Population-Monotonicity of the Nucleolus on a Class of Public Good Problems

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Abstract

Sprumont (1990) shows that the Shapley value (Shapley 1957) is population-monotonic (Thomson 1983) on the class of convex games (Shapley 1971). In this paper we study the population-monotonicity of the nucleolus (Schmeidler 1969). We show that the nucleolus is not population-monotonic on the class of convex games. Our main result is that the nucleolus is population-monotonic on a class of public good problems which is formalized in Littlechild and Owen (1973) under the name of airport games. We also provide a recursive formula for the nucleolus of the airport game.

1 Introduction

In axiomatic game theory, most of the earlier studies pertained to situations where the population is fixed. In recent literature, however much attention has been given to situations where the population is variable. *Population-monotonicity*, introduced by Thomson (1983) in the context of bargaining theory is a property defined on classes of problems of variable size (See Thomson 1992 for a survey). It requires everybody initially present to lose upon the arrival of new agents, if opportunities do not expand.

This paper studies the population-monotonicity of certain solutions to a class of *transferable utility* games. We consider situations where the arrival of new agents is accompanied by an *expansion* of opportunities and we require everybody initially present to gain. We concentrate on the solution known as the *nucleolus* (Schmeidler 1969). One of the basic properties of the nucleolus is that it is in the *core* whenever the core is non-empty. There has been an increasing interest in this solution since Sobolev's (1975) axiomatization. One of the most interesting recent results concerning it is the discovery that the 2000 year old Talmud prescribes solutions to bankruptcy problems that coincide with the nucleoli of games associated with such problems in a natural way. (Aumann and Maschler 1985).

The class of *convex* games (Shapley 1971) is a rich class of games which exhibit "increasing returns to cooperation". Sprumont (1990) shows that the *Shapley value* (Shapley 1957) is

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population-monotonic on the class of convex games. We first ask whether this is also true for the nucleolus. The answer is unfortunately negative.

We then consider the class of *public good problems* illustrated by the following example. A group of airlines N share the cost of a runway. To serve the planes of a particular airline i, the length of the runway (proportional to its cost) must be at least c_i. If a coalition of airlines S want to use the runway together, the length of the runway should be $max_{j \in S} c_j$. Any solution for this problem is interpreted as a specification of fees to be paid by the airlines to cover the cost of the runway when it is used by the grand coalition N. Our main result is that the nucleolus is population-monotonic on this class of public good problems.

We then study the population-monotonicity of two other solutions, on this class of public good games. The solutions are the $\tau - value$ (Tijs 1981) and the *separable cost remaining benefit (SCRB)* solution (Federal Inter Agency River Basin Committee 1950). We obtain negative results for both.

2 Solutions

There is an infinite number of potential agents, indexed by the positive integers Z. Let $\mathcal{P}$ be the class of all finite subsets of Z, with generic elements N, N' etc. We denote the cardinality of N by $|N|$.

A **TU game** is a vector $v \in R^{2^{|N|}-1}$. Given a **coalition** $S \subset N$, $v(S) \in R$ represents what S can achieve on its own, its **worth**. Let Γ^N be the class of all games involving the group N. Let $\Gamma = \bigcup_{N \in \mathcal{P}} \Gamma^N$. A **solution on** Γ is a correspondence that associates with every $N \in \mathcal{P}$ and every $v \in \Gamma^N$ a nonempty set of vectors $x \in R^{|N|}$ such that $\sum_{i \in S} x_i \leq v(N)$. To introduce the solutions that we will study, we need some preliminary definitions.

The **imputation set** is given by

$$I(v) \equiv \{x \in R^{|N|} | \sum_{i \in N} x(i) = v(N),\ x(i) \geq v(i)\ \forall i \in N\}$$

Definition: Given $v \in R^{2^{|N|}-1}$, the **core of** v is given by

$$C(v) \equiv \{x \in I(v) | \sum_{i \in S} x_i \geq v(S)\ \forall S \subset N\}$$

Consider the game v and the payoff vector x. The **excess of a coalition** S **with respect to** x, and the **excess of** v **with respect to** x are given by

$$e^v(S, x) \equiv v(S) - \sum_{i \in S} x_i$$
$$e^v(x) \equiv (e^v(S, x))_{S \subseteq N}$$

Let $\Theta(e^v(x))$ be the vector whose coordinates are the excesses arranged in decreasing order. We write $x \leq_L y$ if there is an integer m, $1 \leq m \leq 2^n - 1$ such that, $\Theta_k(e^v(x)) = \Theta_k(e^v(y))$ for all $k < m$ and $\Theta_m(e^v(x)) < \Theta_m(e^v(y))$.

