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Abstract

This note introduces a model of contests with random noise and a shared prize that combines features of Tullock (1980) and Lazear and Rosen (1981). Similar to results in Lazear and Rosen, as the level of noise decreases the equilibrium effort rises. As the noise variance approaches zero, the equilibrium effort of the shared-prize contest approaches that of a Tullock lottery contest.

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1. Introduction

A wide variety of competitions arise in economic life, and new competitions are regularly introduced to attract effort and reward achievement. Such competitions are commonly modeled as contests, in which players compete over a prize by expending costly resources. There are an enormous variety of possible contests (Konrad, 2009), but the three canonical models are based on Tullock (1980), Lazear and Rosen (1981), and Hillman and Riley (1989).

Tullock (1980) models a probabilistic contest between two players, in which player i's probability of winning is defined by a contest success function, $p_i(x_i, x_j) = x_i^r/(x_i^r + x_j^r)$, where x_i and x_j are the efforts of players i and j. The player expending the highest effort has a higher probability of winning, but the other player still has a chance to win. The most popular version of the Tullock contest is a simple lottery, in which $r = 1$. In this contest a unique pure strategy Nash equilibrium exists where players earn positive payoffs.

Lazear and Rosen (1981) examine rank-order tournaments, in which players with greater achievements always win. They show that when players' cost of effort is sufficiently convex and is translated into achievement with random noise, rank-order tournaments can also generate pure strategy Nash equilibria. Hillman and Riley (1989) study a closely related version of the Tullock contest, in which $r = \infty$. This is known as a first-price all-pay auction, where the winner is always the player who expends the highest effort. Instead of a pure strategy Nash equilibrium, in this contest a symmetric mixed strategy Nash equilibrium exists in which players choose efforts randomly over some interval.

A number of studies have tried to establish common links between these three building blocks of contest theory. For example, Che and Gale (2000) provide a link between the rank-order tournament of Lazear and Rosen and the all-pay auction of Hillman and Riley. They partially characterize the equilibrium and show that with insufficient noise no pure strategy equilibrium exists in the rank-order tournament. Hirshleifer and Riley (1992) show how an R&D race between two players that is modeled as a rank-order tournament is equivalent to a lottery contest for certain assumptions on the noise distribution. Baye and Hoppe (2003) identify conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race games are strategically equivalent to the lottery contest.

This note introduces a model of contests with random noise that combines features of Tullock and Lazear and Rosen. We interpret this contest as a shared-prize contest in which all contestants receive a prize share that is proportional to their achievement (Long and Vousden, 1987). This type of contest imitates some forms of competition between firms, whose marketing or lobbying effort may be rewarded through a share of industry profit. Shared-prize contests may also be used within firms to reward workers, or as a type of procurement contract to elicit effort among suppliers (Zheng and Vukina, 2007).

The analytical results in this paper are restricted to the case of two symmetric players, and multiplicative random noise with a uniform distribution, although we also consider different noise distributions using numerical methods. We show that, similar to the rank-order tournament of Lazear and Rosen, in the shared-prize contest the equilibrium effort increases as the noise variance becomes smaller. Furthermore, as the noise variance approaches zero, the equilibrium effort with the shared prize approaches the equilibrium effort in a Tullock lottery contest.

2. The Model

Consider a simple contest in which two risk-neutral players i and j compete for a prize V. Both players expend individual efforts x_i and x_j. The output y_i of player i is determined by a production function

$$y_i(x_i|\varepsilon_i) = x_i\varepsilon_i, \tag{1}$$

where ε_i is a random variable that is drawn from the distribution F. The random component ε_i, can be thought of as production luck or measurement error, and is not observable to either of the players. Player i's probability of winning the prize is defined by a contest success function:

$$p_i(x_i, x_j|\varepsilon_i, \varepsilon_j) = y_i^r/(y_i^r + y_j^r). \tag{2}$$

Every player who exerts effort x has to bear cost $C(x)$. The expected payoff for player i can be written as:

$$E(\pi_i) = p_i V - C(x_i). \tag{3}$$

The Nash equilibrium depends on the specific conditions of the contest.

2.1. Equilibria in Standard Contests

To obtain the standard lottery contest in Tullock (1980) we set $r = 1$, $y_i = x_i$, and $C(x_i) = x_i$. In this case the Nash equilibrium is unique and it is given by

$$x^* = V/4. \tag{4}$$

For the all-pay auction of Hillman and Riley (1989), we set $r = \infty$, $y_i = x_i$, and $C(x_i) = x_i$. The complete characterization of the equilibrium can be found in Baye et al. (1996). In the all-pay auction no pure strategy equilibrium exists, unlike in the lottery contest. The mixed strategy Nash equilibrium is characterized by the cumulative distribution function,

