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Abstract

This paper studies equilibria of second price auctions when valuations and participation costs are both private information with general distribution functions. We consider the existence and uniqueness of equilibrium in this general framework of two-dimensional types. It is shown that there always exists an equilibrium, and further there exists a unique symmetric equilibrium when all bidders are ex ante homogeneous. Moreover, we identify a sufficient condition under which there is a unique equilibrium in a heterogeneous economy with two bidders. Our general result includes many existing results as special cases.

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Key Words: Two-Dimensional Types, Private Values, Private Participation Costs, Second Price Auctions, Existence and Uniqueness of Equilibrium.

1 Introduction

While an auction is an effective way to exploit private information by increasing the competition among buyers and thus can increase allocation efficiency, it is not freely implemented actually. In order to participate, bidders may have to incur some participation costs[1] that arise from

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[1]Some related terminology includes participation cost, participation fee, entry cost or opportunity cost. See Laffont and Green (1984), Samuelson (1986), McAfee and McMillan (1987), Harstad (1990), Levin and Smith (1994), etc.

many sources. For instance, sellers may require that bidders pay some participation fee; bidders themselves may have transportation costs to go to an auction spot; or they need spend some money to learn the rules of the auction and how to submit bids. Bidders may even have opportunity costs to attend an auction.

With participation costs, not all bidders are willing to participate. If a bidder's expected revenue from participating in the auction is less than the participation cost, he will not participate. Otherwise, the bidder participates and submits a bid accordingly. Even if a bidder decides to participate in an auction, since he may expect some other bidders will not participate in the auction, his bidding behavior may not be the same as that in the standard auction without participation costs.

The study of participation costs in auctions mainly focuses on second price auctions due to the simplicity of bidding behavior. In a standard second price auction, bidding one's true valuation is a weakly dominant strategy. There are also other equilibria in the standard second price auction as shown in Blume and Heidhues (2004); for example, the bidder with the highest value bids his true value and all others bid zero. This is referred to as the asymmetric bidding equilibrium in the standard second price auction. However, he cannot do better than bidding his true value. This is also true in second price auctions in the presence of participation costs. Therefore in this paper we only consider equilibria in which each bidder uses a cutoff strategy; i.e., bids his true value if participation is optimal, and does not participate otherwise.

Laffont and Green (1984) were the first to study the second price auction with participation costs when bidders' valuations and participation costs are both private information. However, their proof on the existence and uniqueness of equilibrium is problematic.They applied the Banach fixed-point theorem (also known as the contraction mapping theorem or contraction mapping principle) to show the existence and uniqueness result, however the condition for applying the theorem is not satisfied.[2] In fact, the study on equilibria in mechanism design in general and auction design in particular is much harder and more complicated than they thought in the setting of multiple dimensional types due to the lack of a natural order on types. As such, one then turns to studying the existence of equilibria in a unidimensional economic environment where either valuations or participation costs are private information, such those studies in Campbell (1998), Tan and Yilankaya (2006), Cao and Tian (2010). Besides, Laffont and Green (1984) imposed a restrictive assumption that values and participation costs are uniformly distributed.

Campbell (1998) and Tan and Yilankaya (2006) studied the existence of equilibria in second

[2]See details in the Appendix B.

price auction in an economic environment when bidders' values are private information and participation costs are common knowledge and the same. They found that asymmetric equilibria may exist when bidders are ex ante homogeneous. Uniqueness of the equilibrium cannot be guaranteed. Some other studies, including Samuelson (1985), Stegeman (1996), Levin and Smith (1994), etc, also assumed that participation costs are the same across players. While the assumption of equal participation costs is stringent and unrealistic, Cao and Tian (2013) investigated the equilibria when bidders may have differentiated participation costs. They introduced the notions of monotonic equilibrium and non-monotonic equilibrium. Cao and Tian (2010) studied similar problems for first price auction in an economic environment with equal participation costs. On the other hand, Kaplan and Sela (2006) simplified the framework of Laffont and Green in another way. They studied equilibria of second price auctions with participation costs when bidders' participation costs are private information and are drawn from the same distribution function, while valuations are common knowledge.

The existence of equilibria in second price auctions with privately-informed valuations and participation costs then remains unanswered. This paper aims to fill the gap. We consider the existence and uniqueness of (Bayesian-Nash) equilibria in sealed-bid second price, or Vickrey auctions with bidder participation costs in a general two-dimensional economic environment. Special cases of this general specification include that either the valuations or participation costs are common knowledge, as those having been investigated in the existing literature.

Under a general two-dimensional distribution of the bidders' participation costs and valuations we prove that equilibria always exist, and further there exists a unique symmetric equilibrium when all bidders are ex ante homogeneous (i.e., bidders have the same distributions). Moreover, we identify the conditions under which we have a unique equilibrium (as such, there is no asymmetric equilibrium) in a simple two-bidder economy. Special cases where multiple equilibria may exist are also discussed. There may exist an equilibrium in which one bidder never participates or an equilibrium in which one bidder always participates.

Thus, our general framework establishes the existence of equilibrium and uniqueness of symmetric equilibrium not only in the two-dimensional uniform setting as studied in Laffont and Green (1984), but also in many other two-dimensional settings such as truncated normal distributions, exponential distributions, etc. Moreover, our framework can deal with asymmetric equilibria as considered in literature with one-dimensional private information, such as those in Campbell (1998), Tan and Yilankaya (2006) and Cao and Tian (2013).

The remainder of the paper proceeds as follows. Section 2 describes a general setting of economic environments. Section 3 establishes the existence of equilibrium. The uniqueness of equilibrium is discussed in Section 4. In Section 5 we give a brief discussion on multiple equilibria. Concluding remarks are provided in Section 6. All the proofs are in appendix A.

2 The Setup

We consider an independent value economic environment with one seller and n buyers. Let $N = \{1, 2, \ldots, n\}$. The seller has an indivisible object which he values at zero to sell to one of the buyers. The auction format is the sealed-bid second price auction (see Vickrey, 1961). In order to submit a bid, bidder i must pay a participation cost c_i. Buyer i's value for the object, v_i, and participation cost c_i are private and independently drawn from the distribution function $K_i(v_i, c_i)$ with the support $[0,1] \times [0,1]$. Let $k_i(v_i, c_i)$ denote the corresponding density function. In particular, when v_i and c_i are independent, we have $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, where $F_i(v_i)$ and $G_i(c_i)$ are the cumulative distribution functions of bidder i's valuation and participation cost, $f_i(v_i)$ and $g_i(c_i)$[3] are the corresponding density functions.