Definition: Given $v \in R^{2^{|N|}-1}$, the **nucleolus of** v is given by

$$Nu(v) \equiv \{x \in I(v) | \nexists y \in I(v), \Theta(e^v(y)) \leq_L \Theta(e^v(x))\}$$

Nucleolus allocations lexicographically minimize coalitional dissatisfaction starting with the most dissatisfied coalition. The nucleolus is a single-valued, continuous solution and it is in the core whenever the core is non-empty.

The **marginal contribution of agent i to the grand coalition N** $b_i(v)$ and the **gap vector** g^v of the game v are given by

$$b_i(v) \equiv v(N) - v(N-i) \quad \text{for all } i \in N$$
$$g^v(S) \equiv \textstyle\sum_{j \in S} b_j(v) - v(S) \quad \text{for all } S \subseteq N$$

Definition: Given $v \in R^{2^{|N|}-1}$, the **separable cost remaining benefit** solution $B(v)$ is given by

$$B_i(v) \equiv b_i(v) - g^v(N)/|N| \text{ for all } i \in N$$

According to the separable cost remaining benefit method, every agent i receives his marginal contribution to the grand coalition $b_i(v)$, and shares the difference between the worth of the grand coalition and the sum of the marginal contributions to the grand coalition equally.

The **concession vector** $\lambda^v \in R^{|N|}$ of the game v is given by

$$\lambda_i^v \equiv min_{S, i \in S}\ g^v(S) \quad \text{for all} \quad i \in N$$

Let Γ^B be the class of games with a non-empty core. We will define the next solution only for this class of games for which it takes a particular form.

Definition: Given $v \in \Gamma^B$, the $\tau -$ ***value*** of v is given by

$$\tau(v) \equiv b(v) - \frac{\lambda^v}{\sum_{j \in N} \lambda_j^v}$$

We now can state the main property we study. Given $N, N' \in \mathcal{P}$ such that $N \subset N'$, and a game $v \in \Gamma^{N'}$ let v_N be the restriction of v to the group N. Formally, $v_N \equiv (v(S))_{S \subseteq N}$

Population-monotonicity requires everybody initially present to gain upon the arrival of new agents.

Definition: A solution $\psi \in \Gamma$ is **population-monotonic** if for all $N, N' \in \mathcal{P}$ with $N \subset N'$, for all $v \in \Gamma^{N'}$, $\psi_i(v) \geq \psi_i(v_N)$ for all $i \in N$.

3 Convex Games

The class of the convex games is a class of games in which the incentives for joining a coalition increases as the coalition grows in an analogous way to the increasing returns to scale associated with convex production functions in economics.

It is shown by Sprumont (1990) that the Shapley value is *population-monotonic* on the class of convex games. Thus, it is natural to ask whether other solutions have that property.

Definition: A game v is **convex** if

$$v(S \bigcup i) - v(S) \leq v(T \bigcup i) - v(T) \quad \text{for all } i \in N, i \notin T \text{ such that } S \subseteq T \subseteq N.$$

For a convex game v

$$\tau_i(v) = b_i(v) - g^v(N)\frac{g^v(i)}{\sum_{j \in N} g^v(j)} \tag{1}$$

Proposition 1: Neither the nucleolus, nor the $\tau - value$, nor the separable cost remaining benefit method is *population-monotonic* on the class of convex games.

Proof: Let $N' = \{1,2,3,4\}$ be the set of players with $v(i) = 0, v(ij) = 1, v(234) = 5$, $v(ijk) = 2$ otherwise and $v(N') = 6$. Note that v is convex.

