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Abstract

This paper provides a tractable continuous-time CARA-Gaussian framework to explore how the interactions of risk aversion and induced uncertainty due to informational frictions determine strategic consumption-portfolio rules, precautionary savings, and consumption dynamics in the presence of uninsurable labor income. Specifically, after solving the model explicitly, we explore the relative importance of the two types of induced uncertainty: (i) model uncertainty due to robustness and (ii) state uncertainty due to limited information-processing capacity as well as risk aversion in determining asset allocation, precautionary savings, and consumption dynamics. Finally, we discuss how the separation between risk aversion and intertemporal substitution affects strategic asset allocation and precautionary savings.

JEL Classification Numbers: *C61, D81, E21.*

Keywords: Robustness, Model Uncertainty, Rational Inattention, Uninsurable Labor Income, Strategic Asset Allocation, Precautionary Savings.

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1 Introduction

For most individual consumers, human wealth, the expected present value of their current and future labor earnings, is a major fraction of their total wealth. The key difference between financial wealth and human wealth is that the latter is a non-tradable asset because it is difficult to sell claims against future labor income due to the moral hazard problem. Recently some studies have examined the effects of non-tradable labor income on the optimal share of financial wealth invested in this risky asset. For example, Heaton and Lucas (2000) studied how the presence of background risks influences portfolio allocations. They found that labor income is the most important source of wealth and labor income risk is weakly positively correlated with equity returns. Viceira (2001) examined the effects of labor income risk on optimal consumption and portfolio choice for both employed and retired investors. Chan and Viceira (2005) showed that when labor income risk is idiosyncratic, endogenous labor supply can have significant positive effects on the share invested in the risky asset relative to the case in which labor income is exogenous. Wang (2009) examined optimal consumption-saving and asset allocation when consumers cannot observe their income growth.[1] These studies mainly consider two key aspects of labor income risk: the variance and persistence of labor income and the correlation between labor income and the equity return. In the presence of labor income, there is a income-hedging demand when the equity return is correlated with labor income. For example, if the labor income risk is positively correlated with the shock to the equity return, the equity is less desirable because it offers a bad hedge against negative labor income shocks. Another important implication of non-tradable labor income is that it leads to precautionary savings by interacting with risk aversion when it is not perfectly correlated with the equity return.

This paper provides a tractable continuous-time constant absolute risk aversion (CARA)-Gaussian framework to explore how induced uncertainty due to informational frictions, the interaction of model uncertainty due to robustness and state uncertainty due to rational inattention, affects strategic asset allocation, precautionary savings, and consumption dynamics in the presence of non-tradable labor income.[2] Model uncertainty and state uncertainty arise from two major types of incomplete information: one is incomplete information about the distribution of the state evolution equation and the other is incomplete information about the true level of the state. Hansen and Sargent (1995) first introduced the preference for robustness (RB, a concern

[1] See Campbell and Viceira (2002) for a recent survey on this topic.

[2] Here we label model uncertainty or state uncertainty "induced uncertainty" because it is induced by the interactions of the preference for robustness or information-processing constraints with fundamental uncertainty (e.g., labor income uncertainty or uncertainty about the return on the risky asset).

for model misspecification) into linear-quadratic-Gaussian (LQG) economic models.[3] In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they choose their decisions as if the subjective distribution over shocks was chosen by an evil agent to minimize their utility.[4] As discussed in Hansen, Sargent, and Tallarini (HST, 1999) and Luo and Young (2010), robustness (RB) models can produce precautionary savings even within the class of LQ models, which leads to analytical simplicity.[5] Sims (2003) first introduced information-processing constraint (rational inattention or RI) into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation agents have finite Shannon channel capacity, limiting their ability to process signals about the true state. As a result, a shock to the state induces only gradual responses by individuals, as their finite capacity requires some time to discover how much the state has moved. Because RI introduces additional uncertainty, the endogenous noise due to finite capacity, into economic models, RI by itself creates an additional demand for robustness. The distinction between these two types of informational frictions can be seen from the following continuous-time transition equation of the true state (s_t):

$$ds_t = (As_t + Bc_t)\,dt + \sigma dB_t, \tag{1}$$

where s_t and c_t are state and control variables, respectively, A, B, and σ are constant coefficients, and B_t is a standard Brownian motion. Under RB, agents do not know the true data generating process driven by the random innovation (B_t), whereas agents under RI cannot observe the true state (s_t) perfectly.

As the first contribution of this paper, we construct a continuous-time theoretical framework in which there are (i) two fundamental risks: uninsurable labor income and the equity return, (ii) two types of induced uncertainty due to informational frictions described above: model uncertainty (MU) due to the preference for robustness and state uncertainty (SU) due to rational inattention, and (iii) CARA-constant intertemporal substitution.[6] We then show that the models with these

[3]See Hansen and Sargent (2007) for a textbook treatment on robustness.

[4]The solution to a robust decision-maker's problem is the equilibrium of a max-min game between the decision-maker and nature.

[5]Luo, Nie, and Young (2012) briefly discussed the differences between CARA and RB within the discrete-time setting. Although both RB and CARA preferences (i.e., Caballero 1990 and Wang 2004, 2009) increase the precautionary savings premium via the intercept terms in the consumption functions, they have distinct implications for the marginal propensity to consume out of permanent income (MPC). Specifically, CARA preferences do not alter the MPC relative to the LQ case, whereas RB increases the MPC. That is, under RB, in response to a negative wealth shock, the consumer would choose to reduce consumption more than that predicted in the CARA model (i.e., save more to protect himself against the negative shock).

[6]The main reason that we adopt the CARA utility specification is that we can obtain explicit solutions under

features can be solved explicitly. In particular, when introducing state uncertainty due to rational inattention, we derive the continuous-time version of the information-processing constraint (IPC) proposed in Sims (2003). In this case, we can obtain the explicit expressions for the steady state conditional variance, the variance of the optimal noise, and the Kalman gain in terms of finite capacity ($\kappa > 0$) and fundamental uncertainty.[7] This paper is therefore closely related to the literature on imperfect information, learning, asset allocation and asset pricing. For example, Gennotte (1986), Detemple (1986), Veronesi (2000), and Wang (2009).

Second, after solving the models explicitly, we can exactly inspect the mechanism through which these two types of informational frictions interact and determine the level of induced uncertainty. Using the closed-form strategic consumption-portfolio rules in the models with the three features described above, we can explore how the informational frictions affect different types of demand for the risky asset (i.e., the standard speculation demand and the income-hedging demand), the precautionary saving demand due to the presence of undiversified labor income, and consumption dynamics.[8] In particular, we find that CARA is more important than RB in determining optimal allocation in the risky asset and the precautionary saving demand. Furthermore, we find that SU is more important than MU when the consumer is not highly information-constrained.

Third, when calibrating the RB parameter using the detection error probabilities (DEP, or p), we find that RI due to finite capacity can affect the calibrated parameter value of RB. Specifically, we find that in the presence of model uncertainty, the correlation between the equity return and undiversified labor income not only affects the hedging demand for the risky asset, but also affects its standard speculation demand. The key reason is that given the same value of the deep RB parameter measured by DEP, the correlation between labor income and the equity return increases the calibrated value of RB and thus reduces the optimal share invested in the risky asset. In addition, we also examine how the separation of risk aversion and intertemporal substitution and the stochastic properties of undiversified labor income affect the calibrated RB parameter, the optimal share invested in the risky asset, and the precautionary saving demand.

This paper is organized as follows. Section 2 presents the setup of a continuous-time consump-

CARA. In the RI literature, some studies assume ex post Gaussian distributions and Gaussian noise but adopt exponential or CARA preferences; for example, Peng (2004), Peng and Xiong (2005), Mondria (2010), and Van Nieuwerburgh and Veldkamp (2009, 2010).

[7] We also show that the RI model is equivalent to the traditional signal extraction (SE) model with exogenously specified noises in the sense that they lead to the same model dynamics when the signal-to-noise ratio and finite capacity satisfy some restriction. In other words, we can provide a microfoundation (limited information-processing constraint) for the exogenously specified SNR in the traditional SE models.

[8] Maenhout (2004, 2006), Uppal and Wang (2002), Cao, Wang, and Zhang (2005), and Liu (2010) examined how model uncertainty affects portfolio choices and/or asset prices.

tion and portfolio choice model with uninsurable labor income. Section 3 introduces RB into the benchmark model and examines the theoretical and empirical implications of RB on consumption-portfolio rules and precautionary savings using calibrated RB parameters. Section 4 examines the interactions of RB and state uncertainty due to limited information-processing constraint. Section 5 discusses the role of intertemporal substitution in an extension with recursive utility. Section 6 concludes.

2 A Continuous-time Consumption-Portfolio Choice Model with Uninsurable Labor Income

In this paper, we follow Wang (2009) and consider a continuous-time version of the Caballero-type model (1990) with portfolio choice. The typical consumer facing uninsurable labor income in the model economy makes optimal consumption-saving-asset allocation decisions. Specifically, we assume that the consumer can access: one risk-free asset and one risky asset, and also receive uninsurable labor income. Labor income (y_t) is assumed to follow a continuous-time AR(1) (Ornstein-Uhlenbeck) process:

$$dy_t = \rho \left(\frac{\mu}{\rho} - y_t \right) dt + \sigma_y dB_{y,t}, \tag{2}$$

where the unconditional mean and variance of income are $\overline{y} = \mu/\rho$ and $\sigma_y^2/(2\rho)$, respectively, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state,[9] $B_{y,t}$ is a standard Brownian motion on the real line $\mathcal{R}$, and σ_y is the unconditional volatility of the income change over an incremental unit of time.

The agent can invest in both a risk free asset with a constant interest rate r and a risky asset (i.e., the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by

$$dr_t^e = (r + \pi) dt + \sigma_e dB_{e,t}, \tag{3}$$

where π is the market risk premium, $B_{e,t}$ is a standard Brownian motion, and σ_e is the standard deviation of the market return. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. When $\rho_{ye} = 0$, the labor income risk is idiosyncratic and is uncorrelated with the risky market return; when $\rho_{ye} = 1$, the labor income risk is

[9] If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The $\rho = 0$ case corresponds to a simple Brownian motion without drift. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0, which means that y can never deviate from $\overline{y}$.

perfectly correlated with the risky market return. The agent's financial wealth evolution is then given by

$$dw_t = (rw_t + y_t - c_t)\,dt + \alpha_t\,(\pi dt + \sigma_e dB_{e,t})\,, \tag{4}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

The typical consumer is assumed to maximize the following expected lifetime utility:

$$E_0\left[\int_{t=0}^{\infty} \exp(-\delta t)\, u(c_t) dt\right], \tag{5}$$

subject to (4). The utility function takes the CARA form: $u(c_t) = -\exp(-\gamma c_t)/\gamma$, where $\gamma > 0$ is the coefficient of absolute risk aversion.[10] To simplify the model, we define a new state variable, s_t:

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r:

$$h_t \equiv E_t\left[\int_t^{\infty} \exp(-r(s-t))\, y_s ds\right].$$

For the given the income process, (2), $h_t = \frac{1}{r+\rho} y_t + \frac{\mu}{r(r+\rho)}$. Following the state-space-reduction approach proposed in Luo (2008) and using this new state variable, we can rewrite (4) as

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma dB_t, \tag{6}$$

where $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$, $\sigma_s = \sigma_y/(r+\rho)$, and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t} \tag{7}$$

is the unconditional variance of the innovation to s_t.[11]

In this benchmark full-information rational expectations (FI-RE) model, we assume that the consumer trusts the model and observes the state perfectly, i.e., no model uncertainty and no

[10] It is well-known that the CARA utility specification is tractable for deriving the consumption function or optimal consumption-portfolio rules in different settings. See Merton (1969), Caballero (1990), Svensson and Werner (1993), Weil (1993), and Wang (2004, 2009).