$$F^*(x) = x/V \quad \text{for} \quad x \in [0, V]. \tag{5}$$

To obtain a rank-order tournament of Lazear and Rosen (1981), we set $r = \infty$, $y_i = x_i\varepsilon_i$, and $C(x_i) = C_x$, $C_{xx} > 0$. Noise in the production function and convexity of the cost function is necessary in order to generate a pure strategy equilibrium. When it exists, the pure strategy Nash equilibrium effort x^* can be obtained from the following expression:

$$C_x(x^*)x^* = V \int \varepsilon[f(\varepsilon)]^2 d\varepsilon. \tag{6}$$

The main difference between a rank-order tournament and the other two contests is the noise component ε. Once a distribution of ε is specified it is easy to analyze the effect of noise on equilibrium effort. For example, if ε is uniformly distributed on the interval $[1- a, 1 + a]$ then $\int \varepsilon[f(\varepsilon)]^2 d\varepsilon = 1/(2a)$. It follows from (6) that the equilibrium effort decreases in the variance of the distribution. This major finding of Lazear and Rosen (1981) agrees with economic intuition: it is less worthwhile to work hard when the output of the effort is noisier.

2.2. Equilibrium with Random Noise and a Shared Prize

The contest we study closely resembles the Tullock lottery, with the difference that payoffs are deterministic and proportional to performance, subject to random noise. That is, the p_i in (2) is interpreted as the proportion of the prize value, rather than the probability of winning the prize. When modeling a conventional Tullock competition with risk neutral agents, adding a noise component would be redundant since the winner of such a contest is already chosen probabilistically (Fullerton and McAfee, 1999). However, in many economic competitions players are rewarded proportionally to some measure of performance which depends on effort and a random component.

To model this type of competition, we consider a contest where all players receive a portion of a fixed and known prize. Such a contest arises when $r = 1$. This is exactly the same restriction as in the standard Tullock (1980) lottery contest, but now the contest success function is interpreted as a deterministic share of the prize. Randomness enters only through the production function (1). The analysis that follows assumes that $y_i = x_i \varepsilon_i$, where ε_i is a random variable that is drawn from the distribution F on the interval $[0, +\infty)$. This multiplicative noise production function has been used by O'Keefe et al. (1984), Hirshleifer and Riley (1992), and Gerchak and He (2003). A contest with this production function can also be interpreted as a contest where players have different, unknown abilities ε_i (Rosen, 1986). More importantly, multiplicative noise implies that the contest success function (2) satisfies the axioms introduced by Skaperdas (1996). In particular, the contest success function satisfies the conditions of a probability distribution: $\sum_i p_i(x_i, x_j | \varepsilon_i, \varepsilon_j) = 1$ and $p_i(x_i, x_j | \varepsilon_i, \varepsilon_j) \geq 0$, for all x_i and x_j.

Multiplicative noise also guarantees that the contest success function is homogeneous, i.e., $p_i(\lambda x_i, \lambda x_j | \varepsilon_i, \varepsilon_j) = p_i(x_i, x_j | \varepsilon_i, \varepsilon_j)$ for all $\lambda > 0$.

Given the restrictions $r = 1$ and $y_i = x_i\varepsilon_i$, the expected payoff (3) can be rewritten as:

$$E(\pi_i) = V \iint \frac{x_i\varepsilon_i}{x_i\varepsilon_i + x_j\varepsilon_j} f(\varepsilon_i) f(\varepsilon_j) d\varepsilon_i d\varepsilon_j - C(x_i). \tag{7}$$

Taking the first order condition and assuming a symmetric equilibrium, the optimal effort can be obtained from the following expression:

$$C_x(x^*)x^* = V \iint \frac{\varepsilon_i\varepsilon_j}{(\varepsilon_i + \varepsilon_j)^2} f(\varepsilon_i) f(\varepsilon_j) d\varepsilon_i d\varepsilon_j \tag{8}$$

The equilibrium effort depends on the value of the prize, the convexity of the cost function, and the distribution of the noise. An increase in the size of the prize increases individual effort. However, it is not straightforward to evaluate how the equilibrium effort in (8) is affected by the variance of the noise distribution. If we assume the cost function is linear, $C(x_i) = x_i$, and that ε_i and ε_j are independent and uniformly distributed on the interval $[1-a, 1+a]$, where $a \in (0,1)$ scales the variance of the distribution, then

$$x^* = V \frac{-2a^2 + a(a-2)\log(1-a) + a(a+2)\log(1+a) + \log(1-a^2)}{4a^2}. \tag{9}$$

The expected payoff at the symmetric equilibrium (9) is positive

$$E(\pi^*) = \frac{V}{2} - x^* > 0, \tag{10}$$

and the second order condition evaluated at the equilibrium is satisfied:

$$\frac{\partial^2 E(\pi)}{\partial x^2} = -V \frac{2}{x^2} \iint \frac{\varepsilon_i^2 \varepsilon_j}{(\varepsilon_i + \varepsilon_j)^3} f(\varepsilon_i) f(\varepsilon_j) d\varepsilon_i d\varepsilon_j = -\frac{4}{x^*} < 0. \tag{11}$$