It can be found that,

$$Nu(v) = (1/2, 11/6, 11/6, 11/6)$$
$$B(v) = (-3/4, 9/4, 9/4, 9/4)$$
$$\tau(v) = (6/13, 24/13, 24/13, 24/13)$$

Now consider the restriction of v to the group $N = \{1,2,3\}$. In this case,

$$Nu(v_N) = B(v_N) = \tau(v_N) = (2/3, 2/3, 2/3)$$

showing that agent 1 gains whereas agents 2 and 3 suffer from agent 4 leaving the game. □

4 A Public Good Problem: The Airport Game

4.1 Nucleolus of the Airport Game

We will consider the following particular **public good problem**. Let N be the set of the agents and $S \subseteq N$ be any coalition. Each coalition S is characterized by a number $c(S)$, which is interpreted as the cost of producing the public good at the right level for the coalition S. For simplicity we assume that no two one-agent coalitions have the same cost. However, this assumption does not affect our results since the nucleolus is a continuous solution. Agents are ordered so that $c(1) < c(2) < \ldots < c(n)$. Let $i(S)$ denote the agent in coalition S who has the highest cost. Formally,

$$i(S) \equiv \{i | i \in S, c(i) > c(j) \; \forall i \in S\}$$

We consider the following class of games c for which $c(S) = c(i(S))$. Thus, the cost of the public good for any coalition depends on the most costly agent in that coalition. This class of games is introduced in Littlechild and Owen (1973) under the name of **airport games**. Note that $-c$ is a convex game. From now on, when we refer to the nucleolus of the airport game, we actually mean the negative nucleolus of the negative airport game. In Littlechild (1974) the nucleolus of the airport game $y = Nu(c)$ is shown to be

$$y_i = r_k \ \ for\ i_{k-1} < i \leq i_k \ and\ k = 1, 2, \ldots, k' \tag{2}$$

where r_k and i_k are defined inductively by

$$r_k = min[min_{i_{k-1}+1,\ldots,n-1} \{\tfrac{c_i - c_{i_{k-1}} + r_{k-1}}{i - i_{k-1} + 1}\}, \tfrac{c_n - c_{i_{k-1}} + r_{k-1}}{n - i_{k-1}}]$$

and i_k denotes the largest value of i for which the above expression attains its minimum. (Beginning with $r_0 = i_0 = c_0 = 0$ and continuing for $k = 1, 2, \ldots, k'$ where $i_{k'} = n$).

In order to study its *population-monotonicity* we provide in the following lemma an alternative formula. (See appendix A)

Lemma 1: The nucleolus of the airport game is given by the following recursive formula:

$$\begin{aligned} &y_0 = 0 \\ &y_i = min_{j=i,\ldots,n-1} \{\tfrac{c_j - \sum_{k=0}^{i-1} y_k}{j-i+2}\} \quad \text{for} \quad i = 1, 2, \ldots, n-1 \\ &y_n = c_n - \textstyle\sum_{k=0}^{n-1} y_k \end{aligned} \tag{3}$$

Population-monotonicity requires the fees imposed on the agents initially present not to increase upon the arrival of a new agent. Now we are ready to present our main result.

Proposition 2: The nucleolus is *population-monotonic* on the class of airport games.

Proof: Initially, $N = \{1, 2, \ldots, n\}$. Consider formula (2). By the derivation of the formula (using the Kopelowits algoritm) it is trivial to see that r_k is nondecreasing in k, which says that players with higher costs do not pay less.

We now add a new agent with cost $\hat{c}$. Let c' be the resulting game, $y' = Nu(c')$ and $L(\hat{c}) \equiv \{i | c_i < \hat{c}\}$. We will study the effect of the new agents for two groups separately.

Case 1: $i \in L(\hat{c})$.

Consider the payoffs in c and c' of the $(i_0 + 1)$th agent (who is agent 1) given by (1).

$$\begin{aligned} y_1 &= r_1 = min\{\tfrac{c_1}{2}, \ldots, \tfrac{c_j}{j+1}, \ldots, \tfrac{c_{n-1}}{n}, \tfrac{c_n}{n}\} \\ y_1' &= r_1' = min\{\tfrac{c_1}{2}, \ldots, \tfrac{c_j}{j+1}, \tfrac{\hat{c}}{j+2}, \tfrac{c_{j+1}}{j+3}, \ldots, \tfrac{c_{n-1}}{n+1}, \tfrac{c_n}{n+1}\} \end{aligned}$$

We have $y_1' \leq y_1$. If strict inequality holds, $y_i' = r_1' < r_1 \leq y_i$ for all i such that $i \in L(\hat{c})$. If equality holds, we consider the payoffs of the $(i_1 + 1)$th agent and the same argument holds. We proceed similarly till we reach the new agent. Thus we have showed that for all agents i who are less costly than the new agent, we have $y_i' \leq y_i$.

Case 2: $i \notin L(\hat{c})$.

