The Golden Rule of Capital Accumulation and Global Recession. Aggregate Production Function and the Cambridge Capital Controversy.

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Abstract
A new macroeconomic model is presented, which makes it possible to take a fresh look both at the long-term equilibrium growth process and at short-term deviations from it. Its key hypothesis is investment-to-profits equality. This hypothesis has classical roots and corresponds to the Ricardian and Marx approach and coincides with Phelps' Golden Rule of capital accumulation as well as with Uzawa's classical hypothesis. Under this assumption the long-term output growth rate is determined by the rate of capital accumulation, which in turn is equal to the net profit rate. The profit rate value is the result of a trade-off between workers and proprietors. The relationship between aggregate output and inputs is analytically derived in this paper where the variable values are measured not in physical units, but in the current monetary cost. It has the Cobb-Douglas functional form but is neither neoclassical production function nor technical relationship, which could specify the maximum output obtainable from a given set of inputs. The exponent of capital in the resulting function is equal to the investment rate, whose current value is not constant in time. So the output is no longer an unalterable function of inputs, and its shape can vary. The 'production function' shift parameter, which is commonly associated with the level of technology, may be expressed in terms of the wage level. The reasons for the 2007–2008 global recession have been clarified.

1 Introduction

The key hypothesis of the presented model is that investment and profits (capital income) are equal. The workers spend their entire wages to buy 'corn', the proprietors reinvest their profits in capital accumulation. This hypothesis has classical roots and corresponds to the Ricardian (1817) and Marxian (1867) approach, to Pazinetti's (1962) "Socialist system", as well as to Uzawa's (1961a) classical hypothesis. It is also one of the possible formulations of the well-known Robinson's (1962) neoclassical theorem and Phelps' (1961) Golden Rule of capital accumulation, which maximizes consumption[1]. The importance of this rule for economics is not completely understood so far, and this rule exhibits one of the fundamental properties of a human being, a desire for maximizing consumption. It will be shown in Appendix D that non-compliance with the Golden Rule for a long time can explain the last occurrence of the 2007 – 2008 recession.

Initially, the Golden Rule appears as an answer to the question: What is the optimal saving (investment) rate[2] in the Solow-Swan model (Solow, 1956, 1957 and Swan, 1956)? This answer was given independently by Phelps, (1961), Desrousseaux (1961), Allais (1962), Weizsäcker (1962), Swan (1963) and Robinson (1962). But on closer examination, it turns out that the Golden Rule is an absolutely independent statement, which does not require the existence of the neoclassical foundations. Indeed, it will be discussed (in Section 2) that the derivation of the Golden Rule (Phelps, 1965) requires only that the aggregate output Y and consumption C be functions of capital stock K, and the

[1] Thus the key hypothesis of the presented study is named 'Golden Rule', but the wording of this rule shifts the emphasis. It is unbelievable that an optimal investment rate value, which maximizes consumption, exists. The latter becomes maximum because of the investment-to-profits equality, and thus the investment rate is equal to capital income share automatically, while both the values can vary simultaneously (more details, see Section 2.2). The idea to compare investment and profits as the criterion of dynamic efficiency (Golden Rule compliance) was also proposed by Abel, Mankiw, Summers, and Zeckhauser (1989).

[2] Since this variable corresponds exactly to the investment rate $s = I/Y$, this term will be used further.

profit maximizing condition be valid, $\partial Y/\partial K= r+\delta$.[3] This presupposes the existence of the relationship between aggregate output and inputs in general, but does not require that this function must be the neoclassical production function, and that its variables must be measured in physical units.

The monetary valuation of capital and other variables is applied in the present paper (their cost is measured as a book value in accounting) that allows the re-conciliation of the Cambridge capital controversy[4]. The theoretical foundations of the aggregate production function and the other neoclassical instruments have been criticized during the discussion of this well-known controversy. This dispute has not been completed until now (see Cohen and Harcourt, 2003, Pasinetti, 2000, Solow, 2000 and Bliss, 1975). One of the most fundamental issues of the "English" Cantabrigians' criticism is the meaning and, as a consequence, the unit of capital measurement when industrial capitalist societies are analyzed. The heterogeneous capital goods aggregation in physical units poses serious difficulties. Therefore, it has been argued that the existence of any aggregate production function is not possible in rigorous approach, except in unlikely special cases (e.g., for one-commodity models), see Fisher (1969) and (1971), Felipe (2001), Felipe and McCombie (2002) and (2003), McCombie (2000-2001) and Hahn (1972). On the other hand, the neoclassical approach criticized above and the neoclassical aggregate production function (mostly Cobb–Douglas), which has been used both in micro- and macroeconomics, yield good results (see Cobb and Douglas, 1928 and Douglas, 1948) in describing both economic growth and convergence in standards of living (Mankiw, Romer and Weil, 1992 and Fraser, 2002).

It appears that both the English and American (or the so-called neoclassical) parties each makes sense. The analytical derivation of the relationship between output and inputs (capital stock and labor), which has the Cobb-Douglas functional form, is presented in Section 3. This relationship implies the use of not physical but monetary units for the valuation of capital, output, etc, thereby it cannot be properly termed as a "production function". Thus, a new term is coined, "the monetized production function". Many years back, Wicksell (1911) noted that heterogeneous capital goods cannot be measured and aggregated in physical units. Instead, capital valuation must be used. Such valuation is commonly used by classical economists as a unit of labor. In this study, monetary units (the book values) are used for this purpose. This approach (which can be termed 'accounting') inevitably limits the model. Adam Smith (1776) also noticed the fact that the monetary valuation is meaningful only in a certain place at a certain time. Therefore the aggregate monetized production function obtained in this study is not identical for different economies with different capital stock values, nominal wages, institutions, etc. Moreover, this function is neither the neoclassical production function nor a technical one, which could specify the maximum output obtainable from a given set of inputs. In this sense, the resulting function may seem similar to the functional form, which was obtained with the help of income accounting identity by Felippe, McCombie (2010) and Shaikh (1974), but it will be shown in Section 3 that the "monetized" one has a lot of economic sense. Despite the lack of universality and limited domain, it describes short-term processes much better than the "humbug" production function (see Section 3.4). The correlation coefficient between the actual and calculated changes in the GDP of the US economy is equal to 0.98. This offers a possibility to practically use the function despite the fact that it is correct only when deviations from the equilibrium are small. The result can better explain the fact that the monetized aggregate production function "works" while the neoclassical aggregate production function do not exist.

The equality between investment and profits is assumed to be held in this study not only along an equilibrium growth path[5], but also during deviations from such a way. This hypothesis is termed the Dynamic Golden Rule; its justification is presented below in Section 2. Under the Dynamic Golden Rule assumption, the problems of the aggregate production function's existence and the problems of its aggregation are resolved in Section 3. This assumption, together with the condition for maximizing profit, is a sufficient condition for the existence of such a function. The problem of aggregation of the production function is related to the non-additivity of the Cobb–Douglas function. But, if the Dynamic Golden Rule holds, both for different industries and for the aggregate economy, then all of the corresponding monetized production functions have the Cobb–Douglas form.

[3] The profit rate $r+\delta$ (the sum of the net profit rate r and depreciation rate δ) is the ratio of profits to the capital stock.
[4] This problem was widely discussed in the 1960s by Harcourt (1969), Samuelson (1962) and (1966), Sraffa (1960), Robinson (1953-1954), Pasinetti. (1966), (1969) and (1970), Blaug (1975) and Champernowne (1953-54).
[5] The meaning of the term 'long-term equilibrium growth path' in the work is clarified in Section 2.

Moreover, the aggregate monetized production function exhibits a lot of very interesting properties. It is a function of not only the capital stock K and the labor L, but also of the investment rate s and the wages w (all four variables are mutually independent and can vary with time). The curvature of the Cobb–Douglas function is determined by the exponent of capital, which is equal to the investment rate s in this instance. Since the current value of s can vary with time, the resulting monetized production function is no longer a static function of inputs with permanent shape. Consequences from this property will be discussed in Section 3.

Another property of the monetized production function is even more important. It will be shown in Section 3, that the value of wages raised to the power of $(1 - s)$ determines this function shift parameter. This shift parameter is commonly associated with the level of technology. However, in the neoclassical case, it is not clear why the growth rate of labor productivity (which must be proportional to the level of technology) is remarkably stable in the developed countries (about 1.5 – 2% per year) while the exogenous technological progress moves irregularly. The last conclusion resolves the well-known Solow computer paradox (the growth rate of output is remarkably stable in developed countries, while the technological progress moves irregularly). However, if the labor productivity is determined in terms of relatively smoothly varying wages (as is the case in the present model), the contradiction is absent. This does not mean that the technological progress does not matter. This means that the technological progress confronts the economy with the choice of the technology based on the cost and efficiency. Also, the existing wage level determines the economic expediency of one technology or the other. In fact, low wages are advantageous to a cheap low productivity technology and vice versa (see Appendix A).

The proposed model reproduces processes occurring both along a long-term equilibrium growth path and during short-term deviations from it, which are associated with business cycles or transition processes[6]. Kalecki's (1968) methodological conclusion, asserting that "the long-run trend is only a slowly changing component of a chain of short-period situations; it has no independent entity", is fully confirmed in the work. The model makes it possible to take a fresh look at the equilibrium growth process (commonly termed the balanced growth in the Solow model). The question about the stability of such a path has been raised when discussing the Harrod-Domar framework (Harrod, 1939 and Domar, 1946), the well-known knife-edge problem. The fundamental equation obtained by Harrod (1939) determines this path and links at equilibrium three variables: the growth rate of real output g^*, the capital-to-output ratio $(K/Y)^*$, and the investment rate s^*, where the asterisk indicates the equilibrium value. If these three variables are exogenous and governed by different independent factors, then the existence of an equilibrium growth path is accidental and should occur rarely.

However, the surprising constancy of the average values of the real output growth rate g^*, the capital-to-output ratio $(K/Y)^*$, and the rate of profit r^*, at least for the developed economies[7], testifies to the existence of the concept of equilibrium growth. It is commonly considered that the existence of such a path follows from the neoclassical growth model due to the Solow (1988) adjustment process. The neoclassical production function has a mechanism to ensure the balance: the capital intensity $k = K/L$ (and therefore, K/Y) changes adjusting to the factors considered to be exogenous in the model (the growth rate of real output g^* and the investment rate s^*).

Nevertheless, the neoclassical conception of equilibrium as the end of an economic process was brought into question during the debates around the Cambridge capital controversy. Robinson (1933) indicated that equilibrium is not the outcome of this process, and therefore it is not an adequate tool for analyzing the processes of capital accumulation and growth. She claimed that the equilibria must be path-dependent (Robinson, 1975). Moreover, the general equilibrium approach has also demonstrated the lack of results supporting the stability of the equilibrium path (Hahn, 1984, Fisher, 1989, Irlgrao and Israel, 1990).

The present model does not adopt the neoclassical approach. The equilibrium growth path is not some special case, and it is not unique here; it is simply the consequence of the constancy of two of the three mentioned above variables (g^*, s^*, and $(K/Y)^*$). If their average values are stable, then the

[6] In this study the term 'transition process' refers to the process of moving the economy from one equilibrium path to another.

[7] This constancy was formulated as the Kaldor (1963) facts 50 years ago, and it is still observed today (see Acemoglu, 2008, Barro, Sala-i Martin, 2004 and Jones, 2002).

third should also tend to the stable value, which is defined by Equation (1) in Section 2. This equation is derived in Appendix B and is identical to the steady-state definition by Solow and to the Harrod-Domar fundamental equation. It will be shown in Section 3 that the independent exogenous variables here are g^* and $(K/Y)^*$, while the variable s^* is an adjustment factor. When the average $(K/Y)^*$- or g^*-values are slowly moving during a transition process, then s^* adjusts to their changes according to Equation (1), and the economy moves towards a new equilibrium growth path. Such a transition process from one equilibrium path to another is considered in Appendix E within the framework of the two-sector economy growth model.

It turns out that the long-term output growth rate is determined by the long-term rate of capital accumulation, which in turn is equal to the long-term net profit rate (Golden Rule). The profit rate value is the result of a trade-off between workers and proprietors.

While the long-term equilibrium growth rate g^* and the profit rate r^* are considered to be constant, their current values (g and r) can vary during a short-term time period. However, it is difficult to understand in details the mechanism of such deviations associated with short-term business cycles and transition processes by using only the resulting aggregate monetized production function. For this purpose, an additional analysis is carried out in Appendix E by invoking the more complicated two-sector model of economic growth, where the Uzawa (1961b and 1963) capital-intensity condition is clarified. The theoretical description of short-term processes obtained there shows a good qualitative agreement with observed data.

2 Model

Both the long-term and short-term processes occurring in a simple two sector economy are considered in the present model, and the markets are supposed to be perfectly competitive. Without loss of generality, the households are assumed to be only consuming, and the business to be only investing.

The output, wages, investment, profits, etc., are measured not in physical units, but in the current monetary cost (book value). Therefore, the relationship between aggregate output and inputs is not identical for different economies. This relationship which is analytically derived here cannot be properly termed the production function, and therefore the term the monetized production function is adopted. It describes the relation between the total output and the inputs in a certain place at a certain time.

Constant returns to scale are expected. Indeed, if capital and labor are doubled, the output is expected to double, despite the monetary valuation of the variable cost.

The often criticized assumption of diminishing returns is not necessary for the derivation of the production function, as well as the other Inada (1963) conditions.

2.1 Equilibrium Growth Path

The aggregate output Y is determined by a set of mutually independent variables (capital stock K, labor L, wages w, and investment rate s). Their quantities vary over time during business cycles, but the average values of the growth rate of real output g^*, of the investment rate s^*, of the capital-to-output ratio $(K/Y)^*$, and of the depreciation rate δ may remain rather stable. Such stability is considered to be the criterion of the equilibrium path in this study. It turns out that the three averages mentioned above are governed by the following equation:

$$(K/Y)^* = s^*/(g^*+\delta), \tag{1}$$

which corresponds to the Solow steady state definition, as well as to Harrods's (1939) fundamental equation. Equation (1) is derived in Appendix B without any assumptions about the nature of the three variables mentioned above. Only algebra is applied; if the two variables are stable, then the third should tend to the value given by Equation (1)[8]. The variables g^* and $(K/Y)^*$ are exogenous to the

[8] It is shown in Appendix B that making practical use of the averages instead of the equilibrium values is quite justified.

model[9], while s^* adjusts its value according to Equation (1). Thus, the equilibrium growth path is not something special or unique, but it is any long-term trend with stable exogenous average values of g^* and $(K/Y)^*$, while s^* is considered to be an adjustment factor.

2.2 Golden Rule

Despite the fact that the Golden Rule of capital accumulation has been derived by Phelps in the framework of the neoclassical approach, such an approach can be shown not to be necessary, and the Golden Rule is an independent statement. The solution to the problem of consumption C maximization is rather simple (see Phelps, 1965):

$$C^* = Y^*(1 - s^*) = Y^* - s^*Y^*$$

where C is consumption. Substituting the s^*-value from Equation (1) yields

$$C^* = Y^* - (g^* + \delta)K^*.$$

Equating the derivative to zero to find the maximum gives

$$dC^*/dK = dY^*/dK - (g^* + \delta) = 0,$$

and substituting $dY/dK = (r + \delta)$ from Euler's theorem, leads to

$$r^* = g^* \tag{2}$$

Thus, the Golden Rule derivation does not require the neoclassical framework, only the validity of Equation (1) is necessary.

Combining Equations (1) and (2) and assuming that the Golden Rule is valid give the equilibrium value of the investment rate (the "optimal" investment rate), which turns out to be equal to the capital income share,

$$s^* = (r^* + \delta)(K/Y)^* \tag{3}$$

The last equation can be rewritten in other form, which corresponds to the Uzawa (1961b) classical hypothesis about the equality between investment and profits (see Equation 4),

$$s^*Y^* = (r^* + \delta)K^* \tag{4}$$

Equation (2) states that the profit rate and the growth rate of real output are equal. Equation (3) states that the investment rate and the capital income share are equal. All three equations ((2) – (4)) are mathematically equivalent. But only one of the three statements can be the "cause", while the other two are the consequences. According to the logic of the present model, there is no "optimal" investment rate; both the right-hand and the left-hand sides of Equation (3) can vary. Therefore the equality between investment and profits (Equation (4)) is considered in the present work as the Golden Rule formulation and as the primary assumption.

The importance of the Golden Rule validity along the equilibrium growth path is to be justified. Equating the aggregate income and the expenditures for the two sector economy yields the obvious equation:

$$C^* + I^* = w^*L^* + (r^* + \delta)\, K^*,$$

or

[9] This statement will be justified in Section 3.

$$(C^* - w^*L^*)/Y^* = (r^*+\delta)\ K^*/Y^* - s^*,$$

substituting $(K/Y)^*$ from Equation (1) gives

$$(C^* - w^*L^*)/Y^* = (r^* - g^*)\ s^*/(g^*+\delta) \tag{5}$$

where C^* is household consumption expenditures, $(wL)^*$ is labor income, and $(r^* + \delta)K^*$ is the capital income (profits) along the equilibrium growth path.

If the Golden Rule is satisfied, then $r^* = g^*$ (Equation (2)), and both the left-hand and right-hand sides of (5) reduce to zero. Therefore, the consumption becomes equal to the labor income $C^* = w^*L^*$, and the investment is equal to the profits. If the Golden Rule is not valid for a long time, then there are two alternatives:

1. If $C^* > w^*L^*$, then the households consume not only the labor income, but also a portion of the profits. Consequently, the right-hand side of Equation (5) must be positive, as well as the left-hand side, which means that $g^* < r^*$, i.e., the rate of real output growth is lower than the rate of profit. This scenario may happen, and society has to submit oneself to the lower output rates because of excessive consumption, small investment, and a low capital stock growth rate.

It is readily seen from Equation (5) that g^* should reduce to zero when

$$(C^* - w^*L^*)/Y^* = r^*s^*/\delta$$

Putting the following values characteristic of developed economies: s^*=0.3; r^*=0.025; δ =0.05 into the above equation, an estimate of $(C^* - w^*L^*)/Y^* = 0.15$ can be obtained, with the result that the real output growth should stop when the consumption share of aggregate output exceeds 15% labor income share. The net profits are to be consumed completely, and the investment is to be equal to depreciation. The capital stock growth, which is necessary for output growth, should stop.

