On the stability of recursive least squares in the Gauss-Markov model

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Problem and Motivation
Consider the classical model $y_n = X_n \beta + \varepsilon_n$ where $X_n$ is an $n \times p$ real matrix of fixed regressors, $y_n$ ($n \times 1$) a response vector, $\beta$ a $p \times 1$ vector of unknown coefficients, $\text{rk}(X_n) = p$ for $n \geq p$. Let $\hat{\beta}(n)$ denote the ordinary least squares estimate of $\beta$ obtained from $n$ observations, with $n \geq p$, and assume $\varepsilon_n$ ($n \times 1$) is a vector of non-observable random disturbances with expectation $0$ and variance $\sigma^2 I_n$.

An updating formula for $\hat{\beta}(n + 1)$ as a function of $\hat{\beta}(n)$ is

$$\hat{\beta}(n + 1) - \beta = W^{-1}V(\hat{\beta}(n) - \beta) + w, \ n = p, p + 1, \ldots \tag{1}$$

where $V \equiv X_n'X_n$, $W \equiv X_{n+1}'X_{n+1}$, $w \equiv W^{-1}x\varepsilon_{n+1}$, and $x$ denotes the vector of new observations at the values of the explanatory variables. Eq. (1) arises for example in Kalman filtering and recursive least squares theories, where the unknown $\beta$ is considered as time-varying states of dynamic system (see the discussion in Kianifard and Swallow, 1996) and $W^{-1}V$ is often developed as $I_p - (1 + c)^{-1}V^{-1}xx'$; $c$ equals $x'V^{-1}x$.

This exercise provides some properties of $W^{-1}V$, with all its eigenvalues and eigenvectors. Let $A \equiv W^{-1}V$ have eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$.

Show that

(i) these eigenvalues are real, and that

(ii) $\lambda_1 = 1/(1 + c)$, $\lambda_2 = \lambda_3 = \cdots = \lambda_p = 1$. 1
Solution and Discussion

(i) $A$ is the product between two real symmetric matrices. Let $\lambda$ be an eigenvalue of $A$, and $u + iv$ an associated eigenvector, where $i^2 = -1$. Then

$$A(u + iv) = \lambda(u + iv).$$

Premultiplying both sides of this equation with $W$ leads to

$$V(u + iv) = \lambda W(u + iv).$$

As $W = V + xx'$ therefore the previous equation becomes

$$(1 - \lambda)V(u + iv) = \lambda xx'(u + iv).$$

Premultiply both sides with $(u - iv)'$. Because of the symmetry of $V$ we obtain

$$(1 - \lambda)(u'Vu + v'Vv) = \lambda((u'x)^2 + (v'x)^2).$$

This implies that $\lambda$ is real.

\[\Box\]

(ii) The following determinant

$$|I_p - A| = |I_p - W^{-1}V|$$

$$= |W^{-1}(W - V)|$$

$$= |W^{-1}| \cdot |xx'|$$

$$= |W|^{-1} \cdot 0$$

$$= 0$$

shows $\lambda = 1$ is a root of the characteristic equation $|\lambda I_p - A| = 0$. Now, let $z$ be an eigenvector of $A$ associated with the eigenvalue 1; therefore
\( W^{-1}Vz = z \) or \( Vz = Wz \), which from the definition of \( W \) implies

\[
\begin{aligned}
0_{(p \times 1)} &= xx'z, \\
\end{aligned}
\]

showing \( z \) is orthogonal to \( x \). Remaining eigenvalues of \( A \) are given using Wolkowicz and Styan’s inequalities. We need \( \text{trace}(A) \).

\[
\begin{aligned}
\text{trace}(A) &= \text{trace}(W^{-1}V) \\
&= \text{trace}(W^{-1}(W - xx')) \\
&= \text{trace}(I_p - W^{-1}xx') \\
&= p - xx'W^{-1}x.
\end{aligned}
\]

Moreover, premultiplying \( W = V + xx' \) by \( xx'W^{-1} \) and postmultiplying it by \( V^{-1}x \) implies \( xx'W^{-1}x = c/(1 + c) \). Consequently

\[
\text{trace}(A) = p - c/(1 + c),
\]

and it can be shown \( x \) is an eigenvector of \( A \) and \( 1/(1 + c) \) the associated eigenvalue. Premultiplying \( A \) with \( x' \) gives

\[
\begin{aligned}
x'A &= x'(I_p - W^{-1}xx') \\
&= x' - (xx'W^{-1}x)x' \\
&= (1 - \frac{c}{1 + c})x' \\
&= \frac{1}{1 + c}x'.
\end{aligned}
\]

As \( A \) has real eigenvalues we can apply the inequalities of Wolkowicz and Styan reproduced in Magnus and Neudecker (1991, p. 239) to find the order of multiplicity of previously found eigenvalues:

\[
\begin{aligned}
m - s(p - 1)^{1/2} &\leq \lambda_1 \leq m - \frac{s}{(p - 1)^{1/2}} \\
m + \frac{s}{(p - 1)^{1/2}} &\leq \lambda_p \leq m + s(p - 1)^{1/2},
\end{aligned}
\]
where \( m = (1/p)\text{trace}(A) \) and \( s^2 = (1/p)\text{trace}(A^2) - m^2 \).

We obtain

\[
\frac{1}{1 + c} \leq \lambda_1 \leq 1 - \frac{2c}{p(1 + c)} \tag{2}
\]

\[
1 \leq \lambda_p \leq 1 + \frac{(p - 2)}{p} \frac{c}{1 + c}. \tag{3}
\]

From Theorem 4 in Magnus and Neudecker (1991, p. 203),

\[
\lambda_1 \leq \frac{x'Ax}{x'x} \leq \lambda_p
\]

\[
\Leftrightarrow \lambda_1 \leq \frac{x'(I_p - W^{-1}xx')x}{x'x} \leq \lambda_p
\]

\[
\Leftrightarrow \lambda_1 \leq 1 - x'W^{-1}x \leq \lambda_p
\]

\[
\Leftrightarrow \lambda_1 \leq 1 - \frac{c}{1 + c} \leq \lambda_p
\]

\[
\Leftrightarrow \lambda_1 \leq \frac{1}{1 + c} \leq \lambda_p.
\]

Combination of Eq. (2) and this result gives \( \lambda_1 = 1/(1 + c) \), which implies equality holds on the left of Eq. (3), that is \( \lambda_p = 1 \) and the \( p - 1 \) largest eigenvalues are equal (Magnus and Neudecker, 1991, p. 239).

□

References