Now find the nucleolus for the agents in c and c' with formula (3). Let $c_j < \hat{c} < c_{j+1}$. We introduce agent 0 with cost 0 which will not change anything other than simplifying the algebra. Without the new player,

$$y_0 = min\{\tfrac{0}{2}, \tfrac{c_1}{3}, \dots, \tfrac{c_{n-1}}{n+1}\} = 0$$
$$y_1 = min\{\tfrac{c_1 - y_0}{2}, \dots, \tfrac{c_{n-1} - y_0}{n}\}$$
$$\vdots$$
$$y_j = min\{\tfrac{c_j - \sum_{k=0}^{j-1} y_k}{2}, \tfrac{c_{j+1} - \sum_{k=0}^{j-1} y_k}{3}, \dots, \tfrac{c_{n-1} - \sum_{k=0}^{j-1} y_k}{n-j+1}\}$$
$$\vdots$$
$$y_{n-1} = \tfrac{c_{n-1} - \sum_{k=0}^{n-2} y_k}{2}$$
$$y_n = c_n - \sum_{k=0}^{n-1} y_k$$

With the new player,

$$y'_0 = min\{\tfrac{0}{2}, \tfrac{c_1}{3}, \dots, \tfrac{\hat{c}}{j+3}, \dots, \tfrac{c_{n-1}}{n+2}\} = 0$$
$$y'_1 = min\{\tfrac{c'_1 - y'_0}{2}, \dots, \tfrac{c_j - y'_0}{j+1}, \tfrac{\hat{c} - y'_0}{j+2}, \dots, \tfrac{c_{n-1} - y'_0}{n+1}\}$$
$$\vdots$$
$$y'_j = min\{\tfrac{c_j - \sum_{k=0}^{j-1} y'_k}{2}, \tfrac{\hat{c} - \sum_{k=0}^{j-1} y'_k}{3}, \dots, \tfrac{c_{n-1} - \sum_{k=0}^{j-1} y'_k}{n-j+2}\}$$
$$\hat{y}' = min\{\tfrac{\hat{c} - \sum_{k=0}^{j} y'_k}{2}, \tfrac{c_{j+1} - \sum_{k=0}^{j} y'_k}{3}, \dots, \tfrac{c_{n-1} - \sum_{k=0}^{j} y'_k}{n-j+1}\}$$
$$\vdots$$
$$y'_{n-1} = \tfrac{c_{n-1} - \sum_{k=0}^{n-2} y'_k - \hat{y}'}{2}$$
$$y'_n = c_n - \sum_{k=0}^{n-1} y'_k - \hat{y}'$$

Comparing y and y' we have, $y'_1 > 0$ therefore, $y'_1 + y'_0 > y_0$
This in turn implies,

$$y'_2 + y'_1 + y'_0 = min\{\tfrac{c_2 + y'_1 + y'_0}{2}, \dots, \tfrac{c_j + (j-1)(y'_1 + y'_0)}{j}, \tfrac{\hat{c} + j(y'_1 + y'_0)}{j+1}, \dots, \tfrac{c_{n-1} + (n-1)(y'_1 + y'_0)}{n}\}$$
$$> min\{\tfrac{c_1 + y_0}{2}, \dots, \tfrac{c_{j-1} + (j-1)y_0}{j}, \tfrac{c_j + jy_0}{j+1}, \dots, \tfrac{c_{n-1} + (n-1)y_0}{n}\} = y_0 + y_1$$

Similarly $y'_3 + y'_2 + y'_1 + y'_0 > y_2 + y_1 + y_0$ and so on.
At some step j we have $\hat{y}' + \sum_{k=0}^{j} y'_k > \sum_{k=0}^{j} y_k$, implying $y'_{j+1} < y_{j+1}$.
Similarly at step $j+1$, $\hat{y}' + \sum_{k=0}^{j+1} y'_k > \sum_{k=0}^{j+1} y_k$, implying $y'_{j+2} < y_{j+2}$.
Proceeding in this way, we obtain $y'_i < y_i$ for all i such that $c(i) > \hat{c}$.

□

4.2 Separable Cost Remaining Benefit Payoffs of the Airport Game

We now show that SCRB of the airport game is not *population-monotonic.*

Proposition 3: The separable cost remainining benefit method is not *population-monotonic* on the class of airport games.

Proof: Let $N' = \{1,2,3\}$ be the set of players with $c(1) = 4, c(2) = 9, c(3) = 10$.