Each bidder knows his own value and participation cost before he makes his entrance decision and does not know others' decisions when he makes his own. If bidder i decides to participate in the auction, he pays a non-refundable participation cost c_i and submits a bid. The bidder with the highest bid wins the object and pays the second highest bid. If there is only one person in the auction, he wins the object and pays 0. If there is a tie, the allocation is determined by a fair lottery. The bidder who wins the object pays his own bid.

In this second price auction mechanism with participation costs, the individually rational action set for any type of bidder is :$\{No\} \cup (0, 1]$[4], where "$\{No\}$" denotes not participating in the auction. Bidder i incurs the participation cost if and only if his action is different from "$\{No\}$". Bidders are risk neutral and they will compare the expected payoffs from participating with participation costs to decide whether or not to participate. If the expected payoff from participating is less than the costs, they will not participate. Otherwise, they will participate and submit bids. Further if a bidder finds participating in this second price auction optimal, he cannot do better than bidding his true valuation (i.e., bidding his true value is a weakly

[3]When v_i or c_i takes discrete values, their density functions $f_i(v)$ and $g_i(c_i)$ are reduced to the discrete probability distribution functions, which can be represented by the Dirac delta function. The density at the discrete point is infinity.

[4]For completeness, we assume a bidder with valuation 0 and participation cost 0 does not participate in the auction. The strategy of $\{No\}$ will be denoted by 0.

dominant strategy).[5]

Given the equilibrium strategies of all others, a bidder's expected payoff from participating in the auction is a non-decreasing function of his valuation. Putting it differently, the maximum one would like to pay to participate in an auction is a non-decreasing function of his valuation. Therefore, we can focus on Bayesian-Nash equilibria in which each bidder uses a cutoff strategy[6] denoted by $c_i^*(v_i)$, i.e., one bids his true valuation if his participation cost is less than some cutoff and does not participate otherwise. An equilibrium strategy of each bidder i is then determined by the expected payoff of participating in the auction $c_i^*(v_i)$ when his value is v_i[7]. Let $b_i(v_i, c_i)$ denote bidder i's strategy. Then the bidding decision function can be characterized by

$$b_i(v_i, c_i) = \begin{cases} v_i & \text{if } 0 \leq c_i \leq c_i^*(v_i) \\ \text{No} & \text{otherwise.} \end{cases}$$

Remark 1 At an equilibrium, $c_i^*(v_i) > 0$ is a cost cutoff (critical) point such that individual i is indifferent from participating in the auction or not. Bidder i will participate in the auction whenever $0 < c_i \leq c_i^*(v_i)$. Note that at equilibrium, we have $c_i^*(v_i) \leq v_i$.

The description of the equilibria can be slightly different under different informational structures on $K_i(v_i, c_i)$:

(1) v_i is private information and c_i is common knowledge to all bidders. In this case, $K_i(v_i, c_i) = F_i(v_i)$. Campbell (1998), Tan and Yilankaya (2006) and Cao and Tian (2013) studied this special case. The equilibrium is described by a valuation cutoff v_i^* for each bidder i. Bidder i submits a bid when $v_i \geq v_i^*$.

(2) c_i is private information and v_i is common knowledge to all bidders. In this case, $K_i(v_i, c_i) = G_i(c_i)$. Kaplan and Sela (2006) investigated this kind of economic environment. The equilibrium is described by a cost cutoff point c_i^* for each bidder i. Bidder i submits a bid when $c_i \leq c_i^*$.

3 The Existence of Equilibrium

Suppose, provisionally, there exists an equilibrium in which each bidder i uses $c_i^*(v_i)$ as his entrance decision making. Then for bidder i with value v_i, when his participation cost $c_i \leq c_i^*(v_i)$,

[5]There may exist an equilibrium in which bidders do not bid their true value when they participate. See the special example constructed in Cao and Tian (2013).

[6]Lu and Sun (2007) showed that for any auction mechanism with participation costs, the participating and nonparticipating types of any bidder are divided by a nondecreasing and equicontinuous shutdown curve.

[7]In equilibrium, $c_i^*(v_i)$ depends on the distributions of all bidders' valuations and participation costs.

the bidder will participate in the auction and submit his weakly dominant bid, or else he will stay out[8]. For bidder i, to submit a bid v_i, he should participate in the auction first; i.e., $c_i \leq c_i^*(v_i)$. So the density of submitting a bid v_i is

$$f_{c_i^*(v_i)}(v_i) = \int_0^{c_i^*(v_i)} k_i(v_i, c_i) dc_i.$$

Remark 2 When v_i and c_i are independent, bidder i with value v_i will submit the bid v_i with probability $G_i(c_i^*(v_i))$ and stay out with probability $1 - G_i(c_i^*(v_i))$.

$f_{c_i^*(v_i)}(0)$ refers to the probability (density) that bidder i does not submit a bid. Let $F_{c_i^*(v_i)}(v_i)$ be the corresponding cumulative probability. Note that there is a mass at $v_i = 0$ for $F_{c_i^*(v_i)}(v_i)$. For each bidder i, let the maximal bid of the other bidders be m_i. Note that, if $m_i > 0$, at least one of the other bidders participates in the auction. If $m_i = 0$, no other bidders participates in the auction.

The payoff of participating in the auction for bidder i with value v_i is given by $\int_0^{v_i}(v_i - m_i)d\prod_{j\neq i} F_{c_j^*}(m_i)$, and thus the zero expected net-payoff condition for bidder i to participate in the auction when his valuation is v_i requires that

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j\neq i} F_{c_j^*}(m_i).$$

With some algebra derivations, we have

Lemma 1

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i. \tag{1}$$

Remark 3 When v_i and c_i are independent, $K_i(v_i, c_i) = F_i(v_i)G_i(c_i)$ and $k_i(v_i, c_i) = f_i(v_i)g_i(c_i)$, we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j\neq i} [1 - \int_{m_i}^1 G_j(c_j(\tau)) f_j(\tau) d\tau] dm_i.$$

Taking derivative of equation (1) with respect to v_i, we have

$$c_i^{*\prime}(v_i) = \prod_{j\neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau]. \tag{2}$$

Notice that $c_i^*(0) = 0$, thus the above equation is a functional differential equation with the initial condition. Specially when v_i and c_i are independent,

$$c_i^{*\prime}(v_i) = \prod_{j\neq i} [1 - \int_{v_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau].$$

[8] $c_i^*(v_i)$ can be interpreted as the maximal amount that bidder i would like to pay to participate in the auction when his value is v_i.