From (9), it is straightforward to show that $\partial x^*/\partial a < 0$, i.e. as the level of noise increases the equilibrium effort decreases. This result is similar to the result obtained by Lazear and Rosen using $r = \infty$.[1] The crucial difference in our model is that we model the success function (a share of the prize) as the Tullock lottery (r = 1). We can solve for equilibrium as the variance of noise approaches to zero, by evaluating x^* at the limit as $a \to 0$:

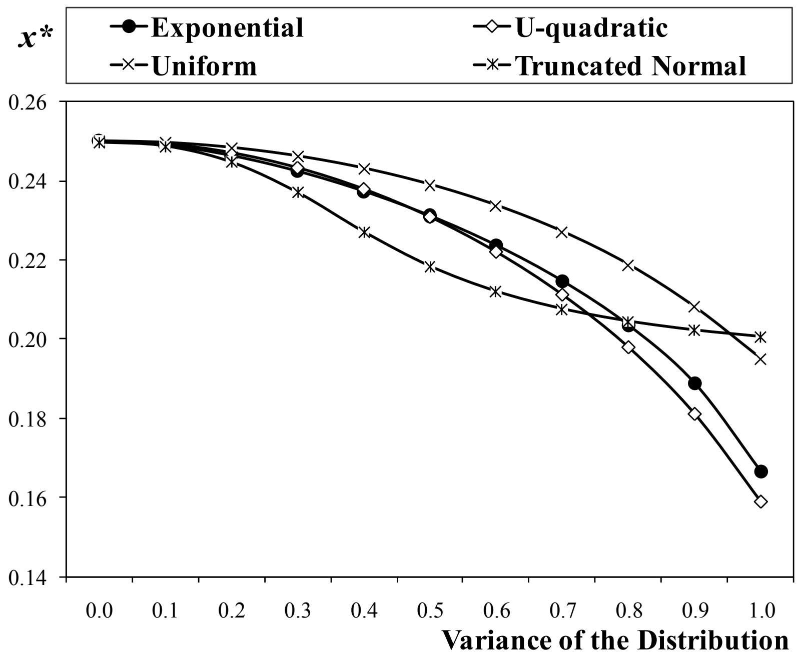
$$\lim_{a\to 0} x^* = \lim_{a\to 0} V \frac{1}{4}\left[-2 + \frac{(a-2)\log(1-a)}{a} + \frac{(a+2)\log(1+a)}{a} + \frac{\log(1-a^2)}{a^2}\right] \quad (12)$$

With L'Hopital's rule we can show that $x^* \to V/4$ as $a \to 0$. Therefore, as the variance of noise approaches zero, the equilibrium of this shared-prize contest approaches the equilibrium of a simple lottery contest without noise (4). A smooth transition exists between this type of contest with a random noise and a lottery contest. There is no such transition between a rank-order tournament and an all-pay auction (Che and Gale, 2000).

The assumption that the error term is uniformly distributed permits a closed form solution for the equilibrium effort. Our main conclusions are also robust to other noise distributions. To examine robustness we computed numerical solutions for three extreme cases: a (truncated) normal distribution, a U-shaped quadratic distribution, and the exponential distribution. We only restricted the mean of the distribution to equal 1. Figure 1 displays the equilibrium effort as a function of the distribution variance, and the two main conclusions drawn for the equilibrium with the uniform distribution (9) still hold. First, increases in the noise variance decrease equilibrium effort. Second, as the noise variance approaches zero the equilibrium efforts converge to a simple Tullock lottery contest without noise.

[1] Dasgupta and Nti (1998) and Amegashie (2006) also obtain a similar result, but in those studies the noise enters the contest success function as a constant term instead of a random variable.

Figure 1 – Equilibrium Effort as a Function of Noise Variance (*V* is normalized to 1)



3. Conclusions

This note presents a contest in which agents are rewarded proportionally to their achievement, where this achievement depends on both effort and random noise. Our approach offers a structural model of contests in which a Tullock success function is linked to an explicit source of random noise. Similar to Lazear and Rosen, the equilibrium effort increases as the variance of noise decreases. As the noise variance approaches zero, it approaches the equilibrium effort of a Tullock lottery contest.

Our restrictions on the production function and the distribution of noise were chosen in order to obtain a closed form solution for equilibrium effort. Using numerical simulations we demonstrate that our results are robust to three very different noise distributions, but as shown by Gerchak and He (2003) even in a standard rank-order tournament, changing the production function and the distribution of noise can significantly alter results. Therefore, a natural extension of this study is to examine how the production function and the noise distribution affect equilibrium behavior in this type of contest. Other extensions could evaluate the effect of asymmetry, incomplete information, and number of players. We leave these issues for future research.

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