[11] The main advantage of this state-space-reduction approach is to allow us to solve the model with model uncertainty and state uncertainty explicitly and help better inspect the mechanism by which the informational frictions interact and affect optimal consumption-portfolio rules. It is worth noting that the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption-portfolio rules. The detailed proof is available from the author by request.

state uncertainty. The value function is denoted by $J(s_t)$. The Hamilton-Jacobi-Bellman (HJB) equation for this optimizing problem can be written as:

$$0 = \sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t)\right],$$

where

$$\mathcal{D}J(s_t) = J_s(rs_t - c_t + \pi\alpha_t) + \frac{1}{2}J_{ss}\left(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right). \tag{8}$$

Solving the above HJB subject to (6) leads to the following optimal portfolio-consumption rules:[12]

$$\alpha = \frac{\pi}{r\gamma\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{9}$$

and

$$c_t = rs_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} - \Gamma, \tag{10}$$

where

$$\Gamma \equiv \frac{1}{2}r\gamma\left(1 - \rho_{ye}^2\right)\sigma_s^2, \tag{11}$$

is the investor's precautionary saving demand. The first term in (9) is the standard speculation demand for the risky asset, which is positively correlated with the risk premium of the risky asset over the risk free asset and is negatively correlated with the degree of risk aversion and the variance of the return to the risky asset. The second term in (9) is the labor income hedging demand of the risky asset. When $\rho_{ye} \neq 0$, i.e., the income shock is not purely idiosyncratic, the desirability of the risky asset depends not only on its expected excess return relative to its variance, but also on its ability to hedge consumption against bad realizations of labor income. Following the literature of precautionary savings, we measure the demand for precautionary saving as the amount of saving due to the interaction of the degree of risk aversion and non-diversifiable labor income risk.[13] If labor income is perfectly correlated with the return to the risky asset (i.e., $\rho_{ye} = \pm 1$), the market is complete and the consumer can fully hedge his labor income risk; consequently, his demand for precautionary saving is 0.

3 Incorporating Model Uncertainty

3.1 Modeling Robustness

As argued in Hansen and Sargent (2007), a simplest version of robustness considers the question of how to make optimal decisions when the decision maker does not know the true probability

[12] This solution is similar to that obtained in Model I of Wang (2009).

[13] Note that hedging with the risky asset ($\rho_{ye} \neq 0$) reduces the consumer's precautionary saving demand.

model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the reference (or approximating) model governing the evolution of the state variables is the true model, but also perform reasonably well when there is some type of model misspecification. To introduce robustness into our model proposed above, we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004) to assume that consumers are concerned about the model misspecifications and take Equation (6) as the approximating model.[14] The corresponding distorting model can thus be obtained by adding an endogenous distortion $\upsilon(s_t)$ to (6):

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma(\sigma\upsilon(s_t)\,dt + dB_t)\,. \tag{12}$$

As shown in AHS (2003), the objective $\mathcal{D}J$ defined in (8) plays a crucial role in introducing robustness. $\mathcal{D}J$ can be thought of as $E[dJ]/dt$ and is easily obtained using Itô's lemma. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable. The consumer accepts the approximating model, (6), as the best approximating model, but is still concerned that it is misspecified. He therefore wants to consider a range of models (i.e., the distorted model, (12)) surrounding the approximating model when computing the continuation payoff. A preference for robustness is then achieved by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (12) and (ii) an entropy penalty:

$$\inf_{\upsilon}\left[\mathcal{D}J + \upsilon(s_t)\,\sigma^2 J_s + \frac{1}{2\vartheta}\upsilon(s_t)^2\,\sigma^2\right], \tag{13}$$

where the first two terms are the expected continuation payoff when the state variable follows (12), i.e., the alternative model based on drift distortion $\upsilon(s_t)$.[15] ϑ_t is fixed and state independent in AHS (2003), whereas it is state-dependent in Maenhout (2004). The key reason of using a state-dependent counterpart ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem with the CRRA utility function.[16] In this paper, we also specify that ϑ_t is state-dependent ($\vartheta(s_t)$) in the CARA-Gaussian setting. The main reason for this specification is to guarantee the homotheticity, which makes robustness not wear off as the value of the total

[14] As argued in Hansen and Sargent (2007), the agent's committment technology is irrelevant under RB if the evolution of the state is backward-looking. We therefore do not specifiy the committment technology of the consumer in the RB models of this paper.

[15] Note that the $\vartheta = 0$ case corresponds to the standard expected utility case.

[16] See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

wealth increases.[17]

Applying these results in the above model yields the following HJB equation under robustness:

$$\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(s_t\right)+\mathcal{D}J\left(s_t\right)+\upsilon\left(s_t\right)\sigma^2 J_s+\frac{1}{2\vartheta\left(s_t\right)}\upsilon\left(s_t\right)^2\sigma^2\right], \tag{14}$$

where the last term in the HJB above is due to the agent's preference for robustness and reflects a concern about the quadratic variation in the partial derivative of the value function weighted by ϑ.[18] Solving first for the infimization part of (14) yields:

$$\upsilon\left(s_t\right)^*=-\vartheta\left(s_t\right)J_s,$$

where $\vartheta\left(s_t\right)=-\vartheta/J\left(s_t\right)>0$. (See Appendix 7.1 for the derivation.) Because $\vartheta\left(s_t\right)>0$, the perturbation adds a negative drift term to the state transition equation because $J_s>0$. Substituting for υ^* in (14) gives:

$$\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(s_t\right)+\left(rs_t-c_t+\pi\alpha_t\right)J_s+\frac{1}{2}\sigma^2 J_{ss}-\frac{1}{2}\vartheta\left(s_t\right)\sigma^2 J_s^2\right]. \tag{15}$$

3.2 Theoretical Implications

Following the standard procedure, we can then solve (15) and obtain the optimal consumption-portfolio rules under robustness. The following proposition summarizes the solution:

Proposition 1 *Under robustness, the optimal consumption and portfolio rules under robustness are*

$$c_t^*=rs_t+\frac{\delta-r}{r\gamma}+\frac{\pi^2}{2r\gamma\sigma_e^2}-\frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}-\Gamma, \tag{16}$$

and

$$\alpha^*=\frac{\pi}{r\widetilde{\gamma}\sigma_e^2}-\frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{17}$$

respectively, where the effective coefficient of absolute risk aversion $\widetilde{\gamma}$ *is defined as:* $\widetilde{\gamma}\equiv\left(1+\vartheta\right)\gamma$ *and the precautionary savings demand,* Γ*, is*

$$\Gamma=\frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\sigma_s^2. \tag{18}$$

Finally, the worst possible distortion can be written as

$$\upsilon^*=-r\gamma\vartheta. \tag{19}$$

[17] In the detailed procedure of solving the robust HJB proposed in Appendix 7.1, it is clear that the impact of robustness wears off if we assume that ϑ_t is constant.

[18] See AHS (2003) and Maenhout (2004) for detailed discussions.

Proof. See Appendix 7.1. ■

From (16), it is clear that robustness does not change the marginal propensity to consume out of permanent income (MPC), but affects the amount of precautionary savings (Γ). In other words, in the continuous-time setting, consumption is not sensitive to unanticipated income shocks. This conclusion is different from that obtained in the discrete-time robust-LQG permanent income model in which the MPC is increased via the interaction between RB and income uncertainty and consumption is more sensitive to unanticipated shocks.[19]

Expression (17) shows that RB reduces the optimal speculation demand by a factor, $1+\vartheta$, but does not affect the hedging demand of the risky asset. In other words, RB increases the relative importance of the income hedging demand to the speculation demand by increasing the effective coefficient of absolute risk aversion ($\widetilde{\gamma}$). Expression (18) shows that the precautionary savings premium increases with the degree of robustness (ϑ) by increasing the value of $\widetilde{\gamma}$ and interacting with two types of fundamental uncertainty: labor income uncertainty (σ_s^2) and the correlation between labor income and the equity return (ρ_{ye}). An interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings premium, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance. Specifically, using (18), we have the following proposition:

Proposition 2 *The relative importance of robustness (RB, ϑ) and CARA (γ) in determining precautionary saving can be measured by:*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{1+\vartheta}{\vartheta} > 1, \tag{20}$$

where $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta} =$ and $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$ are the elasticities of the precautionary saving demand to RB and CARA, respectively. (20) means that absolute risk aversion measured by γ is more important than RB measured by ϑ in determining the precautionary savings demand. Note that Expression (20) can also measure the relative importance of RB and CARA in determining portfolio choice.

Proof. Using (17) and (18), the proof is straightforward. ■

Using (20), it is straightforward to show that $\partial\mu_{\gamma\vartheta}/\partial\vartheta > 0$, which means that $\mu_{\gamma\vartheta}$ is increasing with the degree of RB, ϑ. To fully explore the quantitative effects of robustness on portfolio choice and precautionary saving, we need to calibrate ϑ using the detection error probability approach (DEP) proposed in Hansen, Sargent, and Wang (henceforth, HSW, 2002), AHS (2003), and Hansen and Sargent (Chapter 9, 2007). In the next subsection, we will examine the relative importance of RB to CARA quantitatively after calibrating ϑ using the U.S. data.

[19] See HST (1999) and Luo and Young (2010) for detailed discussions on how RB affects consumption and precautionary savings in the discrete-time robust-LQG models.

Proposition 3 *The observational equivalence between the discount factor and robustness established in the discrete-time Hansen-Sargent-Tallarini (1999) does not hold in our continuous-time CARA-Gaussian model.*

Hansen, Sargent, and Tallarini (1999) (henceforth, HST) show that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG permanent income model. The reason is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or RB on consumption and investment.[20] In contrast, in the continuous-time CARA-Gaussian model discussed in this section, the observational equivalence between the discount rate and the RB parameter no longer holds. This result can be readily obtained by inspecting the explicit expressions of consumption, precautionary savings, and portfolio choice, (16)-(18). First, ϑ affects optimal portfolio choice via increasing $\widetilde{\gamma}$, whereas δ does not enter the portfolio rule. Second, although both the discount factor ($\exp(-\delta)$) and γ reduce the constant term in the consumption function and their observational equivalence can be established in the sense that they generate the same value of the constant term, they imply different portfolio choices.[21]

3.3 Empirical Implications

3.3.1 Calibrating the Robustness Parameter

To fully explore how RB affects the joint behavior of portfolio choice, consumption, and labor income, we adopt the calibration procedure outlined in HSW (2002) and AHS (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the consumer to distinguish the two models. The value of p is

[20] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would change as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

[21] Note that the precautionary saving demand is included in the constant term.

determined by the following procedure. Let model P denote the approximating model, (6):

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma dB_t,$$

and model Q be the distorted model, (12):

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma\left(\sigma\upsilon\left(s_t\right)dt + dB_t\right).$$

Define p_P as

$$p_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P\right), \tag{21}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. When model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as

$$p_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q\right). \tag{22}$$

Given initial priors of 0.5 on each model and that the length of the sample is N, the detection error probability, p, can be written as:

$$p\left(\vartheta; N\right) = \frac{1}{2}\left(p_P + p_Q\right), \tag{23}$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1 - p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of ϑ such that $p\left(\vartheta; N\right)$ equals a given value (for example, 10%) after simulating model P, (6), and model Q, (12).[22] In the continuous-time model with the iid Gaussian specification, $p\left(\vartheta; N\right)$ can be easily computed. Because both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions and are normally distributed random variables. Specifically, the logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = -\int_0^N \overline{\upsilon} dB_s - \frac{1}{2}\int_0^N \overline{\upsilon}^2 ds, \tag{24}$$

where

$$\overline{\upsilon} \equiv \upsilon^*\sigma = -r\gamma\vartheta\sqrt{\sigma_e^2\alpha^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha}. \tag{25}$$

[22] The number of periods used in the calculation, N, is set to be the actual length of the data we study. For example, if we consider the post-war U.S. annual time series data provided by Robert Shiller from $1946 - 2010$, $T = 65$.