Lemma 2 *$c_i^*(v_i)$ has the following properties:*

(i) $c_i^*(0) = 0$.

(ii) $0 \leq c_i^*(v_i) \leq v_i$.

(iii) $c_i^{*\prime}(1) = 1$.

(iv) $\frac{dc_i^*(v_i)}{dn} < 0$.

(v) $\frac{dc_i^*(v_i)}{dv_i} \geq 0$ *and* $\frac{d^2c_i^*(v_i)}{dv_i^2} \geq 0$.

(i) means that, when bidder i's value for the object is 0, the value of participating in the auction for bidder i is zero and thus the cost cutoff point for the bidder to enter the auction is zero. Then, as long as the bidder has participation cost bigger than zero, he will not participate in the auction.

(ii) means that a bidder will not be willing to pay more than his value to participate in the auction.

(iii) means that, when a bidder's value is 1, the marginal willingness to pay to enter the auction is 1. The intuition is that when his value for the object is 1, he will win the object almost surely. Then the marginal willingness to pay is equal to the marginal increase in the valuation.

(iv) states that the participation cutoff point is a nondecreasing function in the number of bidders. As the number of bidders increases, the probability to win the object will decrease, holding other things constant. More bidders will increase the competition among the bidders and thus reduce the expected payoff.

(v) states that the marginal willingness to pay is positive and increasing. The intuition is that when a bidder's value increases, the probability of winning the auction increases. The willingness to pay increases and so does the marginal willingness to pay.

Definition 1 For the economic environment under consideration, a cutoff curve equilibrium is an n-dimensional plane comprised of $(c_1^*(v_1), c_2^*(v_2), ...c_n^*(v_n))$ that is a solution of the following equation system:

$$(P1)\begin{cases} c_1^*(v_1) = \int_0^{v_1} \prod_{j\neq 1}[1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_1 \\ c_2^*(v_2) = \int_0^{v_2} \prod_{j\neq 2}[1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_2 \\ \vdots \\ c_n^*(v_n) = \int_0^{v_n} \prod_{j\neq n}[1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_n. \end{cases}$$

The differential equation system above is a partial functional differential equation system, but not a partial differential equation system. The derivative of $c_i^*(v_i)$ at v_i depends not only on v_i itself, but also on the future path of $c_i^*(v_j)$ with $j \neq i$ and $v_j \geq v_i$. Besides, we have multiple variables in the functional differential equation system, which increases the difficulty to study the existence of equilibrium. To overcome this multiple variable problem, we transfer the original differential equation system to the following integral equation system

$$(P2)\begin{cases} c_1^*(v) = \int_0^v \prod_{j\neq 1}[1 - \int_{m_1}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_1 \\ c_2^*(v) = \int_0^v \prod_{j\neq 2}[1 - \int_{m_2}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_2 \\ \vdots \\ c_n^*(v) = \int_0^v \prod_{j\neq n}[1 - \int_{m_n}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_n. \end{cases}$$

Lemma 3 Problem (P1) and problem (P2) are equivalently solvable in the sense that
(1), if $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1), then $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to problem (P2).
(2), if $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to problem (P2), then $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1).

Thus we have reduced the multiple variables functional differential equation system to a single variable functional equation system. We then have the following result on the existence of equilibrium $(c_1^*(v), c_2^*(v), ...c_n^*(v))$:

Proposition 1 (The Existence Theorem) *For the general economic environment under consideration, the integral equation system (P2) has at least one solution $(c_1^*(v), c_2^*(v), ...c_n^*(v))$; i.e., there is always an equilibrium in which every bidder i uses his own cutoff curve $c_i^*(v)$.*

Remark 4 When v_i and c_i are independent, the equilibrium is an n-dimensional plane composed of $(c_1^*(v), c_2^*(v), ...c_n^*(v))$ that is a solution of the following integral equation system:

$$(P3)\begin{cases} c_1^*(v) = \int_0^v \prod_{j\neq 1}[1 - \int_{m_1}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_1 \\ c_2^*(v) = \int_0^v \prod_{j\neq 2}[1 - \int_{m_2}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_2 \\ \vdots \\ c_n^*(v) = \int_0^v \prod_{j\neq n}[1 - \int_{m_n}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_n. \end{cases}$$

The above setting with two-dimensional private values and participation costs with general distribution functions is very general and contains many existing results as special cases, as discussed in Appendix C.

4 Uniqueness of Equilibrium

To investigate the uniqueness of equilibrium $c^*(v)$, we first consider the case where all bidders are ex ante homogeneous in the sense that they have the same joint distribution functions of valuations and participation costs and focus on the symmetric equilibrium in which all bidders use the same cutoff curve.

(P2) can be rewritten as

$$c^*(v) = \int_0^v [1 - \int_m^1 \int_0^{c^*(\tau)} k_j(\tau, c) dc d\tau]^{n-1} dm, \tag{3}$$

and correspondingly we have

$$c^{*\prime}(v) = [1 - \int_v^1 \int_0^{c^*(\tau)} k(\tau, c) dc d\tau]^{n-1}, c^*(0) = 0. \tag{4}$$

We the have the following result.

Proposition 2 (Uniqueness of Symmetric Equilibrium) *Suppose that all bidders have the same distribution function $K(v, c)$. There is a unique symmetric equilibrium at which each bidder uses the same cutoff strategy.*

Remark 5 Uniqueness of the symmetric equilibrium has been established in some special cases.

1) In Campbell (1998) and Tan and Yilankaya (2006), when bidders have the same participation cost and continuously differentiable valuation distribution function, there is a unique symmetric equilibrium in which each bidder uses a same cutoff point v^s for his entrance decision making.

2) In Kaplan and Sela (2006), when all bidders have the same valuations for the object and continuously differentiable participation cost distribution functions, there is a unique symmetric cutoff point c^*.

3) Earlier, Laffont and Green (1984) investigated the existence of equilibria when both valuations and participation costs are uniformly distributed. They got the uniqueness of the symmetric equilibrium under the simple two-dimensional economic environment. However, their proof is incomplete.

Remark 6 Note that the above proposition only shows the uniqueness of symmetric equilibrium when bidders are ex ante homogeneous. It does not exclude the possibility of asymmetric equilibrium. As shown by counter-example in Campbell (1998), Tan and Yilankaya (2006), Kaplan and Sela (2006), there are asymmetric equilibria where ex ante homogeneous bidders use different cutoff strategy, which means the equilibria are not unique.

As such, generally the uniqueness of equilibrium cannot be guaranteed (we will also discuss this in the next section). However, for an economy with only two bidders and independent participation costs and valuations, we can have the uniqueness result.

