On the stability of recursive least squares in the Gauss-Markov model

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6. September 2004

Online at http://mpra.ub.uni-muenchen.de/58036/
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Problem and Motivation
Consider the classical model \( y_n = X_n \beta + \varepsilon_n \) where \( X_n \) is an \( n \times p \) real matrix of fixed regressors, \( y_n \) (\( n \times 1 \)) a response vector, \( \beta \) a \( p \times 1 \) vector of unknown coefficients, \( \text{rk}(X_n) = p \) for \( n \geq p \). Let \( \hat{\beta}(n) \) denote the ordinary least squares estimate of \( \beta \) obtained from \( n \) observations, with \( n \geq p \), and assume \( \varepsilon_n \) (\( n \times 1 \)) is a vector of non-observable random disturbances with expectation 0 and variance \( \sigma^2 I_n \).

An updating formula for \( \hat{\beta}(n + 1) \) as a function of \( \hat{\beta}(n) \) is

\[
\hat{\beta}(n + 1) - \beta = W^{-1} V (\hat{\beta}(n) - \beta) + w, \ n = p, p + 1, \ldots
\]

where \( V \equiv X_n' X_n \), \( W \equiv X_{n+1}' X_{n+1} \), \( w \equiv W^{-1} x \varepsilon_{n+1} \), and \( x \) denotes the vector of new observations at the values of the explanatory variables. Eq. (1) arises for example in Kalman filtering and recursive least squares theories, where the unknown \( \beta \) is considered as time-varying states of dynamic system (see the discussion in Kianifard and Swallow, 1996) and \( W^{-1} V \) is often developed as \( I_p - (1 + c)^{-1} V^{-1} x x' \); \( c \) equals \( x' V^{-1} x \).

This exercise provides some properties of \( W^{-1} V \), with all its eigenvalues and eigenvectors. Let \( A \equiv W^{-1} V \) have eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p \).

Show that

(i) these eigenvalues are real, and that

(ii) \( \lambda_1 = 1/(1 + c) \), \( \lambda_2 = \lambda_3 = \cdots = \lambda_p = 1 \).
Solution and Discussion

(i) \( A \) is the product between two real symmetric matrices. Let \( \lambda \) be an eigenvalue of \( A \), and \( u + iv \) an associated eigenvector, where \( i^2 = -1 \). Then

\[ A(u + iv) = \lambda(u + iv). \]

Premultiplying both sides of this equation with \( W \) leads to

\[ V(u + iv) = \lambda W(u + iv). \]

As \( W = V + xx' \) therefore the previous equation becomes

\[ (1 - \lambda)V(u + iv) = \lambda xx'(u + iv). \]

Premultiply both sides with \( (u - iv)' \). Because of the symmetry of \( V \) we obtain

\[ (1 - \lambda)(u'Vu + v'Vv) = \lambda((u'x)^2 + (v'x)^2). \]

This implies that \( \lambda \) is real.

\( \square \)

(ii) The following determinant

\[
|I_p - A| = |I_p - W^{-1}V| = |W^{-1}(W - V)| = |W^{-1}| |xx'| = |W|^{-1} \cdot 0 = 0
\]

shows \( \lambda = 1 \) is a root of the characteristic equation \( |\lambda I_p - A| = 0 \). Now, let \( z \) be an eigenvector of \( A \) associated with the eigenvalue 1; therefore
\( W^{-1}Vz = z \) or \( Vz = Wz \), which from the definition of \( W \) implies

\[
\begin{align*}
0 \\ (p \times 1)
\end{align*}
= xx'z,
\]
showing \( z \) is orthogonal to \( x \). Remaining eigenvalues of \( A \) are given using Wolkowicz and Styan’s inequalities. We need \( \text{trace}(A) \).

\[
\text{trace}(A) = \text{trace}(W^{-1}V) = \text{trace}(W^{-1}(W - xx')) = \text{trace}(I_p - W^{-1}xx') = p - x'W^{-1}x.
\]

Moreover, premultiplying \( W = V + xx' \) by \( x'W^{-1} \) and postmultiplying it by \( V^{-1}x \) implies \( x'W^{-1}x = c/(1 + c) \). Consequently

\[
\text{trace}(A) = p - c/(1 + c),
\]
and it can be shown \( x \) is an eigenvector of \( A \) and \( 1/(1 + c) \) the associated eigenvalue. Premultiplying \( A \) with \( x' \) gives

\[
x'A = x'(I_p - W^{-1}xx') = x' - (x'W^{-1}x)x' = (1 - \frac{c}{1 + c})x' = \frac{1}{1 + c}x'.
\]

As \( A \) has real eigenvalues we can apply the inequalities of Wolkowicz and Styan reproduced in Magnus and Neudecker (1991, p. 239) to find the order of multiplicity of previously found eigenvalues:

\[
m - s(p - 1)^{1/2} \leq \lambda_1 \leq m - \frac{s}{(p - 1)^{1/2}}
\]

\[
m + \frac{s}{(p - 1)^{1/2}} \leq \lambda_p \leq m + s(p - 1)^{1/2},
\]
where \( m = (1/p)\text{trace}(A) \) and \( s^2 = (1/p)\text{trace}(A^2) - m^2 \).

We obtain

\[
\frac{1}{1+c} \leq \lambda_1 \leq 1 - \frac{2}{p(1+c)} \frac{c}{1+c} \tag{2}
\]

\[
1 \leq \lambda_p \leq 1 + \frac{(p-2)}{p} \frac{c}{1+c} \tag{3}
\]

From Theorem 4 in Magnus and Neudecker (1991, p. 203),

\[
\begin{align*}
\lambda_1 & \leq \frac{x'Ax}{x'x} \leq \lambda_p \\
\Leftrightarrow \lambda_1 & \leq \frac{x'(I_p - W^{-1}xx')x}{x'x} \leq \lambda_p \\
\Leftrightarrow \lambda_1 & \leq 1 - \frac{x'W^{-1}x}{x'x} \leq \lambda_p \\
\Leftrightarrow \lambda_1 & \leq 1 - \frac{c}{1+c} \leq \lambda_p \\
\Leftrightarrow \lambda_1 & \leq \frac{1}{1+c} \leq \lambda_p.
\end{align*}
\]

Combination of Eq. (2) and this result gives \( \lambda_1 = 1/(1+c) \), which implies equality holds on the left of Eq. (3), that is \( \lambda_p = 1 \) and the \( p-1 \) largest eigenvalues are equal (Magnus and Neudecker, 1991, p. 239).

\[\square\]

References

