a sophisticated one. This result is in line with O'Donoghue and Rabin (1999), who in a simpler framework find a "sophistication effect": when the reward of an action is immediate, naive players suffer less from the time inconsistency problem than sophisticated players.

Obviously, in our application, the rewards from consumption are also immediate. Here, the sophistication effect can be explained as follows. Different from naive players, sophisticated players are pessimistic about their future selves; they know that future selves will be impatient and squander most of their wealth quickly. As a consequence, sophisticated agents restrict the tendency of the future self to over-consume by increasing immediate consumption, which restricts the availability of the resource in the future. In other words, rather than allowing future selves to squander the wealth, current selves prefer to do this themselves. Hence, in our example, first period consumption is higher if there is a sophistication effect. Of course, if current selves can commit to a future consumption path, this result does not necessarily hold. In the presence of a commitment device, sophisticated agents benefit from their knowledge because it enables them to restrict future consumption by committing to a certain consumption path.

4 The one-player case: T periods

We now generalize the two-period decision problem we described in Section 2, to one with T periods. This complicates the problem. Consider the simplest case, with $T = 3$. Then the decision made by our player at $T = 1$ will be influenced by her perceptions concerning her type at $T = 2$. We will denote these perceptions as $\mu_{12}^{\mathbf{A}}$, where the first subscript reflects the time period in which perceptions are formed, and the second superscript reflects the time period that these perceptions apply to. But the decision made at $T = 1$ will also be influenced by her perceptions concerning her type at $T = 3$, denoted $\mu_{13}^{\mathbf{A}}$. Complicating matters further, the optimal decision at $T = 1$ will be influenced by her perception of the action that the future self will make at $T = 2$, which will in turn be determined by the perceptions that the future self at $T = 2$ will have, or rather, the perceptions that the current self at $T = 1$ will perceive that future self to have. Denote these perceptions as $\mu_1^{\mathbf{A}}\left(\mu_{23}^{\mathbf{A}}\right)$; these are the perceptions that at $T = 1$, player A perceives her future self at $T = 2$ to have concerning her type at $T = 3$.

To simplify matters, we make the following assumptions[5]

[5]Note that these assumptions also implicitly made by O'Donoghue and Rabin (1999). They assume that a naive player not only beliefs that she will be time-consistent in the next period, but also in any future period. Effectively, this is our perception consistency. Also, they implicitly rule

Assumption 1 ***Perception consistency***. *Perceptions concerning the type of a future self are identical for all future selves:* $\mu_{ij}^{\mathbf{A}} = \mu_{ik}^{\mathbf{A}}$ *for all* $i < T$, $j, k \in \{i+1, \ldots, T\}$.

Assumption 2 ***Intraplayer perception naivety.*** *Perceptions of a future self are assumed identical to perceptions of the current self:* $\mu_i^{\mathbf{A}}(\mu_{jk}^{\mathbf{A}}) = \mu_{ik}^{\mathbf{A}}$ *for all* $T \geq k > j > i$.

Note that there is a subtle difference between these two assumptions. Perception consistency implies that a player rules out that her type will change at some point in the future; if she perceives herself to be time-consistent at some point in the future, then she should perceive herself to be time-consistent at any point in the future. This seems a natural assumption to make; it is hard to justify a case in which, say, a player is naive concerning her future self in even periods but sophisticated concerning herself in odd periods.[6] Intraplayer perception naivety implies that a person rules out that her future self will change her opinion about selves that are even further in the future. Thus, we rule out that a player perceives today that her future self in two weeks is sophisticated, but maintains the possibility that one week from now she perceives that same future self to be naive.

Note that this also implies that we assume that a naive person will never learn to be more sophisticated through e.g. some kind of Bayesian updating. This greatly simplifies the analysis and seems consistent with casual observation. Still, it is feasible to enrich our framework to allow for such learning, but we leave that for future research.

At time t, define history $\mathbf{H}_t \equiv (a_1, \ldots, a_{t-1})$. Similar to (2) and (3), lifetime utility

out complications that may be caused by, say, a sophisticated player that maintains the possibility that he may be naive in the future. This is explicitly ruled out by our intraplayer perception naivety.

[6]It is conceivable though that a player is sophisticated concerning the near future (say, up to some $t \leq t^*$), but naive concerning the more distant future ($t > t^*$). It is straightforward to extend the analysis to allow for such a possibility. That, however, is beyond the scope of this paper.

at time $t \leq T$ can then be written

$$U_1^A\left(\mathbf{a};\beta^A\right) = u_1(a_1) + \beta^A \sum_{k=2}^{T} \delta^k u_k^A(\mathbf{H}_k, a_k) + \beta^A \delta^{T+1} u_{t+1}^A(\mathbf{H}_{t+1}),$$
$$U_t^A\left(\mathbf{a};\gamma^A\right) = u_t\left(\mathbf{H}_t, a_t\right) + \gamma^A \sum_{k=t+1}^{T} \delta^k u_k^A(\mathbf{H}_k, a_k) + \gamma^A \delta^{T+1} u_{t+1}^A(\mathbf{H}_{t+1}) \; \forall 1 < t \leq T,$$

with $\mathbf{a}$ the vector of all decisions: $\mathbf{a} \equiv (a_1, a_2, \ldots, a_k)$, and where we allow felicity in period $T+1$ to also play a role, just as we did in the case that $T=2$. Given the assumptions above, $\mu^{\mathbf{A}}$ now reflects the perceptions at any time t concerning the type of the future self at any time $k > t$. More precisely $\mu^{\mathbf{A}}(\gamma) = \Pr\left(\gamma^A = \gamma | \beta^A = \beta\right)$ with γ^A the time inconsistency at any future period.[7]

Definition 3 *In the T-period one-person game, a perception-perfect strategy at time τ for a time-inconsistent player, given her perceptions $\mu^{\mathbf{A}}$ and history $\mathbf{H}_t$ is a strategy profile $(a_\tau^*, a_{\tau+1}^*, \ldots a_T^*)$ such that*

$$a_T^*(\mathbf{H}_T;\mu^{\mathbf{A}}) = \arg \max_{a_T \in \mathcal{A}_T(\mathbf{H}_T)} \sum_{\gamma \in \{\beta,1\}} \mu^A(\gamma)\, u_T^A\left(\mathbf{H}_T, a_T; \gamma\right); \tag{7}$$
$$a_t^*(\mathbf{H}_t;\mu^{\mathbf{A}}) = \arg \max_{a_t \in \mathcal{A}_t(\mathbf{H}_t)} \sum_{\gamma \in \{\beta,1\}} \mu^A(\gamma)\, U_t^A\left(\mathbf{H}_t, a_t, a_{t+1}^*\left(\mathbf{H}_{t+1};\mu^{\mathbf{A}}\right), \ldots, a_T^*(\mathbf{H}_T;\mu^{\mathbf{A}});\gamma\right) \; \forall \tau < t < T;$$
$$a_\tau^*(\beta;\mu^{\mathbf{A}}) = \arg \max_{a_\tau \in \mathcal{A}_\tau(\mathbf{H}_\tau)} U_\tau^A\left(\mathbf{H}_\tau, a_1, a_2^*\left(\mathbf{H}_2;\mu^A\right), \ldots, a_T^*(\mathbf{H}_T;\mu^{\mathbf{A}});\beta\right).$$