It will be shown in Appendix D that unduly grown consumption could be the reason for the 2007 – 2008 global recession.

2. If $C^* < w^*L^*$, then this means that society accumulates capital at a faster rate without consuming the capital income and only partially consuming labor income. This case does not conform to saving logic because the saving is now made in order to consume more later on. This 'later on' does not ensue ever if both sides of Equation (5) are smaller than zero.

The rate of real output growth ($g^* > r^*$) does not remain high continuously without sufficient growth in consumption. Phelps, (1965) termed such a situation as the "dynamic inefficiency". He showed that if investment constantly exceeds profits, then the capital to output ratio increases and consumption is less than it is in the case when the Golden Rule is satisfied.

In order to illustrate the argument put forward, the USSR economy in 1970–1990 should be recalled. In compliance with the policy of the ruling Communist Party, the growth rate of the investment goods production should be higher than the growth rate of consumer goods production (i.e., the growth rate of investment had to be higher than consumption growth). The more and more workers in the Soviet Union produced fixed assets, but consumer goods production naturally was falling behind. The result was a scarcity of consumer goods and a growth of prices.

Both of the scenarios described above do not look attractive. Then it means that the Golden Rule validity (when Equations (2) – (4) are satisfied) is realistic enough to be a model assumption in itself (without binding to the Solow model).

This hypothesis will be tested on actual data sets in the next section.

2.3 Dynamic Golden Rule

Phelps's formulation of the Golden Rule (Equation (2)), the validity of which has been discussed in the preceding section, is applicable only along an equilibrium growth path (since the equality between the current r- and g-values is not evident). However, the balance between the current investment and profits is observed for the developed economies (see Figures 6 – 8 in Appendix C for United Kingdom (UK), Japan, and U.S.). Therefore, it seems rather reasonable to assume a permanent investment to profits equality in economy, thereby extending the Golden Rule to the general case including transition processes. This hypothesis is termed here the Dynamic Golden Rule.

Then, rewriting Equation (4) for the general case gives:

$$sY = (r + \delta)K \tag{6}$$

Like the Golden Rule, the Dynamic Golden Rule is expected to be not a special case of the Solow model, but an independent rule that is the main model assumption and is observed both along the equilibrium growth path and when deviations from this path occur.

2.4 Model Assumptions

The variables, which have dimension of cost (Y, K, w, I, C), are measured in terms of the current monetary value.

The assumptions of the model are as follows.

(1) The following assumption of constant returns to scale is adopted: $F(aK,aL) = aF(K,L)$, therefore, the 'production function' can be expressed in the intensive form $y = y(k)$, which relates the values of labor productivity $y \equiv Y/L$ and the capital intensity $k \equiv K/L$. The profit maximization condition is satisfied, i.e., the marginal product of capital is equal to the profit rate $(r + \delta) = \partial y(k)/\partial k$. The diminishing returns are not expected, as well as the other Inada conditions. However, the analytically obtained monetized production function exhibits these properties, as we shall see below.

(2) The main model assumption is a permanent investment and profits equality, the Dynamic Golden Rule, that applies both along the long-term equilibrium growth path (Equations (2) – (4)) and when deviations from this path occur during short-term business cycles and under transition processes (Equation (6)). The last equation can be rewritten in the intensive form:

$$sy = (r + \delta)k \tag{6i}$$

(3) The current k-, s-, and wage w-values are considered to be mutually independent[10], but not constant in time.

3 The Derivation of the Cobb–Douglas Monetized Production Function

3.1 Analytical derivation

Theorem 1. If model assumptions stated above are true (the Dynamic Golden Rule is valid, returns to scale are constant, and the current capital intensity k-, investment rate s-, and wage w-values are considered to be mutually independent), then the dependence of output from inputs (capital stock and labor) takes the Cobb–Douglas form where the investment rate s is the exponent of capital, and $(1 - s)$ is the exponent of labor:

$$Y = \alpha K^s L^{1-s}\,. \tag{7}$$

Or in the intensive form:

$$y = y(k) = \alpha k^s \tag{7i}$$

where α is a shift parameter in the production function independent of k.

Proof: Assumption (2) means Equation (6i) validity, and substituting the profit maximization condition $(r + \delta) = \partial y(k)/\partial k$, the following relation can be obtained

[10] These variables linearly independence is well confirmed by correlation analysis of the data in Appendix F.

$$\partial y\,(k)/\partial k = sy\,(k)/k$$

where s and k are independent according to the assumption (3). Therefore, the last differential equation has the analytical solution (Equation (7i)). ■

In the (k, y) plane, the growth of output along an equilibrium path is represented by a straight line defined by Equation (1), which can be rewritten in the intensive form:

$$k/y = (K/Y)^* = s^*/(r^* + \delta)$$

or

$$y = k/(K/Y)^* = (r^* + \delta) / s^* \qquad (1i)$$

In turn, the monetized production function (7i) represents deviations from such a path, e.g., during business cycles. The value of the shift parameter α is independent of k, but in general, it is not a constant. This parameter corresponds to the labor productivity level, which in turn is determined by the level of technology.

The value of α can be calculated for the equilibrium point (k^*, y^*) where the corresponding production function (7i) and the condition (1i) for equilibrium growth intersect. Figure 1 shows (points a and b) such intersections of the straight line (equilibrium path) and two intensive production functions with different shift parameters. By equating the right-hand sides of (1i) and (7i), the following relations can be obtained:

$$\alpha (k^*)^{s^*} = k^*(g^* + \delta)/s^*,$$
$$\alpha = (k^*)^{1-s^*}(g^* + \delta)/s^*$$

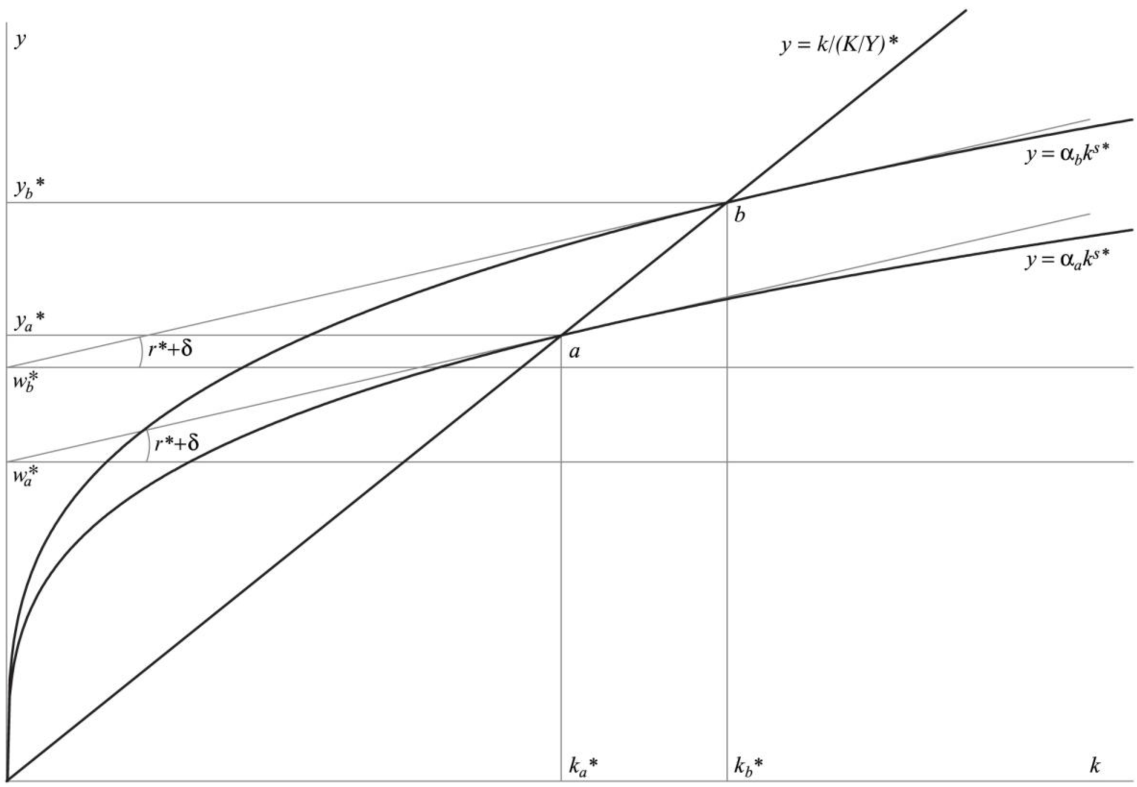

Figure 1 Two intensive production functions of different shift parameters. Equilibrium growth path

In the general case, the quantity α is not only the function of the level of technology, but it is also the function of the investment rate s, $\alpha = \alpha\,(s)$, remaining independent of k. In the last equation, the α-value is obtained at the equilibrium point $\alpha = \alpha\,(s^*)$ where $s = s^*$. Next, the derivation of the quantity α follows for the case when $s \neq s^*$.

Imagine a hypothetical situation in which the investment rate originating at the equilibrium point have instantaneously and discontinuously changed from the value of s^* to s_1. At the same time, the values of k and y cannot change instantly ($k = k^*$, $y = y^*$), and the monetized production function then changes its curvature only (see Figure 2). So, the point c representing the equilibrium in the (k, y) plane remains stationary. It means that:

$$y(k^*, s^*) = y(k^*, s),$$

or

$$\alpha(s^*)(k^*)^{s^*} = \alpha(s)(k^*)^s$$
$$\alpha(s) = \alpha(s^*)(k^*)^{s^*-s} = (k^*)^{1-s}(g^* + \delta)/s^* \tag{8i}$$

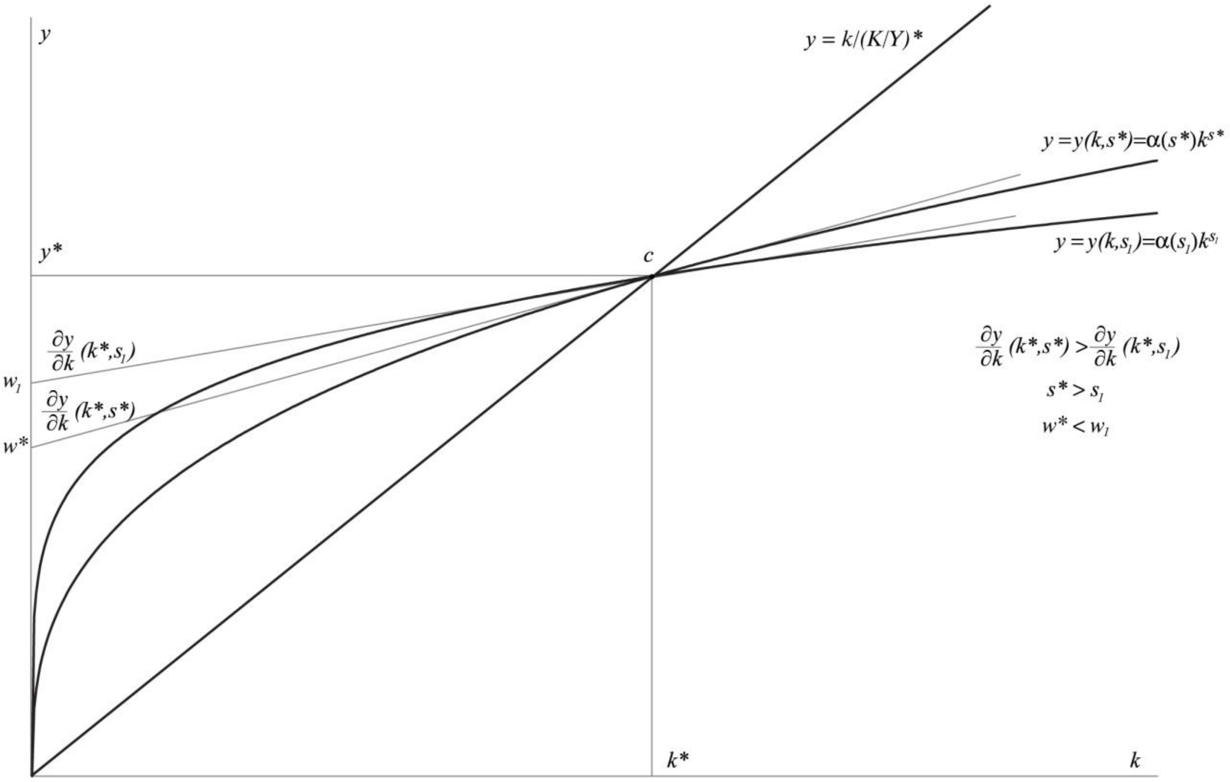

Figure 2 Change in the curvature of the monetized production function that has occurred due to an investment rate variation

The shift parameter α in Equation (8i) is expressed in terms of the equilibrium value of k^*. It will be shown later, that it is more convenient to use the equilibrium wage level w^* instead of k^*. The values of k^* and w^* are directly proportional to each other when the Golden Rule is valid. Really, substituting $s^*y^*=(r^*+\delta)k^*$ (Golden Rule) into the obvious relation $y^*=w^*+(r^*+\delta)k^*$ yields:

$$k^* = w^* \frac{s^*}{(r^* + \delta)(1 - s^*)} \tag{9}$$

Substituting this relation into Equation (8i) gives:

$$\alpha(s) = \frac{(w^*)^{1-s}(r^* + \delta^*)^s}{(s^*)^s(1 - s^*)^{1-s}} \tag{10}$$

The dynamics of aggregate output can be represented as a sum of the following two components:

(a) The component of growth, i.e., the movement along the equilibrium path in accordance with Equation (1), during which the production function shift parameter is growing. Such a path exists if the

average variable values in Equation (1) are stable. Component (a) is responsible for long-term processes.
(b) The cyclical component when the point, which represents the economy, is departing from the equilibrium trend along the monetized production function. Component (b) is responsible for short-term processes.

These two components are independent because they are governed by different groups of independent variables (the growth component depends on the equilibrium values of the variables, while the cyclical component on their current values). Such an approach is evidenced by the observed data, e.g., Hodrick and Prescott (1997) represent the time series as the sum of a smoothly varying growth trend component and a cyclical component. They found that the nature of the co-movements of the cyclical components of the macroeconomic time series is very different from the co-movements of the slowly varying components of the corresponding variables.

Similarly, the correlation analysis in Appendix F shows that the two groups of variables corresponding to the growth and cyclical components are linearly independent, and the growth component is strongly connected with the current wage rate. The last empirical observation indicates the surprising fact that the current wage value is linearly independent of the variables that determine the cyclical component. This means wage inelasticity during short-term processes. This property is used as an additional assumption when considering business cycles and transition processes in Appendix E.

Since the averaging is performed over the cyclical component, then the equality $w = w^*$ can be used in Equation (10) for performing calculations in applied econometrics. Indeed, the production function, which is responsible for the cyclical component of output, is governed by the values of k and s during transition processes. Then, the independence of w from k and s means the independence of w from the cyclical component, i.e., w and w^* are the same[11].

Consequently,

$$\alpha = \frac{(w)^{1-s}(r^* + \delta^*)^s}{(s^*)^s (1-s^*)^{1-s}} \tag{11}$$

and as a result, Equation (7) can be rewritten as

$$Y = \gamma K^s (wL)^{1-s} \tag{12}$$

where

$$\gamma = \frac{(r^* + \delta)^s}{(s^*)^s (1-s^*)^{1-s}} = \frac{(g^* + \delta)^s}{(s^*)^s (1-s^*)^{(1-s)}}$$

When deriving coefficient γ, we assume the presence of an equilibrium growth, thence the average values $g^* = r^*$, $(K/Y)^*$, and then s^* should be stable. This assumption allows this coefficient to be explicitly expressed in terms of the equilibrium variables in Equation (12). However, the equilibrium growth is not necessary to derive Equation (12), only the Dynamic Golden Rule and the profit maximization condition are required. Then, in invoking a rigorous approach, γ cannot be expressed in such a simple form, but in any way it retains memory of the previous history of output, which is a very important property. Hence, the current monetized production function "remembers" the past path of the economy. Robinson (1974 and 1975) insisted upon the necessity for such a "memory", putting back on the agenda what we now call path-dependent equilibria.

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[11] In general, the substitution of w for w^* is not necessary, and it has no significant effect on the subsequent arguments. Such a substitution is suitable for performing applied calculations, since the value of w is well defined in contrast to the value of w^*.

3.2 Properties of the Monetized Production Function

3.2.1 Cobb–Douglas form

The analytically derived in this study monetized production function (Equation (12)) takes the Cobb–Douglas form where the exponent of capital is equal to the investment rate s, which in this case is equal to the capital income share. Hence, Douglas's claim (1948) that output is described by exactly such a function finds confirmation.

The Cobb–Douglas function is the best known type of production function, which was empirically constructed to approximate the output of American manufacturing since 1899 to 1922 (Cobb, Douglas, 1928). More than 60 years ago, Douglas (1948) claimed that "Laws of production" exist and output is described by Cobb–Douglas production function both for different industries and for the aggregate economy. His claim was substantiated by good agreement between the function under consideration and the observed time series (Douglas, 1976). Later, a large number of studies have been devoted to the causes of and conditions for the existence of the Cobb–Douglas aggregate production function (e.g. Jones, 2005 and Chilarescu, Vaneecloo, 2007). On the contrary, other authors (see Felipe, Fisher, 2003 and Temple, 2006) took the view that if the output of different industries is described by different Cobb–Douglas functions, then the aggregate production function is not necessarily has the Cobb–Douglas form. The sum of two Cobb–Douglas functions with different exponents of capital does not have the Cobb–Douglas form, inasmuch as this function is nonadditive in general.

The approach presented in this paper gives a simple explanation for the existence of the Douglas "Laws of production". When the Dynamic Golden Rule is valid, the constant-return production function necessarily has the Cobb–Douglas form where the exponent of capital is equal to the capital share (if the investment rate s and the capital intensity k are mutually independent).

This approach clarifies the question why the production functions have the Cobb–Douglas form both for the aggregate economy and for different industries simultaneously (Douglas, 1948) despite of their different capital income shares. The Cobb–Douglas function nonadditivity does not result in a contradiction when investment and profits are equal (Dynamic Golden Rule) for different industries separately.