To show this, write corresponding functional differential equation system as:

$$(P6)\begin{cases} c_1^{*\prime}(v) = [1 - \int_v^1 G_2(c_2^*(\tau))f_2(\tau)d\tau], c_1^*(0) = 0, \\ c_2^{*\prime}(v) = [1 - \int_v^1 G_1(c_1^*(\tau))f_1(\tau)d\tau], c_2^*(0) = 0. \end{cases}$$

We then have the following result.

Proposition 3 (Uniqueness of Equilibrium) In the two-bidder economy where $G_i(c)$ is continuously differentiable on $[0,1]$ and $\delta_i = \max_c g_i(c)$, there is a unique equilibrium when $\delta_i \int_0^1 (1 - F_i(s))ds < 1$.

When $G_i(c_i)$ is uniform on $[0,1]$, $\delta_i = 1$ and $\int_0^1(1-F_i(s))ds < 1$, we have a unique equilibrium. Especially, when bidders are ex ante homogeneous, the unique equilibrium is symmetric. To see this, consider the following examples.

Example 1 Now we assume $G_i(c)$ and $F_i(v)$ are both uniform on $[0,1]$. At equilibrium we have

$$c_1^{*\prime}(v) = 1 - \int_v^1 c_2^*(\tau)d\tau,$$

$$c_2^{*\prime}(v) = 1 - \int_v^1 c_1^*(\tau)d\tau.$$

Then $c_1^{*\prime\prime}(v) = c_2^*(v)$ and $c_2^{*\prime\prime}(v) = c_1^*(v)$. Thus we have $c_1^{*(4)}(v) = c_1^*(v)$ and $c_2^{*(4)}(v) = c_2^*(v)$ with $c_1^*(0) = 0$, $c_1^{*\prime}(1) = 1$, $c_2^*(0) = 0$ and $c_2^{*\prime}(1) = 1$. One can check that the only equilibrium is $c_1^*(v) = c_2^*(v) = ae^v - ae^{-v}$, where $a = \frac{e}{e^2+1}$.

5 Discussions

There are in general multiple equilibria. Examples can be found in Campbell (1998), Tan and Yilankaya (2006), Cao and Tian (2013) and Kaplan and Sela (2006) where either participation costs or valuations are common knowledge. In this section we provide an example for the multiplicity of equilibria when both participation costs and valuations are private information.

Suppose the support of v_i and c_i to be $[0,1] \times [\epsilon, \delta]$, where $[\epsilon, \delta]$ is a subset of $[0,1]$ and $\epsilon > 0$. To investigate the existence of equilibrium, we construct a new density function $\widetilde{k_i}(v_i, c_i)$ with support $[0,1] \times [0,1]$ which has the same density as $k_i(v_i, c_i)$ on the interval $[0,1] \times [\epsilon, \delta]$ and

0 otherwise and $\widetilde{K_i}(v_i, c_i)$ is the corresponding cumulative density function. The same as in Section 3, the equilibrium cutoff curve for individual i, $i \in 1, 2, ...n$, is given by

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau] dm_i,$$

with

$$c_i^{*\prime}(v_i) = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau]. \tag{5}$$

By the fixed point theorem, an equilibrium exists. However, the uniqueness of the equilibrium cannot be guaranteed. Specially, when bidders are ex ante homogeneous, asymmetric equilibria may exist.

One special type of asymmetric equilibrium is that some bidders may never participate in the auction. This can happen when the support of participation costs, c, has non-zero lower bound. One implication of such equilibrium is that in this economic environment, some of the bidders can form a collusion to enter the auction regressively so that they can prevent some others entering the auction and thus can reduce the competition among those who participate in the auction, which in turn will increase the benefits from participating.

The expected payoff of participating in the auction is a non-decreasing function of one's true value. Thus the sufficient and necessary condition for a bidder to never participate is that when his value is 1, participating in the auction still gives him an expected payoff that is less than the minimum participation cost, ϵ, given the strategies of other bidders. Formally, suppose in equilibrium, a subset $A = \{1, 2, \ldots, k\} \subset \{1, 2, 3 \ldots, n\}$ of bidders choose to participate in the auction when their valuations are large enough and bidders in $B = \{k+1, \ldots, n\}$ choose never to participate in the auction. Then for all $i \in A$ we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i, j \in A} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} \widetilde{k}_j(\tau, c_j) dc_j d\tau] dm_i.$$

For bidders in B to never participate, it is required that for all $j \in B$,

$$c_j^*(1) = \int_0^1 \prod_{i \in A} [1 - \int_{m_j}^1 \int_0^{c_i^*(\tau)} \widetilde{k}_i(\tau, c_j) dc_j d\tau] dm_j < \epsilon,$$

which raises a requirement for the lower and upper bound of the participation costs and the distributions of valuations and participation costs. To see this, we assume that there are only two bidders and v_i and c_i are independent. The distribution functions are $F(v_i)$ and $G(c_i)$ separately.

Suppose bidder 2 never participates, then bidder 1 enters if and only if $v_1 \geq c_1$ and thus we have $c_1^*(v_1) = v_1$. Given this, the expected payoff of bidder 2 when he participates in the auction is

$$F(\epsilon) + \int_{\epsilon}^{\delta} [(1 - v_2)G(v_2) + (1 - G(v_2))]dF(v_2) + \int_{\delta}^{1} (1 - v_2)dF(v_2)$$

when $v_2 = 1$. We have three terms in the above equation. When bidder 1's value is less than ϵ, he will not enter the auction and bidder 2 will get payoff 1, and the probability is $F(\epsilon)$. The second term is the payoff when bidder 1's value is between ϵ and δ. For any $v_2 \in (\epsilon, \delta)$, bidder 2's payoff is $1 - v_2$ when bidder 1 participates and is 1 when bidder 1 does not participate, and the probabilities are $G(v_2)$ and $1 - G(v_2)$ separately. The third term is the payoff when bidder 1's value is greater than δ and in this case bidder 1 participates for sure.

In order to have a corner equilibrium, we need

$$F(\epsilon) + \int_{\epsilon}^{\delta} [(1 - v_2)G(v_2) + (1 - G(v_2))]dF(v_2) + \int_{\delta}^{1} (1 - v_2)dF(v_2) < \epsilon. \tag{6}$$

It can be seen that in the two homogeneous bidders economy, when $F(\cdot)$ is concave, there is no corner equilibrium. To see this, note that when $F(\cdot)$ is concave, we have $F(v_i) \geq v_i$, equation (6) cannot hold; i.e., corner equilibrium does not exist.