Trivially, a perception-perfect strategy for a time-consistent player is a strategy profile $(a_\tau^*, a_{\tau+1}^*, \ldots a_T^*)$ such that

$$a_T^*\left(\mathbf{H}_T;(0,1)\right) = \arg \max_{a_T \in \mathcal{A}_T(\mathbf{H}_T)} U_T^A\left(\mathbf{H}_T;1\right)$$
$$a_t^*\left(\mathbf{H}_t;(0,1)\right) = \arg \max_{a_t \in \mathcal{A}_t(\mathbf{H}_t)} U_t^A\left(\mathbf{H}_t, a_{t+1}^*\left(\mathbf{H}_{t+1};(0,1)\right), \ldots, a_T^*(\mathbf{H}_T;(0,1));1\right)$$
$$\forall \tau \leq t < T.$$

The perception-perfect strategy for the time-inconsistent player can be understood

[7]Hence, we do not need a subscript t on either γ or γ^A.

much along the same lines as that for the case $T = 2$. We solve with backward induction. First, given $\mathbf{H}_T$, the current self assumes that the future self is going to take the action that maximizes the future self's utility. In the current self's perception, with probability $\mu^A(\beta)$, the future self's utility is given by $U_T^A(\mathbf{H}_t, a_t; \beta)$, while with probability $\mu^A(1)$, it is given by $U_T^A(\mathbf{H}_t, a_t; 1)$. The maximizer is thus given by (7) and denoted $a_T^*(\mathbf{H}_T; \mu^{\mathbf{A}})$. In period $T-1$, with probability $\mu^A(\beta)$, the future self's utility is given by $U_{T-1}^A\left(\mathbf{H}_{T-1}, a_{T-1}, a_T^*\left(\mathbf{H}_T; \mu^{\mathbf{A}}\right); \beta\right)$, with probability $\mu^A(1)$, it is given by $U_{T-1}^A\left(\mathbf{H}_{T-1}, a_{T-1}, a_T^*\left(\mathbf{H}_T; \mu^{\mathbf{A}}\right); 1\right)$. In both cases, $\mathbf{H}_T = (\mathbf{H}_{T-1}, a_{T-1})$. For ease of exposition, this dependence of future history on current action is not explicitly taken into account in our notation above. Again, the current self assumes the future self at $t = T-1$ to take the action that maximizes her utility. This process unravels until period 1, where the current self chooses the a_1 that maximizes her lifetime utility given her perceptions about future selves and given her true β^A in period 1.

Definition 4 *In the T-period one-person game, a perception-perfect equilibrium is a strategy profile $(a_1^*, a_2^*, \ldots, a_T^*)$ such that a_τ^* is part of a perception-perfect strategy at time τ for all $\tau = 1, \ldots, T$.*

Note again that there is a crucial difference between a perception-perfect strategy and a perception-perfect equilibrium; a perception-perfect strategy is a strategy profile that a player *perceives* to be played, while a perception-perfect equilibrium is the strategy profile that actually *will* be played.

It is relatively straightforward to extend the analysis to a case with infinitely many periods. Solving such a model would be similar to solving an infinite-horizon maximization problem in the case of time-consistent preferences, but under the assumption that all future selves have the type the current self perceives them to have.

5 Example: intertemporal consumption T periods

To give a flavor of the analysis, we consider the same consumption example as above, but now with T periods;

$$U_1^A(\mathbf{a}) = \sqrt{a_1} + \sqrt{a_2} + \ldots \sqrt{a_T} + \sqrt{1 - \sum_{t=1}^{T} a_t}.$$

In this case, a time-consistent player would set $a_1^* = \ldots = a_T^* = \frac{1}{T+1}$

We now solve for the perception-perfect strategy of the time-inconsistent player. Define total consumption in the past at time τ as $h_\tau = \sum_{t=1}^{\tau-1} a_t$. At $t = T$, and given first-period consumption a_1 and future time inconsistency γ, the player will choose a_2 as to maximize

$$U_T^A\left(\mathbf{H}_T, a_T; \gamma^A\right) = \sqrt{a_T} + \gamma^A\sqrt{1 - h_T - a_T}.$$

This yields

$$a_T^*\left(\mathbf{H}_t; \mu^{\mathbf{A}}\right) = \frac{1 - h_T}{1 + \left[\beta\mu^A(\beta) + \mu^A(1)\right]^2} = \frac{1 - h_T}{1 + \tilde{\beta}^2},$$

where again $\tilde{\beta}$ is given by (6). Now move back to $T - 1$.

$$U_{T-1}^A\left(\mathbf{H}_{T-1}, a_{T-1}, a_T^*\left(\mathbf{H}_t; \mu^{\mathbf{A}}\right); \gamma^A\right) = \sqrt{a_{T-1}} + \gamma^A\sqrt{\frac{1 - h_{T-1} - a_{T-1}}{1 + \tilde{\beta}^2}}$$
$$+ \gamma^A\sqrt{1 - h_{T-1} - a_{T-1} - \frac{1 - h_{T-1} - a_{T-1}}{1 + \tilde{\beta}^2}}$$

Take advantage of perception consistency to note that the future self at $t = T - 2$ is thus expected to maximize

$$U_{T-1}^A = \sqrt{a_{T-1}} + \tilde{\beta}\sqrt{\frac{1 - h_{T-1} - a_{T-1}}{1 + \tilde{\beta}^2}} + \tilde{\beta}\sqrt{1 - h_{T-1} - a_{T-1} - \frac{1 - h_{T-1} - a_{T-1}}{1 + \tilde{\beta}^2}}$$

This yields

$$a_{T-1}^* = \frac{1 + \tilde{\beta}^2}{\tilde{\beta}^2\left(1 + \tilde{\beta}\right)^2 + 1 + \tilde{\beta}^2}\left(1 - h_{t-1}\right).$$

Solving the model further is conceptually straightforward but analytically tedious.

6 Two-player case: two periods

We now come to the main aim of this paper: to extend the analysis above to a case with multiple players. Needless to say, this will greatly complicate the analysis. The current decisions of a player will now not only depend on her perceptions concerning her own future type, but also on her perceptions concerning the other player's future type, and possibly even about her perceptions of the other player's perceptions, plus how those perceptions will affect her own and the other player's future actions.

For simplicity, we start with the case of two periods, so $T = 2$, and two players, denoted A and B. For ease of exposition, in what follows we will refer to player A as being female, and to player B as being male. Again, player i's present-bias is denoted $\beta^i \in \{1, \beta\}$. The true present-bias of the future self of player i (i.e. player i's type) is $\gamma^i \in \{1, \beta\}$. There are 3 periods, $t = 1, 2, 3$. In the first two periods both A and B make a simultaneous decision. In $t = 1$, player A chooses action $a_1 \in \mathcal{A}_1$, while B chooses action $b_1 \in \mathcal{B}_1$. At $t = 2$, players learn the actions taken at $t = 1$, and player A chooses action $a_2 \in \mathcal{A}_2(a_1, b_1)$, while B chooses $b_2 \in \mathcal{B}_2(a_1, b_1)$. We now have

$$\begin{aligned} U_1^i\left(a_1, b_1, a_2, b_2; \beta^i\right) &= u_1^i(a_1, b_1) + \beta^i \delta u_2^i\left(a_1, b_1, a_2, b_2\right) + \beta^i \delta^2 u_3^i\left(a_1, b_1, a_2, b_2\right) \\ U_2^i\left(a_1, b_1, a_2, b_2; \gamma^i\right) &= u_2^i\left(a_1, b_1, a_2, b_2\right) + \gamma^i \delta u_3^i\left(a_1, b_1, a_2, b_2\right), \end{aligned}$$

$i \in \{A, B\}$.