Since the value of s is less than unity, the monetized production function (Equation (12)) exhibits diminishing marginal returns of capital and the validity of other Inada conditions. These properties have not been expected earlier, but they are inherent in the resulting production function.

3.2.2 The Shift Parameter of the Monetized Production Function is Proportional to the Wage Level Raised to the Power of $(1 - s)$

It is commonly considered that the production function shift parameter is determined by the level of technology. Here, this parameter is defined by the wage value in Equation (12). Then, the level of technology may reasonably be expressed in terms of the wage level. This does not mean that the technological progress does not matter. This means that the technological progress provides the economy with the choice of technology based on the cost and efficiency.

The exponents of the labor L and the wage w in the monetized production function (Equation (12)) are equal. Equality of these exponents can be interpreted as augmentation of the technical change by the labor, according to the Uzawa (1961a) theorem.

3.2.3 The Monetized Production Function Varies its Curvature Depending on Changes in the Investment Rate s

The exponent of capital in the obtained Cobb–Douglas production function is equal to the investment rate s. This variable is not constant with time; therefore, the function should modify its curvature when s varies (see Figure 2). This property fundamentally distinguishes the aggregate production function derived here from the neoclassical aggregate function traditionally used.

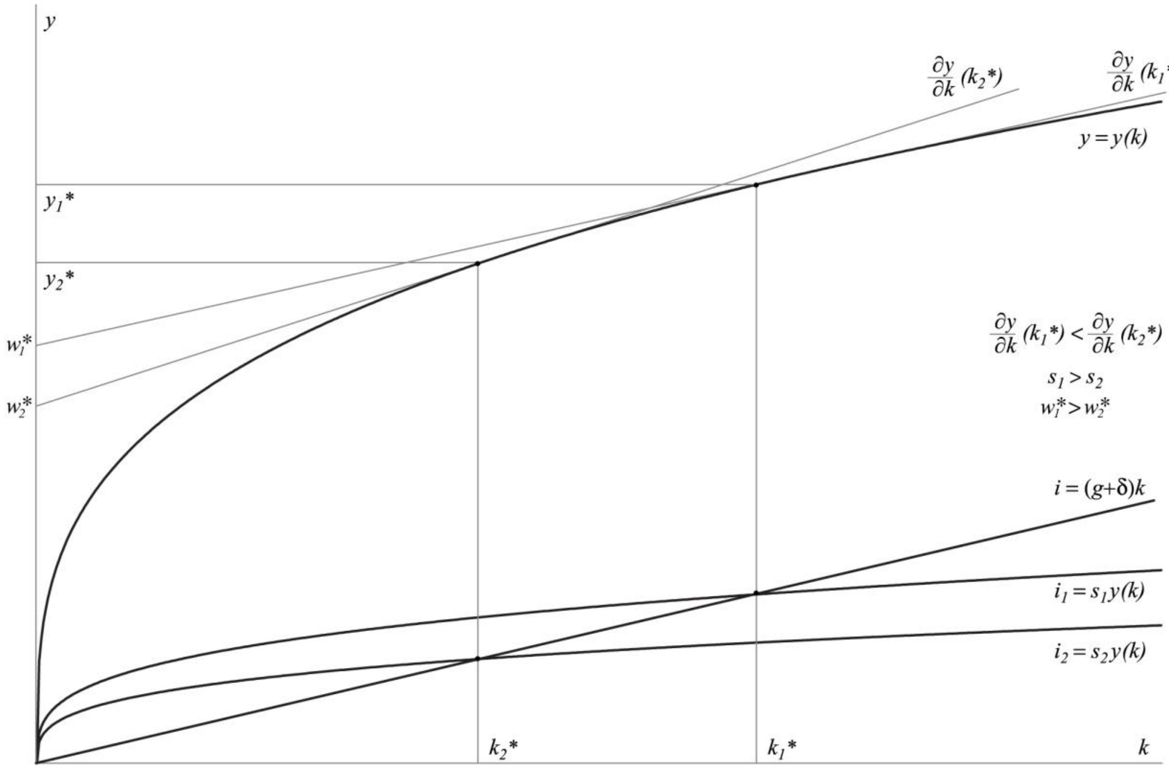

Figure 3 Change in the equilibrium point position that has occurred due to the investment rate variation, Solow model

The Solow model predicts a reduction in the capital intensity k and a growth in the marginal product of capital (MPC) $\partial y/\partial k$ if the investment rate s declines (see Figure 3). The actual statistical data from developed countries suggests that changes in s and r have the same direction. Figures 6 – 8 (Appendix C) show that the profits and investment time series nearly match. Figure C9 (Appendix C) demonstrates that the changes in the capital income (profits) share $((r + d)K/Y)$ occur mostly due to fluctuations in the profit rate, rather than due to changes in K/Y. Combining the last two observations leads to the conclusion that the variations in r and s have the same direction. The production function derived in this paper (Equation (12)) demonstrates precisely such a relationship between the variables. Figure 2 shows the co-movement of the profit rate and the investment rate, while the point c representing the state of the economy in the (y, k) plane remains stationary.

The variations in profits share observed occur before those in the investment rate; however, the synchronism in their changes seen in Figure 6 – 8, Appendix C, can be interpreted as profits being the source of investment: when the first decline, the second should drop as well. Then, this means the lower profit rate if external circumstances result in a short-term decline in profits. In a while, the investment rate will decline as well, so the curvature of the production function has to change in such a way that the tangent slope becomes lower adjusting to the initially changed marginal product of capital (Figure 2). The dynamics described above corresponds to the Dynamic Golden Rule: $s = (r + d)K/Y$.

3.2.4 The Nature of the Average Value of the Investment Rate s^*

The relationship between the co-movement of s and r is observed not only in business cycles, but also in transition processes, when a successfully developing economy reaches developed status, as it was in Japan in the 1970s. In this case, a simultaneous decline in the growth rate of real output g^*, the profit rate r^*, and in the investment rate s^* is observed, while the labor income share $(wL/Y)^*$ grow (see Table 1 and Figure 4).

Table 1 Japan. Average variable values. See Data Sources (Appendix G).

	Period	1956 – 1970	1971 – 1975	1976 – 1990	1991 – 2007
g^*	Real output growth rate (%)	9.7	4.5	4.6	1.3
r^*	Net rate of profit (%)	8.5	2.2	2.7	2.3
s^*	Investment rate (%)	21.1	19.7	16.0	15.1
$(wL/Y)^*$	Labor income share (%)	42.2	50.1	53.4	53.2
$(K/Y)^*$	Capital-to-output ratio	0.91	1.11	1.12	1.185

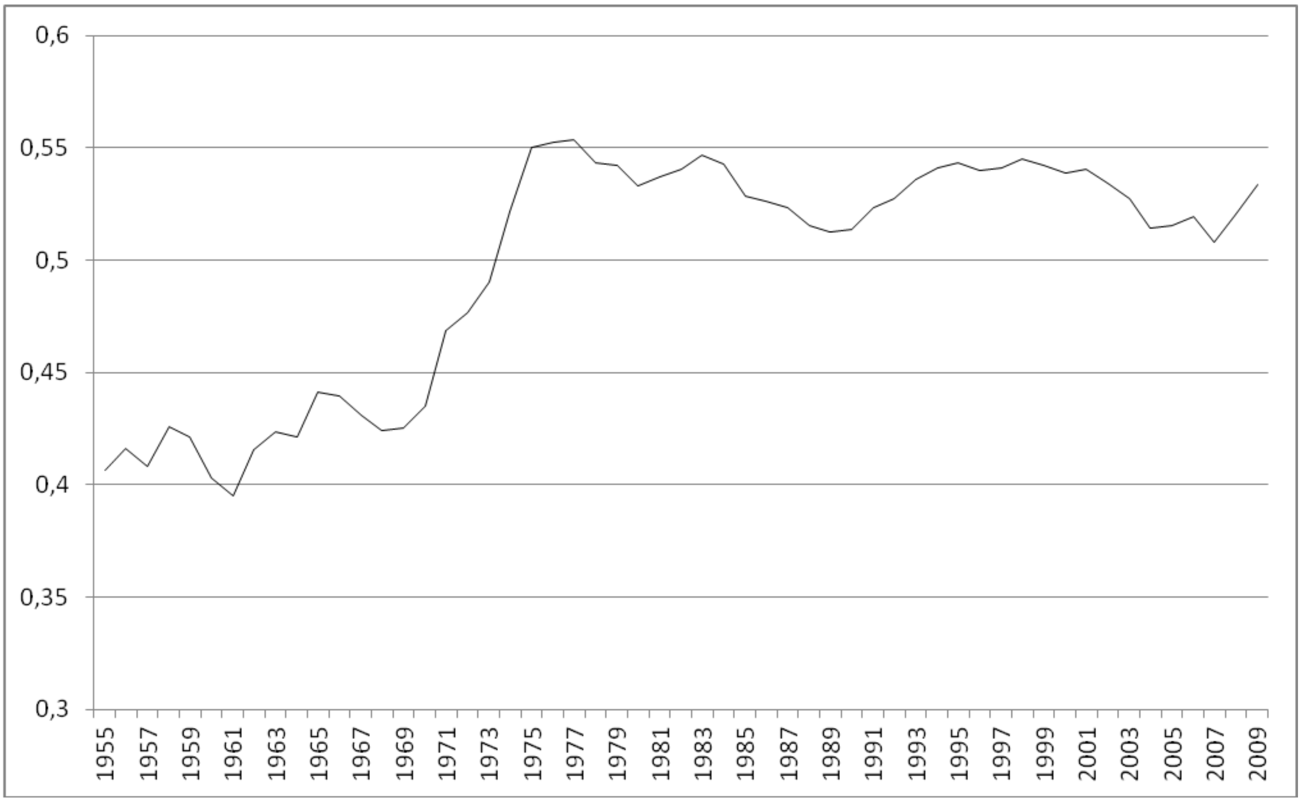

Figure 4 Japan. Labor income share, *wL*/*Y*. See Data Sources (Appendix G)

Figure 4 illustrates the process mentioned above. It shows the dynamics of the labor income share. This share was considerably growing during 1970 – 1976, which means that wages were increasing more rapidly than output. Such a rapid wage expansion leads to an accelerated growth in aggregate consumer demand, which is necessary for the development of the internal market.

If the labor share grows, then the capital share should decrease, consequently, both r^* and s^* (by Golden Rule) reduce, see Table 1. These arguments clarify the essence of the investment rate average value s^*. Its value is the result of a currently reached consensus between workers and proprietors (in other words, between the labor income share and the capital income share). For successfully developing economies (e.g., Japan 1955 – 1970), the higher rate of profit and the growth rate of real output $r^* = g^*$ are typical, as well as the higher investment rate s^*, the higher capital income share, and then the lower labor income share (see Table 1, Figure 4). But at some point, when labor productivity reaches a sufficiently high level, and when the demand for skilled labor increases, both these factors would intensively push the rate of wage growth up. Consequently, s^*, $r^* = g^*$, $(r^*+\delta)K/Y$ decrease, while the labor income share grows (see Table 1, Figure 4). The model presented predicts such a behavior (see Figure 2)[12].

[12] It should be noted that, in this case, the growth of the *K*/*Y* (and *K*/*L*) ratio along an unchanging production function may partly explain a reduction in the marginal product of capital *r* (see Figure 3), but it cannot explain the decline in the capital income share (which is equal to *s*). This share is the exponent of capital in the Cobb–Douglas function, and thus it should not vary if the production function does not change its curvature. Then, the share variation observed in Figure 3 should be associated only with the displacement of trade-off between workers and proprietors.

3.3 Additional Assumption of the Existence of a Minimum Value of the Rate of Profit r^*

What level does the rate of profit decrease to in the process discussed above, when the economy achieves developed status? Under any circumstances, the rate of profit should be greater than the rate of depreciation since the net profits must be greater than zero ($r^* > 0$). Euthanasia of the capitalists, which Keynes (1936) had predicted proceeding with the Ricardian (1817) approach, will not take place. The growth in labor productivity should prevent this. On the other hand, the value of r^* should not be too large, such a view was held by many of the famous economists, such as Adam Smith (1776), John Maynard Keynes (1936), and Milton Friedman (1956).

Hence, an additional assumption can be made: (4) Suppose that the minimum value of the net marginal product of capital $r^* = \tau$ exists, which still encourages entrepreneurs to invest. Then, for a developed economy, the value of r^* should attain this level. This hypothesis is not necessary for the present model, but it complements this model, allowing an explanation of the surprising constancy of the growth rate of the real output g^* for various developed economies. Indeed, if r^* tends to a constant τ, while $g^* = r^*$ according to the Golden Rule, then $g^* = r^* = \tau$, and g^* is a constant as well.

3.3.1 The Knife-edge Problem

Taking into account assumption (4) in section 3.3 above, a new inquiry into the well-known knife-edge problem is possible. This is a question about the existence of an equilibrium growth path. Such a path is described by Equation (1). The impression might be left that the three variables in this equation (g^*, $(K/Y)^*$, and s^*) are governed by unrelated groups of factors. In particular, the following is expected to be true: (a) the growth rate of the real output g^* is the sum of the growth rates of the labor force and of the labor productivity (the latter is determined by an increase in the level of technology), (b) the $(K/Y)^*$ ratio is determined by microeconomic quantities, specifically, by the ratios $(K/Y)_i$ maximizing profits in different industries and by shares of these industries in aggregate output, (c) the investment rate s^* is determined by households and business preferences (“to consume” or “to invest”). And, if these three variables are really independent, then Equation (1), as well as the existence of an equilibrium growth path, is fortuitous. But, in this case, it is not clear why the growth rate of real output, as well as the capital and labor income shares, are so stable, at least in developed countries.

To resolve this problem, only two variables of the three should be considered as exogenous. Kaldor (1955-56) proposed the capital-to-output ratio $(K/Y)^*$ and the investment rate s^* to be exogenous, as in the classical models. Unlike this, the output growth rate g^* and the investment rate s^* are considered to be exogenous in the Solow model, and the capital to output ratio $(K/Y)^*$ (as well as $(K/L)^*$) is a factor, which adjusts to changes in s^* or g^*, and thus brings the existence of the balanced growth path (see Figure 3).

A fresh approach is offered in the present study. The Solow production function is not used, as well as the adjustment process described above. Another adjustment process is suggested where exogenous factors are g^* and $(K/Y)^*$, while the growth rate of the real output g^* is a constant in compliance with assumption (4) made above for developed economies, $g^* = r^* = \tau$. The adjustment factor in this case is the investment rate s^*.

The model under consideration does not make a claim to find a solution to the problem of the equilibrium path stability, such stability is considered to be ensured by the stability of the two exogenous factors, g^* and $(K/Y)^*$. These two quantities really seem to be quite conservative, and they vary slowly.

3.4 The Domain of the Monetized Aggregate Production Function

In the preceding subsection, the equilibrium growth path has been shown to be defined by two exogenous factors, the capital-to-output ratio $(K/Y)^*$ and the real output growth rate, which is equal to the profit rate $g^*=r^*$ when the Golden Rule is valid. The investment rate s^* is an adjustment factor, and it is determined from Equation (1). Therefore, when the exogenous factors $(K/Y)^*$ and $g^*=r^*$ are set, then the only one equilibrium point in the (k, y) plane exists for any wage level w^*. This point, which represents the economy, is determined as the intersection of the ray from the origin $y = k/(K/Y)^*$ and the tangent to the monetized aggregate production function that crosses the y-axis at the point $(0, w^*)$ at an angle of $(r^*+\delta)$ (see Figure 1, points a and b for wages w_a^* and w_b^*). The monetized aggregate production function (Equation (12)) also become specified in this case, and Figure 1 shows the unperturbed function when $s = s^*$. When the labor productivity is growing simultaneously with the wage level, the point is moving upwards along this ray ($y = k/(K/Y)^*$), which governs the equilibrium growth path where the profit rate remains unchanged. The shift parameter of the monetized aggregate production function is also growing in accordance with increasing wages. Hence, the wage w^* determines the function shift parameter, and the investment rate s determines its curvature, while the change of labor L (and hence the capital intensity k) shifts the point that represents the economy along the production function during short-term business cycles.

It will be shown in Appendix E, that any transition process, during which the profit rate changes, is not consistent with an invariable aggregate production function in a two-sector model of economic growth (except for the special case when the capital intensities in both sectors are equal). The reason for such inconsistency is an inevitable change in the ratio of the investment goods price to the price of consumption goods during this process, and consequently a change of the aggregate production function. Thereby, the monetized aggregate production function (Equation (12)) is correct only when the economy is not far from the equilibrium growth path, i.e., when the current values of the variables s, r, K/Y, k, w, do not differ significantly from the equilibrium ones.

In order to compare the calculated and observed output changes, Equation (12) can be differentiated with respect to t to give:

$$\frac{1}{Y}dY/dt = \frac{(1-s)}{w}dw/dt + \frac{(1-s)}{L}dL/dt + \frac{s}{K}dK/dt + \ln\{\frac{(K/(wL))}{(K/(wL))^*}\}ds/dt \tag{13}$$

The GDP changes for the US economy have been calculated using the last equation. A comparison of the calculated and observed data (Figure 5) provides a very good result. The correlation coefficient between these two time series is equal to 0.98.