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = \int_0^N \overline{\upsilon} dB_s + \frac{1}{2}\int_0^N \overline{\upsilon}^2 ds. \tag{26}$$

Using (21)-(26), it is straightforward to derive $p(\vartheta; N)$:

$$p(\vartheta; N) = \Pr\left(x < \frac{\overline{\upsilon}}{2}\sqrt{N}\right), \tag{27}$$

where x follows a standard normal distribution. From the expressions of $\overline{\upsilon}$, (25), and $p(\vartheta; N)$, (27), we can show that the value of p is decreasing with the value of ϑ because $\partial\alpha^*/\partial\vartheta < 0$.

3.3.2 Some Results

Using the data set documented in Campbell (2003), we set the parameter values for the processes of returns, volatility, and consumption as follows: $\mu = 0.08$, $r = 0.02$, $\delta = 0.02$, $\sigma_e = 0.156$, and $\sigma_c = 0.011$. For the labor income process, we follow Wang (2009) and set that $\sigma_y = 0.1$. When $\rho = 0$, i.e., when labor income follows a Brownian motion, we can compute that $\sigma_s = 5$. Figure 1 illustrates how DEP (p) varies with the value of ϑ for different values of γ. We can see from the figure that the stronger the preference for robustness (higher ϑ), the less the p is. For example, when $\gamma = 2$, $p = 10\%$ when $\vartheta = 1.4$, while $p = 16\%$ when $\vartheta = 1$.[23] Both values of p are reasonable as argued in AHS (2002), HSW (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007).

Figures 2 and 3 illustrate how p varies with the value of ϑ for different values of σ_s and ρ_{ye}, respectively.[24] These figures also show that the higher the value of ϑ, the less the p is. In addition, to calibrate the same value of p, less values of σ_s (i.e., more volatile or higher persistent labor income processes) or higher values of ρ_{ye} lead to higher values of ϑ. The intuition behind this result is that σ_s and ϑ have opposite effects on $\overline{\upsilon}$. (This is clear from (25).) To keep the same value of p, if one parameter increases, the other one must reduce to offset its effect on $\overline{\upsilon}$. As emphasized in Hansen and Sargent (2007), in the robustness model, p is the deep model parameter governing the preference for RB, and ϑ reflects the effect of RB on the model's behavior. Combining these facts with the expression for robust portfolio rule, (17), we can see that an increase in ρ_{ye} not only reduces the hedging demand directly, but also reduces the standard speculation demand of the risky asset by affecting the calibrated values of ϑ using the same values of p. In contrast, an increase in σ_s reduces the hedging demand, but increases the speculation demand.

[23] Caballero (1990) and Wang (2004) also set $\gamma = 2$.

[24] Note that since $\sigma_s = \sigma_y / (r + \rho)$, the value of σ_s can measure the persistence (ρ) and volatility (σ_y) of the labor income process.

From the expression for robust portfolio rule, (17), we can see that plausible values of RB can significantly affect the share invested in the risky asset. Figure 4 illustrates how robust portfolio rule varies with the degree of RB for different values of ρ_{ye}. It clearly shows that α^* decreases with the value of ϑ for different values of ρ_{ye}. In addition, it is also clear from the same figure that α^* decreases with ρ_{ye} for a given value of ϑ. The intuition behind this result is the same as that in the FI-RE case: When the labor income risk becomes more positively correlated with the shock to the equity return, the equity is less desirable and the agent thus invests less in it.

Figure 4 also illustrates how the precautionary saving demand (Γ) varies with the degree of RB for different values of ρ_{ye}. It clearly shows that Γ increases with the value of ϑ for different values of ρ_{ye} and RB has a very significant impact on precautionary savings. For example, when $\rho_{ye} = 0.35$, $\Gamma = 0.63$ when $\vartheta = 1$, while $\Gamma = 0.96$ when $\vartheta = 1.4$. Furthermore, we can see that Γ decreases with ρ_{ye} for a given value of ϑ. The intuition for this result is that the higher the value of ρ_{ye}, the more important the hedging demand for the equity, and thus the less demand for precautionary savings.

4 Incorporating State Uncertainty

4.1 Information-Processing Constraint

So far we have considered the case in which the consumer can observe the state perfectly. In this section, we consider a situation in which the typical consumer with the preference for robustness cannot observe the state (s) perfectly due to finite information-processing capacity (rational inattention, or RI). That is, the typical consumer can neither observe s_t, nor can he observe the source of innovation dB_t, included in the state transition equation, (6):

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma dB_t. \tag{28}$$

Following Kasa (2006) and Reis (2011), we assume that the consumer observes only a noisy signal containing imperfect information about s_t:

$$ds_t^* = s_t dt + d\xi_t, \tag{29}$$

where ξ_t is the noise shock, and is a Brownian motion with mean 0 and variance Λ. (In the RI setting, the variance, Λ, is a choice variable for the agent.) Note that here we assume that the consumer receives signals on $s_t dt$ rather than on ds_t. As emphasized in Sims (1998) and discussed in Kasa (2006) and Reis (2011), the latter specification is not suitable to model state uncertainty due to finite capacity because this specification means that in any finite interval, arbitrarily large amounts of information can be passed through the consumer's channel. In addition, following the RI literature, we assume that ξ_t is independent of the Brownian motion governing the fundamental shock, B_t.[25]

To model RI due to finite capacity, we impose the following constraint on the consumer's information-processing ability:

$$\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_t\right) - \mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_{t+\Delta t}\right) \leq \kappa \Delta t, \tag{30}$$

where κ is the consumer's information channel capacity, $\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_t\right)$ denotes the entropy of the state prior to observing the new signal at $t+\Delta t$, and $\mathcal{H}\left(s_{t+\Delta t}|\mathcal{I}_{t+\Delta t}\right)$ is the entropy after observing the new signal. κ imposes an upper bound on the amount of information – that is, the change in the entropy – that can be transmitted in any given period. To apply this information constraint to the state transition equation, we first rewrite (28) in the time interval of $[t, t+\Delta t]$:[26]

$$s_{t+\Delta t} = \rho_{0,t} + \rho_1 s_t + \rho_2 \sqrt{\Delta t} \varepsilon_{t+\Delta t}, \tag{31}$$

where $\rho_{0,t} = \left(-c_t + \pi \alpha_t\right)\left(1 - \exp\left(r\Delta t\right)\right) / \left(-r\Delta t\right)$, $\rho_1 = \exp\left(r\Delta t\right)$, $\rho_2 = \sigma\sqrt{\left(1 - \exp\left(2r\Delta t\right)\right) / \left(-2r\Delta t\right)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to permanent income. Taking conditional variances on both sides of (31) and substituting it into (30), we have

$$\ln\left(\rho_1^2 \Sigma_t + \rho_2^2\right) - \ln\left(\Sigma_{t+\Delta t}\right) \leq 2\kappa\Delta t, \tag{32}$$

which reduces to

$$\dot{\Sigma}_t = 2\left(r - \kappa\right)\Sigma_t + \sigma^2,$$

as $\Delta t \to 0$, where $\Sigma_t = E_t\left[\left(s_t - \widehat{s}_t\right)^2\right]$ the conditional variance at t. (See Appendix 7.2 for a proof.) In the steady state in which $\dot{\Sigma}_t = 0$, the steady state conditional variance can be written as:[27]

$$\Sigma = \frac{\sigma^2}{2\left(\kappa - r\right)}. \tag{33}$$

[25] In the traditional signal extraction literature, sometimes it is assumed that the fundamental shock and the noise shock (or measure errors) are correlated. In real systems, we do observe correlated shocks and noises. For example, if the system is an airplane and winds are buffeting the plane, the effects of the random wind on the airplane' acceleration are complex. This disturbance effect could be modeled as dB_t. Sensors on the airplane can measure its angle and velocity relative to the air, and these measurements modeled as $d\xi_t$ are corrupted by the same wind field that forces the airplane. See Stengel (Chapter 4, 1994) for a discussion on correlated process and measurement noise.

[26] Note that here we use the fact that $\Delta B_t = \varepsilon_t \sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

[27] Note that here we need to impose the restriction $\kappa - r > 0$. If this condition fails, the state is not stabilizable and the conditional variance diverges.

To make optimal decisions, the consumer is required to filter in the optimal way the value of s_t using the observed s_t^*. Although the setting of our CARA-Gaussian model is not a typical tracking problem, the filtering problem in this model could be similar to the tracking problem proposed in Sims (2003, 2010). Specifically, we may think that the model with imperfect state observations can be decomposed into a two-stage optimization procedure:[28]

1. The optimal filtering problem determines the optimal evolution of the perceived (estimated) state;

2. The optimal control problem in which the decision makers treat the perceived state as the underlying state when making optimal decisions.

Here we assume ex post Gaussian distributions and Gaussian noise but adopt exponential or CARA preferences. See Peng (2004), Mondria (2010), and Van Nieuwerburgh and Veldkamp (2009, 2010) for this specification. Because both the optimality of ex post Gaussianity and the standard Kalman filter are based on the linear-quadratic-Gaussian (LQG) specification, the applications of these results in the RI models with CARA preferences are only approximately valid. Specifically, following the Kalman-Bucy filtering method, the optimal estimate for s_t given $\mathcal{F}_t = \left\{ s_j^* \right\}_{j=0}^t$ in the mean square sense coincides with the conditional expectation: $\widehat{s}_t = E_t\left[s_t\right]$, where $E_t\left[\cdot\right]$ is based on $\mathcal{F}_t$. The filtering equations for $\widehat{s}_t$ and Σ_t can be written as[29]

$$d\widehat{s}_t = \left(r\widehat{s}_t - c_t + \pi\alpha_t\right)dt + \theta_t d\eta_t, \tag{34}$$

$$\dot{\Sigma}_t = -\Lambda\theta_t^2 + 2r\Sigma_t + \sigma^2, \tag{35}$$

given $\widehat{s}_0$ and Σ_0, where

$$\theta_t = \frac{\Sigma_t}{\Lambda} \tag{36}$$

is the Kalman gain and

$$d\eta_t = \sqrt{\Lambda} dB_t^*, \tag{37}$$

with mean $E\left[d\eta_t\right] = 0$ and $\operatorname{var}\left(d\eta_t\right) = \Lambda dt$, where B_t^* is a standard Brownian motion. Note that η_t is a Brownian motion with mean 0. Although the Brownian variable, ξ_t, is not observable, the innovation process, η_t, is observable because it is derived from observable processes (i.e., ds_t^* and $\left(r\widehat{s}_t - c_t + \pi\alpha_t\right)dt$). In this case, the path of the conditional expectation, $\widehat{s}_t$, is generated by the path of the innovation process, η_t. In the steady state, we have the following proposition:

[28] See Liptser and Shiryayev (2001) for a textbook treatment on this topic and an application in a precautionary saving model in Wang (2004).

[29] See Liptser and Shiryaev (2001) for deriving the filtering equations updating the conditional mean and variance of the state.