In period 1, what player A expects to happen in period 2 will depend on her perceptions concerning her own future type, and on her perceptions concerning B's future type. For simplicity, we will assume that players can observe each other's current type, so both A and B can observe β^A and β^B. This simplifies the exposition, but it is conceptually straightforward to relax this assumption and also allow players to have perceptions concerning their competitor's current type.

A straightforward extension of the one-person case is as follows. In the perception of person A we have $\mu^{AA}(\gamma) = \Pr^A\left(\gamma^A = \gamma | \beta^A = \beta\right)$, where the first superscript denotes perceptions held by player A, and the second denotes perceptions concerning player A. The superscript on Pr denotes that this is the probability as perceived by player A. Similarly, we have $\mu^{AB}(\gamma) = \Pr^A\left(\gamma^B = \gamma | \beta^B = \beta\right)$. Naturally, $\mu^{BA}(\gamma) = \Pr^B\left(\gamma^A = \gamma | \beta^A = \beta\right)$ and $\mu^{BB}(\gamma) = \Pr^B\left(\gamma^B = \gamma | \beta^B = \beta\right)$.

In principle we now have to be concerned about what A perceives B to perceive about A, for example, i.e. we need to be concerned about $\mu^{\mathbf{AB}}\left(\mu^{\mathbf{BA}}\right)$. We also assume naivety in this respect, in the sense that what A perceives B to perceive about A is the same what A perceives about herself, thus $\mu^{\mathbf{AB}}\left(\mu^{\mathbf{BA}}\right) = \mu^{\mathbf{AA}}$. More generally, we assume

Assumption 3 ***Current interplayer perception naivety.*** *Perceptions of the other player are assumed identical to one's own perceptions:* $\mu^{\mathbf{ij}}(\mu^{\mathbf{jk}}) = \mu^{\mathbf{ik}}$ *for all* $i, j, k \in \{A, B\}$.

Note that this is a natural extension of the intraplayer perception naivety we assumed in the one-person case. That assumption implies that a person rules out that her future self will change her opinion about selves that are even further in the future. This assumption implies that, say, player A rules out that player B has perceptions about the future self of player A that are different from what player A herself has. In other words, player A is so convinced about the type of her future self that she cannot perceive that the other player has different perceptions.

Again, we solve the game using backward induction. For ease of exposition, we restrict attention to the case where both players are time-inconsistent. Consider player A. When deciding upon her first-period action, she again has to form some perception as to what will happen in period 2, given the actions taken in period 1. In the one-person case, she could simply derive the action her future self would be taking in period 2, given her perceptions about her future self. Now the analysis becomes more involved, as she also has to take the type and possible actions of player B into account. Suppose that the actions taken in period 1 are (a_1, b_1). Given these actions, we now look for a Nash equilibrium for the subgame at $t = 2$ as perceived by player A. As an example suppose player A perceives both players to be time-consistent in the future, so $\mu^{AA}(1) = \mu^{AB}(1) = 1$. She will then expect a Nash equilibrium (a_2^A, b_2^A) to be played which is such that a_2^A maximizes her future self's utility given b_2^A and given her perception that her future self is time-consistent, and such that b_2^A maximizes the future self's utility of player B, given A's perception that B's future self is time-consistent. Thus

$$a_2^A = \arg\max_{a_2} U_2^A \left(a_1, b_1, a_2, b_2^A; 1\right)$$
$$b_2^A = \arg\max_{b_2} U_2^A \left(a_1, b_1, a_2^A, b_2; 1\right)$$

where superscripts A denote the fact that we are considering the perceptions of player A. More generally,

Definition 5 *Consider the two-period two-person game played by time-inconsistent* *players. In period 2, given (a_1, b_1) an equilibrium as perceived by player $i \in \{A, B\}$ is an outcome $\left(a_2^i(a_1, b_1; \mu^{\mathbf{iA}}), b_2^i(a_1, b_1; \mu^{\mathbf{iB}})\right)$ that forms a Nash equilibrium of the second-stage game, given the perceptions of player i. Hence*

$$\begin{aligned} a_2^i &= \arg \max_{a_2 \in \mathcal{A}_2(a_1,b_1)} \sum_{\gamma \in \{\beta,1\}} \mu^{iA}(\gamma)\, U_2^A\left(a_1, b_1, a_2, b_2^A; \gamma\right) \\ b_2^i &= \arg \max_{b_2 \in \mathcal{B}_2(a_1,b_1)} \sum_{\gamma \in \{\beta,1\}} \mu^{iB}(\gamma)\, U_2^B\left(a_1, b_1, a_2^A, b_2; \gamma\right) \end{aligned}$$

Moving back to period 1, given that player A has a perception of the play that will ensue in period 2 for any (a_1, b_1) in period 1, it is straightforward to write down the conditions for a subgame perfect Nash equilibrium as perceived by player A. We will refer to this simply as an equilibrium as perceived by player A.

Definition 6 *In period 1, an equilibrium as perceived by player i is an outcome $\left(a_1^i\left(\beta; \mu^{\mathbf{iA}}, \mu^{\mathbf{iB}}\right), b_1^i\left(\beta; \mu^{\mathbf{iA}}, \mu^{\mathbf{iB}}\right)\right)$ that is part of a subgame perfect Nash equilibrium of the entire game, given the perceptions of player i. Thus,*

$$\begin{aligned} a_1^i &= \arg \max_{a_1 \in \mathcal{A}_1} U_1^A\left(a_1, b_1^i, a_2^i(a_1, b_1^i; \mu^{\mathbf{iA}}), b_2^i(a_1, b_1; \mu^{\mathbf{iB}}); \beta\right) \\ b_1^i &= \arg \max_{b_1 \in \mathcal{B}_1} U_1^B\left(a_1^i, b_1, a_2^i(a_1, b_1; \mu^{\mathbf{iA}}), b_2^i(a_1, b_1; \mu^{\mathbf{iB}}); \beta\right). \end{aligned} \tag{8}$$

Using these definitions, and considering play in period 1, we thus expect player A to take an action that she perceives to be part of a subgame perfect equilibrium for the entire game, while we expect player B to take an action that he perceives to be part of a subgame perfect equilibrium for the entire game.