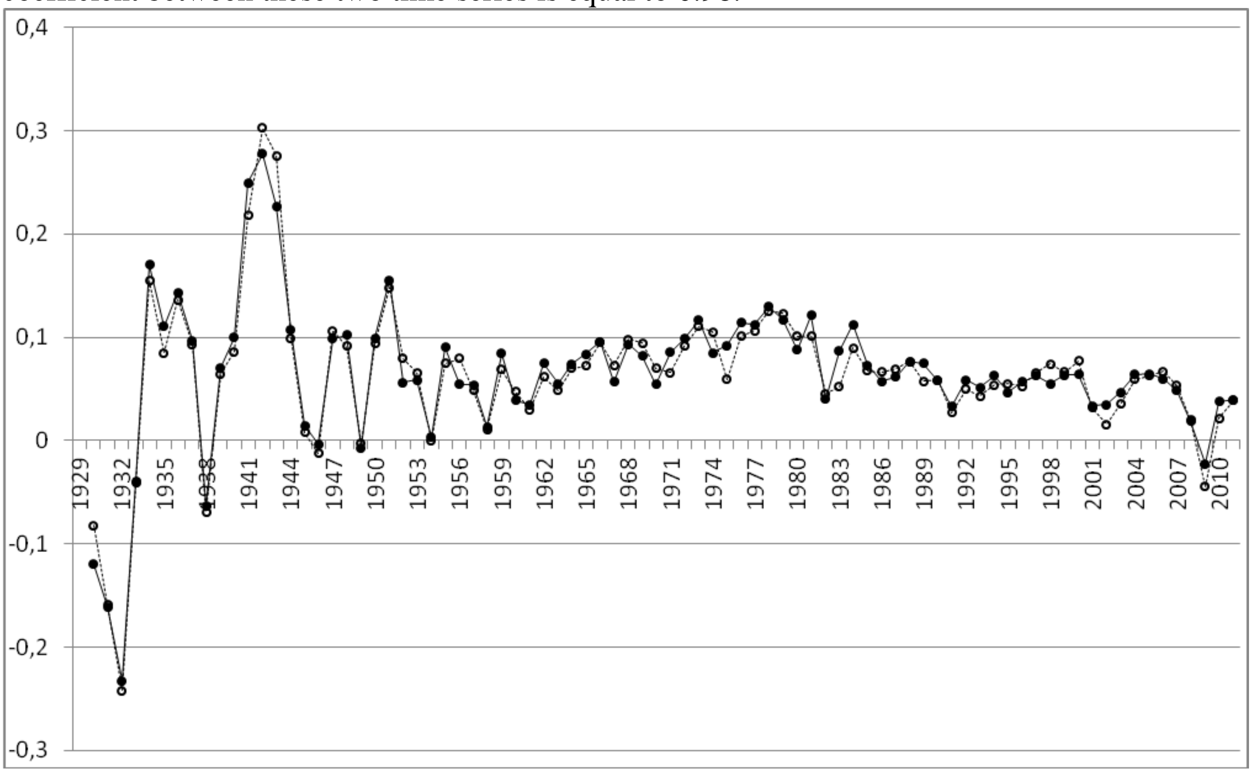

Figure 5 US GDP changes (○ curve) and calculated US GDP changes according to Equation (13) (● curve). See Data Sources (Appendix G)

This demonstrates the practical applicability of the present model despite the limited domain of the monetized production function.

Such a good agreement between the observed GDP changes and the changes that are calculated using the Cobb-Douglas production function some authors (see Shaikh, 1974, Felipe and McCombie, 2010) have explained using the accounting identity. Equation (3) in their work is an analogue of Equation (13) here, and it can be written in terms of the presented study notations as

$$\frac{1}{Y}dY/dt = \frac{(1-s)}{w}dw/dt + \frac{(1-s)}{L}dL/dt + \frac{s}{K}dK/dt + \frac{s}{r}dr/dt \qquad (13a)$$

This equation differs from Equation (13) only in the last term. This term in Equation (13) is very small when the economy is not far away from the equilibrium, and $(K/(wL))/(K/(wL))^{*}\approx 1$. On the contrary, the last term in Equation (13a), s/r (dr/dt), could be extremely large. This is clear because the constancy of the capital income share is required for deriving Equation (13a), and therefore it is more suitable for considering a long-term equilibrium growth path. In the presented paper, the more general case is under consideration when the changes of capital income share in time are permitted. Thus, the resulting monetized production function describes the relationship between output and inputs during short-term deviations from the equilibrium path more precisely than the humbug production function.

3.5 Short-term Business Cycles and Transition Processes

Examination of the changes expected to occur in the wage level w during short-term deviations from the equilibrium growth path trend, using the monetized aggregate production function, shows rather mixed and contradictory results. Consider, e.g., the period of contraction in the business cycle. The rate of profit and the investment rate decline. It would seem that these changes should lead to a growth of wage (see Figure 2, change in investment rate from s^* to s_1). The same interdependence between profit rate and wages demonstrates the neoclassical production function (see Figure 3). But such a behavior of w (growth while a recession occurs and, consequently, labor demand declines) is in conflict both with common sense and with observed statistical data. In this instance, a reduction in output prices can explain the constancy of the wage level observed during a contraction period. Thus, a detailed examination of the short-term processes requires the more complicated model to account for prices and employment. Such a more complicated examination is carried out in Appendix E where an additional assumption of wage inelasticity in the short-term period has been used. The transition process is considered there within the two-sector model of economic growth (similar to the Uzawa 1961b model), the consideration is the same for business cycles. The onset of these processes arises from a primary impetus due to changes in prices and profit rate. If the expansion phase is considered when prices become higher, business makes additional profit and should reinvest it according to the Dynamic Golden Rule. Unexpected changes in prices lead to disequilibrium when the profit maximization condition is violated, $(r + \delta) \neq \partial y(k)/\partial k$. Business aspirations to maximize profits and to reinvest it lead to a sharp increase in employment and in investment goods production when it is impossible to increase the capital stock instantly. An anticipatory growth in labor force (and consequent reduction in the capital intensity) encourages the further growth of demand (mostly for the investment goods) and therefore the development of the expansion cycle phase. The outpacing growth in investment goods production results in an excessive growth in the capital stock, overproduction and in a transformation with time of the cycle phase into the contraction phase. During the latter phase, labor, investment, capital stock, etc., should reduce to a new equilibrium values.

The nature of the primary impetus that causes the cycle onset may vary wildly, from changes in exogenous factors (the population growth rate is considered in Appendix E) to unjustified fluctuations in aggregate demand due to stock market volatility. It is shown in Appendix F that US stock market fluctuations often occur about a year ahead of the changes in aggregate output. It can be assumed that these fluctuations change the level of expectations. The heightened expectations, which households have, lead to the growth in their purchases, which occurs in advance of the business investment[13]. This in turn increases prices, sales, and profits of business.

[13] This proposition is suggested by Gomme, Kydland, and Rupert (2001). They have found two anomalies that have plagued all household production models: the positive correlation between investments by the business and household sectors, and household investment's leading business investment over the business cycle.

The behavior of investment, prices, wages, profit rate, and capital intensity that model predicts during short-term cycles is substantially in agreement with observed data; more detailed consideration of this statement is presented in Appendix E.

4 Discussions and Conclusions

The main tools of the presented model (the equilibrium path, the Golden Rule, and the Cobb–Douglas production function) seem very similar to the neoclassical ones, but it is not the case.

The neoclassical economics envisions the lifetime utility-maximizing consumption decisions of individuals as the driving force of economic activity, with the allocation of given, scarce resources as the fundamental economic problem. Unlike this, in the present study, there is a call for a return to classical political economy vision. There, the profit-making decisions of capitalist business are the driving force (Walsh and Gram, 1980). Similarly, in Marxian (1867) economic theory, the maximization of profits corresponds to the accumulation of capital, which is the driving force behind economic activity within capitalist economic systems, while movable trade-offs between wages and profits are made (Sraffa, 1960).

However, the suggested model and the traditional classical approach, as well as the neoclassical one, show a significant difference. The monetary valuation of capital stock, output, investment, profits, etc., is adopted here (their cost is measured as a book value in accounting). Such an approach removes many of the problems associated with the Cambridge capital controversy (Wicksell effects, reswitching, and capital-reversing): the capital stock value is determined by the value of investments already made, and afterwards it does not change if the rate of profit changes.

On the other hand, the monetary cost valuation is relevant only at a certain time in a certain place that essentially limits the model. However, only this approach makes sense to invoke the main hypothesis of the model, the equality between investment and profits at all times (the Dynamic Golden Rule). This rule is an extension of well-known Phelps's Golden Rule to the general case, which includes deviations from the equilibrium growth path. The Golden Rule can be formulated as follows: spend wages for consumption and profits for investment, and, in this case, the capital income share remains unchanged in the future. Such a statement corresponds to the Bible dictum: "treat people the same way you want them to treat you", which means "do not waste profits that are earned as a return of investment made by the previous generation, and then the next generation will also earn healthy profits".

An analogy can be made to the fact that many of physical laws can be derived from the so-called "common sense", the principle of least action (or Hamilton's principle), for instance. The Golden Rule must be the same "common sense" principal in competitive economics, which maximizes consumption and welfare. Indeed, Golden Rule non-compliance means that sectors borrow from one another. If such non-compliance happens systematically, then one of the sectors falls deeply into debt.

The implementation of the specified Dynamic Golden Rule permits an analytical derivation of a relationship linking output and inputs, which has the Cobb–Douglas form and is termed here "the monetized production function". Certainly, this function is not identical for different economies. Moreover, the function is correct only near the equilibrium growth path. The deviations from this path lead to disequilibrium when the profit maximization condition $(r + \delta) = \partial y\,(k)/\partial k$ is violated, which is one of the conditions in Theorem 1 proof. Nevertheless, Figure 5 shows very good agreement (with the correlation coefficient of 0.98) between the actual time series and the time series calculated using the monetized aggregate production function. This justifies the practical application of this function despite the limitations mentioned above. Anyway, it exhibits a lot of interesting properties.

First, the aggregation problem (because of non-additivity of the Cobb–Douglas function) has been resolved. If the Dynamic Golden Rule is valid, then the Cobb–Douglas production function necessarily exists.

Furthermore, unlike the neoclassical production function, the resulting monetized production function is the function of four variables (K, L, s, and w), that can vary over the time. Their mutual independence is assumed and validated with observed US data sets (see Appendix F). Indeed, the correlation analysis performed shows both the linear independence of variables and their sufficiency for describing output changes.

In this study, the production function shift parameter, which is commonly considered to be determined by the level of technology, could be reasonably expressed in terms of the wage level w. This does not mean that the technological progress does not matter. This means that the technological progress provides an economy with the choice of technology based on the cost and efficiency. Since the level of wage tends to rise in the developed economies, this in turn, promotes the usage of the more and more efficient and expensive machines. Therefore, the labor productivity should also grow, but this process will proceed smoothly in line with the growth in wages. Such a pattern explains a seamless increase in labor productivity despite the fact that the technologies often change in leaps and bounds.

The obtained monetized production function turns out to be path-dependent, in accordance with Robinson's (1953 and 1975) claim, since the value of the coefficient γ accumulates information about the previous history of the economy through averaged g^*- and s^*-values (see Equation (12), Section 3).

The exponent of capital in the derived Cobb–Douglas function is equal to the investment rate s, which can vary with time. Then, the curvature of the monetized production function should be modified together with variations in s. This is an unexpected feature of the function. The function is not already a static relationship between output and factors of production. Using this property, the unidirectional changes in r and s observed both during business cycles and under transition processes can be explained (see Figure 2). Furthermore, the profits are considered to be a source for investment, and the investment rate s is the factor that adjusts the curvature of the production function if the rate of profit r changes during short-term business cycles and transition processes. However, it is difficult to understand the causes and the nature of such processes in terms of the monetized aggregate production function. Therefore, a transition process from one equilibrium path to another is considered in Appendix E under the more complicated two-sector framework. It is shown that any dynamic model must take into account a relative change in prices in different sectors. The meaning of the Uzawa capital-intensity condition is clarified, which states that the equilibrium path is stable when the capital intensity in the consumption goods production sector is greater than in the capital goods production sector. It follows from this condition that for two different equilibrium paths the greater profit rate means the lower aggregate capital intensity, $\partial k^*/\partial r^* < 0$, or the diminishing returns of capital. Short-term transition processes and business cycles are considered in Appendix E by using an additional assumption of wage inelasticity. A non-contradictory description of such cyclic processes is given, which qualitatively agrees with observed data. The intention of business to maximize profits is the driving force of such a process, which moves through disequilibrium when the profit maximization condition is violated.

The presented in this paper model reproduces both short-term economic processes and a long-term output growth along the equilibrium path. The equilibrium growth path is described by Equation (1), which is identical to the Solow steady state definition and to Harrod-Domar's fundamental equation. The relation links three variables (g^*, $(K/Y)^*$ and s^*). In the present work, this equation means only that if two of the three variables are stable, then the third variable tends to the value given by (1). Unlike the neoclassical approach, the variables g^* and $(K/Y)^*$ are exogenous in the present model, while the average value of s^* is considered to be the adjustment factor when reproducing long-term processes. The s^*-value reflects the current consensus between the workers who want to increase their salaries and the proprietors who want to increase profits. Such a trade-off between the income shares of labor and capital would be renegotiated under an alteration along the equilibrium path (when a successfully developing country achieves developed status, e.g., Japan in the early 1970's, see Figure 4). During such a transition process, the labor income share grows, and the capital income share relatively decreases. The changes in the capital income share occur mostly due to the variations in the rate of profit r^* (see Figure 9 in Appendix C). Subsequently, the capital income share reduction under the transition process mentioned above occurs due to a decrease in the rate of profit. There should be a limit (designated by τ) for such a reduction in r^*, below which the investment would seem useless. Information about this limit in the developed economies gives the values of the real interest rates and the growth rates of real output. Those values are surprisingly stable (nearly 2.5 – 3%). Hence, for such economies the exogenous factor g^* can be taken as $g^* = r^* = \tau \approx 2.5 – 3\%$ ($g^* = r^*$ according to the Golden Rule).

With the help of the presented model, it is possible to clarify the reasons for the 2007 – 2008 global recession[14]. It was argued in Section 2.2, that if the Golden Rule is violated in a two-sector economy because of consumption exceeding the labor income ($C^* > w^*L^*$ and $I^* < (r^* + \delta)K^*$), then the growth rate of real output g^* should inevitably decline. Actual data sets demonstrate such a violation of the Golden Rule during the last decades in the economies of US, UK, and Japan (see Appendix C, Figures 6 – 7; and Appendix D, Figure 10). However, the real output growth rate in the US and Great Britain did not decline until 2007. This was made possible because the actually existing economies have more than two sectors; hence, the public sector and the external world should be included into consideration. For such a more complicated review, consumption exceeding labor income is possible not only because of insufficient investment (when $I^* < (r^* + \delta)K^*$), but also because of budget and foreign trade deficits. Thus, the households were stimulated to additionally consume by the additional loans and government subsidies. This additional consumption caused the additional aggregate demand, thus it was managed to maintain temporarily the real output growth rate. The consumption share C/Y increased more than once. The maintaining of the real output growth rate mentioned above entailed a continuous growth in consumption share up to 71% in the US GDP (see Appendix D, Figure 10), almost as big as it was observed in 1929, at the beginning of the Great Depression. Such a large consumption share does not allow the investment rate and government expenditures to have acceptable values without huge budget and foreign trade deficits. Subsequently, the global recession is an inevitable result of such a macroeconomic policy, and the only remedy is a painful reduction in consumption share.

[14] The more detailed examination of reasons for this recession is presented in Appendix D.

Appendix A Appropriateness of Different Technologies Depending on the Level of Wages

The nominal wage level in a given economy is strongly connected with the technologies used in this economy. The high wages necessitate the application of high-performance techniques, even if this is expensive. Low wages fit cheap (and usually low-performance) equipment.

Let two countries with significantly different nominal wage levels to be examined: a "poor" country A where wages, w, are \$5,000 per annum, and a "rich" one B with \$10,000 per annum. The same commodity (one unit sales value of \$1) is produced in both countries, and two possible technologies can be used. The low-performance technology 1 requires equipment worth $K_1 = 20000$ dollars, using which one worker is able to produce $Q_1 = 10\,000$ units of goods.

The high performance equipment for technology 2 costs $K_2 = 100\,000$ dollars and can produce $Q_2 = 20\,000$ units of goods per year employing one worker. Other inputs are free. The markets are globalized; therefore, the same equipment or goods have the same prices in different countries. The rate of profit is calculated as the profit-to-capital ratio, $r = (Q - w)/K$. The results are presented in Table A1:

Table A1 Rate of profit, r, calculations.

	w	Q_1	Q_2	K_1	K_2	r_1	r_2
A	5000	10000	20000	20000	100000	0.25	0.15
B	10000	10000	20000	20000	100000	0	0.10

The results given in the table support the arguments presented in this appendix: Since the proprietors in both economies should maximize the rate of profit, the low-performance and cheap technology 1 will be used in "poor" country A, and the high-performance costly technology 2 in "rich" country B, i.e., the technology used is strongly influenced by the nominal wages.

Appendix B Derivation of Equation (1) Governing the Equilibrium Growth Path

Suppose that the value of the aggregate output Y is growing at a constant rate of g^* starting at time $t = T$: $Y(t)=Y_0\exp(g^*(t-T))$ where Y_0 is the output initial value. Let a stock value of a certain type of capital asset, j, be designated as K_j. Assume that the share of the total output, which is invested in assets of type j ($s_j^* \equiv I_j(t)/Y(t)$), is a constant, then, taking into account depreciation (with the coefficient δ_j) yields

$$K_j(t+\Delta t) = K_j(t)(1-\delta_j) + s_j^* Y(t)\Delta t\,,$$

thus, if $\Delta t \rightarrow 0$

$$dK_j(t)/dt = s_j^* Y(t) - \delta_j K_j(t)$$

Introducing the change of variables $X(t)=K_j(t)/Y(t)$ yields

$$dX/dt = (1/Y)dK_j/dt - (K_j/Y^2)dY/dt = (1/Y)\{s_j^* Y - \delta_j K_j\} - (K_j/Y^2)g^* Y$$

or

$$dX/dt = s_j^* - (g^* + \delta_j)X$$

The solution of the last differential equation is

$$X = s_j^*(g^* + \delta_j) + \text{constant} \times \exp(-(t-T)\times(g^* + \delta_j))$$

For large $(t-T)$-values (when $(g^*+\delta_j)(t-T) >> 1$), given that the second term on the right-hand side can be neglected, the ratio K_j/Y asymptotically reduces to

$$(K_j/Y)^* = s_j^* / (g^* + \delta_j) \qquad \text{(B.1)}$$

If predictable price inflation is running at a constant rate, then stocks revaluation should be considered:

$$dK_j(t)/dt = s_j^* Y(t) - \delta_j K_j(t) + i_j K_j(t)$$

where K_j and Y are currently cost valuated, and i_j is the inflation rate of asset j.
Similar to deriving Equation (B.1), denoting $X=K_j/Y$ yields:

$$dX/dt = (1/Y)dK_j/dt - (K_j/Y^2)dY/dt = (1/Y)\{s_j^* Y - (\delta_j - i_j)K_j\} - (K_j/Y^2)(g^* + i)Y$$

where g^* is the growth rate of the real output, and i is the inflation rate of output, whence

$$dX/dt = s_j^* - (g^* + \delta_j + i - i_j)X\,,$$

and

$$X = s_j^*(g^* + \delta_j + i - i_j) + \text{constant} \times \exp(-t(g^* + \delta_j + i - i_j))$$

Thus, the ratio K_j/Y asymptotically reduces to

$$(K_j/Y)^* = s_j^* / \{(g^* + \delta_j) + (i - i_j)\} \qquad \text{(B.2)}$$

If the output and capital inflation rates are equal ($i_j = i$), then Equation (B.2) reduces to Equation (B.1), where Y and K_j are measured in terms of the current cost valuation, and g^* is the growth rate of real output, similar to that in Equation (B.1).

