How to divide things fairly

Brams, Steven and Kilgour, D. Marc and Klamler, Christian

New York University, Wilfrid Laurier University, University of Graz

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Steven J. Brams
Department of Politics
New York University
New York, NY 10012
USA
steven.brams@nyu.edu

D. Marc Kilgour
Department of Mathematics
Wilfrid Laurier University
Waterloo, Ontario N2L 3C5
CANADA
mkilgour@wlu.ca

Christian Klamler
Institute of Public Economics
University of Graz
A-8010 Graz
AUSTRIA
christian.klamler@uni-graz.at

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Abstract

We analyze a simple sequential algorithm (SA) for allocating indivisible items that are strictly ranked by $n \geq 2$ players. It yields at least one Pareto-optimal allocation which, when $n = 2$, is envy-free unless no envy-free allocation exists. However, an SA allocation may not be maximin or Borda maximin—maximize the minimum rank, or the Borda score—of the items received by a player. Although SA is potentially vulnerable to manipulation, it would be difficult to manipulate in the absence of one player's having complete information about the other players' preferences. We discuss the applicability of SA, such as in assigning people to committees or allocating marital property in a divorce.

How to Divide Things Fairly

1. Introduction

The fair division of items, especially if they are indivisible or cannot be shared, is an age-old problem. In this paper, we describe a simple sequential algorithm, called SA, which seems to have been overlooked in earlier studies, for carrying out this division when the players strictly rank items from best to worst. It is less demanding in the information it elicits than are algorithms that ask players to indicate their utilities for items or to rank bundles of items.

SA works in stages. We illustrate it with several examples in section 2, where we also define four properties of fair division—Pareto-optimality, envy-freeness, maximinality, and Borda maximinality—that we use to assess the fairness of SA allocations.

Assume that there are $n \geq 2$ players. In section 3, we prove that at least one allocation given by SA is Pareto-optimal for all values of n, but it need not be maximin or Borda maximin, even when $n = 2$. When $n = 2$, if there exists a Pareto-optimal allocation that is envy-free, it will be among the allocations that SA yields, although SA may not find all Pareto-optimal, envy-free allocations.

If $n > 2$, SA may not find an allocation that maximizes the number of players who receive envy-free allocations, even when there is a non-SA allocation that is envy-free for all players. On the other hand, a non-SA envy-free allocation may not satisfy other properties, such as maximinality or Borda maximinality.

We compare SA with a 2-person algorithm, AL, that yields envy-free allocations included in those produced by SA (Brams, Kilgour, and Klamler, 2014). Both algorithms satisfy an item-wise definition of envy-freeness, which we spell out in section 2. But unlike SA, AL requires a test for envy-freeness when both players want an item at the same time. If there is no envy-free allocation of this item, AL puts it into a "contested pile."

By contrast, SA allocates a pair of items at every stage—whether they cause envy or not—until all items are exhausted. Although not strategyproof, SA would be difficult to manipulate unless one player has complete information about the preference rankings of the other players.

In section 4, we draw several conclusions. We also discuss the applicability of SA, such as in assigning people to committees or in allocating the marital property in a divorce.

2. The Sequential Algorithm (SA) and Examples

We make three assumptions:

1. There are $n \geq 2$ players and $m = kn$ distinct items to be allocated, where $k \geq 1$ and integral.[1]
2. Each player has a positive utility for each item and ranks the items strictly.
3. The utility of a set of items for a player is the sum of the utilities of the items that comprise it.

Although SA uses only players' rankings of the items, we use the *Borda score* of an item as one measure of its utility to a player: A lowest-ranked item receives 0 points, the next-

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[1] Given this assumption, SA will produce an *equal* allocation, in which each player receives the same number, k, of items. If m is not an integral multiple of n (e.g., when $n = 2$ and m is odd), the "extra" items can be distributed to the players at random—with a maximum one to each player—after SA has been applied.

lowest 1 point, and so on. A player's Borda score is the sum of its points for the subset of items it receives.