Proposition 4 *Given finite capacity κ, in the steady state, the evolution of the perceived state can be written as*

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t)\, dt + \widehat{\sigma} dB_t^*, \tag{38}$$

where

$$\widehat{\sigma} \equiv \Sigma/\sqrt{\Lambda} = f(\kappa)\,\sigma, \tag{39}$$

$f(\kappa) = \sqrt{\frac{\kappa}{\kappa - r}} > 1$ *(i.e., the standard deviation of the estimated state is greater than that of the true state,)*

$$\Lambda = \frac{\sigma^2}{4\kappa(\kappa - r)} \tag{40}$$

is the steady state conditional variance, and

$$\theta = 2\kappa \tag{41}$$

is the corresponding Kalman gain.

Proof. Using $\Sigma = \frac{\sigma^2}{2(\kappa - r)}$ and (35), we can easily obtain that:

$$\Lambda = \frac{\sigma^2}{4\kappa(\kappa - r)} \text{ and } \theta = 2\kappa.$$

■

It is worth noting that the above RI case can be observationally equivalent to the traditional signal extraction case in which the steady state variance of the noise (Λ) or the signal-to-noise ratio (SNR, σ^2/Λ) are specified exogenously. Dividing Λ on both sides of (35), we obtain the following differential Riccati equation governing the evolution of θ_t:

$$\dot{\theta}_t = -\theta_t^2 + 2r\theta_t + \frac{\sigma^2}{\Lambda}, \tag{42}$$

where σ^2/Λ is the signal-to-noise ratio (SNR) in this problem. In the steady state, we have the following proposition for this signal extraction case with exogenous noises:

Proposition 5 *Given* SNR $\left(\sigma^2/\Lambda\right)$, *in the steady state, the evolution of the perceived state can be written as*

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t)\, dt + \widehat{\sigma} dB_t^*,$$

where

$$\widehat{\sigma} \equiv \theta\sqrt{\Lambda} = g(\tau)\,\sigma, \tag{43}$$

$$\theta = r + \sqrt{r^2 + \frac{\sigma^2}{\Lambda}}, \tag{44}$$

$g(\tau) \equiv r\sqrt{\tau} + \sqrt{1 + r^2\tau} > 1$, and $\tau \equiv 1/\,\mathrm{SNR} = \Lambda/\sigma^2$. Furthermore, if SNR *and κ satisfy the following equality:*

$$\mathrm{SNR} = 4\kappa(\kappa - r),$$

then the RI and SE cases are observationally equivalent in the sense that they lead to the same model dynamics.

Proof. In the steady state in which $\dot{\theta}_t = 0$, solving the following algebraic Riccati equation,

$$-\theta_t^2 + 2r\theta_t + \frac{\sigma^2}{\Lambda} = 0,$$

yields the steady state Kalman gain:

$$\theta = r + \sqrt{r^2 + \frac{\sigma^2}{\Lambda}}. \tag{45}$$

and steady state conditional variance: $\Sigma = \theta\Lambda$. ■

4.2 Interaction between Model Uncertainty and State Uncertainty

In this section, we assume that the typical consumer not only cannot observe the state perfectly, but also has concerns about the innovation to perceived permanent income. In the model with both state uncertainty and model uncertainty, the prior variance of s, σ^2, is affected by the optimal portfolio choice, α^*, which is to be determined after solving the whole model with both model uncertainty (ϑ) and state uncertainty (κ or SNR). Given the value of κ, the value of the variance of the noise (Λ) should also be endogenously determined by α^*. The following is the two-stage procedure to solve the optimization problem of the consumer under both model uncertainty (MU) and state uncertainty (SU):

1. First, given finite SNR, we guess that the optimal portfolio choice under MU and SU is time-invariant, i.e., $\alpha_t = \alpha$. Consequently, $\sigma = \sqrt{\sigma_e^2\alpha^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha}$ is also time-invariant. The consumer with imperfect information about the state (SNR > 0) understands that he cannot observe s_t perfectly and needs to use the Kalman filter, (34), to update the perceived state when making decisions. In other words, (34) is regarded as the approximating model in this MU-SU model. The consumer solves the following HJB:
$$\sup_{c_t,\alpha}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(\widehat{s}_t) + \mathcal{D}J(\widehat{s}_t) + \upsilon(\widehat{s}_t)\sigma^2 J_s + \frac{1}{2\vartheta(\widehat{s}_t)}\upsilon(\widehat{s}_t)^2\sigma^2\right], \tag{46}$$
subject to the distorted model:
$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha)\,dt + \widehat{\sigma}\left(\widehat{\sigma}\upsilon(\widehat{s}_t)\,dt + d\widetilde{B}_t\right), \tag{47}$$
where $\widehat{\sigma} \equiv f(\kappa)\sigma$ and $f(\kappa) = \sqrt{\frac{\kappa}{\kappa - r}} > 1$.

2. Second, after solving for optimal consumption and portfolio rules under RB and RI, we can verify whether the resulting portfolio rule is time-variant or not. If yes, our guess in the first step is correct and can thus rationalize the above procedure we used to derive the stochastic property of the endogenous noise, Λ. The key reason is that time-invariant α yields time-invariant variance of the fundamental shock (σ).

The following proposition summarizes the solution to (46)-(47):

Proposition 6 *Given ϑ and θ, the optimal consumption and portfolio rules under robustness are*

$$c_t^* = r\widehat{s}_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} - \Gamma \tag{48}$$

and

$$\alpha^* = \frac{\pi}{r\widetilde{\gamma}f(\kappa)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{49}$$

respectively, where $\widehat{s}_t$ is governed by (47), we use the fact that $\widehat{\sigma}^2 = f(\kappa)^2\sigma^2$, and $\widetilde{\gamma} \equiv (1+\vartheta)\gamma$, and the precautionary savings premium, Γ, is

$$\Gamma = \frac{1}{2}r\widetilde{\gamma}f(\kappa)^2\left(1-\rho_{ye}^2\right)\sigma_s^2. \tag{50}$$

Finally, the worst possible distortion can be written as

$$\upsilon^* = -r\gamma\vartheta. \tag{51}$$

Proof. See Appendix 7.3. ■

From (48), it is clear that robustness does not change the marginal propensity to consume (MPC) out of perceived permanent income ($\widehat{s}_t$), but affects the amount of precautionary savings. In other words, in the continuous-time setting, consumption is not sensitive to unanticipated income shocks. This conclusion is different from that obtained in the discrete-time robust-LQG permanent income model in which the MPC is increased via the interaction between RB and income uncertainty, and consumption is more sensitive to unanticipated shocks.[30]

Given (48)-(50), it is straightforward to show that the observational equivalence between the discount rate and the RB parameter no longer holds in the MU-SU model. The reason is the same as that in the model without SU: ϑ affects optimal portfolio choice via increasing $\widetilde{\gamma}$, whereas δ does not enter the portfolio rule. That is, although both the discount factor ($\exp(-\delta)$) and γ increase the precautionary savings premium and their observational equivalence can be

[30] See HST (1999) and Luo and Young (2010) for detailed discussions on how RB affects consumption and precautionary savings in the discrete-time LQG permanent income models.

established in the sense that they generate the same value of Γ, they imply different portfolio choices. Comparing (16) with (48), it is clear that the certainty equivalence principle holds in this model, i.e., the consumption function under SU can be easily obtained by replacing the true state with the perceived state.

Expression (49) shows that finite capacity (κ) affects the speculation demand invested in the risky asset (the first term in (49)). Given that $f(\kappa) = \sqrt{\kappa/(\kappa - r)} > 1$, we can see that SU reduces the share invested in the risky asset. The intuition behind this result is that consumption reacts to the income and asset return shocks gradually and with delay due to extracting useful information about the true state from noisy observations. In other words, SU and MU affect the optimal portfolio choice in the same direction. Figure 5 illustrates how strategic portfolio rule (α^*) varies with the degree of SU (κ) for different plausible values of γ. It clearly shows that α^* decreases with the value of κ for any given value of γ. In addition, it is also clear from the same figure that α^* decreases with γ for a given value of κ, which is consistent with the result obtained in the model without SU.

Expression (50) shows that the precautionary savings demand is increases the degree of SU governed by $f(\kappa)$. Figure 5 also illustrates how the precautionary saving demand (Γ) varies with the degree of SU for different plausible values of γ. It clearly shows that Γ increases with the value of κ for different values of γ and SU has a significant impact on precautionary savings.

As in the previous section, we use the elasticities of precautionary saving to changes in the degrees of MU and SU as a measure of their importance. Specifically, using (50), we have the following proposition:

Proposition 7 *The relative importance of model uncertainty (MU, ϑ) and state uncertainty (SU, κ) can be measured by:*

$$\mu_{\vartheta\kappa} \equiv -\frac{e_\vartheta}{e_\kappa} = \frac{\vartheta}{1+\vartheta}\frac{\kappa - r}{r}, \tag{52}$$

where $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$ and $e_\kappa \equiv \frac{\partial\Gamma/\Gamma}{\partial\kappa/\kappa}$ are the elasticities of precautionary saving to model uncertainty and state uncertainty, respectively. Furthermore, when

$$\kappa < (\geq)\, r\left(2 + \frac{1}{\vartheta}\right), \tag{53}$$

state uncertainty is more (less) important than model uncertainty in determining the precautionary saving demand. In other words, when finite capacity is sufficiently low, state uncertainty becomes relatively important in determining the precautionary saving demand.

Proof. The proof is straightforward using (52) and the facts that $\mu_{\vartheta\kappa} \equiv \left(\frac{\partial\Gamma/\Gamma}{\partial\widetilde{\vartheta}/\widetilde{\vartheta}}\frac{\partial\widetilde{\vartheta}/\widetilde{\vartheta}}{\partial\vartheta/\vartheta}\right) / \left(\frac{\partial\Gamma/\Gamma}{\partial f/f}\frac{\partial f/f}{\partial\kappa/\kappa}\right)$, where $\widetilde{\vartheta} = 1 + \vartheta$. ■

Using (52), it is straightforward to show that

$$\frac{\partial \mu_{\vartheta\kappa}}{\partial \vartheta} > 0 \text{ and } \frac{\partial \mu_{\vartheta\kappa}}{\partial \kappa} > 0,$$

which mean that $\mu_{\vartheta\kappa}$ is increasing with the degree of RB, ϑ, while is decreasing with the degree of SU (i.e., less values of κ).