The derivation of those equations is based only on the assumption that the values of s_j^*, i_j, i, δ_j, and g^* are stable. This is actually the case, at least for the developed economies. The growth rates of the real output g^* and the investment rate s_j^* are considered as stable exogenous factors, whereas the corresponding value of $(K_j/Y)^*$ is obtained as a result. It can be easily shown that as exogenous factors can serve any two of the three variables. Let us prove the next proposition, viz., if g^* and $(K_j/Y)^*$ are exogenous, then s_j^* adjusts to them:

Proposition: If g^* and $(K_j/Y)^*$ are constant with time, and when $t \geq T$, $Y(t)=Y_0\exp(g^*(t-T))$, then the s_j^*-value should be determined from Equation (B.1). For simplicity, the case without inflation is under consideration.

Proof: Indeed, if the growth rate of output is constant, as well as the $(K_j/Y)^*$ ratio is, then K_j should grow at the same rate g^*, i.e., $K_j(t) = K_{j0}\exp(g^*(t-T))$.[15] Therefore, $dK_j/dt = g^*K_j$ and the equation

$$dK_j(t)/dt = s_j^*Y(t) - \delta_j K_j(t)$$

reduces to

$$g^*K_j(t) = s_j^*Y(t) - \delta_j K_j(t)$$

and then the relation (B.1) holds

$$(K_j/Y)^* = s_j^*/(g^* + \delta_j) \ \blacksquare$$

In Table B1, the calculated and actual $(K_j/Y)^*$ values are compared for a few types of capital assets in the US economy. The average values of the variables are taken as equilibrium ones in Equation B.2. The averaging is performed over the 1949 – 2011 period. A good agreement between the calculated and actual values of $(K_j/Y)^*$ in Table B1 suggests the possibility of such using in applied researches.

Table B1 Actual and calculated $(K_j/Y)^*$ values from Equation (B.2).

j	Type of asset	Averaged over 1949-2011 values						$(K_j/Y)^*$ calculated
		s_j	i_j	δ_j	g	i	(K_j/Y) fact	
1	Durable goods	0.088	0.012	0.243	0.032	0.033	0.339	0.298
2	Residential fixed capital	0.046	0.037	0.015	0.032	0.033	1.100	1.082
3	Total capital	0.192	0.028	0.039	0.032	0.033	2.896	2.504

[15] The proposition about the equality of the growth rates of output and the aggregate stock of capital is proved in the framework of the Uzawa theorem (1961a). This theorem was reverted to later, for example, by Schlicht (2006), Jones and Scrimgeour (2006). Acemoglu (2008) considered more strong conditions, when the growth rates are not constants, but tend to constants.

Appendix C Profits and Investment

Figures 6 – 8 are for non-financial corporate business in UK, Japan, and the U.S. The graphs represent the time series of the investment rate and the capital income (profits) share.

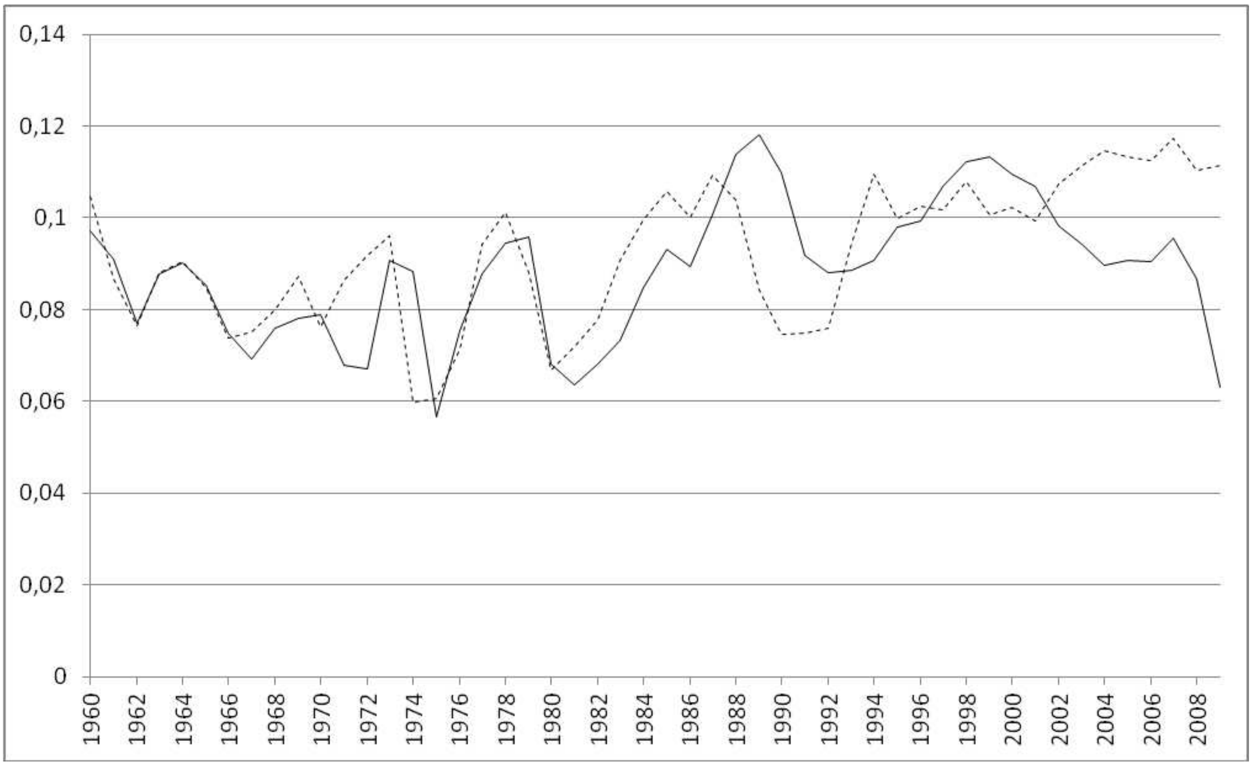

Figure 6 Great Britain, Non-Finance Corporate Business, investment-to-GDP ratio or investment rate (solid curve) and profits-to-GDP ratio, or capital income share (dashed curve)

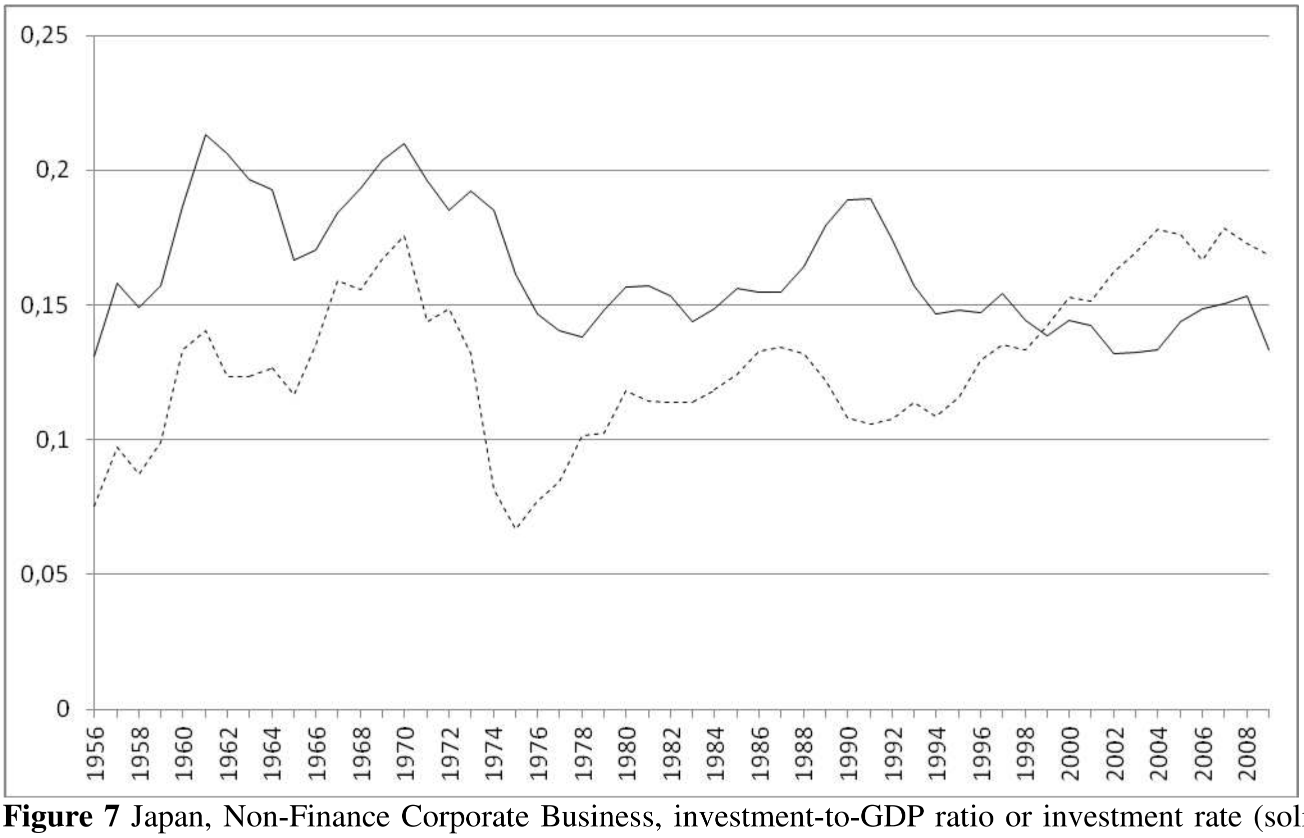

Figure 7 Japan, Non-Finance Corporate Business, investment-to-GDP ratio or investment rate (solid curve) and profits-to-GDP ratio, or capital income share (dashed curve)

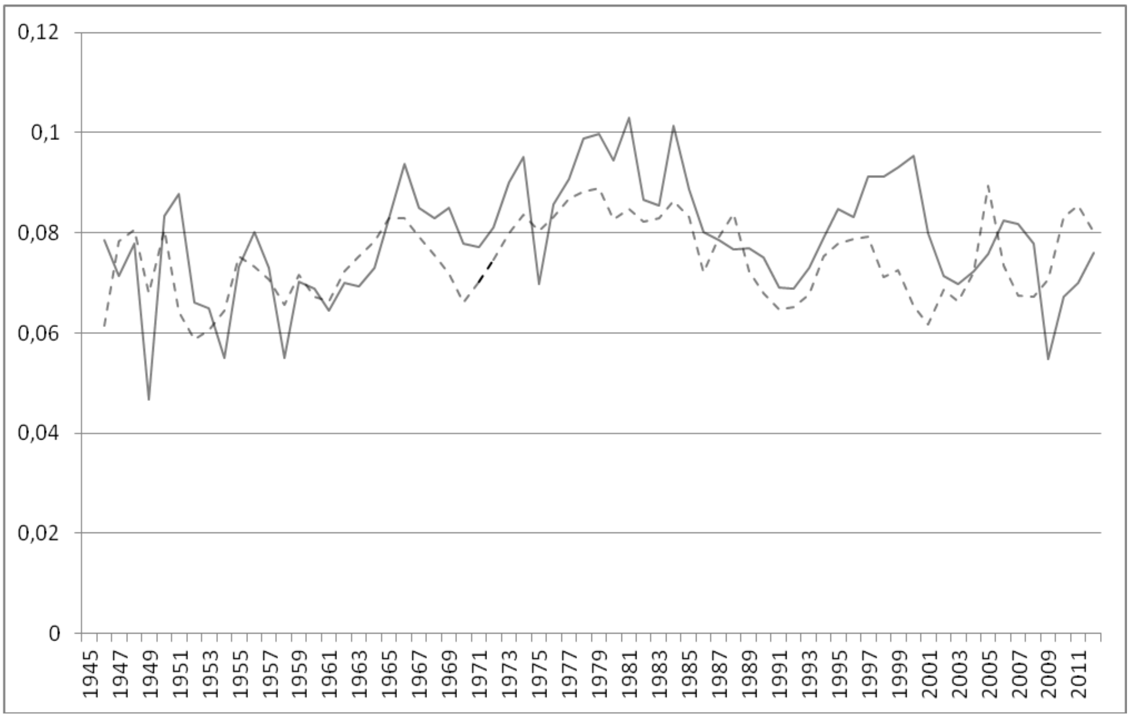

Figure 8 USA, Non-Finance Corporate Business, investment-to-GDP ratio or investment rate (solid curve) and profits-to-GDP ratio, or capital income share (dashed curve)

The agreement between profits and investment observed in Figures 6 – 8 supports the assumption (2), Dynamic Golden Rule, or a permanent equality between profits and investment. Such an agreement is equivalent to the unidirectionality of changes in the investment rate *s* and the profit rate *r*, since the changes in profits during short-term period of time are largely due to changes in *r* (Figure 9).

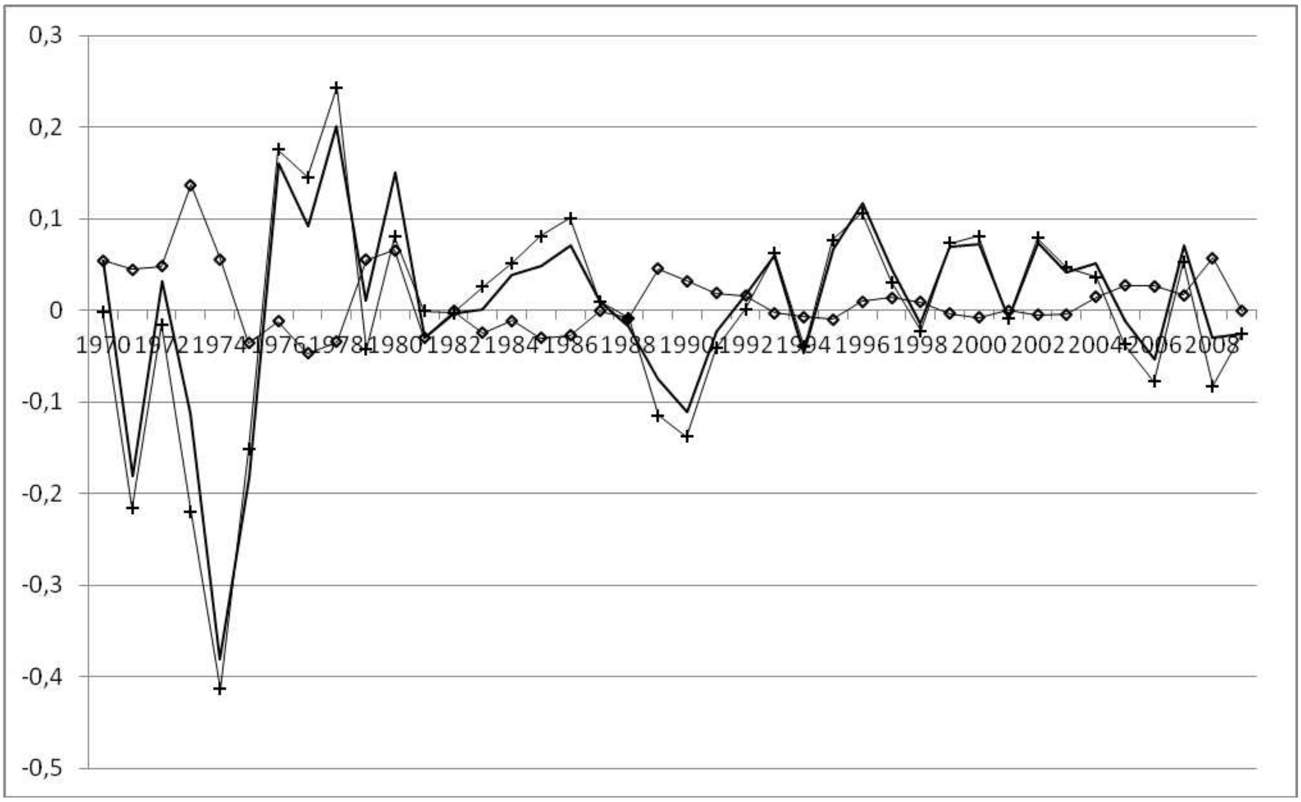

Figure 9 Japan. Non-financial Incorporated Enterprises. Changes in the capital income share $(r+\delta)\times K/Y$ (the curve without markers), and in its two reasons, the rate of profit $(r+\delta)$ (+ curve) and capital-to-output ratio K/Y (◇ curve)

Figure 9 shows that the changes in the rate of profit (not in *K/Y*) are the main cause of profits fluctuations.

Figures 6 – 8 show a certain, approximately one year, advance in time of the profits share with respect to the investment rate. This observation supports the assertion that profits are the reason for and source of the investment rate. The violation of the balance between the profits and the investment during the last decades observed in Figures 6 – 8 indicates non-compliance with the Golden Rule, which is examined at the end of the conclusion and in Appendix E as the cause of the 2007 – 2008 global recession.

Appendix D The Reasons for 2007 – 2008 Global Recession

Changes have taken place during last decades in the macroeconomic policy of many developed countries, of which the U.S.A., the UK, and Japan are under consideration here. Their aggregate output growth was stimulated by an increase in aggregate demand, which in turn was encouraged by a rise in household loans and in government social benefits to persons. Consequently, an anticipatory growth in household consumption occurred, and as a result, the Golden Rule was violated. That is why this time period is termed "abnormal" in Table D1. The consumption becomes greater than the labor income and the profits greater than the investment (see Figures 6 – 7 in Appendix C and Figure 10 in Appendix D). In Section 2.2, it has been shown that the output growth rate should decline, $g^* < r^*$, in the case of the two-sector economy.