The properties of SA allocations that we analyze are the following:

- *Efficiency or Pareto-Optimality (PO):* There is no other allocation that is at least as preferred by all players and strictly preferred by at least one.
- *Envy-freeness (EF)*: Each player values the set of items it receives at least as much as the set of items received by any other player.
- *Maximinality (MX)*: The allocation maximizes the minimum rank of the items received by any player.
- *Borda Maximinality (BMX)*: The allocation maximizes the minimum Borda score of the items received by any player.

MX ensures that the rank of the least-preferred item that any player receives is as high as possible, whereas BMX ensures that the Borda score of the player with the lowest score is as high as possible. As we will show, different allocations may satisfy each of these properties.

Because SA requires only that players rank items, we need a definition of EF, consistent with that given above, that makes only item-by-item ordinal comparisons. We say that a player, say *A*, does not envy another player, say *B*, iff there is an injection (a 1-1 mapping) from *A*'s items onto *B*'s items such that *A* prefers each of its items to the item of *B* to which it is mapped (Brams, Kilgour, and Klamler, 2014). An allocation is *item-wise envy-free* (EF) iff no player envies any other.[2]

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[2] Note that if *A* does not envy *B* in the item-wise sense, then *A*'s utility for its own subset must be greater than *A*'s utility for *B*'s subset, no matter what the numerical values of the utilities are as long as they are

The allocation rules of SA, which give one item to each player on each round, are the following:

(i) On the first round, descend the ranks of the players, one rank at a time, stopping at the first rank at which each player can be given a different item (at or before this rank). This is the *stopping point* for that round; the rank reached is its *depth*, which must be the same for each player. Assign one item to each player in all possible ways that are at or above this depth (there may be only one), which may give rise to one or more SA allocations.

(ii) On subsequent rounds, continue the descent, increasing the depth of the stopping point on each round. At each stopping point, assign items not yet allocated in all possible ways until all items are allocated.

At the conclusion of the iterative process defined by rules (i) and (ii), we recommend that a third rule be applied:

(iii) At the completion of the descent, if SA gives more than one possible allocation, choose one that is PO and, if possible, EF.

The process of descent is the same as that under "fallback bargaining" (Brams and Kilgour, 2001), but its purpose is fair division of items, not reaching an outcome acceptable to some (e.g., a simple majority) or all players. While the fallback descent is

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consistent with *A*'s ranking of items. Because each player receives the same number of items, item-wise envy-freeness, which we henceforth assume, implies EF based on players' values of the subsets of items they receive. The converse is not true, as we illustrate with a later example, making item-wise envy-freeness the more stringent definition.

also used in Brams, Kilgour, and Klamler (forthcoming), the fair-division algorithm proposed there allocates "minimal bundles" of items to players, from which the subtraction of any item would cause the bundle to be less than proportional (i.e., worth less than $1/n$ to a player).

We next give several examples that illustrate rules (i) – (ii) when $n = 2$; later we analyze an example in which $n = 3$. The players are A, B, …, and the items they rank are 1, 2, …. Players rank items in descending order of preference.

Example 1:

A: $\underline{1}$ 2 $\underline{3}$ 4

B: $\underline{2}$ 3 $\underline{4}$ 1

The stopping point of round 1 is depth 1, where A obtains item 1 and B obtains item 2. At depth 2, we cannot give different items to the players, because item 2 has already been given to B, so in round 2 we must descend to depth 3 to give the players different items (item 3 to A and item 4 to B).

We have underscored the items that each player receives. Because this exhausts the items, we are done, which yields the unique SA allocation of (13, 24) to (A, B). Henceforth, we list the players in alphabetical order, and the items that each receives in the same order.

Observe that on each round, each player prefers the item it receives to the item that the other player receives (A prefers item 1 to item 2 and item 3 to item 4; B prefers item 2 to item 3 and item 4 to item 1).[3] Hence, there is a 1-1 matching of A's items to B's, and

[3] Thus we can conclude that an allocation is EF if it gives an EF allocation at every stopping point. A test like this one is useful, because SA does not elicit utilities.

B's items to *A*'s, such that each player prefers its items to the other player's items. Therefore, the allocation (13, 24) is EF.