Using (48) and (49), we can obtain the stochastic properties of the joint dynamics of consumption, labor income, and the equity return. The following proposition summarizes the major results on the effects of RB on the joint behavior of consumption, labor income, and the equity return:

Proposition 8 *Given ϑ and κ, the expected growth of consumption is*

$$g_c \equiv \frac{E\left[dc_t^*\right]}{dt} = -\frac{\delta - r}{r\gamma} + \frac{1}{2} r f(\kappa)^2 \widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{2 r f(\kappa)\widetilde{\gamma}\sigma_e^2}, \tag{54}$$

the volatility of consumption growth is $\operatorname{var}\left(dc_t^*\right) = r^2 f(\kappa)^2 \sigma^2$*, where σ is given in (7) and α^* is given in (49), the relative volatility of consumption growth to income growth is*

$$\mu \equiv \frac{\operatorname{sd}\left(dc_t^*\right)}{\operatorname{sd}\left(dy_t\right)} = \frac{r f(\kappa)}{r + \rho}\sqrt{1 - \rho_{ye}^2 + \left[\frac{\pi (r + \rho)}{\widetilde{\gamma}\sigma_e\sigma_y}\right]^2}, \tag{55}$$

and the contemporaneous correlation between consumption growth and the equity return is

$$\rho_{cy} \equiv \operatorname{corr}\left(dc_t^*, dy_t\right) = f(\kappa)\frac{\left(1 - \rho_{ye}^2\right)\sigma_s + \pi\rho_{ye}/\left(r\widetilde{\gamma}\sigma_e\right)}{\sqrt{\left(1 - \rho_{ye}^2\right)\sigma_s^2 + \pi^2/\left(r\widetilde{\gamma}\sigma_e\right)^2}}. \tag{56}$$

Proof. See Appendix 7.3. ■

Expression (54) clearly shows that both RB and RI can affect the expected consumption growth by interacting with two sources of fundamental uncertainty: (i) labor income uncertainty $\left(\sigma_s^2\right)$ and (ii) asset return uncertainty $\left(\sigma_e^2\right)$. Specifically, we have

$$\frac{\partial g_c}{\partial \vartheta} > 0 \text{ if } \vartheta > \frac{\pi}{r\gamma\sqrt{f(\kappa)^3}\sqrt{1 - \rho_{ye}^2}\sigma_s\sigma_e} - 1. \tag{57}$$

Using the same parameter values above, we can compute that RB can increase the expected growth rate if ϑ is greater than 0.4 when $\kappa = 0.1$. (Here we set $\rho_{ye} = 0.18$.) In contrast, RB can increase the expected growth rate if ϑ is greater than 0.76 when $\kappa = 0.3$. Furthermore, we have

$$\frac{\partial g_c}{\partial \kappa} < 0 \text{ if } \kappa > \underline{\kappa} \equiv \frac{r}{1 - \left\{\pi^2/\left[2\left(r\widetilde{\gamma}\right)^2\left(1 - \rho_{ye}^2\right)\sigma_s^2\sigma_e^2\right]\right\}^{-2/3}}. \tag{58}$$

Because $\underline{\kappa}$ is negative for the plausible parameter values, SU (less κ) can always increase the expected growth rate. (See Figure 6.)

From Expression (55), we can see that RB reduces the relative volatility of consumption growth to income growth by increasing $\widetilde{\gamma}$ and reducing the optimal share invested in the risky asset. This result is different from that obtained in the permanent income model in which RB increases the relative volatility of consumption growth to income growth by strengthening the consumption sensitivity to income shocks.[31] It is also clear from (55) that SU measured by $f(\kappa)$ increases the relative volatility. This effect is similar to that obtained in the discrete-time permanent income model. (See Luo (2008) for a proof on how SU increases the relative volatility of consumption growth to income growth at the individual level.) Figure 6 illustrates how g_c varies with the degree of SU (κ) for different plausible values of γ. It is clear from the figure that the quantitative impact of SU on g_c is much stronger than that of MU on g_c.

Since $|\rho_{ye}| \leq 1$, we have

$$\frac{\partial \rho_{cy}}{\partial \vartheta} > 0,$$

which means that RB raises the contemporaneous correlation between consumption growth and income growth. In addition, $\rho_{cy} = \frac{1}{\sqrt{1+\pi^2/(r\widetilde{\gamma}\sigma_s\sigma_e)^2}}$ when labor income is purely idiosyncratic, i.e., $\rho_{ye} = 0$, while $\rho_{cy} = 1$ when the income risk and the return risk are perfectly correlated. From (56), it is obvious that SU increases ρ_{cy}. (See Figure 6.)

4.3 Empirical Implications

In this section, we adopt the same calibration procedure in the last section to calibrate the value of ϑ for a given DEP, p, in the MU-SU model. In this case, let model P denote the approximating model, (38) and model Q be the distorted model, (47). Because both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients under MU-SU, the log-likelihood ratios are normally distributed random variables. Consequently, the logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = -\int_0^N \overline{\upsilon} dB_s - \frac{1}{2}\int_0^N \overline{\upsilon}^2 ds, \tag{59}$$

where

$$\overline{\upsilon} \equiv \upsilon^* \widehat{\sigma} = -r\gamma\vartheta\sqrt{\sigma_e^2\alpha^{*2} + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha^*}. \tag{60}$$

[31] See Luo and Young (2010) for a proof that RB worsens the PIH model's prediction on the relative volatility of consumption growth to income growth.

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = \int_0^N \overline{v} dB_s + \frac{1}{2}\int_0^N \overline{v}^2 ds. \tag{61}$$

Given (59) and (61), it is straightforward to derive $p(\vartheta; N)$:

$$p(\vartheta; N) = \Pr\left(x < \frac{\overline{v}}{2}\sqrt{N}\right), \tag{62}$$

where x follows a standard normal distribution.

Using the same parameter values as in the last section, Figure 7 illustrates how p varies with the value of ϑ for different values of κ. We can see from the figure that the stronger the preference for robustness (higher ϑ), the less the p is. Tables 1 and 2 report how different values of κ affect calibrated values of ϑ, optimal allocation in the risky asset (α^*), the relative importance of the income hedging demand to the speculation demand ($|\alpha_h^*| / \alpha_s^*$), and precautionary saving demand (Γ) for different values of ρ_{ye} and σ_s, respectively. Specifically, for given values of σ_s and ρ_{ye}, when κ decreases (i.e., more information-constrained), the calibrate value of ϑ increases; consequently, the optimal share invested in the risky asset decreases and the relative importance of the income hedging demand to the speculation demand increases. In addition, the precautionary saving demand decreases with the value of κ.

From Table 1, we can see that for given values of κ, the precautionary saving demand decreases with the correlation between the equity return and labor income risk (ρ_{ye}), holding other factors constant, which is consistent with that we obtained in the MU model. The intuition is that the higher the correlation coefficient, the less demand for the risky asset and thus precautionary saving. In Table 2, we can see that as labor income becomes more volatile, the optimal allocation in the risky asset increases and the precautionary saving demand decreases. The reason is that the higher the value of σ_s, the less the calibrated value of ϑ, holding other factors fixed; consequently, the effective coefficient of absolute risk aversion decreases, and thus the optimal share increases and the precautionary saving demand decreases.

5 Extension: The Role of Intertemporal Substitution

5.1 Separation of Risk Aversion and Intertemporal Substitution

In the previous sections, we discussed how the interaction of risk aversion and informational frictions affects strategic asset allocation and consumption-saving dynamics in the presence of uninsurable labor income. However, given the time-separable utility setting, we cannot examine how intertemporal substitution affects optimal consumption-saving and portfolio rules. In this section, we consider a continuous-time recursive utility (RU) model with iso-elastic intertemporal substitution and exponential risk aversion. This recursive utility specification is proposed in Weil (1993) in a discrete-time consumption-saving model. In our continuous-time setting, the Bellman equation for the optimization problem can be written as:

$$J\left(\widehat{s}_t\right)^{1-1/\varepsilon}=\max_{c_t,\alpha_t}\left\{\left(1-\exp\left(-\delta dt\right)\right)c_t^{1-1/\varepsilon}+\exp\left(-\delta dt\right)\mathcal{C}E_t^{1-1/\varepsilon}\right\}, \tag{63}$$

subject to

$$d\widehat{s}_t=\left(r\widehat{s}_t-c_t+\pi\alpha_t\right)dt+\widehat{\sigma}dB_t^*, \tag{64}$$

where $\widehat{\sigma}\equiv f\left(\kappa\right)\sigma$ and $f\left(\kappa\right)=\sqrt{\frac{\kappa}{\kappa-r}}>1$,[32] ε is the intertemporal elasticity of substitution, δ is the discount rate, γ is the coefficient of absolute risk aversion, and

$$\mathcal{C}E_t\equiv-\frac{1}{\gamma}\ln\left(E_t\left[\exp\left(-\gamma J\left(\widehat{s}_{t+dt}\right)\right)\right]\right)$$

denotes the certainty equivalent in terms of period-t consumption of the uncertain total utility in the future periods. (Note that as κ goes to $+\infty$, $\widehat{\sigma}=\sigma$, $\widehat{s}_t=s_t$, this model reduces to the FI-RE case in which the consumer can observe the state perfectly.)

Furthermore, (63) can be reduced to

$$0=\max_{c_t,\alpha_t}\left\{\delta c_t^{1-1/\varepsilon}-\delta\widetilde{J}\left(\widehat{s}_t\right)+\left(r\widehat{s}_t-c_t+\pi\alpha_t-\frac{1}{2}A\gamma\widehat{\sigma}^2\right)\widetilde{J}_s\left(\widehat{s}_t\right)\right\},$$

where $\widetilde{J}\left(\widehat{s}_t\right)=\left(A\widehat{s}_t+A_0\right)^{1-1/\varepsilon}$, A and A_0 are undetermined coefficients.[33] (See Appendix 7.4 for the derivation.)

5.2 Optimal Consumption-Portfolio Rules under MU and SU

To introduce robustness into the above recursive utility model, we follow the same procedure as in the previous section and write the distorting model by adding an endogenous distortion $\upsilon\left(s_t\right)$ to the law of motion of the state variable $\widehat{s}_t$, (64):

$$d\widehat{s}_t=\left(r\widehat{s}_t-c_t+\pi\alpha\right)dt+\widehat{\sigma}\left(\widehat{\sigma}\upsilon\left(\widehat{s}_t\right)dt+d\widetilde{B}_t\right), \tag{65}$$

[32] Here we use the same procedure as proposed in Section 4.1 to derive this perceived state transition equation.

[33] Note that here we use the fact that

$$\ln\left(E_t\left[\exp\left(-\alpha J\left(\widehat{s}_{t+dt}\right)\right)\right]\right)=-\alpha A\widehat{s}_t-\alpha A_0-\alpha A\left(r\widehat{s}_t-c_t+\pi\alpha_t\right)dt+\frac{1}{2}\alpha^2A^2\widehat{\sigma}^2dt.$$

where the drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of the expected continuation payoff, but adjusted to reflect the additional drift component in the above distortion equation, and an entropy penalty:

$$0 = \max_{c_t, \alpha_t} \inf_{\upsilon_t} \left\{ \delta c_t^{1-1/\varepsilon} - \delta \widetilde{J}(\widehat{s}_t) + \left(r\widehat{s}_t - c_t + \pi\alpha_t - \frac{1}{2}A\gamma\widehat{\sigma}^2 \right) \widetilde{J}_{\widehat{s}}(\widehat{s}_t) + \widehat{\sigma}^2 \upsilon_t \widetilde{J}_{\widehat{s}}(\widehat{s}_t) + \frac{1}{2\vartheta_t}\widehat{\sigma}^2 \upsilon_t^2 \right\},$$

where $\widetilde{J}(\widehat{s}_t) = (A\widehat{s}_t + A_0)^{1-1/\varepsilon}$ and $\widetilde{J}_{\widehat{s}}(\widehat{s}_t) = (1 - 1/\varepsilon) A (A\widehat{s}_t + A_0)^{-1/\varepsilon}$. Following the standard procedure, we can solve for optimal consumption and portfolio rules. The following proposition summarizes the solution:

Proposition 9 *Given ϑ and κ, the optimal consumption and portfolio rules under MU and SU can be written as:*

$$c_t^* = [r + (\delta - r)\varepsilon]\widehat{s}_t + \left[1 + \left(\frac{\delta}{r} - 1\right)\varepsilon\right] \left(\frac{\pi^2}{2A\gamma\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \right) - \Gamma, \tag{66}$$

and

$$\alpha^* = \frac{\pi}{A\widetilde{\gamma} f(\kappa)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{67}$$

respectively, where the effective coefficient of absolute risk aversion $\widetilde{\gamma}$ is defined as: $\widetilde{\gamma} \equiv \gamma + \vartheta$,

$$A = \left[\frac{r + (\delta - r)\varepsilon}{\delta^\varepsilon} \right]^{1/(1-\varepsilon)}, \tag{68}$$

and the precautionary savings demand Γ is

$$\Gamma \equiv \frac{1}{2r}[r + (\delta - r)\varepsilon] A\widetilde{\gamma} f(\kappa)^2 \left(1 - \rho_{ye}^2\right)\sigma_s^2. \tag{69}$$

Proof. See Appendix 7.4. ■

When $\delta = r$, $A = r$ and this RU case is reduced to the benchmark model. The reason is that when the interest rate equals the discount rate, the effect of EIS on consumption growth and saving disappears. When $\delta \neq r$, A is increasing in ε. (We can see this from Figure 8.) From (66) and (69), we can see that EIS affects both the MPC out of perceived permanent income and the precautionary saving demand when $\delta \neq r$. Specifically, both MPC and the precautionary saving demand increases with ε when $\delta > r$. When $\delta < r$, MPC increases with ε, while the precautionary saving demand decreases with ε. (See Figure 9.) From (67) and Figure 9, it is clear that ε reduces the speculation demand invested in the risky asset by increasing the value of A. It is worth noting that although ε reduces the effective coefficient of absolute risk aversion $\widetilde{\gamma} \equiv \gamma + \vartheta$ and thus has the potential to increase the speculation demand, the net effect of ε is to reduce the speculation demand. In summary, risk version and intertemporal substitution shift the optimal allocation in the risky asset in the same direction. However, their impacts on the precautionary saving demand depend on the values of the interest rate and the discount rate.