Definition 7 *A perception-perfect equilibrium of the game is an outcome $(a_1^*, b_1^*, a_2^*, b_2^*)$ such that a_1^* is part of an equilibrium as perceived by player A; b_1^* is part of an equilibrium as perceived by player B; a_2^* is an equilibrium as perceived by player A given (a_1^*, b_1^*); and b_2^* is an equilibrium as perceived by player B given (a_1^*, b_1^*).*

Needless to say, the actions a_1^* and b_1^* do not have to be consistent with each other, in the sense that they do not have to be part of the same equilibrium. Also, we assume that players do not learn anything about the perceptions or type of the other player upon observing first-period actions. Of course, we do allow a player to adapt her strategy in the second period upon observing the other player's action in period 1. In other words, we assume that, say, player A takes the action that she feels is part of an equilibrium of the second stage based on the actions that she actually observed to be played in period 1, rather than the actions that she expected to be played in period 1.

It is straightforward to extend the analysis above to a case where, for example, one player is known to be time-consistent,[8] or to a case where players cannot observe the other player's current type.[9]

7 Application to the common pool problem

We consider a common pool problem similar to the example that we gave for the one-person model. Consider two players, A and B, that live for 3 periods, and start out with joint wealth 1. Instantaneous utility in each period is given by $u_t^i(c) = \sqrt{c}$, $i \in \{A, B\}$. For simplicity, the discount factor δ equals 1. In each of the 2 periods each player takes some amount for immediate consumption out of the common pool. Whatever is left in the last period will be equally shared among the two.

To get some feel for the problem, we first consider the case in which both players

[8]Suppose that player B is known to be time-consistent. In that case, his future self will necessarily also be time-consistent, so $\mu^{AB}(1) = \mu^{BB}(1) = 1$. Moreover, the conditions (8) then modify to

$$a_1^i = \arg\max_{a_1 \in \mathcal{A}_1} U_1^A\left(a_1, a_2^i(a_1, b_1^i; \mu^{\mathbf{AA}}), b_1^i, b_2^i(a_1, b_1; \mu^{\mathbf{iB}}); \beta\right)$$
$$b_1^i = \arg\max_{b_1 \in \mathcal{B}_1} U_1^B\left(a_1^i, a_2^i(a_1, b_1; \mu^{\mathbf{AA}}), b_1, b_2^i(a_1, b_1; \mu^{\mathbf{iB}}); 1\right)$$

[9]The conditions (8) then modify to

$$a_1^i = \arg\max_{a_1 \in \mathcal{A}_1} \sum_{\gamma \in \{\beta, 1\}} \mu^{iA}(\gamma)\, U_1^A\left(a_1, a_2^i(a_1, b_1^i; \mu^{\mathbf{AA}}), b_1^i, b_2^i(a_1, b_1; \mu^{\mathbf{iB}}); \beta\right)$$
$$b_1^i = \arg\max_{b_1 \in \mathcal{B}_1} \sum_{\gamma \in \{\beta, 1\}} \mu^{iA}(\gamma)\, U_1^B\left(a_1^i, a_2^i(a_1, b_1; \mu^{\mathbf{AA}}), b_1, b_2^i(a_1, b_1; \mu^{\mathbf{iB}}); 1\right)$$

are time-consistent. Their respective lifetime utility functions in period 1 then equal

$$U_1^A(a_1, b_1, a_2, b_2) = \sqrt{a_1} + \sqrt{a_2} + \sqrt{\frac{1 - a_1 - a_2 - b_1 - b_2}{2}} \tag{9}$$
$$U_1^B(a_1, b_1, a_2, b_2) = \sqrt{b_1} + \sqrt{b_2} + \sqrt{\frac{1 - a_1 - a_2 - b_1 - b_2}{2}}$$

Using backward induction, in period 2 player A will set a_2 to maximize

$$U_2^A(a_1, b_1, a_2, b_2) = \sqrt{a_2} + \sqrt{\frac{1 - a_1 - a_2 - b_1 - b_2}{2}}$$

which yields the reaction function

$$a_2 = \frac{2}{3}(W_2 - b_2),$$

with $W_2 \equiv 1 - a_1 - b_1$ the amount of wealth left at the start of period 2. Imposing symmetry, this yields the Nash equilibrium $a_2^* = b_2^* = \frac{2}{5}W_2$. Plugging this back into (9), maximizing with respect to a_1 and imposing symmetry, we have $a_1^* = a_2^* = 10/29$.

Now consider the case of time-inconsistent and possibly naive players. First consider the case in which both players are sophisticated with respect to both their own future self and that of the other player. For simplicity, we set $\beta = 1/2$. At $t = 1$, the current self of player A thus perceives an equilibrium in period 2 to satisfy

$$a_2^A = \arg\max \sqrt{a_2} + \frac{1}{2}\sqrt{\frac{W_2 - a_2 - b_2}{2}}$$
$$b_2^A = \arg\max \sqrt{b_2} + \frac{1}{2}\sqrt{\frac{W_2 - a_2 - b_2}{2}}.$$

This yields reaction functions $a_2^A = \frac{8}{9}\left(W_2 - b_2^A\right)$ and $b_2^A = \frac{8}{9}\left(W_2 - a_2^A\right)$ so the perceived equilibrium has $a_2^A = b_2^A = 8W_2/17$. Moving back to period 1, the equilibrium perceived by player A should satisfy[10]

$$a_1^A = \arg\max_{a_1} \sqrt{a_1} + \frac{1}{2}\sqrt{\frac{8\left(1-a_1-b_1^A\right)}{17}} + \frac{1}{2}\sqrt{\frac{1-a_1-b_1^A}{34}}$$
$$b_1^A = \arg\max_{b_1} \sqrt{b_1} + \frac{1}{2}\sqrt{\frac{8\left(1-a_1^A-b_1\right)}{17}} + \frac{1}{2}\sqrt{\frac{1-a_1^A-b_1}{34}}$$

Maximizing and solving for the equilibrium yields $a_1^A = b_1^A = \frac{136}{297} \approx 0.458$, so $a_2^A = b_2^A = \frac{200}{5049} \approx 0.0396$. As player B faces the same problem and the same perceptions, he will have the same perceived equilibrium in periods 1 and 2 as player A. Moreover, as players' perceptions turn out to be correct, what they perceive to be played in period 2 is also what is actually played in period 2. Hence, the perception-perfect equilibrium is $(a_1^*, a_2^*, b_1^*, b_2^*) = \left(\frac{136}{297}, \frac{200}{5049}, \frac{136}{297}, \frac{200}{5049}\right)$.

Now consider the case in which both players are naive concerning all future selves. At $t = 1$, player A perceives the equilibrium in stage 2 to satisfy

$$a_2^A = \arg\max \sqrt{a_2} + \sqrt{\frac{W_2 - a_2 - b_2}{2}}$$
$$b_2^A = \arg\max \sqrt{b_2} + \sqrt{\frac{W_2 - a_2 - b_2}{2}},$$

which yields reaction functions $a_2^A = \frac{2}{3}\left(W_2 - b_2^A\right)$ and $b_2^A = \frac{2}{3}\left(W_2 - a_2^A\right)$, so the equilibrium has $a_2^A = b_2^A = 2W_2/5$. Moving back to period 1, the equilibrium perceived by player A should satisfy[11]

$$a_1^A = \arg\max_{a_1} \sqrt{a_1} + \frac{1}{2}\sqrt{\frac{2\left(1-a_1-b_1^A\right)}{5}} + \frac{1}{2}\sqrt{\frac{1-a_1-b_1^A}{10}}$$
$$b_1^A = \arg\max_{b_1} \sqrt{b_1} + \frac{1}{2}\sqrt{\frac{2\left(1-a_1^A-b_1\right)}{5}} + \frac{1}{2}\sqrt{\frac{1-a_1^A-b_1}{10}}$$

Maximizing and solving for the equilibrium yields $a_1^A = b_1^A = \frac{40}{89} \approx 0.449$, so $a_2^A =$

[10] After period 1, W_2 is left. In period 2, player A perceives both players to consume $8W_2/17$, hence in period 3 there is $W_2/17$ left, which is equally shared among both players. Plugging in $W_2 = 1 - a_1^A - b_1^A$ and using 9 implies these expressions.