Remark 7 If $\epsilon = \delta$; i.e., c_i is common knowledge to all bidders, (6) can be simplified to $F(\epsilon) + \int_{\epsilon}^{1} (1 - v_2)dF(v_2) < \epsilon$; i.e., $\epsilon F(\epsilon) + \int_{\epsilon}^{1} F(v_2)dv_2 < \epsilon$.

Example 2 Assume v_i and c_i to be jointly uniformly distributed (then they are independent) and there are only two bidders. Suppose bidder 2 never participates. We have $c_1^*(v_1) = v_1$, and thus

$$c_2^{*\prime}(v_2) = 1 - \int_{v_2}^{1} G(c_1(\tau))d\tau = 1 - \int_{v_2}^{1} min\{1, max\{\frac{\tau - \epsilon}{\delta - \epsilon}, 0\}\}d\tau,$$

which leads to

$$c_2^{*\prime}(v_2) = \begin{cases} 1 - \int_{\epsilon}^{\delta} \frac{\tau-\epsilon}{\delta-\epsilon} d\tau - \int_{\delta}^{1} d\tau = \frac{\epsilon+\delta}{2} \quad \text{if } v_2 < \epsilon \\ \delta - \frac{\delta^2 - 2\epsilon\delta - v_2^2 + 2\epsilon v_2}{2(\delta-\epsilon)} = \frac{\delta^2 + v_2^2 - 2\epsilon v_2}{2(\delta-\epsilon)} \quad \text{if } \epsilon \leq v_2 < \delta \\ v_2 \quad \text{if } v_2 \geq \delta \end{cases}.$$

Given the above and the initial condition $c_2^*(0) = 0$, we have

$$c_2^*(v_2) = \begin{cases} \frac{\epsilon+\delta}{2} v_2 \quad \text{if } v_2 < \epsilon \\ \frac{v_2^3 - 3\epsilon v_2^2 - \epsilon^3 + 3\delta^2 v_2}{6(\delta-\epsilon)} \quad \text{if } \epsilon \leq v_2 < \delta \\ \frac{\delta^2 + \delta\epsilon + \epsilon^2 + 3v_2^2}{6} \quad \text{if } v_2 \geq \delta \end{cases}.$$

For bidder 2 to never participate, we need $c_2^*(1) = \frac{\delta^2+\delta\epsilon+\epsilon^2+3}{6} \leq \epsilon$, which is equivalent to $\epsilon^2 + (\delta - 6)\epsilon + \delta^2 + 3 \leq 0$. Therefore, when

$$\frac{(6 - \delta) - \sqrt{-3(\delta^2 + 4\delta - 8)}}{2} \leq \epsilon \leq \frac{(6 - \delta) + \sqrt{-3(\delta^2 + 4\delta - 8)}}{2},$$

the required condition is satisfied. For this to be true, we need $\frac{(6-\delta)-\sqrt{-3(\delta^2+4\delta-8)}}{2} < \delta$ and thus $\delta^2 - 2\delta + 1 < 0$, which cannot be true.

However, when $F(\cdot)$ is strictly convex, given proper ϵ and δ, there may be an equilibrium in which one bidder never participates while the other enters the auction whenever his valuation is greater than his participation cost. As an illustration, suppose $F(v_i) = v_i^2$ and $G(c_i)$ is uniformly distributed on $[\epsilon, \delta]$. (6) becomes

$$\frac{\delta^3 + \delta\epsilon^2 + \delta^2\epsilon + \epsilon^3 + 2}{6} < \epsilon.$$

One can check that when $\epsilon = 0.5$ and $\delta = 0.744$, there exists an asymmetric equilibrium.

The other special type of asymmetric equilibrium is that when the lower bound of valuation is positive, one bidder always participates. To see this, suppose the c_i is distributed on $[c_l, c_h]$ with distribution $G_i(c_i)$ and v_i is distributed on $[v_l, v_h]$ with distribution $F_i(v_i)$, assuming $v_h > v_l > c_h > c_l$. Suppose also we have an equilibrium in which bidder 1 always enters and bidder 2 never participates. Then bidder 1 always participates is a best response. For bidder 2's strategy to be a best response, we need

$$\int_{v_l}^{v_h} (v_h - v_1) dF_1(v_1) - c_l < 0,$$

the maximum expected payoff is less than the lowest participation cost. Using integration by parts we have

$$\int_{v_l}^{v_h} F_1(v_1) dv_1 < c_l.$$

One sufficient condition for this to be true is $v_h - v_l < c_l$.

6 Conclusion

This paper investigates equilibria of second price auctions in economic environments with general distribution functions when values and participation costs are both privately informed. We show that there always exists an equilibrium cutoff strategy for each bidder. Moreover, when all bidders are ex ante homogeneous, there is a unique symmetric equilibrium. In a simple two-bidder economy, a sufficient condition for the uniqueness of the equilibrium is identified.

We also show that multiple equilibria may exist. Specifically, when bidders are ex ante homogeneous, there may also exist an asymmetric equilibrium at which one bidder always participates or never participates. Future research may be focused on identifying sufficient conditions to guarantee the uniqueness of equilibrium for more general economic environments.

Appendix A

Proof of Lemma 1:

If $m_i = 0$, none of the other bidders will participate, the probability of which is

$$\prod_{j\neq i} F_{c_j^*}(0) = \prod_{j\neq i} \int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau.$$

Otherwise, at least one other bidder submits a bid. Then

$$\prod_{j\neq i} F_{c_j^*}(m_i) = \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau].$$

Thus, the cutoff curve for individual i, $i \in 1, 2, ...n$, can be characterized by

$$c_i^*(v_i) = \int_0^{v_i} (v_i - m_i) d \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j\neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau].$$

With integrating by parts, we have

$$\begin{aligned} c_i^*(v_i) &= \int_0^{v_i} (v_i - m_i) d \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j\neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\ &= (v_i - m_i) \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] \mid_0^{v_i} + v_i \prod_{j\neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\ &\quad + \int_0^{v_i} \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \\ &= -v_i \prod_{j\neq i} [1 - \int_0^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] + v_i \prod_{j\neq i} [\int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau] \\ &\quad + \int_0^{v_i} \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i. \end{aligned}$$

Since

$$\int_0^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau + \int_0^1 \int_{c_j^*(\tau)}^1 k_j(\tau, c_j) dc_j d\tau = \int_0^1 \int_0^1 k_j(\tau, c_j) dc_j d\tau = 1,$$

we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i.$$

Proof of Lemma 2:

Proof: (i) Letting $v_i = 0$ in the expression of $c_i^*(v_i)$, we have the result.