As noted earlier (ftn. 2), this allocation does not depend on a player's specific utilities for items. Other two-item allocations, such as (12, 34), are not item-wise EF, because there is no 1-1 matching of *B*'s items to *A*'s such that *B* prefers each of its items to the items to which it is matched. In particular, notice that items 1 and 2 bracket items 3 and 4 in *B*'s ranking, so *B* may prefer the combination of items 1 (best) and 4 (worst) to the combination of items 3 and 4 (two middle-ranked items).

For example, if *B*'s utilities for items (1, 2, 3, 4) are (1, 5, 3, 2), then *B*'s utility for its subset of items, 34, is 5, and its utility for *A*'s subset of items, 12, is 6, so *B* will envy *A*. But if *B*'s utilities are (1, 6, 5, 4), then it values its subset at 9 and *A*'s subset at only 7, so in this case *B* will not envy *A*.

In Example 1, only allocation (13, 24) is EF for all possible utilities of either player that are consistent with its ranking. Given assumption (3)—that utilities are additive, so sets of items do not exhibit synergies, either positive or negative—our item-wise definition of EF implies the utility-based definition,

It is easy to see that (13, 24) is PO, because there is no allocation that is at least as preferred by both players.[4] It is also MX, because the only other allocation of two items to each player that gives neither player a worst item is (12, 34), rendering it also MX. However, (12, 34) is *not* BMX, because it gives Borda scores of (5, 3) to (*A*, *B*), so *B*'s score is less than the score of 4 that each player receives from (13, 24).

[4] Formally, we test for Pareto-optimality using the necessary and sufficient condition of Brams and King (2005): An allocation is PO iff it is the product of a *sincere sequence* of item choices by the players, or one consistent with their preferences. Thus, if the players choose items in the order *ABAB*, they obtain (12, 34); if they choose in the order *AABB*, they obtain (12, 34), so both these allocations are PO. By comparison, no sincere sequence yields the allocation (24, 31), so it is not PO.

If SA gives two (or more) allocations, only one may be MX, as our next example illustrates (note that the two allocations given are both SA allocations):

Example 2: A: $\underline{1}$ $\underline{2}$ 3 4 $\underline{5}$ / 6 $\underline{7}$ 8 A: $\underline{1}$ $\underline{2}$ 3 4 $\underline{5}$ / $\underline{6}$ 7 8

B: $\underline{3}$ $\underline{4}$ 5 $\underline{6}$ 7 / $\underline{8}$ 1 2 B: $\underline{3}$ $\underline{4}$ 5 6 $\underline{7}$ / $\underline{8}$ 1 2

In the first two rounds, SA gives (12, 34) to (A, B), reaching depth 2. In round 3, the stopping level is depth 5 (just to the left of the slashes in each ranking), where not only can we give different items to A and B, but we also have a choice:

(i) in round 3, the left-hand allocation gives items (5, 6) to (A, B) at depth 5, followed in round 4 by items (7, 8) at depth 7;

(ii) in round 3, the right-hand allocation gives items (5, 7) to (A, B) at depth 5, followed in round 4 by items (6, 8) at depth 6.

Clearly, the right-hand allocation, with a maximum depth of 6, is MX.[5]

Allocation (1256, 3478) is also BMX, giving (A, B) Borda scores of (18, 18)—a minimum score of 18—whereas the left-hand allocation, (1257, 3468), gives them scores of (17, 19), for a minimum score of 17. Thus, while both allocations are EF and PO, only the right-hand allocation is MX and BMX.

So far we have illustrated how SA can pick more than one allocation that is PO and EF, but only one may be MX or BMX (Example 2). Our next example illustrates that if both players rank the same item last, there cannot be an EF allocation:

Example 3: A: $\underline{1}$ 2 $\underline{3}$ $\underline{4}$ 5 6 A: $\underline{1}$ 2 $\underline{3}$ 4 5 $\underline{6}$

[5] All allocations must have depth 6 or greater; otherwise, item 8 would not be assigned to either player.

B: $\underline{2}$ 3 $\underline{5}$ 4 1 $\underline{6}$ *B*: $\underline{2}$ 3 $\underline{5}$ $\underline{4}$ 1 6

In rounds 1 and 2, with stopping points at depth 1 and depth 3, both allocations agree, giving 13 to *A* and 25 to *B*. At these stopping points, neither player envies the other player for the item it receives, so the allocations are EF up to depth 3.