[11] After period 1, W_2 is left. In period 2, player A perceives both players to consume $2W_2/5$, hence in period 3 there is $W_2/5$ left, which is equally shared among both players. Plugging in $W_2 = 1 - a_1^A - b_1^A$ and using 9 implies these expressions.

$b_2^A = \frac{18}{445} \approx 0.0404$. As player B faces the same problem and the same perceptions, he will have the same perceived equilibrium in periods 1 and 2 as player A. However, players' perceptions turn out to be incorrect: the equilibrium in the second period has them both consuming $8W_2/17$ (as we saw in the previous analysis) rather than $2W_2/5$. Hence, actual consumption in period 2 will turn out to be $a_2^* = b_2^* = \frac{72}{1513} \approx 0.0476$, and the perception-perfect equilibrium is $(a_1^*, a_2^*, b_1^*, b_2^*) = \left(\frac{40}{89}, \frac{72}{1513}, \frac{40}{89}, \frac{72}{1513}\right)$. Just as in the one-person case, sophisticated players strategically consume more in period 1 than naive players do.

Perhaps the most interesting case is the one in which both players perceive themselves to be time-consistent in the future, but perceive their competitor to be time-inconsistent in the future. In other words, each player is naive concerning her own future self, but sophisticated concerning the future self of the other player. As noted in the introduction, Kahneman (2011) argues that this is the typical situation. Player A will then perceive a second-period equilibrium

$$
\begin{aligned}
a_2^A &= \arg\max \sqrt{a_2} + \sqrt{\frac{W_2 - a_2 - b_2}{2}} \\
b_2^A &= \arg\max \sqrt{b_2} + \frac{1}{2}\sqrt{\frac{W_2 - a_2 - b_2}{2}},
\end{aligned}
$$

which yields reaction functions $a_2^A = \frac{2}{3}\left(W_2 - b_2^A\right)$ and $b_2^A = \frac{8}{9}\left(W_2 - a_2^A\right)$, so $a_2^A = 2W_2/11$ and $b_2^A = 8W_2/11$. Thus, player A perceives to consume much less in period 2 than player B does. Note that the reaction functions are strategic substitutes, in the sense of Bulow et al. (1985). Player A expects player B to be very aggressive in period 2, due to A's perception that B will be very impatient and thus claim a high amount of then current consumption. As A (again in her perception) will be much more patient, she will claim a very low share of the available wealth then as future consumption is still important for her.

Plugging this into the perceived first-period problems[12]

$$a_1^A = \arg\max_{a_1} \sqrt{a_1} + \frac{1}{2}\sqrt{\frac{2\left(1-a_1-b_1^A\right)}{11}} + \frac{1}{2}\sqrt{\frac{1-a_1-b_1^A}{22}}$$
$$b_1^A = \arg\max_{b_1} \sqrt{b_1} + \frac{1}{2}\sqrt{\frac{8\left(1-a_1^A-b_1\right)}{11}} + \frac{1}{2}\sqrt{\frac{1-a_1^A-b_1}{22}}$$

yielding best-reply functions $a_1^A = \frac{88}{97}\left(1-b_1^A\right)$ and $b_1^A = \frac{88}{113}\left(1-a_1^A\right)$ so $a_1^A = \frac{2200}{3217} \approx 0.684$ and $b_1^A = \frac{792}{3217} \approx 0.246$. Thus, player A is going to claim a lot of consumption in period 1; she perceives that she will receive very little consumption in the future as player B will be very aggressive then.

However, player B has perceptions similar to player A and solve a similar problem, yielding $b_1^B = \frac{2200}{3217}$ and $a_1^B = \frac{792}{3217}$. Hence, in period 1, each players aims to consume $\frac{2200}{3217}$. However, these consumption plans are incompatible as they more than exhaust the available stock of wealth. If we put as an additional restriction that the available wealth is proportionally rationed if players want to claim more than what is available, then the perception-perfect equilibrium becomes $(a_1^*, b_1^*, a_2^*, b_2^*) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$, i.e. all consumption is done in period 1. Thus, these perceptions exacerbate the common pool problem.

8 Two-player case: T periods

We now extend the two-person two-period model that we analyzed above, to a setting with T periods, $T > 2$. This is conceptually straightforward, but notationally tedious. We solve with backward induction. In period T, what the current player A expects to be played is a game between herself and player B, with both players having the type that she currently expects them to have. Moving back to period 1, and given her perceptions concerning the players in period $T-1$, she can then derive her perceived equilibrium play in that period. Continuing in this manner yields a perceived equilibrium in period 1, and hence a course of action for player A in period 1, with a similar analysis for player B.

[12] In period 2, player A perceives total consumption to be $a_2^A + b_2^A = 10W_2/11$, hence in period 3 there is $W_2/11$ left, which is equally shared among both players. Plugging in $W_2 = 1 - a_1^A - b_1^A$ and using 9 implies these expressions.

To analyze this problem, we again need to make simplifying assumptions concerning the perceptions of players. Not only do we need that A has to believe that she has the same perceptions as B concerning future types, we also need that higher-order perceptions are perceived to be equal. In other words we also need that the perceptions that A has in period l concerning the perceptions of B in period m concerning the perceptions of A in period n, equal the perceptions that A thinks she herself has in period l concerning herself in period n. Thus

Assumption 4 ***Future interplayer perception naivety.*** *Perceptions of the otherplayer are assumed identical to one's own perceptions:* $\mu_{lm}^{\mathbf{ij}}(\mu_{mn}^{\mathbf{jk}}) = \mu_{lm}^{\mathbf{ii}}(\mu_{mn}^{\mathbf{jk}})$ *for all* $i, j, k \in \{A, B\}$ *.*

Without this assumption, we would have to allow for the possibility that, at any time in the future player A maintains the possibility that player B has different perceptions concerning future types than she herself has. This possibility would force player A to also form higher order beliefs concerning perceptions – a possibility that would highly complicate the analysis. Together with the previous assumptions, future interplayer perception naivety implies that all perceptions are always constant – and are always assumed to be constant.