(ii) Since

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j\neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j) dc_j d\tau] dm_i \leqslant \int_0^{v_i} dv_i = v_i$$

by the nonnegativity of $\int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j$ and

$$\int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau \leq \int_{m_i}^1 \int_0^1 k_j(\tau, c_j)dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j)dc_j d\tau = 1,$$

we have $0 \leq c_i^*(v_i) \leq v_i$.

(iii) Letting $v_i = 1$ in (5), we have the result.

(iv) Since n is the number of bidders, as n increases, say, from n to $n+1$, the product term inside the integral will be increased by one more term. Also, note that $0 < 1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau < 1$. So given more bidders, $c_i^*(v_i)$ will decrease.

(v)

$$\frac{dc_i^*(v_i)}{dv_i} = \prod_{j \neq i} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \geq 0$$

by noting that

$$\int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau \leq \int_0^1 \int_0^1 k_j(\tau, c_j)dc_j d = 1.$$

We then have

$$\frac{d^2 c_i^*(v_i)}{dv_i^2} = \sum_{k \neq i} \prod_{j \neq i, j \neq k} [1 - \int_{v_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau] \int_0^{c_k^*(v_i)} k_j(\tau, c_j)dc_j d\tau \geq 0.$$

Proof of Lemma 3:

Proof: Suppose $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to problem (P1), then we have for any $i \in \{1, 2, ...n\}$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i,$$

then by changing the variable v_i to v we have

$$c_i^*(v) = \int_0^{v} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i$$

for all $i \in \{1, 2, ...n\}$. So $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to (P2). On the contrary, if $(c_1^*(v), c_2^*(v), ..., c_n^*(v))$ is a solution to (P2), then we have for any $i \in \{1, 2, ...n\}$,

$$c_i^*(v) = \int_0^{v} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i.$$

Then by changing the variable v to v_i in the i^{th} equation we have

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j \neq i} [1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c_j)dc_j d\tau]dm_i.$$

Thus $(c_1^*(v_1), c_2^*(v_2), ..., c_n^*(v_n))$ is a solution to (P1).

Proof of Proposition 1:

Let $h_i(m_i, c^*) = \prod_{j\neq i}[1 - \int_{m_i}^1 \int_0^{c_j^*(\tau)} k_j(\tau, c) dc d\tau]$. Since $k_j(\tau, c)$ is integrable over c as it is a density function, there exists a continuous function $\gamma_j(\tau, c)$ with $\frac{\partial \gamma_j(\tau,c)}{\partial c} = k_j(\tau, c)$ such that $\int_0^{c_j^*(\tau)} k_j(\tau, c)dc = \gamma_j(\tau, c_j^*(\tau)) - \gamma_j(\tau, 0)$. Thus $h_i(m_i, c^*) = \prod_{j\neq i}[1 - \int_{m_i}^1 [\gamma_j(\tau, c_j^*(\tau)) - \gamma_j(\tau, 0)]d\tau]$, which is a continuous mapping from $[0, 1] \times [0, 1]^n \to [0, 1]$.

Let $H(m, c^*) = (h_1(m_1, c^*), h_2(m_2, c^*), \cdots , h_n(m_n, c^*))'$, which is a continuous mapping from $[0, 1]^n \times [0, 1]^n \to [0, 1]^n$. By Lemma 2, H is bounded above by one. Define

$$M = \{c \in \varphi \mid: \|c\| \leq 1\},$$

where φ is the space of continuous function ϕ defined on $[0, 1]^n \to [0, 1]^n$ with $\|c\| = \sup_{0\leq v\leq 1} c(v)$. Then by Ascoli Theorem, M is compact. M is clearly convex. Define an operator $P : M \to M$ by

$$(Pc)(v) = \int_0^v H(s, c(.))ds$$

Then, by Lemma 2, P is a continuous function from M to itself. Thus, by Schauder-Tychonoff Fixed-point Theorem, there exists a fixed point; i.e., a solution for the functional differential equation system exists.

Proof of Proposition 2:

Proof: The existence of the symmetric equilibrium can be established by the Schauder-Tychonoff Fixed-point Theorem. Here we only need to prove the uniqueness of the symmetric equilibrium. Suppose not, by way of contradiction, we have two different symmetric equilibria $x(v)$ and $y(v)$ to the economic environment we consider. Then we have

$$x'(v) = [1 - \int_v^1 \int_0^{x(\tau)} k(\tau, c)dcd\tau]^{n-1}$$

$$y'(v) = [1 - \int_v^1 \int_0^{y(\tau)} k(\tau, c)dcd\tau]^{n-1}.$$

Suppose $x(1) > y(1)$, then by the continuity of $x(v)$ and $y(v)$ we can find a v^* such that $x(v^*) = y(v^*) = c(v^*)$ and $x(v) > y(v)$ for all $v \in (v^*, 1]$ by noting that $x(0) = y(0)$.

Case 1: If $k(v, c) > 0$ with positive probability measure on $(v^*, 1) \times (c(v^*), 1)$, then for $\tau \in (v^*, 1]$ we have

$$\int_0^{x(\tau)} k(\tau, c)dc > \int_0^{y(\tau)} k(\tau, c)dc$$

for $\tau \in (v^*, 1)$. Then we have $x'(v^*) < y'(v^*)$ which is a contradiction to $x(v) > y(v)$ for $v > v^*$. So we have $x(1) = y(1)$.

Now suppose there exists an interval $[\alpha, \beta] \subset [0, 1]$ such that $x(\alpha) = y(l\alpha)$ and $x(\beta) = y(\beta)$ while for all $v \in (\alpha, \beta)$, $x(v) > y(v)$ and for all $v \in [\beta, 1]$, $x(v) = y(v)$, by the same logic above, we have $x(\beta) = y(\beta)$ and $x'(v) < y'(v)$ for $v \in (\alpha, \beta)$, which is inconsistent with $x(v) > y(v)$ for all $v \in (\alpha, \beta)$. Thus we can prove that $x(v) = y(v)$ for all $v \in [0, 1]$ and so the symmetric equilibrium is unique.

Case 2: If $k(v, c) > 0$ with zero probability measure on $(v^*, 1) \times (c(v^*), 1)$, then we have $x'(v) = y'(v)$ for all $v \in (v^*, 1]$. By $x(v^*) = y(v^*)$ we have $x(v) = y(v)$ for all $v > v^*$, which is a contradiction to $x(v) > y(v)$. Thus there is a unique symmetric equilibrium.