In round 3, the stopping point must be at depth 6, the lowest possible. Items 4 and 6 can be assigned in two different ways. The player who receives item 6 must be envious, because no 1-1 mapping can map item 6 to a less-preferred item. Thus, the allocation of items in Example 3 is not EF, although the allocation of the first four items to both players at depths 1 and 3 is. Both allocations are MX (maximum depth of 6) and BMX (minimum Borda score of 8).

Our final example in this section illustrates that an SA allocation may fail to be EF even when the players rank every item differently:

Example 4: *A*: $\underline{1}$ 2 3 $\underline{4}$ $\underline{5}$ 6 *A*: $\underline{1}$ 2 $\underline{3}$ $\underline{4}$ 5 6

B: $\underline{2}$ $\underline{3}$ 1 5 $\underline{6}$ 4 *B*: $\underline{2}$ 3 1 $\underline{5}$ $\underline{6}$ 4

Here the problem arises in round 2, with the stopping point at depth 4, where the left-hand allocation gives 14 to *A* and 23 to *B*, whereas the right-hand allocation gives 13 to *A* and 25 to *B*. Whichever player receives the lower-ranked item in round 2 will be envious because, again, no 1-1 mapping maps every item of that player to a lower-ranked item of the other player. Despite the fact that each player ranks the six items differently, and no player receives a worst item in either SA allocation, neither allocation in Example 4 is EF.[6]

In Example 4, both allocations are MX (common depth 5); moreover, they are equal according to the BMX, with Borda scores of 8 to the advantaged player and 10 to the disadvantaged player, so the minimum Borda score of each allocation is 8. According to both MX and BMX, therefore, the two allocations in Example 4, unlike Example 3, are equally fair to the players.

To summarize, SA may give a unique PO-EF allocation to both players (Example 1) or multiple PO-EF allocations (Example 2), only one of which—possibly not the same one—is MX or BMX. In addition, SA may not produce an EF allocation (Examples 3 and 4), although at least one allocation it does produce will be MX, BMX, or both.

3. Properties of the Sequential Algorithm

If all items can be allocated in an EF way, we say there is a *complete EF allocation*. For the $n = 2$ case, Brams, Kilgour, and Klamler (2014) provide an algorithm, AL, which finds at least one complete PO-EF allocation if one exists (though not necessarily all of them, as we will show). Furthermore, when there is no complete EF allocation, as in Examples 3 and 4, AL finds the largest and most preferred subset of items that can be allocated in an EF way. Items that cannot be so allocated (e.g., items 4 and 6 in Example 3; items 3 and 6 in Example 4) are placed in a "contested pile."

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[6] When $n = 2$, there is a simple condition, called "Condition D" (Brams, Kilgour, and Klamler, 2014), for determining whether an allocation is EF: For every odd k, the two players' top k items are not identical. In Example 3, A's and B's top 5 items are identical, whereas in Example 4, their top 3 items are identical, so neither example yields an EF allocation. This is not true for the top 1 and top 3 items in Example 1, nor the top 1, 3, 5, and 7 items in Example 2, so in both examples an EF allocation exists—which, as we showed, SA finds.

By contrast, SA always allocates all items. As illustrated in Examples 3 and 4, SA yields an allocation that may be EF on some rounds, but is not a complete EF allocation.

Although SA may not give an EF allocation, it always produces at least one PO allocation. Moreover, this is true however many players there are (i.e., for all $n \geq 2$).