History at time t is now defined as $\mathbf{H}_t \equiv (a_1, b_1; \ldots; a_{t-1}, b_{t-1})$. Lifetime utility at time $t \leq T$ for player i can be written

$$U_1^i\left(\mathbf{a}, \mathbf{b}; \beta^i\right) = u_1^i(a_1, b_1) + \beta^i \sum_{k=t+1}^{T} \delta^k u_k^i(\mathbf{H}_k, a_k, b_k) + \beta^i \delta^{T+1} u_{t+1}^i(\mathbf{H}_{t+1})$$

$$U_t^i\left(\mathbf{a}, \mathbf{b}; \gamma^i\right) = u_t^i(\mathbf{H}_t, a_t, b_t) + \gamma^i \sum_{k=t+1}^{T} \delta^k u_k^i(\mathbf{H}_k, a_k, b_k) + \gamma^A \delta^{T+1} u_{t+1}^A(\mathbf{H}_{t+1})$$

$\forall 1 < t \leq T$ for $i \in \{A, B\}$, $\mathbf{a} =(a_1, \ldots, a_T)$ and $\mathbf{b} =(b_1, \ldots, b_T)$.

The analysis for $T = 2$ naturally extends to one with more periods. Consider period T. In an equilibrium as perceived by player i, actions taken in the last period will be mutual best responses given the perceptions player i has about the future type of both players, and given the history of play up to period T. Again, we can write player i's perceptions about player j's future type as $\mu^{\mathbf{ij}}$, the only difference with the analysis in Section 6 being that this now refers to perceptions about types in any future period, rather than just the next. Given perceived play in period T, player i can then move back to period $T-1$ and derive a perceived equilibrium for that period. This process unravels until period 1, allowing us to write down the conditions for a subgame perfect Nash equilibrium as perceived by player i. We will refer to this simply as an equilibrium as perceived by player i.

Definition 8 *In the T-period, 2-person game with time-inconsistent players, an equilibrium at time τ as perceived by player i, given her perceptions $\mu^{\mathbf{i}}$ and history $\mathbf{H}_t$ is a sequence $(a_\tau^i, b_\tau^i, a_{\tau+1}^i, b_{\tau+1}^i, \ldots, a_T^i, b_T^i)$ such that*

1. *For period T*

$$a_T^i = \arg \max_{a_T \in \mathcal{A}_T(\mathbf{H}_T)} \sum_{\gamma \in \{\beta, 1\}} \mu^{iA}(\gamma)\, U_T^A \left(\mathbf{H}_T, a_T, b_T^A; \gamma\right)$$
$$b_T^i = \arg \max_{b_T \in \mathcal{B}_T(\mathbf{H}_T)} \sum_{\gamma \in \{\beta, 1\}} \mu^{iB}(\gamma)\, U_T^B \left(\mathbf{H}_T, a_T^A, b_T; \gamma\right)$$

2. *For periods t with $\tau < t < T$*

$$\begin{aligned}
a_t^i \left(\mathbf{H}_{\tau+1}; \mu^{\mathbf{A}}\right) = \arg \max_{a_t \in \mathcal{A}_t(\mathbf{H}_t)} \sum_{\gamma \in \{\beta, 1\}} & \mu^{iA}(\gamma)\, U_t^A \Big(\mathbf{H}_t, a_t, b_t^i, a_{t+1}^i \left(\mathbf{H}_{t+1}; \mu^{\mathbf{A}}\right), \\
& b_{t+1}^i \left(\mathbf{H}_{t+1}; \mu^{\mathbf{A}}\right), \ldots, a_T^i(\mathbf{H}_T; \mu^{A}), b_T^i \left(\mathbf{H}_T; \mu^{\mathbf{A}}\right); \gamma\Big) \\
b_t^i \left(\mathbf{H}_{\tau+1}; \mu^{\mathbf{A}}\right) = \arg \max_{b_t \in \mathcal{B}_t(\mathbf{H}_t)} \sum_{\gamma \in \{\beta, 1\}} & \mu^{iB}(\gamma)\, U_t^B \Big(\mathbf{H}_t, a_t^i, b_t, a_{t+1}^i \left(\mathbf{H}_{t+1}; \mu^{\mathbf{A}}\right), \\
& b_{t+1}^i \left(\mathbf{H}_{t+1}; \mu^{\mathbf{A}}\right), \ldots, a_T^i(\mathbf{H}_T; \mu^{\mathbf{A}}), b_T^i \left(\mathbf{H}_T; \mu^{\mathbf{A}}\right); \gamma\Big)
\end{aligned}$$

3. *For $t = \tau$*

$$\begin{aligned}
a_\tau^i = \arg \max_{a_\tau \in \mathcal{A}_\tau(\mathbf{H}_\tau)} & U_\tau^A \Big(\mathbf{H}_\tau, a_\tau, b_\tau^i, a_{\tau+1}^i \left(\mathbf{H}_{\tau+1}; \mu^{\mathbf{A}}\right), b_{\tau+1}^i \left(\mathbf{H}_{\tau+1}; \mu^{\mathbf{A}}\right), \\
& \ldots, a_T^i(\mathbf{H}_T; \mu^{\mathbf{A}}), b_T^i \left(\mathbf{H}_T; \mu^{\mathbf{A}}\right); \beta\Big) \\
b_\tau^i = \arg \max_{b_\tau \in \mathcal{B}_\tau(\mathbf{H}_\tau)} & U_\tau^B \Big(\mathbf{H}_\tau, a_\tau^i, b_\tau, a_{\tau+1}^i \left(\mathbf{H}_{\tau+1}; \mu^{\mathbf{A}}\right), b_{\tau+1}^i \left(\mathbf{H}_{\tau+1}; \mu^{\mathbf{A}}\right), \\
& \ldots, a_T^i(\mathbf{H}_T; \mu^{\mathbf{A}}), b_T^i \left(\mathbf{H}_T; \mu^{\mathbf{A}}\right); \beta\Big)
\end{aligned}$$

Using these definitions, and considering play in period 1, we thus expect player A to take an action that she perceives to be part of a subgame perfect equilibrium for the entire game, while we expect player B to take an action that he perceives to be part of a subgame perfect equilibrium for the entire game.

Definition 9 *A perception-perfect equilibrium of the game is an outcome* $(a_1^*, b_1^*, a_2^*, b_2^*, \ldots, a_T^*, b_T^*)$ *such that* $\forall \tau \in \{1, \ldots, T\}$ a_τ^* *is part of an equilibrium at time* τ *as perceived by player* A; b_τ^* *is part of an equilibrium at time* τ *as perceived by player* B.

Note again that players do not learn anything about the perceptions or type of the other player upon observing her actions. Of course, we do allow a player to adapt her strategy upon observing the other player's action in a previous period. In other words, we assume that, say, player A takes the action that she feels is part of an equilibrium of the second stage based on the actions that she actually observed to be played in period 1, rather than the actions that she expected to be played in period 1. Also, it is again straightforward to extend the analysis above to a case where, for example, one player is known to be time-consistent, or to a case where players cannot observe the other player's current type.

9 Application to Sequential Bargaining

9.1 Rubinstein bargaining with time-consistent players

In this section, we apply our framework to a dynamic bargaining game as proposed by Stahl (1972) and Rubinstein (1982).[13] In a Rubinstein bargaining game, two players, A and B, bargain over the division of a pie of size 1. There are T periods. In odd-numbered periods ($t = 1, 3, 5, \ldots$) player A proposes a sharing rule $(x_t, 1 - x_t)$ that player B can accept or reject. The first number in the sharing rule always represents the share that A obtains, while the second number is the share that B obtains. If player B accepts an offer, the game ends and the proposed division is implemented. If B rejects, he makes a counteroffer in the next period that player A can accept

[13] As noted in the introduction, we are not the first to consider Rubinstein bargaining with possibly naive hyperbolic discounters. Akin (2007) and Sarafidis (2006) derive comparable results by imposing restrictions on possible types and without explicitly modeling systems of perceptions

or reject. In this standard specification of the game, both players have the usual time-consistent preferences. Suppose that player A uses discount factor δ_A, while B uses δ_B. Hence if $(x, 1-x)$ is accepted at time t, the payoffs to the players are $(\delta_A^t x, \delta_B^t (1-x))$.