It can be found that,

$$B(c) = (3, 3, 4)$$

Now consider the restriction of c to the group $N = \{1,3\}$. In this case,

$$B(c) = (2, 8)$$

showing that the agent 1 gains whereas agent 3 lose from agent 2 leaving the game. $\square$

4.3 $\tau - value$ of the Airport Game

The $\tau - value$ of the airport game is given in Driessen(1985) by

$$\tau_i(c) = \frac{c(n-1)c(i)}{\sum_{k=1}^{n-1} c(k) + c(n-1)} \quad i = 1, 2, \ldots n-1 \qquad (4)$$
$$\tau_n(c) = \tau_{n-1}(c) + (c_n - c_{n-1})$$

Proposition 4: The $\tau - value$ is not *population-monotonic* on the class of airport games.

Proof: Let $N' = \{1,2,3,4\}$ be the set of players with $c(1) = 1, c(2) = 2, c(3) = 11, c(4) = 15$.

It can be found that,

$$\tau(c) = (11/25, 22/25, 121/25, 221/25)$$

Now consider the restriction of c to the group $N = \{1,2,4\}$. In this case,

$$\tau(c_N) = (2/5, 4/5, 69/5)$$

showing that agents 1 and 2 gain whereas agent 4 lose from agent 3 leaving the game. $\square$

5 Appendix

5.1 Appendix A

Proof of Lemma 1:

Claim 1: $c_{i_l} - r_l = \sum_{j=1}^{l}(i_j - i_{j-1})r_j \quad l = 1, \ldots, k'$
Proof of Claim 1: Since $c_{i_1} - r_1 = (i_1 - i_0)r_1$ the claim is correct for $l = 1$.
Suppose the claim is correct for $l = k$. Then,

$$c_{i_k} - r_k = \sum_{j=1}^{k}(i_j - i_{j-1})r_j \tag{5}$$

Further,

$$r_{k+1} = \frac{c_{i_{k+1}} - c_{i_k} + r_k}{i_{k+1} - i_k + 1} \tag{6}$$

Thus, (5) and (6) together imply, $(i_{k+1} - i_k)r_{k+1} + c_{i_k} - r_k = c_{i_{k+1}} - r_{k+1}$
Replacing (5) once more we have,
$(i_{k+1} - i_k)r_{k+1} + \sum_{j=1}^{k}(i_j - i_{j-1})r_j = c_{i_{k+1}} - r_{k+1}$
or equivalently, $\sum_{j=1}^{k+1}(i_j - i_{j-1})r_j = c_{i_{k+1}} - r_{k+1}$
showing that the claim is correct for $l = k + 1$ □

Combining Claim 1 with the inequality $c_n > c_{n-1}$, the payoff formula simplifies to,

$$y_i = r_k \quad \text{for } i_{k-1} < i < i_k \quad \text{and} \quad k = 1, \ldots, k' - 1$$

$$r_k = min_{i=i_{k-1}+1,\ldots,n-1} \left\{\frac{c_i - \sum_{j=0}^{k-1}(i_j - i_{j-1})r_j}{i - i_{k-1} + 1}\right\}$$

$$y_n = c_n - \sum_{j=1}^{k'-1}(i_j - i_{j-1})r_j$$

where $r_0 = i_0 = i_{-1}$.

Define $I_l = \{i_{l-1} + 1, \ldots, i_l\}$. With this procedure minimization is done by only the first agent for each I_k. Here we have $y_{i_{l-1}+1} = \ldots = y_{i_l} = r_l$, where,

$$r_l = \frac{c_{i_l} - \sum_{j=1}^{l-1}(i_j - i_{j-1})r_j}{i_l - i_{l-1} + 1} \tag{7}$$

$$r_l \leq \frac{c_m - \sum_{j=1}^{l-1}(i_j - i_{j-1})r_j}{m - i_{l-1} + 1} \quad \text{forall } m \geq i_{l-1} + 1 \tag{8}$$

In the last set of inequalities we have strict inequality for $m > i_l$.