Theorem 1. *SA rules (i) and (ii) produce at least one allocation that is PO.*

Proof. Under SA, all items are allocated one at a time to the players and ranked at or above each stopping point in the descent process. Because each allocation gives equal numbers of items to the players, a non-SA allocation must give at least one player an item it ranks below some item it would receive under SA. This proves that no non-SA allocation can be Pareto-superior to any SA allocation. Because Pareto-superiority is irreflexive and transitive, at least one of the SA allocations—say, X—must be maximal with respect to Pareto-superiority within the set of SA allocations. Because no non-SA allocation can be Pareto-superior to X, X must be PO. ■

Theorem 1 guarantees that at least one allocation must survive the application of rule (iii). Our next example, however, shows that rule (iii) has bite—not every SA allocation need be PO:[7]

Example 5:	A: $\underline{1}\ \underline{2}\ 3\ \underline{4}\ 5\ /\ \underline{6}\ 7\ 8$	A: $\underline{1}\ \underline{2}\ 3\ 4\ \underline{5}\ /\ \underline{6}\ 7\ 8$
	B: $\underline{7}\ \underline{8}\ 1\ 2\ \underline{3}\ /\ \underline{5}\ 4\ 6$	B: $\underline{7}\ \underline{8}\ 1\ 2\ \underline{3}\ /\ 5\ \underline{4}\ 6$

At the completion of round 2, SA gives (12, 78) to (A, B), stopping at depth 2. The next stopping point is at depth 5, indicated by the slashes, when B must receive item 3 in

[7] Example 5 refutes a claim (Theorem 5) in Brams, Kilgour, and Klamler (2014) that AL, which gives the same complete EF allocations in Example 5 as SA does, always yields allocations that are "locally Pareto-optimal."

round 3. There is a choice: The left-hand allocation gives item 4 to *A*, and the right-hand allocation gives item 5 to *A*.

Continuing, the left-hand allocation gives items 6 and 5, respectively, to *A* and *B* in round 4, with stopping point at depth 6, whereas the right-hand allocation gives items 6 and 4 to *A* and *B* in round 4, with the stopping point at depth 7. Because both players prefer the last two items they receive in the left-hand allocation to those that they receive in the right-hand allocation, the left-hand allocation Pareto-dominates the right-hand allocation, so only the left-hand allocation is PO.

Interestingly enough, both SA allocations in Example 5 are complete EF allocations, even though only one is PO, showing that EF does not imply PO. The converse also fails, because an allocation that gives one player only its top items will generally make another player envious. Thus, PO and EF are independent properties.

In Example 5, the left-hand allocation, (1246, 7835) is MX. That it Pareto-dominates the right-hand allocation, (1256, 7834), is reflected in its greater Borda scores, (19, 18) versus (18, 17), so the former allocation is also BMX.

But SA does not invariably find a PO-EF allocation that—based on the properties of MX or BMX—is superior to a non-SA allocation, as our next example illustrates:

Example 6: A: $\underline{1}\ \underline{2}\ \underline{3}\ \underline{4}\ 5\ 6\ 7\ 8$ $\quad$ A: $\underline{1}\ \underline{2}\ 3\ \underline{4}\ \underline{5}\ 6\ 7\ 8$

B: $\underline{8}\ \underline{7}\ \underline{6}\ 3\ 2\ 1\ \underline{5}\ 4$ $\quad$ B: $\underline{8}\ \underline{7}\ \underline{6}\ \underline{3}\ 2\ 1\ 5\ 4$

The SA allocation is shown on the left. In the first three rounds, at depths 1, 2, and 3, SA allocates (123, 876) to (*A*, *B*). On round 4 and at depth 7, *A* and *B* receive, respectively, items 4 and 5, producing the allocation (1234, 8765).

But the non-SA allocation (1245, 8763) goes only to depth 5. Moreover, it is not only MX but also BMX, giving Borda scores of (20, 22), compared with (22, 19) for the SA allocation. Both the left-hand and the right-hand allocations are EF and PO. In fact. we return to this example later to show that there are seven distinct complete EF allocations, but only the aforementioned two are PO.

Both the left-hand (SA) and right-hand (non-SA) allocations in Example 6 are complete EF and PO (*A* prefers the former, and *B* the latter, when each player obtains its four best items).[8] But only the non-SA allocation is MX and BMX.

Although at least one SA allocation is PO by Theorem 1, it may not be EF, as we showed in Examples 3 and 4. But if there is an EF allocation when $n = 2$, we have the following:

Theorem 2. *Let $n = 2$. If an EF allocation exists, then SA will give at least one allocation that is EF and PO.*

Proof. We earlier alluded to Condition D (see ftn. 6)—that an EF allocation exists iff, for all odd k, at least one of *A*'s k most preferred items is not one of *B*'s k most preferred items (Brams, Kilgour, and Klamler, 2014). Also, an allocation is EF iff, for each player, the j^{th} item received is among its top $2j – 1$ items.