To fix ideas, we first consider the well-known solution to the standard model. Consider the case that T is even. We look for a subgame perfect equilibrium. In stage T, A will accept any proposal. Player B will thus offer $(x_{T-1}, 1-x_{T-1}) = (0, 1)$. Knowing this, in stage $T-1$, player A claims the highest share that would still make player B be willing to accept. Hence, she plays $(x_{T-1}, 1-x_{T-1}) = (1-\delta_B, \delta_B)$. With the same logic, in period $T-2$, player B offers A the lowest share she is still willing to accept, so $(x_{T-2}, 1-x_{T-2}) = (\delta_A (1-\delta_B), 1-\delta_A (1-\delta_B))$, etc. The equilibrium then has player A making an offer in period 1 that is immediately accepted. In the remainder, in a T-period game with T odd where it is common knowledge that A and B use discount factors δ_A and δ_B respectively, we denote the equilibrium sharing rule proposed in period t as $(x_t^* (\delta_A, \delta_B), 1-x_t^* (\delta_A, \delta_B))$.

9.2 Rubinstein bargaining with time-inconsistent players

Now consider our framework with hyperbolic and possibly naive discounters. Our solution concept, perception-perfect equilibrium, requires that in each period each player chooses the action that is part of a subgame-perfect equilibrium, given her perceptions concerning the future types of both players. In a sequential move game as we have here, this concept is relatively easy to implement.

Suppose that both players use the discount factor δ, but may differ in the extent to which they are time-consistent. We first derive the equilibrium as perceived by player A. She perceives the future player B to have type $\gamma^{AB} \in \{\beta, 1\}$ and the future player A to have type $\gamma^{AA} \in \{\beta, 1\}$. In other words, she perceives the future player B to use discount factor $\gamma^{AB}\delta$, and the future player A to use discount factor $\gamma^{AA}\delta$. Importantly, she also perceives all other players, present and future, to have those same perceptions. In stage T, player A will accept anything. Player B will thus offer $(x_T, 1-x_T) = (0, 1)$. Knowing this, in stage $T-1$, player A claims the highest share she perceives she can get and that would still make player B be willing to accept. Hence, she plays $(x_{T-1}, 1-x_{T-1}) = (1-\gamma^{AB}\delta, \gamma^{AB}\delta)$. For period $T-2$, player A perceives player B to have the same perceptions concerning how play will continue in $T-1$. Hence, in period $T-2$, the current player A perceives B to offer A the lowest share she is still willing to accept, so $(x_{T-2}, 1-x_{T-2}) = \left(\gamma^{AA}\delta\left(1-\gamma^{AB}\delta_B\right), 1-\gamma^{AA}\delta\left(1-\gamma^{AB}\delta\right)\right)$. Hence, player A perceives future selves to act as if player A's true discount factor is $\gamma^{AA}\delta$, while player B's true discount factor is $\gamma^{AB}\delta$. Thus, the equilibrium as perceived by player A is $\left(x_t^*\left(\gamma^{AA}\delta, \gamma^{AB}\delta\right), 1-x_t^*\left(\gamma^{AA}\delta, \gamma^{AB}\delta\right)\right)$, for all $t \in \{1, \ldots T\}$. Similarly, the equilibrium as perceived by player B is $\left(x_t^*\left(\gamma^{BA}\delta, \gamma^{AB}\delta\right), 1-x_t^*\left(\gamma^{AA}\delta, \gamma^{AB}\delta\right)\right)$ for all $t \in \{1, \ldots T\}$.[14]

9.3 Infinite horizon

To derive some qualitative predictions, we look at the case with an infinite horizon. From the literature on Rubinstein bargaining, we know the following. Suppose that players are time-consistent and have discount factors δ_A and δ_B. In a period where it is player A's turn to make an offer, the unique equilibrium then has equilibrium payoff to player A that equal

$$\pi_A(\text{A moves first}) = \frac{1-\delta_B}{1-\delta_A\delta_B}.$$

If it is player B's turn to make an offer, the equilibrium payoff to player A is

$$\pi_A(\text{B moves first}) = \frac{\delta_A\left(1-\delta_B\right)}{1-\delta_A\delta_B}.$$

Of course, expressions for π_B are similar. A straightforward proof can be found in Shaked and Sutton (1984) or Fudenberg and Tirole (1991), chapter 4.

Now consider our model with possibly time-inconsistent players. Again consider the equilibrium as perceived by player A. For the finite-horizon case, we saw that that equilibrium is equivalent to one with time-consistent players where $\delta_A = \gamma^{AA}\delta$ and $\delta_B = \gamma^{AB}\delta$. It is straightforward to see that that also applies to the infinite horizon case.[15] Thus, for any future period where A moves first, $i \in \{A, B\}$ perceives the

[14] Note that we also need that player A prefers her current offer above what she will get from B in the future, properly discounted. It is easy to show, however, that that is always satisfied.

[15] The proof is identical to that in Shaked and Sutton (1984) or Fudenberg and Tirole (1991), but using discount factors $\gamma^{AA}\delta$ and $\gamma^{AB}\delta$ rather that δ_A and δ_B. Hence, we do not repeat it here.

continuation payoffs of player A to be

$$\pi_A^i(\text{A moves first}) = \frac{1-\gamma^{iA}\delta}{1-\gamma^{iA}\gamma^{iB}\delta^2}.$$

and those of player B

$$\pi_B^i(\text{A moves first}) = \frac{\gamma^{iA}\left(1-\gamma^{iB}\right)}{1-\gamma^{iA}\gamma^{iB}}$$

More generally, for any future period where j moves first, i perceives the continuation payoffs of player k to be

$$\pi_k^i(j \text{ moves first}) = \begin{cases} \frac{1-\gamma^{im}\delta}{1-\gamma^{iA}\gamma^{i/B}\delta^2} & j=k, m \neq j \\ \frac{\gamma^{ik}\delta\left(1-\gamma^{ij}\delta\right)}{1-\gamma^{iA}\gamma^{i/B}\delta^2} & j \neq k \end{cases}$$

for $i, j, k, m \in \{A, B\}$.

Note however that these expressions apply to any future period. When one player makes an offer to another player in the current time period, she will not base that offer on the perceived future type of that player, but rather on the current type. By assumption, she can observe the true current type β^B of the other player. If player A makes an offer in period 1, she will thus offer B the lowest amount he is willing to accept, given that if B can make a counteroffer in the next period, B's continuation payoff will be $\left(1-\gamma^{AA}\delta\right) / \left(1-\gamma^{AA}\gamma^{AB}\delta^2\right)$. Thus, A will offer

$$1 - x_t\left(\gamma^{AA}, \gamma^{AB}\right) = \frac{\beta^B\delta\left(1-\gamma^{AA}\delta\right)}{1-\gamma^{AA}\gamma^{AB}\delta^2}. \tag{10}$$

A similar analysis holds if it is player B's turn to move.