To explore the quantitative effects of ε on asset allocation and precautionary saving, we follow the same procedure to calibrate the values of ϑ using p. Table 3 reports these results for $p = 10\%$ and $\kappa = 0.5$. For given values of κ, we can see from the table that the optimal share (α^*) decreases with ε. Because the income hedging demand is independent of ε, the relative importance of the income hedging demand to the speculation demand increases with ε. Furthermore, it is also clear from the table that the precautionary saving demand increases with ε for different values of κ.

6 Conclusion

This paper has developed a tractable continuous-time CARA-Gaussian framework to explore how induced uncertainty due to informational frictions affects strategic consumption-portfolio rules, precautionary savings, and consumption dynamics in the presence of uninsurable labor income. Specifically, we explored the relative importance of the two types of induced uncertainty: (i) model uncertainty due to robustness and (ii) state uncertainty due to limited information-processing capacity as well as risk aversion in determining strategic consumption-portfolio rules and precautionary savings. In addition, we find that both MU and SU reduce the optimal share invested in the risky asset. Finally, we find that MU measured by the interaction of fundamental uncertainty and calibrated values of robustness not only affects the hedging demand for the risky asset, but also affects the standard speculation demand.

7 Appendix

7.1 Solving the MU Model

The Bellman equation associated with the optimization problem is

$$J(s_t) = \sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) + \exp(-\delta dt)\, J(s_{t+dt})\right],$$

subject to (12), where $J(s_t)$ is the value function. The HJB equation for this problem is then

$$0 = \sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t)\right],$$

where $\mathcal{D}J(s_t) = J_s(rs_t - c_t + \pi\alpha_t) + \frac{1}{2}J_{ss}\left(\sigma_{et}^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right)$. Under RB, the HJB can be written as

$$\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + \upsilon(s_t)\,\sigma^2 J_s + \frac{1}{2\vartheta(s_t)}\upsilon(s_t)^2\,\sigma^2\right]$$

subject to the distorting equation, (12). Solving first for the infimization part of the problem yields

$$\upsilon^*(s_t) = -\vartheta(s_t) J_s.$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for υ^* in the robust HJB equation gives:

$$\sup_{c_t, \alpha_t} \left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + (rs_t - c_t + \pi\alpha_t) J_s + \frac{1}{2}\sigma^2 J_{ss} - \frac{1}{2}\vartheta(s_t)\sigma^2 J_s^2 \right]. \tag{70}$$

Performing the indicated optimization yields the first-order conditions for c_t and α_t:

$$c_t = -\frac{1}{\gamma}\ln(J_s), \tag{71}$$

$$\alpha_t = \frac{\pi J_s + \rho_{ye}\sigma_s\sigma_e\left(J_{ss} - \vartheta J_s^2\right)}{\left(\vartheta J_s^2 - J_{ss}\right)\sigma_e^2}. \tag{72}$$

Substitute (71) and (72) back into (70) to arrive at the partial differential equation

$$0 = -\frac{J_s}{\gamma} - \delta J + \left(rs_t + \frac{1}{\gamma}\ln(J_s) + \pi\alpha_t\right) J_s + \frac{1}{2}\left(J_{ss} - \vartheta_t J_s^2\right)\sigma^2, \tag{73}$$

where $\sigma^2 = \sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t$. Conjecture that the value function is of the form

$$J(s_t) = -\frac{1}{\alpha_1}\exp(-\alpha_0 - \alpha_1 s_t),$$

where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain that $J_s = \exp(-\alpha_0 - \alpha_1 s_t) > 0$ and $J_{ss} = -\alpha_1\exp(-\alpha_0 - \alpha_1 s_t) < 0$. Further more, we guess that $\vartheta(s_t) = -\frac{\vartheta}{J(s_t)} = \frac{\alpha_1\vartheta}{\exp(-\alpha_0 - \alpha_1 s_t)} > 0$. (73) can thus be reduced to

$$-\delta\frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left[rs_t - \left(\frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma}s_t\right) + \frac{\pi\left(\pi - \rho_{ye}\sigma_s\sigma_e\alpha_1(1+\vartheta)\right)}{(1+\vartheta)\alpha_1\sigma_e^2}\right] - \frac{1}{2}\alpha_1(1+\vartheta)\left(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right)$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \tag{74}$$

$$\alpha_0 = \frac{\delta}{r} - 1 + \frac{\pi\left[\pi - \rho_{ye}\sigma_s\sigma_e r\gamma(1+\vartheta)\right]}{(1+\vartheta)r\sigma_e^2} - \frac{1}{2}(1+\vartheta)r\gamma^2\left(\sigma_e^2\alpha_t + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right), \tag{75}$$

where $\alpha^* = \frac{\pi}{(1+\vartheta)\gamma r\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}$. Substituting these back into the first-order condition (71) yields the consumption function, (16), in the main text. Using (75) and $\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t = \left(1 - \rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{(r\tilde{\gamma})^2\sigma_e^2}$, we can obtain the expression for the precautionary savings premium, (18), in the main text.

7.2 Deriving Continuous-time IPC

The IPC,

$$\ln\left(\rho_1^2\Sigma_t+\rho_2^2\right)-\ln\Sigma_{t+\Delta t}=2\kappa\Delta t, \tag{76}$$

can be rewritten as

$$\ln\left(\exp\left(2r\Delta t\right)\Sigma_t+\frac{1-\exp\left(2r\Delta t\right)}{-2r\Delta t}\Delta t\sigma^2\right)-\ln\Sigma_{t+\Delta t}=2\kappa\Delta t, \tag{77}$$

which can be reduced to

$$\Sigma_{t+\Delta t}-\Sigma_t=\left[\exp\left(2\left(r-\kappa\right)\Delta t\right)-1\right]\Sigma_t+\frac{\exp\left(2\left(r-\kappa\right)\Delta t\right)-\exp\left(-2\kappa\Delta t\right)}{2r}\sigma^2.$$

Dividing Δt on both sides of this equation and letting $\Delta t\to 0$, we have the following updating equation for Σ_t:

$$\dot{\Sigma}_t=\lim_{\Delta t\to 0}\frac{\Sigma_{t+\Delta t}-\Sigma_t}{\Delta t}=2\left(r-\kappa\right)\Sigma_t+\sigma^2.$$

7.3 Solving the MU-SU Model

The Bellman equation associated with the optimization problem under SU is

$$J\left(\widehat{s}_t\right)=\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)+\exp\left(-\delta dt\right)J\left(\widehat{s}_{t+dt}\right)\right],$$

subject to (47), where $J\left(\widehat{s}_t\right)$ is the value function. The HJB equation for this problem can thus be written as

$$0=\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(\widehat{s}_t\right)+\mathcal{D}J\left(s_t\right)\right],$$

where

$$\mathcal{D}J\left(\widehat{s}_t\right)=J_{\widehat{s}}\left(r\widehat{s}_t-c_t+\pi\alpha_t\right)+\frac{1}{2}J_{\widehat{s}\widehat{s}}f\left(\kappa\right)^2\left(\sigma_e^2\alpha_t^2+\sigma_s^2+2\rho_{ye}\sigma_s\sigma_e\alpha_t\right).$$

Here we use the facts that $\widehat{\sigma}\equiv f\left(\kappa\right)\sigma$ and $\sigma=\sqrt{\sigma_e^2\alpha^2+\sigma_s^2+2\rho_{ye}\sigma_s\sigma_e\alpha}$. Under MU and SU, the HJB can be written as

$$\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(\widehat{s}_t\right)+\mathcal{D}J\left(\widehat{s}_t\right)+\upsilon\left(\widehat{s}_t\right)\widehat{\sigma}^2J_{\widehat{s}}+\frac{1}{2\vartheta\left(s_t\right)}\upsilon\left(\widehat{s}_t\right)^2\widehat{\sigma}^2\right]$$

subject to the distorting equation, (47). Solving first for the infimization part of the problem yields: $\upsilon^*\left(\widehat{s}_t\right)=-\vartheta\left(\widehat{s}_t\right)J_{\widehat{s}}$. Given that $\vartheta\left(\widehat{s}_t\right)>0$, the perturbation adds a negative drift term to the state transition equation because $J_{\widehat{s}}>0$. Substituting for υ^* in the robust HJB equation gives:

$$\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right)-\delta J\left(\widehat{s}_t\right)+\left(r\widehat{s}_t-c_t+\pi\alpha_t\right)J_{\widehat{s}}+\frac{1}{2}\widehat{\sigma}^2J_{\widehat{s}\widehat{s}}-\frac{1}{2}\vartheta\left(\widehat{s}_t\right)\widehat{\sigma}^2J_{\widehat{s}}^2\right]. \tag{78}$$

Performing the indicated optimization yields the first-order conditions for c_t and α_t:

$$c_t = -\frac{1}{\gamma}\ln\left(J_{\widehat{s}}\right), \tag{79}$$

$$\alpha_t = \frac{\pi J_{\widehat{s}}/f(\kappa) + \rho_{ye}\sigma_s\sigma_e\left(J_{\widehat{s}\widehat{s}} - \vartheta J_{\widehat{s}}^2\right)}{\left(\vartheta J_{\widehat{s}}^2 - J_{\widehat{s}\widehat{s}}\right)\sigma_e^2}. \tag{80}$$

Substitute (79) and (80) back into (78) to arrive at the partial differential equation

$$0 = -\frac{J_{\widehat{s}}}{\gamma} - \delta J + \left(r\widehat{s}_t + \frac{1}{\gamma}\ln\left(J_{\widehat{s}}\right) + \pi\alpha_t\right)J_{\widehat{s}} + \frac{1}{2}f(\kappa)^2\left(J_{\widehat{s}\widehat{s}} - \vartheta_t J_{\widehat{s}}^2\right)\sigma^2.$$

Conjecture that the value function is of the form

$$J\left(\widehat{s}_t\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right),$$

where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain $J_{\widehat{s}} = \exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right) > 0$ and $J_{\widehat{s}\widehat{s}} = -\alpha_1\exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right) < 0$. Furthermore, we guess that $\vartheta\left(\widehat{s}_t\right) = -\frac{\vartheta}{J(\widehat{s}_t)} = \frac{\alpha_1\vartheta}{\exp(-\alpha_0-\alpha_1\widehat{s}_t)} > 0$. Substituting these expressions into (73) yields:

$$-\delta\frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left\{r\widehat{s}_t - \left(\frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma}\widehat{s}_t\right) + \frac{\pi\left[\pi/f(\kappa) - \rho_{ye}\sigma_s\sigma_e\alpha_1(1+\vartheta)\right]}{(1+\vartheta)\alpha_1\sigma_e^2}\right\}$$
$$-\frac{1}{2}f(\kappa)^2\alpha_1(1+\vartheta)\left(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right)$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \tag{81}$$

$$\alpha_0 = \frac{\delta - r}{r} + \frac{\pi\left[\pi/f(\kappa) - \rho_{ye}\sigma_s\sigma_e r\alpha(1+\vartheta)\right]}{(1+\vartheta)r\sigma_e^2} - \frac{1}{2}rf(\kappa)^2(1+\vartheta)\gamma^2\left(\sigma_e^2\alpha_t + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right). \tag{82}$$

Substituting (81) and (82) into (79), (80), and (**??**) yields the optimal portfolio and consumption rule, (49) and (48), respectively, in the main text. Using (82) and that fact that $\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t = \left(1 - \rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{r^2\widetilde{\gamma}^2 f(\kappa)^2\sigma_e^2}$, we can obtain Expression (50) in the main text.