Assume Condition D holds. Taking $k = 1$, it is clear that *A*'s and *B*'s most preferred items are different, so on round $j = 1$, SA must allocate to each player its most preferred item, and the stopping depth is $d_1 = 1$.

[8] AL would also give the SA allocation in Example 6, because, like SA, it immediately allocates the most preferred items at each stage if they are different (Brams, Kilgour, and Klamler, 2014). Hence, like SA, AL does not always find all PO-EF allocations, as incorrectly indicated in Brams, Kilgour, and Klamler (2014), including those that might be MX or BMX.

Now assume that, up to the completion of round j, each player has been allocated by SA j of its top $2j - 1$ items, and that the stopping depth on round j is $d_j \leq 2j - 1$. Consider round $j + 1$. Combined, the preference orderings of A and B account for either 2, 3, or 4 distinct items at depth $2j$ or $2j + 1$. Therefore, to assign an additional item to both A and B from their top $2j + 1$ items, it is necessary to increase the stopping depth to at most $d_j + 2$.

If there is a choice, ensure that a player does not prefer any unassigned item to the item it receives. It follows that $d_{j+1} \leq 2j + 1$, and that the $j + 1^{\text{st}}$ item received by each player is among its $2j + 1$ most preferred items. Therefore, the resulting SA allocation is EF. Moreover, it is PO, because it is the result of a sequence of sincere choices (see ftn. 4). ■

When $n = 2$, it is relatively easy to determine whether a given allocation is EF, PO, MX, or BMX. It is considerably more complex to find *all* allocations that are, say, EF.

To illustrate this calculation, recall from Example 6 that we gave a non-SA equal allocation that improved upon the SA allocation on MX and BMX, but we did not prove that it was the only such allocation, or that there was not another allocation that better satisfied one or both of these properties. To analyze Example 6 in detail, we list all possible ways of item-by-item allocations at each odd depth.

Example 6 (repeated): A: 1 2 3 4 5 6 7 8

B: 8 7 6 3 2 1 5 4

At depth 1, (A, B) must receive items (1, 8). Then, at depth 3, (A, B) must receive, in addition, one of (2, 7), (2, 6), (3, 7), or (3, 6). Finally, at depth 5 and again at depth 7, (A,

B) must receive pairs of items that depend on the items already received. The details are shown in Table 1, which includes all EF allocations for Example 6, as well as their MX depths and Borda scores.

Table 1 about here

As Table 1 shows, there are seven EF allocations, labelled a, b, c, d, e, f, and g, which we call *classes*, that can be reached in a total of 21 different ways. Specifically, there are 7 a's, 4 b's, 2 c's, 1 d, 1 e, 5 f's, and 1 g. The `MX depths and Borda scores depend only on the class, not on the way it was obtained. These scores are shown only for the first member of each class.

The MX depths of the b's and the f's are minimal (i.e., 5), but only the b's have a maximin Borda score (20). This verifies that allocation b, (1245, 8765), is indeed MX and BMX. It, along with the unique SA allocation (allocation 1 in class a), are the only PO allocations.

So far we have not illustrated SA with examples in which $n > 2$. While its application to the division of items among three or more players is straightforward, if more tedious, SA no longer ensures that if there is a complete EF allocation, it will be chosen by SA when $n > 2$.

Example 7: A: $\underline{1}$ $\underline{2}$ 3 4 5 $\underline{6}$ 7 8 9

B: $\underline{5}$ $\underline{8}$ 1 2 6 $\underline{7}$ 3 4 9

C: $\underline{3}$ $\underline{4}$ $\underline{9}$ 1 2 5 6 7 8

SA allocates items (1, 5, 3) to (A, B, C) at depth 1; then (2, 8, 4) at depth 2; and finally (6, 7, 9) at depth 6. Notice that B may envy A for obtaining items {1, 2, 6}, which fall between *B*'s two best items (items 5 and 8) and its 6th-best item (item 7). Because *A*'s items bracket *B*'s, it follows that there is no 1-1 matching of *B*'s items to *A*'s such that *B* always prefers its own item to the item of *A* to which it is matched. Thus, this allocation is not EF.