Yet, player A's offer will not always be accepted. If it is not, there will be delay in bargaining. Consider period 1. Player B perceives his continuation payoff in that period to be

$$\pi_B^B(B \text{ moves first}) = \frac{1-\gamma^{BA}\delta}{1-\gamma^{BA}\gamma^{BB}\delta^2}.$$

He will thus reject A's offer (10) if he perceives it to give him a lower net present value than holding out and making a counteroffer in the next period, thus if

$$\frac{\beta^B \delta \left(1-\gamma^{AA}\delta\right)}{1-\gamma^{AA}\gamma^{AB}\delta^2} < \frac{\beta^B \delta \left(1-\gamma^{BA}\delta\right)}{1-\gamma^{BA}\gamma^{BB}\delta^2}.$$

We thus have:

Theorem 1 *In the perception-perfect equilibrium of the Rubinstein bargaining game with possibly naive hyperbolic discounters, in period t, player i will make an offer*

$$\frac{\beta^j \delta \left(1-\gamma^{ii}\delta\right)}{1-\gamma^{iA}\gamma^{iB}\delta^2}$$

to player j, $i \in \{A, B\}$, $j \neq i$. Player j will accept if and only if

$$\frac{1-\gamma^{ii}\delta}{1-\gamma^{iA}\gamma^{iB}\delta^2} \geq \frac{1-\gamma^{ji}\delta}{1-\gamma^{jA}\gamma^{jB}\delta^2} \tag{11}$$

Note that this expression does not directly depend on β. Thus, if we only have hyperbolic discounting, but no naivety, there will never be a delay in reaching an agreement. More generally, if A and B share the same perceptions (thus $\gamma^{AA} = \gamma^{BA}$ and $\gamma^{BA} = \gamma^{BB}$) the left- and right-hand side of (11) are equal and there is no delay. This is natural: in the equilibrium of this game, any player offers to the other player what she perceives the other player is just willing to accept. As long as those perceptions are shared we get the same qualitative outcome as in the standard Rubinstein model, in the sense that the first offer will be immediately accepted.

9.4 Bargaining breakdown

Above, we derived condition (11) for a delay in bargaining to occur.[16] Note that this immediately implies

Corollary 2 *In the Rubinstein bargaining model with hyperbolic, possible naive discounters, negotiations break down in the sense that an agreement is never reached*

[16] For more reasons why there may be delay in Rubinstein bargaining, see e.g. Yildiz (2004), and the references therein.

whenever the following conditions hold:

$$\frac{1-\gamma^{AA}\delta}{1-\gamma^{AA}\gamma^{AB}\delta^2} < \frac{1-\gamma^{BA}\delta}{1-\gamma^{BA}\gamma^{BB}\delta^2} \tag{12}$$

$$\frac{1-\gamma^{BB}\delta}{1-\gamma^{BA}\gamma^{BB}\delta^2} < \frac{1-\gamma^{AB}\delta}{1-\gamma^{AA}\gamma^{AB}\delta^2}. \tag{13}$$

This result allows us to easily derive whether negotiations will break down in various scenarios. Consider for example the case in which both players are sophisticated about the other player, but naive about themselves. Thus, assume $\gamma^{AB} = \gamma^{BA} = \beta$ and $\gamma^{AA} = \gamma^{BB} = 1$. In that case, the denominators of both (12) and (13) are equal, and both conditions simplify to $1-\delta < 1-\beta\delta$, which is always satisfied. Hence, bargaining breaks down and the two parties never reach an agreement.

The intuition is as follows. If player A makes an offer to player B, she perceives the future B to be time inconsistent. Hence, her offer will be relatively low, as she perceives B to be very impatient. Player B however, perceives his future self to be patient. Therefore, he will not accept the current offer of player A, as he perceives to be able to do better. The same is true in the opposite case where player B makes an offer to A. Hence, players keep rejecting each others' offers and an agreement is never reached. Qualitatively, we thus get a similar result to that in the case of the common pool problem discussed earlier. Also there, the game broke down if players correctly anticipated their competitor's time inconsistency but were naive about their own.

Now suppose that each player is sophisticated about her own time inconsistency, but naive about the other player, so $\gamma^{AB} = \gamma^{BA} = 1$ and $\gamma^{AA} = \gamma^{BB} = \beta$. The conditions then simplify to $1-\beta\delta < 1-\delta$, which is never satisfied. Players immediately reach an agreement, like they do in the standard model. Now, player A perceives a future B to be more patient that B himself perceives his future self to be. Hence, the offer of A is actually better than B was expecting to get, and he will gladly accept.

When players differ in their naivety, the outcome depends on who moves first. Consider a case in which player A is naive about both players, while B is sophisticated about both. Hence $\gamma^{AA} = \gamma^{AB} = 1$ and $\gamma^{BA} = \gamma^{BB} = \beta$. Conditions (12) and (13) then simplify to

$$\frac{1-\delta}{1-\delta^2} < \frac{1-\beta\delta}{1-\beta\delta^2}$$
$$\frac{1-\beta\delta}{1-\beta^2\delta^2} < \frac{1-\delta}{1-\delta^2}$$

It is easy to see that the first condition is always satisfied, while the second never is. We thus get some delay in bargaining: player B rejects the offer of player A, but player A accepts the counteroffer. When A moves first, B accepts immediately, perceiving the offer of A as overwhelmingly generous.

10 Conclusion

In this paper, we proposed a solution concept, perception-perfect equilibrium, for games played between hyperbolic discounters that are possibly naive about their own future time inconsistency, and/or the time inconsistency of their competitor. A perception-perfect equilibrium essentially requires each player in each period to play an action that is consistent with subgame perfection, given the perception of that player concerning the time consistency of each player, and under the assumption that all other present and future players have the same perceptions.

We applied our solution concept to the common pool problem and to Rubinstein bargaining. In both cases, we showed that, if we assume that players are sophisticated about their competitor's future time inconsistency but naive about their own, the perfection-perfect equilibria of those games are disastrous. The common pool is exhausted even more quickly than with standard, rational players, and even more quickly than with time-inconsistent but sophisticated players. Bargaining in the Rubinstein model breaks down completely, as each offer is rejected.

Of course, our approach is just the first step in the analysis of such games. There is much room for further analysis. For example, our perception-perfect equilibrium requires that players are strategically naive, in the sense that they do not take into account the possibility that other players may have different perceptions. Also, they do not learn from past behavior of other players. If offers in a bargaining game are rejected repeatedly, for example, one may expect players to take that into account and choose a somewhat different strategy when making further offers. Also, a highly sophisticated player may take advantage of her knowledge concerning the naivety of the other player to gain a strategic advantage.

Still, our framework is highly flexible and easily allows for extensions and modifications. For example, it is easy to allow for cases in which players are partially naive and realize their future time inconsistency to some limited extent. Also, it is straightforward to extend our perception-perfect equilibrium to a case with more than two types, or with more than two players. Our framework may even be applied to other (mis)perceptions and behavioral biases to which players are possibly unaware.

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