Then in both cases we prove that there is a unique symmetric equilibrium.

Proof of Proposition 3:

Proof: Define a mapping

$$(Pc)(v) = \int_0^v ds - \int_0^v \int_s^1 \begin{pmatrix} 0 & f_1(\tau) \\ f_2(\tau) & 0 \end{pmatrix} \begin{pmatrix} G_1(c_1(\tau)) \\ G_2(c_2(\tau)) \end{pmatrix} d\tau ds,$$

where $c = (c_1, c_2)'$.

Take any $x(v) = (x_1(v), x_2(v))'$ and $y(v) = (y_1(v), y_2(v))'$ with $x(v)$, $y(v) \in \varphi$ where φ is the space of monotonic increasing continuous functions defined on $[0, 1] \to [0, 1]$. Then we have

$$\begin{aligned} |(Px)(v) - (Py)(v)| &\leq \int_0^v \int_s^1 \begin{pmatrix} 0 & g_1(\widehat{x}_1(\tau))f_1(\tau) \\ g_2(\widehat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} | \begin{pmatrix} x_1(\tau) - y_1(\tau) \\ x_2(\tau) - y_2(\tau) \end{pmatrix} | d\tau ds \\ &= \int_0^v \int_s^1 \begin{pmatrix} 0 & g_1(\widehat{x}_1(\tau))f_1(\tau) \\ g_2(\widehat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} d\tau ds \sup_{0<v\leq 1} |x(v) - y(v)| \\ &\leq \int_0^1 \int_s^1 \begin{pmatrix} 0 & g_1(\widehat{x}_1(\tau))f_1(\tau) \\ g_2(\widehat{x}_2(\tau))f_2(\tau) & 0 \end{pmatrix} d\tau ds \sup_{0<v\leq 1} |x(v) - y(v)| \\ &\leq \int_0^1 \begin{pmatrix} 0 & \delta_1(1 - F_1(s)) \\ \delta_2(1 - F_2(s)) & 0 \end{pmatrix} ds \sup_{0<v\leq 1} |x(v) - y(v)|, \end{aligned} \tag{7}$$

where the first equality comes from mean value theorem, $\widehat{x}_i(\tau)$ is some number between $x_i(\tau)$ and $y_i(\tau)$, and δ_i is the maximum of $g_i(c)$, $i = 1, 2$. Thus when $\delta_i \int_0^1 (1 - F_i(s)) ds < 1$, the above mapping is a contraction, so there exists a unique equilibrium.

Appendix B

Problematic Details in Green and Laffont (1984):

In their proof of Lemma 3 (Economics Letters, 16, (1984), 34-35), they first showed the following inequality

$$(a-b)^n \leq (a-b)\frac{n}{2^n} \text{ if } a \leq \frac{1}{2},\ b \leq \frac{1}{2}.$$

After they get the following expression on page 35,

$$F(\lambda^{t+1}(\theta)) - F(\lambda^t(\theta)) = \int_0^\theta \{[1 - \int_m^1 \lambda^{t+1}(\tau)d\tau]^{n-1} - [1 - \int_m^1 \lambda^t(\tau)d\tau]^{n-1}\}dm,$$

they apply the above inequality to reach

$$\|F(\lambda^{t+1}(.)) - F(\lambda^t(.))\| < 1/2\|\lambda^{t+1}(.) - \lambda^t(.)\|$$

which can not pass through since now $a = 1 - \int_m^1 \lambda^{t+1}(\tau)d\tau \geq \frac{1}{2}$ and $b = 1 - \int_m^1 \lambda^t(\tau)d\tau \geq \frac{1}{2}$ and so $a^{n-1} + ba^{n-2} + \cdots + b^{n-2}a + b^{n-1}$ is not necessary less than 1, which implies that the Banach fixed point theorem cannot be applied.

Appendix C

Special Cases of the General Model:

In this part, for simplicity, we assume v_i and c_i are independent to illustrate the generality of our setting in the main body of the paper.

Case 1. Suppose there is a subset, denoted by A, of bidders whose valuations are common knowledge. Then for all $i \in \bar{A} = N \setminus A$, we have

$$c_i^*(v) = \int_0^v \prod_{j\in\bar{A}\setminus\{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j\in A\setminus\{i\}, v_j > v} [1 - G_j(c_j^*(v_j))]$$
$$\times \prod_{j\in\bar{A}\setminus\{i\}, v_j < v} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i.$$

For all $i \in A$,

$$c_i^*(v_i) = \int_0^{v_i} \prod_{j\in\bar{A}\setminus\{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau] \prod_{j\in A\setminus\{i\}, v_j > v_i} [1 - G_j(c_j^*(v_j))]$$
$$\times \prod_{j\in A\setminus\{i\}, v_j < v_i} [1 - \int_{m_i}^1 G_j(c_j^*(\tau))f_j(\tau)d\tau]dm_i.$$

In this case, one needs to distinguish the difference between $v_i > v_j$ and $v_j > v_i$, since under these two situations the expected payoff has different expressions.

Example 3 Suppose $n = 2$ and $v_1 < v_2$ is common knowledge, we have two bidders. Then for the bidder with value v_1,

$$\begin{aligned}
c_2^*(v_2) &= \int_0^{v_2} [1 - \int_{m_2}^1 G_1(c_1^*(\tau)) f_1(\tau) d\tau] dm_2 \\
&= \int_0^{v_1} [1 - \int_{m_2}^1 G_1(c_1^*(\tau)) f_1(\tau) d\tau] dm_2 \\
&\quad + \int_{v_1}^{v_2} [1 - \int_{m_2}^1 G_1(c_1^*(\tau)) f_1(\tau) d\tau] dm_2 \\
&= v_1(1 - G_1(c_1^*(v_1))) + (v_2 - v_1),
\end{aligned}$$

and

$$\begin{aligned}
c_1^*(v_1) &= \int_0^{v_1} [1 - \int_{m_1}^1 G_2(c_2^*(\tau)) f_2(\tau) d\tau] dm_1 \\
&= \int_0^{v_1} [1 - G_2(c_2^*(v_2)] dm_1 = v_1(1 - G_2(c_2^*(v_2))),
\end{aligned}$$

which can be reduced to the formula obtained in Kaplan and Sela (2006) when the cost distribution functions are the same.