However, by switching items 6 and 7 between *A* and *B* in the SA allocation, we obtain the following non-SA allocation:

Example 7 (cont.): *A*: $\underline{1}$ $\underline{2}$ 3 4 5 6 $\underline{7}$ 8 9

B: $\underline{5}$ $\underline{8}$ 1 2 $\underline{6}$ 7 3 4 9

C: $\underline{3}$ $\underline{4}$ $\underline{9}$ 1 2 5 6 7 8

which is EF as well as PO.[9]

To show that the non-SA allocation is EF, observe *C* gets its three best items, so it cannot do better and, therefore, will not be envious. But now it is easy to check that the required 1-1 matchings of *A*'s items to *B*'s, and *B*'s to *A*s all exist, confirming that this allocation is also EF. Alternatively, one can ascertain that Condition D is satisfied (see ftn. 6).

Although not EF, the SA allocation in Example 7 has the advantage of being both MX and BMX. It gives *A* and *B* at worst a 6th-best item, whereas the non-SA allocation gives *A* a 7th-best item. Similarly, the SA allocation gives Borda scores of (18, 18, 21) to (*A*, *B*, *C*), whereas the EF allocation gives them Borda scores of (17, 19, 21).

As a final property of SA, we consider its vulnerability of manipulation. Not surprisingly, if $n = 2$ and one player (say, *A*) has complete information about the

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[9] To demonstrate formally that the SA allocation is PO, use the sequence of sincere choices given by *ABCABCCAB*. To demonstrate that the non-SA allocation is PO, use the sequence of sincere choices given by *ABCABCCBA* (see ftn. 4).

preferences of the other player (*B*), and *B* is sincere, *A* can exploit *B*, as shown in our next example:

Example 8: *A*: $\underline{1}\ \underline{2}\ 3\ \underline{4}\ 5\ 6$

B: $\underline{6}\ \underline{3}\ \underline{5}\ 4\ 2\ 1$

The SA allocation is underscored, with *B* receiving its three top items and *A* not doing quite so well. But now assume that *A* insincerely indicates its preferences to be those shown below, with *B*'s preferences remaining the same:

Example 8 (cont.): *A'*: $\underline{3}\ \underline{1}\ \underline{2}\ 4\ 5\ 6$

B: $\underline{6}\ 3\ \underline{5}\ \underline{4}\ 2\ 1$

This SA allocation shows that *A*'s insincere preferences turn its original disadvantage into an advantage by giving it its three top items, whereas *B* now does worse.

Although not strategyproof, SA seems relatively invulnerable to strategizing in the absence of any player's having complete information about its opponents' preferences. The manipulator's task is further complicated if the other players are aware that an opponent might try to capitalize on its information and, consequently, take countermeasures (e.g., through deception) to try to prevent their exploitation.

To summarize, we have shown that if $n \geq 2$, SA always yields at least one PO allocation and, if $n = 2$, SA always yields an allocation that is PO and EF, provided an EF allocation exists. Although, initially, SA may produce some allocations that are not PO, these will be eliminated by invoking SA rule (iii). The set of PO-EF allocations that SA produces, however, may not include one that satisfies the properties of MX or BMX, although our examples suggest that it may not be far off.

If $n > 2$, SA may not maximize the number of players who receive EF allocations (perhaps all). But a non-SA allocation that does so may be neither MX nor BMX, so an SA allocation may have redeeming properties even if it is not a complete EF allocation. While SA is not strategyproof, in most real-life cases it is unlikely that one player would have sufficient information about other players' preference rankings, not to mention be able to formulate a strategy that would exploit such information.