Using (48) and (49), we have $dc_t^* = rd\widehat{s}_t$ and $\text{var}\left(dc_t^*\right) = r^2 f(\kappa)^2\sigma^2$. The relative volatility of consumption growth to income growth can thus be written as

$$\mu \equiv \frac{\text{sd}\left(dc_t^*\right)}{\text{sd}\left(dy_t\right)} = rf(\kappa)\sqrt{\frac{1-\rho_{ye}^2}{(r+\rho)^2} + \frac{\pi^2}{\widetilde{\gamma}^2\sigma_e^2\sigma_y^2}}.$$

The contemporaneous covariance between consumption growth and income growth is

$$\text{cov}\left(dc_t^*, dy_t\right) = rf(\kappa)\,\text{cov}\left(\alpha^*\sigma_e dB_{e,t} + \frac{1}{r+\rho}\sigma_y dB_{y,t}, \sigma_y dB_{y,t}\right) = rf(\kappa)\left(\frac{1}{r+\rho}\sigma_y^2 + \alpha^*\rho_{ye}\sigma_e\sigma_y\right),$$

which implies that

$$\rho_{cy} \equiv \operatorname{corr}\left(dc_t^*, dy_t\right) = f(\kappa)\,\frac{\sigma_s + \alpha^* \rho_{ye}\sigma_e}{\sigma}.$$

Substituting $\sigma = \sqrt{\left(1-\rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{(r\widetilde{\gamma})^2 f(\kappa)^2 \sigma_e^2}}$ and $\alpha^* = \frac{\pi}{r\widetilde{\gamma} f(\kappa)^2 \sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}$ into this expression leads to (56) in the main text.

7.4 Solving the RU Model with MU and SU

We first guess that the value function is $J(\widehat{s}_t) = A\widehat{s}_t + A_0$. At time $t+dt$, the value function is $J(\widehat{s}_{t+dt}) = A\widehat{s}_{t+dt} + A_0$ and $dJ \equiv J(\widehat{s}_{t+dt}) - J(\widehat{s}_t) = Ad\widehat{s}_t = A(r\widehat{s}_t - c_t + \pi\alpha_t)\,dt + A\theta d\xi_t$, and

$$\begin{aligned} E_t\left[\exp\left(-\gamma J\left(\widehat{s}_{t+dt}\right)\right)\right] &= E_t\left[\exp\left(-\gamma A\widehat{s}_t - \gamma A_0 - \gamma A\left(r\widehat{s}_t - c_t + \pi\alpha_t\right)dt - \gamma A\theta d\xi_t\right)\right] \\ &= \exp\left(-\gamma A\widehat{s}_t - \gamma A_0\right)\exp\left(-\gamma A\left(r\widehat{s}_t - c_t + \pi\alpha_t\right)dt\right)\exp\left(\frac{1}{2}\gamma^2 A^2\widehat{\sigma}^2 dt\right), \end{aligned}$$

where $\widehat{\sigma} = f(\kappa)\,\sigma$. We can therefore obtain:

$$\ln\left(E_t\left[\exp\left(-\alpha J\left(\widehat{s}_{t+dt}\right)\right)\right]\right) = -\alpha A\widehat{s}_t - \alpha A_0 - \alpha A\left(r\widehat{s}_t - c_t + \pi\alpha_t\right)dt + \frac{1}{2}\alpha^2 A^2\widehat{\sigma}^2 dt.$$

Substituting this expression back into the Bellman equation yields:

$$J(\widehat{s}_t)^{1-1/\varepsilon} = \max_{c_t,\alpha_t}\left\{\left(1-\exp\left(-\delta dt\right)\right)c_t^{1-1/\varepsilon} + \exp\left(-\delta dt\right)\begin{bmatrix} J(\widehat{s}_t) + \\ A\left(r\widehat{s}_t - c_t + \pi\alpha_t\right)dt - \frac{1}{2}\alpha A^2\widehat{\sigma}^2 dt \end{bmatrix}^{1-1/\varepsilon}\right\},$$

which can be reduced to

$$0 = \max_{c_t,\alpha_t}\left\{\delta c_t^{1-1/\varepsilon} - \delta\widetilde{J}(\widehat{s}_t) + \left(r\widehat{s}_t - c_t + \pi\alpha_t - \frac{1}{2}A\gamma\widehat{\sigma}^2\right)\widetilde{J}_s(\widehat{s}_t)\right\},$$

where $\widetilde{J}(\widehat{s}_t) = \left(A\widehat{s}_t + A_0\right)^{1-1/\varepsilon}$ and $\widetilde{J}_{\widehat{s}}(\widehat{s}_t) = (1-1/\varepsilon)\,A\left(A\widehat{s}_t + A_0\right)^{-1/\varepsilon}$.

In the presence of robustness, the HJB is

$$0 = \max_{c_t,\alpha_t}\inf_{\upsilon_t}\left\{\delta c_t^{1-1/\varepsilon} - \delta\widetilde{J}(\widehat{s}_t) + \left(r\widehat{s}_t - c_t + \pi\alpha_t - \frac{1}{2}A\gamma\widehat{\sigma}^2\right)\widetilde{J}_{\widehat{s}}(\widehat{s}_t) + \widehat{\sigma}^2\upsilon_t\widetilde{J}_{\widehat{s}}(\widehat{s}_t) + \frac{1}{2\vartheta_t}\widehat{\sigma}^2\upsilon_t^2\right\}.$$

Solving first for the infimization part of the problem yields $\upsilon_t^* = -\vartheta_t\widetilde{J}_{\widehat{s}}$. Here we also assume that $\vartheta_t = -\vartheta A/\widetilde{J}_{\widehat{s}}$ to guarantee the homothecity of the RB problem. Substituting υ_t^* back into the above robust HJB equation yields

$$0 = \max_{c_t,\alpha_t}\left\{\delta c_t^{1-1/\varepsilon} - \delta\widetilde{J}(\widehat{s}_t) + \left[r\widehat{s}_t - c_t + \pi\alpha_t - \frac{1}{2}A\left(\gamma+\vartheta\right)\widehat{\sigma}^2\right]\widetilde{J}_{\widehat{s}}(\widehat{s}_t)\right\}.$$

The FOC with respect to c_t is

$$\delta\left(1-1/\varepsilon\right)c_t^{-1/\varepsilon} = \widetilde{J}_{\widehat{s}}(\widehat{s}_t),$$

which means that $c_t = \left(\frac{A}{\delta}\right)^{-\varepsilon}(A\widehat{s}_t + A_0)$. The FOC for α_t leads to:

$$\alpha = \frac{\pi}{A\theta\left(\gamma+\vartheta\right)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}. \tag{83}$$

Substituting these expressions back into the HJB yields:

$$0 = \delta\left[\left(\frac{A}{\delta}\right)^{1-\varepsilon} - 1\right](A\widehat{s}_t + A_0) + (1-1/\varepsilon)\,A\left[\left(r - \left(\frac{A}{\delta}\right)^{-\varepsilon}A\right)\widehat{s}_t - \left(\frac{A}{\delta}\right)^{-\varepsilon}A_0 + \pi\alpha - \frac{1}{2}A^2\left(\gamma+\vartheta\right)\widehat{\sigma}^2\right]$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$A = \left[\frac{r + (\delta - r)\,\varepsilon}{\delta^{\varepsilon}}\right]^{1/(1-\varepsilon)} \text{ and } A_0 = \frac{A}{r}\left[\pi\alpha - \frac{1}{2}A\left(\gamma+\vartheta\right)\widehat{\sigma}^2\right].$$

The consumption function is

$$c_t = \left[r + (\delta - r)\,\varepsilon\right]\widehat{s}_t + \frac{1}{2}\left[1 + \left(\frac{\delta}{r} - 1\right)\varepsilon\right]A\theta\left(\gamma+\vartheta\right)\left(\sigma_e^2\alpha^2 - \sigma_s^2\right)$$

where $\alpha = \frac{\pi}{A\theta(\gamma+\vartheta)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}$. Substituting the expression for portfolio rule into the above consumption function yields (66) in the main text.

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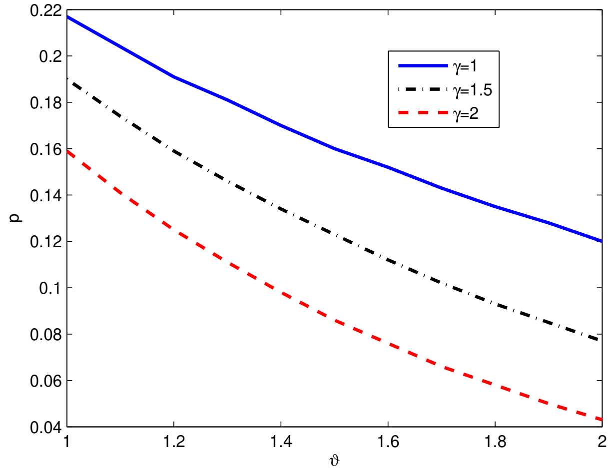
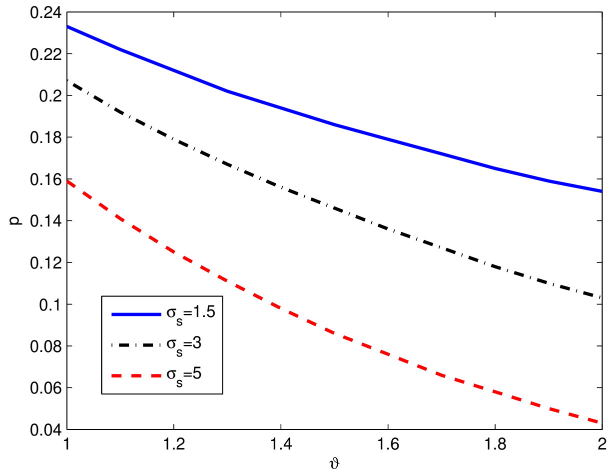