Case 2. On the contrary, suppose there is a subset, denoted by B, of bidders whose participation costs are common knowledge, as discussed in Campbell (1998), Tan and Yilankaya (2006) and Cao and Tian (2007). Let $\bar{A} = N \setminus A$. Then, for all $i \in N$, we have

$$\begin{aligned}
c_i^*(v_i) &= \int_0^{v_i} \prod_{j \in B \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i \\
&= \int_0^{v_i} \prod_{j \in B \setminus \{i\}} [1 - \int_{m_j^*}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_j}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i \\
&= \int_0^{v_i} \prod_{j \in B \setminus \{i\}, m_j > v} [1 - \int_{m_j}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \prod_{j \in B \setminus \{i\}, m_j < v} [1 - \int_{m_j}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] \\
&\quad \times \prod_{j \in \bar{B} \setminus \{i\}} [1 - \int_{m_i}^1 G_j(c_j^*(\tau)) f_j(\tau) d\tau] dm_i,
\end{aligned}$$

where m_j^* is determined by $c_j^*(m_j) = c_j$ for $j \in B$. It may be remarked that $c_i^*(v_i)$ may have different functional forms when v_i is in the different regions of $v_i > m_j^*$ and $v_i \leq m_j^*$.

Example 4 Consider an economic environment with two bidders whose values are drawn from the same continuous distribution function $F(v)$. Bidders' participation costs are common knowledge and the same, $c_1 = c_2 = c$. This is an economy studied in Campbell (1998) and Tan and Yilankaya (2006) for $n = 2$. Let $c_1^*(m_1^*) = c_2^*(m_2^*) = c$.

Then for bidder 1, we have

$$c_1^*(v_i) = \int_0^{v_i} [1 - \int_{m_1}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau.$$

As such, we have

$$c_1^*(v_1) = \int_0^{v_1} [1 - \int_{m_2^*}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau = F(m_2^*)v_1$$

when $v_1 < m_2^*$, and

$$\begin{aligned} c_1^*(v_1) &= \int_0^{m_2^*} [1 - \int_{m_2^*}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau + \int_{m_2^*}^{v_1} [1 - \int_{m_2^*}^1 G(c_2^*(\tau))f(\tau)dm_1]d\tau \\ &= F(m_2^*)m_2^* + \int_{m_2^*}^{v_1} F(m_1)dm_1 \end{aligned}$$

when $v_1 \geq m_2^*$.

Similarly, for bidder 2, we have

$$c_2^*(v_2) = \int_0^{v_2} [1 - \int_{m_2}^1 G(c_1^*(\tau))f(\tau)d\tau]dm_2.$$

Then, we have $c_2^*(v_2) = F(m_1^*)v_2$ when $v_2 < m_1^*$, and $c_2^*(v_2) = F(m_1^*)m_1^* + \int_{m_1^*}^{v_2} F(m_2)dm_2$ when $v_2 \geq m_1^*$.

We can use these equations to find the cutoff points. It is clear that there is a symmetric equilibrium in which both bidders use the same cutoff point $m_1^* = m_2^* = m^*$, which satisfies the equation

$$m^*F(m^*) = c.$$

Indeed, by the monotonicity of $m^*F(m^*)$, the symmetric equilibrium exists and is unique.

Now if we provisionally suppose that $m_1^* < m_2^*$, then we should have

$$c_1^*(m_1^*) = m_1^*F(m_2^*) = c,$$

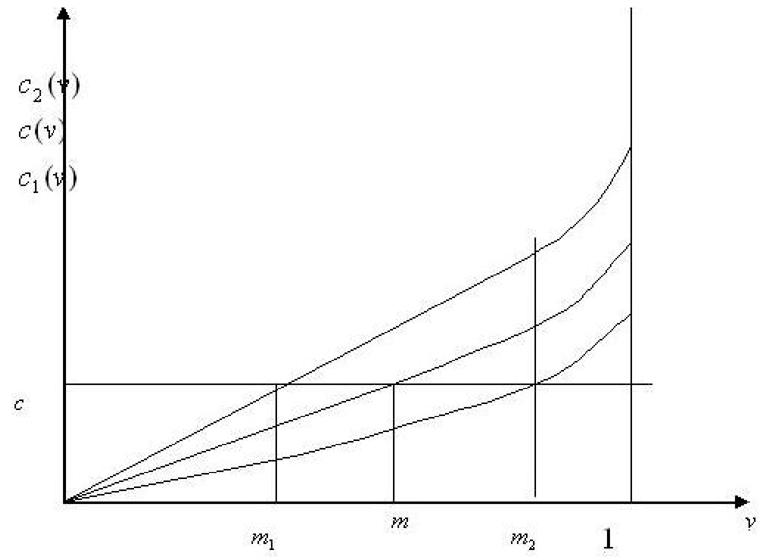
and

$$c_2^*(m_2^*) = m_1^*F(m_1^*) + \int_{m_1^*}^{m_2^*} F(m_2)dm_2 = c.$$

Tan and Yilankaya (2006) showed that when $F(v)$ is strictly convex, there exists $m_1^* < m_2^*$ satisfying the above two equations.

We can use Figure 1 to illustrate the equilibria in Example 4. There are three curves in the graph. The middle curve indicates both bidders use the same cutoff point $c^*(v)$, and then have the same cutoff point m^*. The highest curve is bidder 1's reaction curve $c_1^*(v_1)$. There is a kink at $v_1 = m_2^*$. Before reaching this point, the curve is a straight line passing through the original

Figure 1: Symmetric & Asymmetric Equilibrium with 2 Bidders



point with slope $F(m_2^*)$. Beyond m_2^*, it is a smooth curve with the slope changing along the curve, which is $F(v)$. We can see as $v \to 1$, the slope goes to 1, which is consistent with the properties of the cutoff curves described in Lemma 2. The lowest curve is bidder 2's reaction curve $c_2^*(v_2)$. The equilibrium is the intersection of the horizontal line c and each bidder's cutoff curve.

Case 3. When all participation costs are zero, $G_i(c_i^*(\tau)) = 1$ for all τ and all i. Then

$$c_i^*(v) = \int_0^v \prod_{j \neq i} [1 - \int_{m_i}^1 f_j(\tau) d\tau] dm_i = \int_0^v \prod_{j \neq i} F_j(m_i) dm_i > 0,$$

and thus, a bidder with positive value for the object will always participate in the auction and submit a bid. Under this circumstance the entrance equilibrium curve is unique.

Case 4. When all participation costs are 1, $G_j(c_j^*(\tau)) = 0$ for all $c_j^*(\tau) < 1$, and thus $c_i^{*\prime}(v) = 1$. Considering the initial condition, we have $c_i^*(v_i) = v_i$, i.e., a bidder with value v_i would like to pay at most v_i to enter the auction. Now since the designed participation cost is 1 for all bidders, then there will be no one participating in the auction.

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