Consider any agent i who is not a first agent in any I_l. Thus, $i \in I_l, i \neq i_{l-1} + 1$. We know that $y_i = y_{i_{l-1}+1}$. The next claim shows that we can have a minimization problem also for agent i, similar to that of agent $i_{l-1} + 1$ without changing anything.
Claim 2: Consider agent $i_{l-1} + 2$

$$y_{i_{l-1}+2} = min_{i=i_{l-1}+2,\ldots,n-1}\left\{\frac{c_i - \sum_{j=1}^{l-1}(i_j - i_{j-1})r_j - r_l}{i - (i_{l-1}+1) + 1}\right\} = r_l$$

Proof of Claim 2:
Choose $i = i_l$. Let $\sum_{j=1}^{l-1}(i_j - i_{j-1})r_j = R$. Note that (7) implies

$$c_{i_l} - R - r_l - (i_l - i_{l-1})r_l = 0$$

Thus,

$$c_{i_l} - R - r_l - (i_l - i_{l-1})r_l + (c_{i_l} - R)(i_l - i_{l-1}) = (c_{i_l} - R)(i_l - i_{l-1})$$

Therefore,

$$(c_{i_l} - R - r_l)(i_l - i_{l-1} + 1) = (c_{i_l} - R)(i_l - i_{l-1})$$

or equivalently,

$$\frac{c_{i_l}-R-r_l}{i_l-i_{l-1}} = \frac{c_{i_l}-R}{i_l-i_{l-1}+1} = r_l$$

which means that r_l is attainable.

Next we show that $r_l \leq \frac{c_i-R-r_l}{i-(i_{l-1}+1)+1}$ forall $i \geq i_{l-1}+2$
Suppose not. Thus, for some $m \geq i_{l-1}+2$, $\frac{c_m-R-r_l}{m-i_{l-1}} < r_l$
From (8) we know that $r_l \leq \frac{c_m-R}{m-i_{l-1}+1}$ forall $m \geq i_{l-1}+1$
Therefore there exists an $m \geq i_{l-1}+2$ such that

$$\frac{c_m-R-r_l}{m-i_{l-1}} < r_l \leq \frac{c_m-R}{m-i_{l-1}+1}$$

This may be replaced by

$$(c_m - R - r_l)(m - i_{l-1}) < (c_m - R)(m - i_{l-1})$$

or equivalently

$$-(m - i_{l-1})r_l + c_m - R - r_l < 0$$

Thus there exists $m \geq i_{l-1}+2$ such that $\frac{c_m-R}{m-i_{l-1}+1} < r_l$ contradicting (6), proving Claim 2.□

An analogous result is valid for the next player, and so on. But this means that all players but the last one solve their own minimization problems in the following way

$$y_0 = 0$$

$$y_i = min_{j=i,\ldots,n-1}\{\frac{c_j-\sum_{k=0}^{i-1} y_k}{j-i+2}\} \quad i = 1, 2, \ldots, n-1$$

Whereas for the last player

$$y_n = c_n - \sum_{k=0}^{n-1} y_k$$ □

5.2 Appendix B

Derivations for Proposition 1:

$N' = \{1,2,3,4\}$ with $v(i) = 0, v(ij) = 1, v(234) = 5, v(ijk) = 2$ otherwise and $v(N') = 6$. $Nu_2(v) = Nu_3(v) = Nu_4(v)$ by the anonimity of nucleolus. Let $x = (\alpha, \beta, \beta, \beta)$. The excess of v with respect to x are given by

$$
\begin{aligned}
&e^v(1,x) = -\alpha \\
&e^v(i,x) = -\beta \quad i \in \{2,3,4\} \\
&e^v(1i,x) = 1 - (\alpha + \beta) \quad i \in \{2,3,4\} \\
&e^v(ij,v) = 1 - 2\beta \quad i \neq j \in \{2,3,4\} \\
&e^v(1ij,x) = 2 - (\alpha + 2\beta) \quad i \neq j \in \{2,3,4\} \\
&e^v(234,x) = 5 - 3\beta \\
&e^v(N',x) = 0
\end{aligned}
$$

At $x = Nu(v)$ we have $\alpha \leq \beta$ by the weak coalitional monotonicity and anonimity of the nucleolus which implies

$$-\alpha \geq -\beta > 1 - 2\beta \text{ and } -\alpha > 1 - (\alpha + \beta) > 2 - (\alpha + 2\beta).$$

Therefore we need to minimize $max\{-\alpha, -\beta, 5 - 3\beta\}$ such that $\alpha + 3\beta = 6$ at $x = Nu(x)$. This leads to $\alpha = 1/2, \beta = 11/6$ and $Nu(v) = (1/2, 11/6, 11/6, 11/6)$.