In reality, an economy includes more than two sectors. The government and the external world must to be added. Hence, the growth in household consumption share can occur not only by reducing business investment rate, but can also be due to budget and foreign trade deficits. That is why the output growth rate did not collapsed in the US economy up to 2007.

Actually, the US household consumption share (*C*/*Y*) continued to grow with respect to the labor income share during the last three decades (see Figure 10). On the one hand, such an additional increase in consumption maintains the pace of aggregate output (GDP) growth during abnormal period of time. On the other hand, the observed continuous growth of consumption share requires adequate funding sources. Here, the following three of the sources are under consideration: the funds of households, the government, and the external world.

During the abnormal period, the additional household loans led to reducing the households' net lending and, at the same time, provided funds for additional consumption. This additional consumption maintained the output growth rate. Such an interrelation between households' net lending and output growth is confirmed by statistical data. Table D1 shows the reverse in the sign of the correlation between the GDP-weighted changes in the households' net lending and in the output, consumption, and labor income. The correlation coefficients for the abnormal period change their sign to negative. Indeed, this could mean that the greater additional household loans lead to the greater additional consumption and therefore to the greater additional output. However, the households' lending is a too small source for funding the continuous growth of the consumption share. Figure 11 shows that the growth of this share is accompanied by an increase in the government social contributions and in the foreign trade deficit. The sum of changes in the government social contributions and in the foreign trade deficit is close to the difference between the consumption and the labor income. This observation confirms the suggestion that the budget and foreign trade deficits are the funding sources for the continuous consumption share growth. However, these sources cannot grow without limit, as well as the consumption share cannot do, and the retribution (observed global recession) was inevitable, when the *C*/*Y* ratio had nearly attained the level that was before the Great Depression in 1929 (71% for the U.S.A.). Consequently, the non-compliance with the Golden Rule during the abnormal time period can be regarded as the cause of the last global recession. At such a large level of consumption share it becomes impossible to have a reasonable investment rate and government spending (15% and 20% of the GDP averaged for the U.S.A.) without a huge budget or/and trade deficit. But that is unacceptable during a long time interval. To remedy the economy, a painful medication must be prescribed, a reduction in the consumption share to a level consistent with the labor income share.

Table D1 Correlation coefficients between variables changes relative to GDP

UK

Variable	**Normal period**	**Abnormal period**
	1949 – 1987	**1988 – 2009**
Households saving		
GDP	0.455	–0.316
Consumption	0.396	–0.269
Labor income change	0.579	–0.020

JAPAN

Variable	**Normal period**	**Abnormal period**
	1956–1991	**1992–2009**
Households net lending		
GDP	0.374	–0.352
Consumption	0.313	–0.572
Labor income	0.451	–0.239

U.S.A.

Variable	**Normal period**	**Abnormal period**
	1947 – 1985	**1986 – 2008**
Households saving-investment		
GDP	0.098	–0.552
Consumption	–0.153	–0.575
Labor income	0.179	–0.287

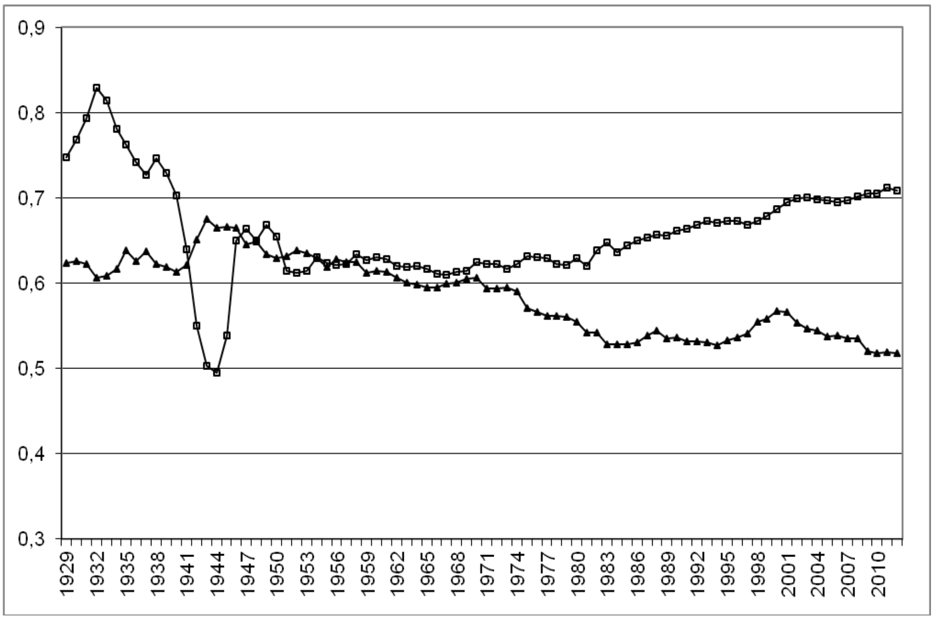

Figure 10 U.S.A.: Consumption share (□ curve) and labor income share (▲ curve)

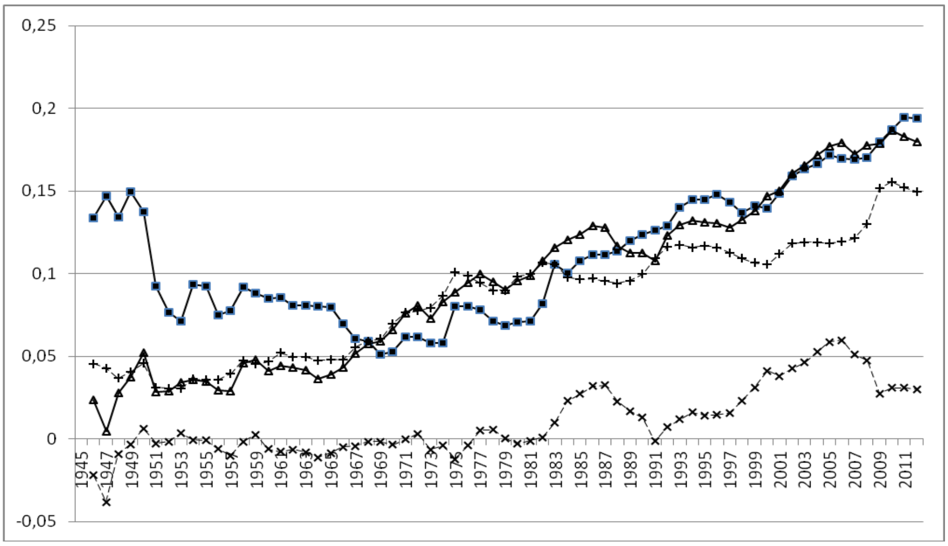

Figure 11 U.S. economy: The ratio of the rest of the world net lending to GDP (× curve). The ratio of the government social contribution to GDP (+ curve). The ×- and +-curve sum (Δ curve). The difference between the consumption share and the labor income share $(C - wL)/Y$ (■ curve)

Appendix E A Transition Process from one Equilibrium Path to Another. Business Cycles

E.1 Comparison of Different Equilibrium Growth Paths. The Meaning of the Uzawa Capital-Intensity Condition

In Appendix B, it was shown that the equilibrium growth path is determined by two exogenous factors, whose average values are quite stable. If at least one of these factors changes its value, then the equilibrium growth path should be changed with time as well. The changes in wages, prices, and the allocation of the labor between the different sectors are examined here within a two-sector model of economic growth similar to that of Uzawa (1961b) and (1963).

It is shown in the present Appendix E that the price of investment goods should inevitably change with respect to the price of consumer goods, and such price variations must be considered in any dynamic model taking into account more than one sector. Then, the aggregate production function cannot be the same for different equilibrium paths.

Let two commodities, consumption goods and investment goods (machines), to be produced. Both these goods are produced by means of capital and labor. The capital stock, $\tilde{K}$, the consumption, $\tilde{C}$, the output of investment and consumption goods, $\tilde{Y}_i, \tilde{Y}_c$, are expressed in physical units, as well as the wage, $\tilde{w}$, and the aggregate output, $\tilde{Y}$ (the last two are in units of consumption goods). The variables expressed in physical units are marked with a tilde. Let us write the obvious equations for output by sector, denoting $q \equiv \tilde{Y}_i / \tilde{K}_i$, $q_c \equiv \tilde{Y}_c / \tilde{K}_c$, $\tilde{k}_i \equiv \tilde{K}_i / L_i$, $\tilde{k}_c \equiv \tilde{K}_c / L_c$. The subscript *i* refers to the investment-goods sector, and the subscript *c* to the consumption-goods sector.

$$\tilde{Y}_c = \tilde{w}L_c + rp\tilde{K}_c,$$
$$p\tilde{Y}_i = \tilde{w}L_i + rp\tilde{K}_i$$

Where $p=P_i/P_c$ is the price of capital goods in units of consumer goods, *r* is the profit rate. Without loss of generality, neglecting the depreciation yields

$$\tilde{Y}_c / \tilde{K}_c = \tilde{w} /(\tilde{K}_c / L_c) + rp$$
$$p\tilde{Y}_i / \tilde{K}_i = \tilde{w} /(\tilde{K}_i / L_i) + rp$$

or

$$q_c = \tilde{w} / \tilde{k}_c + rp \tag{E.1}$$
$$pq_i = \tilde{w} / \tilde{k}_i + rp \tag{E.2}$$

The values of $\tilde{w}$ and *p* can be calculated as the functions of *r* using the solution to the linear system which involves Equations (E.1) and (E.2):

$$\tilde{w} = q_c \tilde{k}_c \tilde{k}_i (q_i - r) /(q_i \tilde{k}_i + r(\tilde{k}_c - \tilde{k}_i)) \tag{E.3}$$
$$p = q_c \tilde{k}_c /(q_i \tilde{k}_i + r(\tilde{k}_c - \tilde{k}_i)) \tag{E.4}$$

Assume that each sector has a unique exogenously given and immutable production technology. Consequently, each sector has the technologically specified number of output (consumption or investment goods) that are produced by a single machine per time unit. This number is given and invariable for each sector, as well as the number of employees needed to service one machine. This means that in an equilibrium state the capital performance (output-to-capital ratio) and capital intensity values are equal to the technologically specified ones, $q_c = q_c^{**}$, $q_i = q_i^{**}$, $\tilde{k}_c = \tilde{k}_c^{**}$, $\tilde{k}_i = \tilde{k}_i^{**}$. These quantities are invariable for the different equilibrium states (equilibrium paths), and they are designated by two asterisks (equilibrium values that can be different for different equilibrium paths are designated by one asterisk). If the exogenous profit rate r^* is also stable, then the wage $\tilde{w}^*$ and price p^* are determined from Equations (E.3) and (E.4) and are unchangeable as well. In that case, both exogenous factors are assigned, and the equilibrium growth without technological progress takes place. The transition from one equilibrium path to another must be accompanied by changes in the profit rate occurring due to different reasons, one of which is addressed at the end of this subsection. The wage and prices should also acquire the new equilibrium values along the new equilibrium path with a new profit rate, $\tilde{w}^* = \tilde{w}^*(r^*)$ and $p^* = p^*(r^*)$, then Equations (E.3) and (E.4) can be rewritten for equilibrium values as:

$$\tilde{w}^* = q_c^{**}\tilde{k}_c^{**}\tilde{k}_i^{**}(q_i^{**} - r^*)/(q_i^{**}\tilde{k}_i^{**} + r^*(\tilde{k}_c^{**} - \tilde{k}_i^{**})) \tag{E.3*}$$

$$p^* = q_c^{**}\tilde{k}_c^{**}/(q_i^{**}\tilde{k}_i^{**} + r^*(\tilde{k}_c^{**} - \tilde{k}_i^{**})) \tag{E.4*}$$

The derivatives of $\tilde{w}^*(r^*)$ and $p^*(r^*)$ can be calculated as:

$$\partial\tilde{w}^*/\partial r^* = -q_c^{**}q_i^{**}\tilde{k}_i^{**}(\tilde{k}_c^{**})^2/(q_i^{**}\tilde{k}_i^{**} + r^*(\tilde{k}_c^{**} - \tilde{k}_i^{**}))^2 < 0 \tag{E.5}$$

$$\partial p^*/\partial r^* = -q_c^{**}\tilde{k}_c^{**}(\tilde{k}_c^{**} - \tilde{k}_i^{**})/(q_i^{**}\tilde{k}_i^{**} + r^*(\tilde{k}_c^{**} - \tilde{k}_i^{**}))^2 \tag{E.6}$$

It follows from Equation (E.5) that $\partial\tilde{w}^*/\partial r^*$ is always less than zero, which means an understandable and unceasing fight between wages and profits.

Equation (E6) reveals that $\partial p^*/\partial r^* = 0$ only when $\tilde{k}_i^{**} = \tilde{k}_c^{**}$. This means that any dynamic more-than-one sector model must account for inevitable changes in prices (except for the special case when the capital intensities in different sectors are equal that corresponds to the one-commodity model). This statement makes it difficult to use the invariable neoclassical aggregate production function in such models.

Note that the model will be closed only if the demand for outputs is specified. Let us follow the Uzawa (1961b) classical hypothesis when all labor income is consumed and all profits are invested. This hypothesis exactly corresponds to the formulation of the Golden Rule of capital accumulation in this paper. Then

$$\tilde{Y}_c = \tilde{C} = \tilde{w}L$$

$$\tilde{Y}_c/\tilde{K}_c = \tilde{w}/[(\tilde{K}_c/L_c)(L_c/L)]$$

Denoting $\lambda_c \equiv L_c/L$ and $\lambda_i \equiv L_i/L$, and using Equations (E.3) and (E.4) yields

$$\lambda_c = \tilde{w}/(q_c\tilde{k}_c) = \tilde{k}_i(q_i - r)/(\tilde{k}_i q_i + r(\tilde{k}_c - \tilde{k}_i))$$

$$\lambda_i = 1 - \lambda_c = r\tilde{k}_c/(\tilde{k}_i q_i + r(\tilde{k}_c - \tilde{k}_i)) \tag{E.8}$$

According with Equation (E.5) $\partial\tilde{w}^*/\partial r^* < 0$, then

$$\partial\lambda_c^*/\partial r^* < 0$$

and

$$\partial\lambda_i^*/\partial r^* = \tilde{k}_c^{**}\tilde{k}_i^{**}q_i^{**}/[\tilde{k}_i^{**}q_i^{**} + r^*(\tilde{k}_c^{**} - \tilde{k}_i^{**})]^2 > 0 \tag{E.9}$$

The aggregate capital intensity $\tilde{k}$ in terms of consumer goods is equal to

$$\tilde{k} = \lambda_i \tilde{k}_i + \lambda_c \tilde{k}_c = q_i \tilde{k}_c \tilde{k}_i / (\tilde{k}_i q_i + r(\tilde{k}_c - \tilde{k}_i))$$

(see Equations (E.7) and (E.8)), then

$$\partial \tilde{k}^* / \partial r^* = -q_i^{**} \tilde{k}_c^{**} \tilde{k}_i^{**} (\tilde{k}_c^{**} - \tilde{k}_i^{**}) / (\tilde{k}_i^{**} q_i^{**} + r^* (\tilde{k}_c^{**} - \tilde{k}_i^{**}))^2 \quad \text{(E.10)}$$

The following two different cases are possible:

$$\text{(a) } \partial \tilde{k}^* / \partial r^* < 0 \text{ if } \tilde{k}_i^{**} < \tilde{k}_c^{**}$$
$$\text{(b) } \partial \tilde{k}^* / \partial r^* > 0 \text{ if } \tilde{k}_i^{**} > \tilde{k}_c^{**}.$$

The inequality $\tilde{k}_i^{**} < \tilde{k}_c^{**}$ in case (a), when the consumer goods sector is more capital intensive then the investment goods sector, coincides precisely with the Uzawa(1961b and 1963) capital-intensity condition, or the condition of the equilibrium path stability. Only case (a) ensures the desired diminishing returns of aggregate capital $\partial \tilde{k}^* / \partial r^* < 0$, while case (b) when $\partial \tilde{k}^* / \partial r^* > 0$ looks much less attractive, and the equilibrium position really is not stable in this case. This circumstance clarifies the sense of the aforementioned Uzawa condition.

Consider an interesting special case of the transition from one equilibrium path to another. Suppose that the population and labor growth rate n changes without any variations in technology that results in an alteration in the profit rate ($r = n$ along the equilibrium path if the classical hypothesis is valid[16]). The faster growing labor requires the greater investment nk to ensure sufficient capital. Then, the labor share λ_i^* in the investment goods sector should grow (see inequality (E.8)). The lesser labor in the consumer goods sector would produce the lesser consumption goods per capita. This means a lower wage in terms of the consumer goods, and therefore the lower consumption (see Equation (E.5)). The result is that the rapid population growth is "disadvantageous" in terms of consumption maximization. Perhaps the intuitive understanding of this circumstance is the cause of the low rate of population growth in developed countries.

E.2 The Transition Process Between Two Equilibrium Paths. The Inelasticity of Wage During a Short-term Period. Business Cycles.

In the end of preceding subsection, the example of a transition process from one equilibrium growth path to another has been considered when the exogenous growth rates of population and labor had been accelerated, and whereby the equilibrium profit rate increased. Comparison of both paths has been made, the new equilibrium has been characterized by a relative increase in the labor λ_i^* in the sector producing investment goods and by a decrease in the wages $\tilde{w}^*$ and in the investment goods price p^* in terms of consumer goods (see Equations (E.5), (E.6), and (E.9)). In present subsection, the short-term process of transition from one equilibrium path to another for this special case is under theoretical consideration directly. Additional assumption is added, the change in the population growth rate is considered to be unexpected. Although this assumption looks implausibly it makes the hypothetical transition process under consideration similar to that of the formation of business cycles. Both processes represent cyclical fluctuations in output and other variables that occur after an unexpected initial external impetus. The resulting theoretical description of this process corresponds well with the observed features of the cyclical fluctuations, first of all with a relatively large investment and labor deviations from the current trend.