Figure 1: Relationship between ϑ and p



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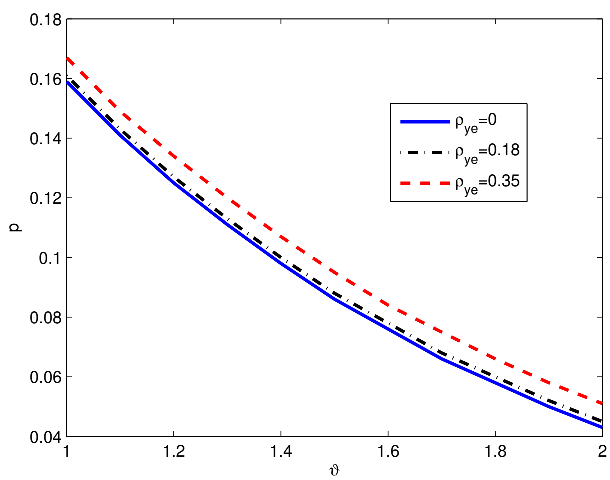

Figure 3: Relationship between ϑ and p

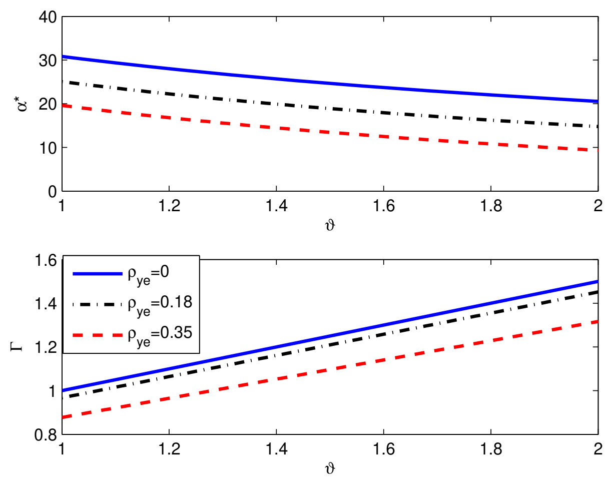

Figure 4: Robust Portfolio Rule and Precautionary Savings

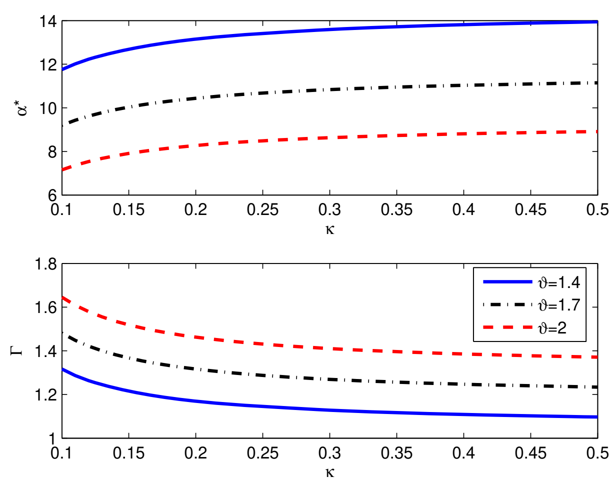

Figure 5: Robust Portfolio Rule and Precautionary Savings under SU

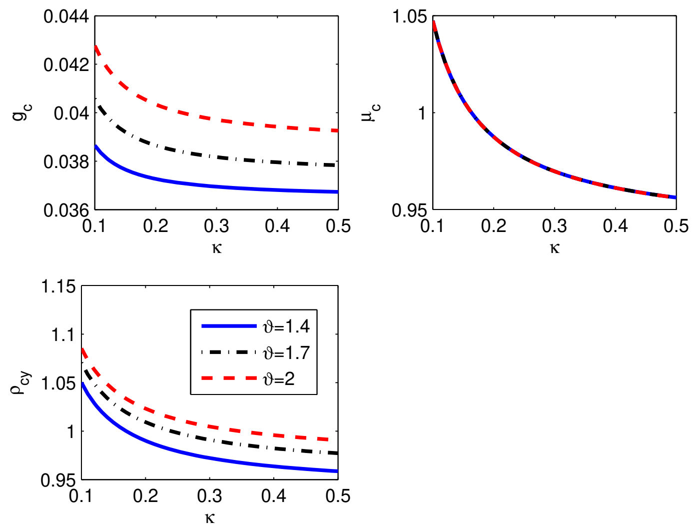

Figure 6: Effects of MU and SU on Consumption Dynamics

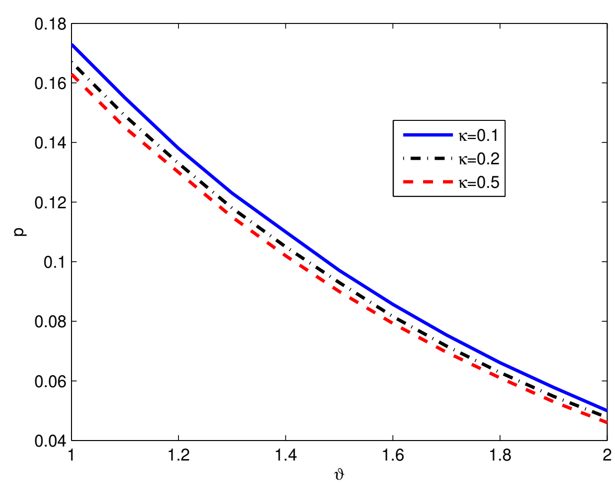

Figure 7: Relationship between ϑ and p under SU

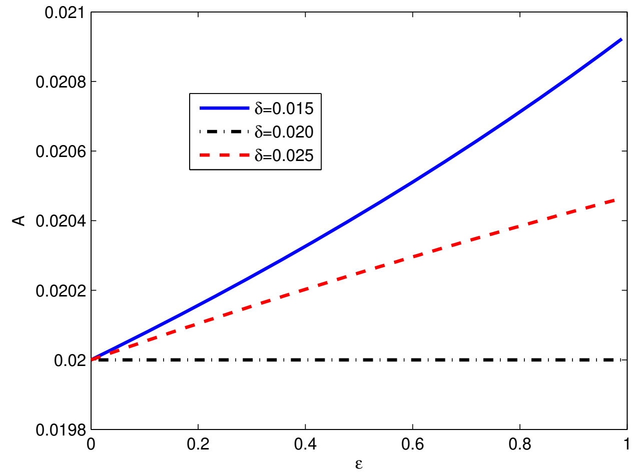

Figure 8: Relationship between ε and A

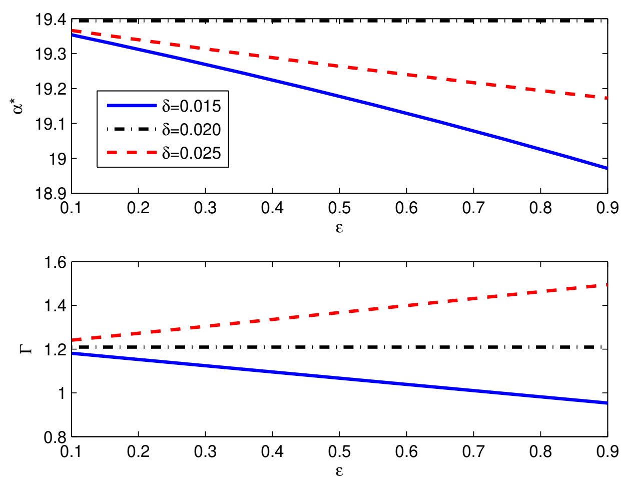

Figure 9: Robust Portfolio Rule and Precautionary Savings under RU

Table 1: Implications of the correlation on α^* and Γ under MU and SU ($p = 10\%, \sigma_s = 5$)

		$\rho_{ye} = 0$	$\rho_{ye} = 0.18$	$\rho_{ye} = 0.35$
$\kappa = 0.1$	ϑ	1.455	1.475	1.536
	α^*	22.46	16.51	10.52
	α_s^*	22.46	22.28	21.74
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.259	0.516
	Γ	1.534	1.497	1.391
$\kappa = 0.2$	ϑ	1.418	1.437	1.495
	α^*	24.18	18.23	12.22
	α_s^*	24.18	23.99	23.44
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.240	0.479
	Γ	1.364	1. 31	1.216
$\kappa = 0.5$	ϑ	1.396	1.415	1.47
	α^*	25.21	19.24	13.23
	α_s^*	25.21	25.01	24.45
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.231	0.459
	Γ	1.279	1.217	1.129
$\kappa = 1$	ϑ	1.389	1.407	1.462
	α^*	25.54	19.58	13.57
	α_s^*	25.54	25.35	24.78
	α_h^*	0	-5.77	-11.22
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0	0.228	0.453
	Γ	1.253	1.188	1.102

Table 2: Implications of income uncertainty on α^* and Γ under MU and SU ($p = 10\%$, $\rho_{ye} = 0.18$)

		$\sigma_s = 4$	$\sigma_s = 5$	$\sigma_s = 6$
$\kappa = 0.1$	ϑ	1.770	1.475	1.265
	α^*	15.29	16.51	17.42
	α_s^*	19.90	22.28	24.34
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.232	0.259	0.284
	Γ	1.675	1.497	1.370
$\kappa = 0.2$	ϑ	1.715	1.437	1.238
	α^*	16.92	18.23	19.21
	α_s^*	21.54	23.99	26.13
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.214	0.240	0.265
	Γ	1.460	1.310	1.203
$\kappa = 0.5$	ϑ	1.682	1.415	1.222
	α^*	17.90	19.24	20.26
	α_s^*	22.52	25.01	27.18
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.205	0.231	0.255
	Γ	1.352	1.217	1.120
$\kappa = 1$	ϑ	1.671	1.407	1.216
	α^*	18.23	19.58	20.61
	α_s^*	22.84	25.35	27.54
	α_h^*	-4.62	-5.77	-6.92
	$\lvert\alpha_h^*\rvert / \alpha_s^*$	0.202	0.228	0.251
	Γ	1.319	1.188	1.094

Table 3: Implications of EIS on α^* and Γ under MU and SU ($p = 10\%$, $\rho_{ye} = 0.18$, $\sigma_s = 5$)

		$\varepsilon = 0.1$	$\varepsilon = 0.5$	$\varepsilon = 0.9$
$\kappa = 0.1$	ϑ/A	1.475	1.48	1.486
	α^*	16. 45	16. 19	15. 94
	α_s^*	22. 22	21. 96	21. 71
	α_h^*	-5.77	-5.77	-5.77
	$\|\alpha_h^*\| / \alpha_s^*$	0.260	0.263	0.266
	Γ	1. 538	1.856	1.881
$\kappa = 0.2$	ϑ/A	1.437	1.443	1.449
	α^*	18.16	17.87	17.61
	α_s^*	23.93	23.64	23.38
	α_h^*	-5.77	-5.77	-5.77
	$\|\alpha_h^*\| / \alpha_s^*$	0.241	0.244	0.247
	Γ	1.346	1.629	1. 647
$\kappa = 0.5$	ϑ/A	1.415	1.423	1.428
	α^*	19.17	18.85	18.58
	α_s^*	24.94	24.62	24.35
	α_h^*	-5.77	-5.77	-5.77
	$\|\alpha_h^*\| / \alpha_s^*$	0.231	0.234	0.237
	Γ	1.251	1.515	1.531
$\kappa = 1$	ϑ/A	1.408	1.414	1.42
	α^*	19.50	19.20	18.92
	α_s^*	25.27	24.96	24.69
	α_h^*	-5.77	-5.77	-5.77
	$\|\alpha_h^*\| / \alpha_s^*$	0.228	0.231	0.234
	Γ	1.221	1.478	1.495