The marginal contributions to the grand coalition are
$b_1(v) = v(N') - v(234) = 1,\ b_2(v) = b_3(v) = b_4(v) = v(N') - v(123) = 4$
Furthermore
$g^v(N') = \sum_{i \in N'} b_i(v) - v(N') = 13 - 6 = 7$
$g_v(1) = b_1(v) - v(1) = 1,\ g_v(2) = g_v(3) = g_v(4) = b_4(v) - v(4) = 4$
Therefore
$B_1(v) = b_1(v) - g^v(N')/|N'| = 1 - 7/4 = -3/4,\ B_2(v) = B_3(v) = B_4(v) = b_4(v) - g^v(N')/|N'| = 4 - 7/4 = 9/4$. Thus $B(v) = (-3/4, 9/4, 9/4, 9/4)$.

Finally $\tau_1(v) = b_1(v) - \frac{g^v(N')g^v(1)}{\sum_{i \in N'} g^v(i)} = 1 - 7 \times 1/13 = 6/13$,
$\tau_2(v) = \tau_3(v) = \tau_4(v) = b_4(v) - \frac{g^v(N')g^v(4)}{\sum_{i \in N'} g^v(i)} = 4 - 7 \times 4/13 = 24/13$.
Thus $\tau(v) = (6/13, 24/13, 24/13, 24/13)$.

Let $N = \{2,3,4\}$, $Nu(v_N) = B(v_N) = \tau(v_N) = (2/3, 2/3, 2/3)$ immediately by the anonimity of the nucleolus, $\underline{SCRB}$ solution and τ-value.

Derivations for Proposition 3:

$N' = \{1,2,3\}$ $c(1) = 4, c(2) = 9, c(3) = 10$.
Marginal contributions to the grand coalition are
$b_1(v) = c(N') - c(23) = 10 - 10 = 0, b_2(v) = c(N') - c(13) = 10 - 10 = 0$
$b_3(v) = c(N') - c(12) = 10 - 9 = 1$.
Furthermore $g^c(N') = \sum_{i \in N'} b_i(c) - c(N') = 1 - 10 = -9$.
Therefore $B_1(c) = B_2(c) = b_2(c) - g^c(N')/|N'| = 0 + 9/3 = 3$

$B_3(c) = b_3(c) - g^c(N')/|N'| = 1 + 9/3 = 4$. Thus $B(c) = (3, 3, 4)$.

Let $N = \{1, 3\}$, then $b_1(c_N) = c(N) - c(3) = 0, b_3(c_N) = c(N) - c(1) = 6$.
Furthermore $g^{c_N}(N) = \sum_{i \in N} b_i(c_N) - c(N) = 6 - 10 = -4$.
Therefore $B_1(c_N) = b_1(c_N) - g^{c_N}(N)/|N| = 2, B_3(c_N) = b_3(c_N) - g^{c_N}(N)/|N| = 8$. Thus $B(c_N) = (2, 8)$.

Derivations for Proposition 4:

$N' = \{1, 2, 3, 4\}$ $c(1) = 1, c(2) = 2, c(3) = 11, c(4) = 15$. By the formula (4)
$\tau_1(c) = \frac{c(3)c(1)}{\sum_{k=1}^{n-1} c(k)+c(n-1)} = 11 \times 1/(1 + 2 + 11 + 11) = 11/25$
$\tau_2(c) = \frac{c(2)}{c(1)}\tau_1(c) = 2 \times 11/25 = 22/25$
$\tau_3(c) = \frac{c(3)}{c(1)}\tau_1(c) = 11 \times 11/25 = 121/25$
$\tau_4(c) = \tau_3(c) + [c(4) - c(3)] = 121/25 + 4 = 221/25$
Thus $\tau(c) = (11/25, 22/25, 121/25, 221/25)$.

Let $N = \{1, 2, 4\}$. Then
$\tau_1(c_N) = \frac{c(2)c(1)}{c(2)+c(2)+c(1)} = 2 \times 1/(2 + 2 + 1) = 2/5$
$\tau_2(c_N) = \frac{c(2)}{c(1)}\tau_1(c_N) = 2 \times 2/5 = 4/5$
$\tau_4(c_N) = \tau_2(c_N) + [c(4) - c(2)] = 4/5 + 13 = 69/5$
Thus $\tau(c_N) = (2/5, 4/5, 69/5)$.

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