Two assumptions are used here, the Dynamic Golden Rule and the inelasticity of the monetary wage level in short-term processes. The former statement is justified in Sections 2.2 and 2.3, and the latter statement is justified in Section 3.1 and in Appendix F. As far as the current monetary wage level

[16] Really, the equilibrium growth without technological progress means that the capital intensity is equal to the equilibrium value and is invariable: $k = k^*$, $dk/dt = 0$. Then $I - \delta K + nK = 0$. The classical hypothesis means that $I = (r + \delta)K$, and hence $r = n$.

w is considered to be a constant during short-term period, this value is more convenient to use as the cost unit instead of the consumer good price P_c. The output in both sectors can be written as:

$$P_c\widetilde{Y}_c = wL_c + rP_i\widetilde{K}_c$$
$$P_i\widetilde{Y}_i = wL_i + rP_i\widetilde{K}_i$$

Then the linear system involving equations (E.1) and (E.2) can be rewritten as:

$$P_c q_c = w / \widetilde{k}_c + rP_i \tag{E.11}$$
$$P_i q_i = w / \widetilde{k}_i + rP_i \tag{E.12}$$

The economy is characterized by discrete-time values, so that the time is indexed by $t = 0, 1, 2, \ldots$. The time periods between two consecutive instants of time here correspond to the duration of manufacturing and commissioning of the investment goods, so the investment goods produced during period m is to be used since period $(m + 1)$ when the capital stock increases. Along the equilibrium path, the growth rate of output, capital stock, and other variables is predictable, which makes it possible to comply with the equality between investment and profits that are planned ahead of time (Golden Rule). If the profits change unexpectedly, as is the case in the considered hypothetical transition process, the investment can respond only in the next period of time, and therefore it lags behind the profits by one time period. Thus, we assume that the Dynamic Golden Rule holds during the transition process in a special way with a slight shift in time equal to one period. Such a delay is actually observed (see Figures 6 – 8 in Appendix C).

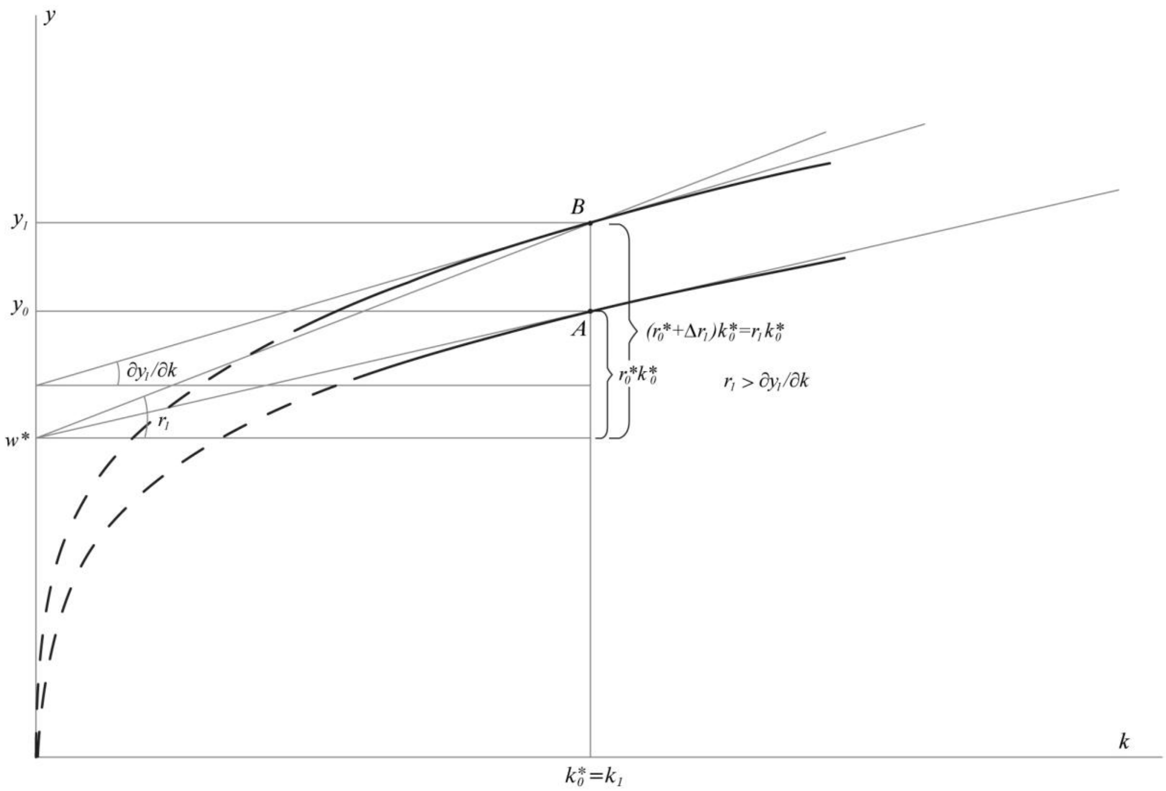

Figure 12 Short-term transition process, 1st time period. The position of the point that characterizes the aggregate economy moves from the point A to the point B

At the beginning of the process theoretically considered here (during the first period), the hypothetical unexpected additional increase in population causes additional demand for consumer goods. Its price increase is the consequence of this since the output in physical units can not grow instantaneously. While the price is going up, the position of the point that characterizes the aggregate economy in the (y, k) plane in Figure 12 (where the output and the capital stock are measured not in physical units, but in the current monetary cost) moves from the point A to the point B. The new position is not an equilibrium because the marginal product of capital (the profit rate from new investment) is less than the profit rate r_1 (which is calculated as the total profit divided by the total capital stock), $\partial y_1/\partial k < r_1 = (y_1 - w^*)/k_1$. The reason for this disequilibrium is due to the fact that the new capital is debited at an increased price ($P_{i,t=1}=P_{i,1}$ during the 1st period, see Equation (E.13) below) whereas the revaluation of the old capital stock is not made, $k_{t=1}= k_1= k_0^*$, where k_0^* is the monetary aggregate capital intensity along the primary equilibrium growth path, $k_0^* = k^*(r_0^*) = \widetilde{k}^*(r_0^*)P_i^*(r_0^*)$, where the equilibrium profit rate is $r^*=r_0^*$, and the investment goods price is $P_i^*(r_0^*)$. This is clearly seen in Figure 12 where the slope of the tangent to the production function at point B is less than the slope of the line segment connecting the point $(0, w^*)$ and the point B.

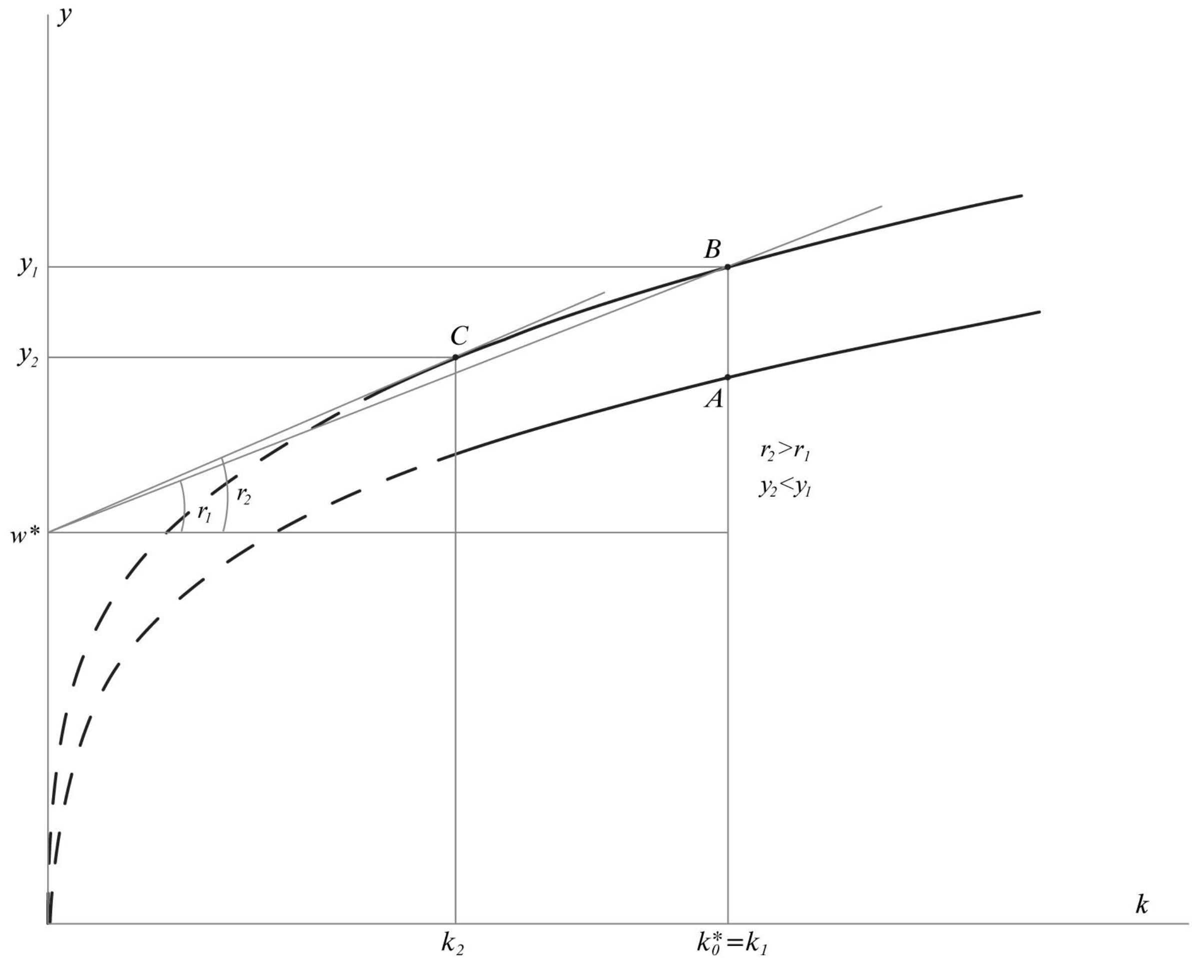

Figure 13 Short-term transition process, 2nd time period. The position of the point that characterizes the aggregate economy moves from the point B to the point C

A growth in consumer demand during the 1st period requires the greater output in physical units during the 2nd period. During the latter period, the capital stock has not yet changed; therefore, the output can be increased only by involving additional labor. As a consequence, the current value of capital intensity decline relative to the technologically specified initial one, $k_2 < k_0^*$. Then, the position of the point that characterizes the aggregate economy in the (y, k) plane moves from B to C (see Figure 13). While the capital intensity reduces, the total labor productivity y also drops, and both the marginal product of capital and the profit rate grow[17], $\partial y_2/\partial k > \partial y_1/\partial k$, $r_2 > r_1$. In such a way, business seeks to improve the relatively low marginal product of capital at the point B and to maximize profits. The

[17] The last assertions imply diminishing returns of aggregate capital, which occur in compliance with the Uzawa capital intensity condition, see Equation (E.10).

numerical estimates presented below show that the investment goods sector exhibits an advance growth in output and labor in this process due to the Dynamic Golden Rule validity.

Along the primary equilibrium growth path (during the zero period of time), the investment goods price $P_i^*(r_0^*)$ is related to the corresponding value of profit rate r_0^* by the following equation:

$$P_i^*(r_0^*)q_i^{**} = w/\tilde{k}_i^{**} + r_0^* P_i^*(r_0^*),$$

which is similar to Equation (E.12).

Suppose that during the 1st period the profit rate grows from the equilibrium value r_0^* to the value of $r_1 = r_0^* + \Delta r_1$ due to the prices growing, while the values of labor productivity and capital intensity in physical units for the investment goods sector do not change and are equal to the equilibrium values: $\tilde{y}_{i1} = \tilde{Y}_{i1}/L_1 = \tilde{Y}_{i0}/L_0 = \tilde{y}_i^*(r_0^*)$ and $\tilde{k}_{i1} = \tilde{k}_{i0} = \tilde{k}_i^{**}$, respectively. The equilibrium values of output $\tilde{Y}_i^*(t) = \tilde{Y}_{it}^*$ and capital stock $\tilde{K}_i^*(t) = \tilde{K}_{it}^*$ grow with time due to the growth of the population and labor, $L^*(t)=L_t^*$ but their values per labor unit are constant along the equilibrium path. The capital stock must not be revaluated during short-term processes due to the accounting principle of prudence, and then the price of investment goods during the 1st period $P_{i,t=1}=P_{i,1}$ can be determined from the equation

$$P_{i1}q_i^{**} = w/\tilde{k}_i^{**} + (r_0^* + \Delta r_1)P_i^*(r_0^*), \text{ then}$$
$$P_{i1} = P_i^*(r_0^*)(1 + \Delta r_1/q_i^{**})$$

or

$$(P_{i1} - P_i^*(r_0^*))/P_i^*(r_0^*) = \Delta P_{i1}/P_i^*(r_0^*) = \Delta r_1/q_i^{**} \tag{E.13}$$

During the zero time period, the profits are equal to $r_0^* k_0^*$ (see Figure 12). During the 1st period they are equal to $(r_0^*+\Delta r_1)k_0^*$. Consequently, the profits during this period grow proportionally to the growth of the profit rate, i.e., $(1+ \Delta r_1/ r_0^*)$ times the initially expected equilibrium value. Hence, according to the Dynamic Golden Rule, the monetary investment Y_{i2} rises in the 2nd period in the same proportion relative to its equilibrium value,

$$Y_{i2}/Y_{i2}^* = (1 + \Delta r_1/r_0^*)$$

or

$$(Y_{i2} - Y_{i2}^*)/Y_{i2}^* = \Delta Y_{i2}/Y_{i2}^* = \Delta r_1/r_0^* \tag{E.14}$$

where Y_{i2}^* is the initially expected equilibrium output of investment goods over the 2nd period, which corresponds to the primary equilibrium growth path,

$$Y_{i2}^* = P_i^*(r_0^*)\tilde{Y}_{i2}^* = P_i^*(r_0^*)q_i^{**}\tilde{k}_i^{**}L_{i2}^* = P_i^*(r_0^*)q_i^{**}\tilde{k}_i^{**}\lambda_i^*(r_0^*)N_2\beta^{**},$$

where β^{**} is the equilibrium employment, N_2 is the population for the 2nd period, and the equilibrium labor value is $L_2^*=N_2\beta^{**}$. Since the output of investment goods in physical units over the 2nd period is planned in advance, therefore their price in the 1st period should be used for this purpose, $Y_{i2} = P_{i1}\tilde{Y}_{i2}$. Then, taking into account the corresponding price growth during the 1st period (see Equation (E.13)), the ratio of the investments during 2nd period in physical units to those initially expected along the primary equilibrium path is equal to

$$\Delta\tilde{Y}_{i2}/\tilde{Y}_{i2}^* = \Delta(Y_{i2}/P_{i2})/(Y_{i2}^*/P_i^*(r_0^*)) = \Delta Y_{i2}/Y_{i2}^* - \Delta P_{i2}/P_i^*(r_0^*) = \Delta r_1/r_0^* - \Delta r_1/q_i^{**},$$

or

$$\Delta\widetilde{Y}_{i2} / \widetilde{Y}_{i2}^{*} = (\Delta r_1 / r_0^{*})(1 - r_0^{*} / q_i^{**}) \tag{E.15}$$

Dividing the numerator and the denominator by the equilibrium labor value $L_2^{*}=N_2\beta^{**}$ and denoting $\widetilde{y}_{it} \equiv \widetilde{Y}_{it} /(N_t\beta^{**})$, Equation (E.15) may be rewritten in the intensive form

$$\Delta\widetilde{y}_{i2} / \widetilde{y}_i^{*}(r_0^{*}) = [\Delta\widetilde{Y}_{i2} /(N_2\beta^{**})] / \widetilde{y}_i^{*}(r_0^{*}) = (\Delta r_1 / r_0^{*})(1 - r_0^{*} / q_i^{**}) \tag{E.15i}$$

The capital stock during the 2nd period is at the initial equilibrium level, thereby the investment goods output growth in this period being possible only due to the growth of labor, and wherein the capital intensity $\widetilde{k}_{i2}$ decreases. Obviously, the labor productivity does not drop with the growth of capital intensity, $d(\widetilde{Y}_i / L_i) / d\widetilde{k}_i \geq 0$, consequently $\Delta(\widetilde{Y}_i / L_i) \leq 0$ due to a reduction in $\widetilde{k}_{i2}$ for the case under consideration. Also, the labor force growth in the investment sector is not less than the physical output growth in this sector,

$$\Delta\widetilde{Y}_i / \widetilde{Y}_i^{*} = \Delta((\widetilde{Y}_i / L_i)L_i) / \widetilde{Y}_i^{*} = \Delta(\widetilde{Y}_i / L_i) /(\widetilde{Y}_i^{*} / L_i^{*}) + \Delta L_i / L_i^{*} \leq \Delta L_i / L_i^{*},$$

therefore

$$\Delta\widetilde{Y}_i / \widetilde{Y}_i^{*} \leq \Delta L_i / L_i^{*} \tag{E.16}$$

For example, if $r_0^{*} = 0.1$, $\Delta r_1 = \Delta n = 0.01$ (per year), $q_i^{**} = 0.4$ and $\lambda_i^{*} = 0.3$, then the profits for the 1st period and the monetary investment over the 2nd one increase by $\Delta r_1 / r_0^{*} = 0.1 = 10\%$ (see Equation (14)). The prices grow by 2.5% (see Equation (13)), and the relative output growth in physical units is equal to 7.5% according to Equation (E.15). Thus, in the present example, the aggregate additional increase in labor over the 2nd period $\Delta L_i / L^{*} = \lambda_i^{*} \Delta L_i / L_i^{*}$ is at least 2% only due to the investment sector, which is significantly higher than the additional population growth rate of 1% ($\Delta n = 0.01$). This means that employment during the 2nd period is significantly higher than its equilibrium level.

The growth in investment and employment over the 2nd period accompany an economic growth, and thus it encourages further price rises. As a consequence, the profits over the 2nd period and the investment during the 3rd period are expected to increase further. This contributes to the illusion of the lack of capital: the capital intensity in the investment sector is significantly lower than the technologically specified value.