4. Conclusions

SA is a remarkably simple algorithm for allocating indivisible items to two or more players that possesses attractive properties. It always yields at least one PO allocation and, when $n = 2$, at least one complete PO-EF allocation if such an allocation exists. While there is no guarantee that SA allocations will be MX or BMX, it seems likely that if they do not satisfy these properties, they will come close to doing so. Thus in Example 6, the maximin Borda score of the SA allocation is 19, close to the maximin Borda score of a non-SA allocation (20).[10]

A potentially more serious problem is that, when $n > 2$, SA may not ensure that as many players as possible (perhaps all) receive an EF allocation, whereas a non-SA EF allocation may not satisfy MX or BMX (as in Example 7). We know of no method, short of exhaustive search, that does a better job of finding PO allocations that, at least when $n = 2$, will be complete EF allocations if this is possible.

SA, as we have illustrated, is not strategyproof. However, it would be difficult to manipulate in practice unless one player had complete information about the rankings of the other players, which is improbable.

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[10] Intuitively, SA allocations will not be far from MX-BMX, because at the end of each round, there is never an unassigned item that any player prefers to the item it just received.

SA seems most applicable to allocation problems in which there are numerous small items, which need not be physical goods but could, for example, be committees on which members of an organization desire to serve. If there is one big item that two players desire (e.g., the house in a divorce), it may not be possible to prevent envy, especially if the procedure specifies that each player receive the same number of items.[11] In such a case, the most practical solution might be to sell the big item—in effect, making it divisible—and divide the proceeds.

Other modifications in SA might include not restricting the allocation of items of one to each player on every round, and relaxing the assumption that the number of items is an integer multiple of the number of players. These modifications would change the problem fundamentally, however, because properties like EF, MX, and BMX would have to be redefined to take into account that players may not receive the same number of items.

[11] Such a stipulation might be viewed as an essential to achieving fairness in some situations.

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Table 1

EF Allocations, MX Depths, and Borda Scores for Example 6

Allocation	Depth ≤ 3	Depth ≤ 5	Depth ≤ 7	Complete	MX Depth	BMX Score
1	(2, 7)	(3, 6)	(4, 5)	(1234, 8765)-*a*	7	(22, 19)
2		(4, 6)	(3, 5)	(1243, 8765)-*a*		
3			(5, 3)	(1245, 8763)-*b*	5	(20, 22)
4		(4, 3)	(5, 6)	(1245, 8736)-*b*		
5			(6, 5)	(1246, 8735)-*c*	7	(19, 18)
6		(5, 6)	(4, 3)	(1254, 8763)-*b*		
7		(6, 3)	(4, 5)	(1264, 8735)-*c*		
8	(2, 6)	(3, 5)	(4, 7)	(1234, 8657)-*a*		
9		(4, 3)	(5, 7)	(1245, 8637)-*b*		
10			(7, 5)	(1247, 8635)-*d*	7	(18, 17)
11	(3, 7)	(2, 6)	(4, 5)	(1324,8765)-*a*		
12		(4, 6)	(2, 5)	(1342, 8765)-*a*		
13		(4, 2)	(6, 5)	(1346, 8725)-*e*	7	(18, 17)
14			(5, 2)	(1345, 8762)-*f*	5	(19, 21)
15		(5, 6)	(4, 2)	(1354, 8762)-*f*		
16		(5, 2)	(4, 6)	(1354, 8726)-*f*		
17	(3, 6)	(2, 7)	(4, 5)	(1324, 8675)-*a*		
18		(2, 5)	(4, 7)	(1324, 8657)-*a*		
19		(4, 2)	(5, 7)	(1345, 8627)-*f*		
20			(7, 5)	(1347, 8625)-*g*	7	(17, 16)
21		(5, 7)	(4, 2)	(1354, 8672)-*f*		

Table 1 (cont.)

Note: At depth #1, the 21 complete EF allocations give items (1, 8) to (A, B). At lower depths, they fall into seven classes (7 *a*'s, 4 *b*'s, 2 *c*'s, 1 *d*, 1 *e*, 5 *f*'s, 1 *g*), each of which gives the same complete allocation but different items at different maximum odd depths. The MX depths, and the BMX scores, are shown only for the first member of each class. The MX depths of the *b*'s and the *f*'s are minimal (#5), but only the *b*'s have a maximin Borda score (20). The *a*'s and the *b*'s are the only two classes that yield PO allocations, with the first *a* allocation (allocation 1) being the unique SA allocation.