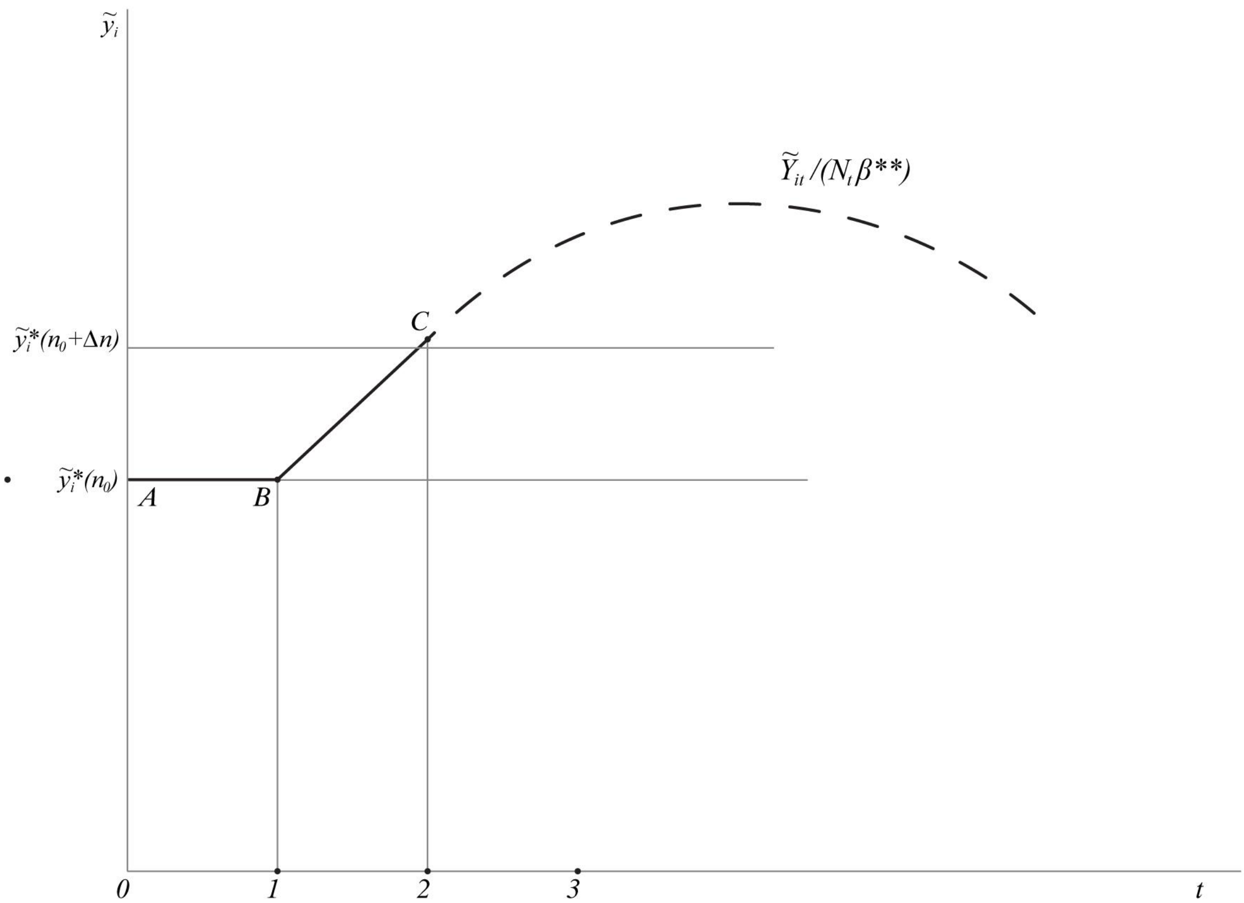

Figure 14 Short-term transition process. Changes in the investment goods sector output in physical units. The inevitability of overproduction

It is interesting to compare the investment growth calculated in Equation (15i) for the 2nd time period with the corresponding ratio along the new equilibrium path (when $r^*=n_0+\Delta n$). It is hardly possible to determine precisely the profit rate growth in the 1st period, Δr_1. For simplicity, assume that this initial shift in the profit rate is equal to its resulting change at the end of the transition process, $\Delta r_1 = \Delta n$. Then Equation (E.15i) can be rewritten as

$$\Delta\tilde{y}_{i2}/\tilde{y}_i^*(r_0^*) = [\Delta\tilde{Y}_{i2}/(N_2\beta^{**})]/\tilde{y}_i^*(r_0^*) = (\Delta n/n_0)(1-n_0/q_i^{**}) \tag{E.15a}$$

Considering the fact that $\tilde{y}_i^*(r^*) \equiv \tilde{Y}_{it}^*/L_t^* = q_i^{**}\tilde{k}_i^{**}\lambda_i^*(r^*)$ by definition, q_i^{**} and k_i^{**} are invariable, then $(\tilde{y}_i^*(r_0^*+\Delta r)-\tilde{y}_i^*(r_0^*))/\tilde{y}_i^*(r_0^*) = \Delta\tilde{y}_i^*/\tilde{y}_i^*(r_0^*) = \Delta\lambda_i^*/\lambda_i^*(r_0^*)$ when the profit rate changes. As a result of the transition process, the profit rate changes from $r^*=n_0$ to $r^*=n_0+\Delta n$, and then using Equations (E.8) and (E.9)

$$\begin{aligned}\Delta\tilde{y}_i^*/\tilde{y}_i^*(n_0) &= \Delta\lambda_i^*/\lambda_i^*(n_0) = (\Delta n/n_0)(\tilde{k}_i^{**}q_i^{**}/[\tilde{k}_i^{**}q_i^{**}+n_0(\tilde{k}_c^{**}-\tilde{k}_i^{**})] \approx \\ &(\Delta n/n_0)[1-(n_0/q_i^{**})(\tilde{k}_c^{**}/\tilde{k}_i^{**}-1)]\end{aligned} \tag{E.17}$$

Since the ratio $n_0/q_i^{**}=r_0^*/q_i^{**}=r_0^*K_i/Y_i$ is the capital income share in the investment sector and thereby is insignificant, $n_0/q_i^{**}<<1$, it gives a small contribution to the result in Equation (15a) as well as the value of $(n_0/q_i^{**})(\tilde{k}_c^{**}/\tilde{k}_i^{**}-1)<<1$ in Equation (E.17). That is, the investment $\tilde{Y}_{i2}/(N_2\beta^{**})$ becomes very close to its final equilibrium value $\tilde{y}_i^*(n_0 + \Delta n)$ already in the 2nd period of the transition process, if the economy's response to the initial impetus is large enough ($\Delta r_1 = \Delta n$). Investment continue to grow during the third period, consequently it becomes higher than it is necessary to provide the technologically specified capital intensity at the equilibrium employment level. This is illustrated in Figure 14. The advance growth in investment and capital accumulation inevitably leads to the effect of overproduction and eventually launches a reverse process during which the employment and other settings should return to the new equilibrium values[18]. Therefore, the transition which is studied here turns out to be a cyclical process. An equilibrium value must be established, since stability is ensured as a consequence of the Uzawa capital intensity condition, $\partial k/\partial r < 0$, similar to the neoclassical stability mechanism.

The transition process from one equilibrium path to another theoretically described above is caused by hypothetical unanticipated changes in an exogenous factor, the population growth rate. Such a transition process is going through disequilibrium and cyclical changes in output, investment, and labor. The initial change in the exogenous factor provides impetus in the form of an increased demand, and then it produces a deviation from the equilibrium employment, investment growth and causes the development of the expansion phase. The initial impetus may occur not only due to the unexpected labor change, this case is unlikely to happen in reality often and is used here due to its clearness. Such impetus may occur also due to other circumstances affecting the profit rate and prices, including biased reasons, for example, due to raised expectations. Thus, the foregoing mechanism can describe not only transition processes, but also short-term business cycles.

The properties of processes described above theoretically are actually observed. During the expansion period, the profit rate, prices, investment, and employment grow, and the growth rate of the latter two factors increases at a pace faster than that of the aggregate output growth rate. The capital-to-output ratio (and hence capital intensity) reduces first, and then gradually returns to its previous equilibrium level, see Figure 9 in Appendix C.

[18] The profit rate along the new equilibrium path, $r^* = n_0 + \Delta n$, will be greater relative to the primary one ($r_0^* = n_0$). Then the wages (and per capita consumption) in physical units should decline according to Equation (E.5), such reduction in living standards looks feasible only during the contraction phase.

Appendix F Correlation Analysis

The linear dependence between the relative changes in the variables observed in the actual US economy is studied in this appendix by submitting the NIPA time series to a correlation analysis. The variations in the data with time are studied on a relative scale. The correlation coefficients are presented in Tables F1 and F2 both for the time series of current values and of real values acquired during the 1949 – 2012 time period.

Table F1 Correlation coefficients (Current variable values)

$\frac{1}{Y}\frac{dY}{dt}$	$\frac{1}{Y}\frac{dY}{dt}$						
$\frac{1}{y}\frac{dy}{dt}$	0.675	$\frac{1}{y}\frac{dy}{dt}$					
$\frac{1}{w}\frac{dw}{dt}$	0.724	0.7912	$\frac{1}{w}\frac{dw}{dt}$				
$\frac{ds}{dt}$	0.573	0.2196	0.126	$\frac{ds}{dt}$			
$\frac{1}{k}\frac{dk}{dt}$	0.230	0.6251	0.5886	-0.0594	$\frac{1}{k}\frac{dk}{dt}$		
$\frac{1}{K}\frac{dK}{dt}$	0.7144	0.6008	0.7317	0.3321	0.7680	$\frac{1}{K}\frac{dK}{dt}$	
$\frac{1}{L}\frac{dL}{dt}$	0.767	0.0445	0.2937	0.5844	-0.2321	0.4444	$\frac{1}{L}\frac{dL}{dt}$
$\frac{1}{P}\frac{dP}{dt}$	0.6207	0.8086	0.830	-0.0568	0.6914	0.7238	0.138

Data Sources: see Appendix G

Table F1 demonstrates that the relative changes in aggregate output (gross domestic product) Y are determined by variations in two groups of factors. First, these are the investment rate s and the labor L, with the correlation coefficient between changes in the saving rate and output being equal to 0.573, between changes in output and labor being equal to 0.767, and between changes in the saving rate and labor being equal to 0.584. These factors are responsible for the cyclic component of output. On the other hand, the GDP changes are determined by variations in the labor productivity y and in the wages w, which are responsible for the labor productivity growth, with the correlation coefficient between changes in output and labor productivity being equal to 0.675, between changes in output and wage being equal to 0.724, and between changes in wage and labor productivity being equal to 0.791. These two groups of factors are considered to be linearly independent because they show weak correlation between themselves (see Table FI).

Unfortunately, the data in Table F1 do not prove linear independence between changes in k, s, and w. This is not obvious because of the substantial correlation between changes in wage and the capital intensity, the coefficient being equal to 0.589. The reason for such a significant correlation coefficient is that both the variables are strongly connected with the price level P. The mutually independence between k, s, and w is confirmed by the correlation analysis of real values (see Table F2), as well as by analyzing the Gram determinants.

Table F2 Correlation coefficients (Real values of Y, y, w, k, K)

$\frac{1}{Y/P}\frac{d(Y/P)}{dt}$	$\frac{1}{Y/P}\frac{d(Y/P)}{dt}$						
$\frac{1}{y/P}\frac{d(y/P)}{dt}$	0.3964	$\frac{1}{y/P}\frac{d(y/P)}{dt}$					
$\frac{1}{w/P_w}\frac{d(w/P_w)}{dt}$	0.4109	0.5642	$\frac{1}{w/P_w}\frac{d(w/P_w)}{dt}$				

$\frac{ds}{dt}$	0.7798	0.4272	0.2348	$\frac{ds}{dt}$			
$\frac{1}{k/P_k}\frac{d(k/P_k)}{dt}$	-0.357	0.3061	0.0752	-0.0272	$\frac{1}{k/P_k}\frac{d(k/P_k)}{dt}$		
$\frac{1}{K/P_k}\frac{d(K/P_k)}{dt}$	0.5135	0.1574	0.2000	0.5987	0.4742	$\frac{1}{K/P_k}\frac{d(K/P_k)}{dt}$	
$\frac{1}{L}\frac{dL}{dt}$	0.8407	-0.1638	0.1076	0.5844	-0.5657	0.4570	$\frac{1}{L}\frac{dL}{dt}$
$\frac{1}{P}\frac{dP}{dt}$	-0.1315	-0.4726	-0.5662	-0.0568	-0.2586	-0.1319	0.138
$\frac{1}{E}\frac{dE}{dt}$	0.5585	0.185	0.3101	0.5204	-0.2123	0.2981	0.4895

Data Sources: see Appendix G

Table F2 reveals an interesting fact, a significant correlation coefficient between the changes in the equity aggregate value shifted by one year and the changes in the aggregate output Y, and the changes in the s- and L- values (these variables are responsible for the cyclical component of output). This observation suggests that the cause of business cycles may be the stock market instability (significant changes in stock prices often lead to changes in expectations and, consequently, they lead to output changes occurring about a year later).

The correlation coefficients in Tables F1 and F2 provide information on the linear relationship between two variables only. To obtain information about the linear dependence in the group of variables, a Gram determinant can be used. The time series of each variable can be associated with the vector $\boldsymbol{r}(r_1, r_2, ..., r_N)$ in an N-dimensional Euclidian space, where N is the number of measurements (time intervals in our case). The vectors are zero mean random variables. Then, the correlation between two variables is the dot product of the corresponding vectors, normalized to the standard deviation, or the cosine of the angle between these two vectors.

If the correlation coefficients between m variables are known, it is possible to make the Gram determinant, which geometrically is the square of the hypervolume V of the m-dimensional parallelepiped spanned by the corresponding vectors, embedded in the N-dimensional space. For example, if $m = 3$, then

$$G(a,b,c) = \begin{vmatrix} 1 & \langle a;b \rangle & \langle a;c \rangle \\ \langle a;b \rangle & 1 & \langle b;c \rangle \\ \langle a;c \rangle & \langle b;c \rangle & 1 \end{vmatrix} = V^2$$

In particular, the vectors are linearly independent if and only if the Gram determinant is nonzero. If the vectors are linearly dependent, then the corresponding parallelepiped degenerates (e.g, a cube shrinks to a square) and so it has a zero hypervolume.

The distance between the $\boldsymbol{x}$-vector and the subspace $\boldsymbol{\Omega(a,b,c)}$ is equal to

$$\|x - \Omega(a,b,c)\| = \sqrt{G(a,b,c,x)/G(a,b,c)}\,.$$

The values of the Gram determinants were calculated for the 1929 –2012 U.S. time series:

If the Gram determinant $G(\boldsymbol{K,L,s,w}) = 0.249$, then the hypervolume $V(K,L,s,w) = \sqrt{G(K,L,s,w)} = 0.5$; if $G(\boldsymbol{k, s, w}) = 0.8666$, then $V(\boldsymbol{k, s, w}) = 0.93$. These values are substantially greater than zero, i.e., the vectors in the group {$\boldsymbol{K, L, s, w}$} are mutually linearly independent, as well as the vectors in the group {$\boldsymbol{k, s, w}$} (this statement supports the assumption (2) in Section 2.4).

If $G(\boldsymbol{Y, K, L, s, w}) = 0.00696$, then $\|Y - \Omega(K,L,s,w)\|^2 = 0.0279$. Both the distance and the volume are rather small and comparable to the error, i.e., the $\boldsymbol{Y}$-changes are determined almost entirely by the changes in the vectors {$\boldsymbol{K, L, s, w}$}.

If $G(\boldsymbol{Y}, \boldsymbol{K}, \boldsymbol{L}, \boldsymbol{w}) = 0.0111$, which is slightly higher than the value of $G(\boldsymbol{Y}, \boldsymbol{K}, \boldsymbol{L}, \boldsymbol{s}, \boldsymbol{w})$, i.e., the parameter s exerts a little additional effect on the Y-value, and the parameter s is linearly independent of the vector group $\{\boldsymbol{Y},\boldsymbol{K},\boldsymbol{L},\boldsymbol{s},\boldsymbol{w}\}$, $\|s-\Omega(Y,K,L,w)\| = 0.797$.

The Gram determinant $G(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{w}) = 0.304$ when $G(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{s}, \boldsymbol{w}) = 0.249$, then $\|s-\Omega(K,L,w)\| = 0.905$, i.e, s is linearly independent of $\{\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{w}\}$.

The Gram determinant $G(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{s}) = 0.688$ when $G(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{s}, \boldsymbol{w}) = 0.249$, then $\|w-\Omega(K,L,s)\| = 0.601$.

The Gram determinant $G(\boldsymbol{k}, \boldsymbol{s}) = 0.936$ when $G(k, s, w) = 0.867$, then $\|w-\Omega(k,s)\| = 0.962$, i.e., w is linearly independent of $\{\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{s}\}$ and $\{\boldsymbol{k}, \boldsymbol{s}\}$ (this statement supports the speculations in Section 3.1).

Appendix G Data Sources

All the numerical data presented in this paper are the result of the calculations done by the author by using the data that can be retrieved from the:
— NIPA Tables and NIPA Fixed Assets tables that are published by the Bureau of Economic Analysis, http://www.bea.gov/, (US statistics);
— Flow of Funds Accounts of the United States (Z.1) that are compiled by the Board of Governors of the Federal Reserve, http://www.federalreserve.gov/, (US statistics);
— Office for National Statistics (ONS), http://www.ons.gov.uk/, (UK statistics);
— Cabinet Office, Government of Japan: Annual report on National Accounts of 2000 (data from 1955 to 1979 is used according to 68 SNA), and Annual report on National Accounts of 2011 (1980-2009 data is used according to 93 SNA), http://www.esri.cao.go.jp/, (statistics of Japan).

The annual calendar year time series are used. All these time series data are at current prices (except dimensionless ones: labor, price indexes, and saving rate time series).

The relative change of a given variable A per a time unit (year) for the year n is calculated as

$$\frac{1}{A}\frac{dA}{dt}(n) = (A(n) - A(n-1))/A(n-1) = A(n)/A(n-1) - 1 \qquad \text{(G.1)}$$

The change of the dimensionless investment rate s-value is calculated as

$$\frac{ds}{dt}(n) = s(n) - s(n-1) \qquad \text{(G.2)}$$

The real variable value is calculated by dividing its nominal (current price) quantity $A(n)$ by the corresponding price index $P_A(n)$, and then its relative change is calculated as

$$\frac{1}{(A/P_A)}\frac{d(A/P_A)}{dt}(n) = A(n)/P_A(n)/(A(n-1)/P_A(n-1)) - 1 \qquad \text{(G.3)}$$

The change of a given variable A for the year n relative to GDP (Y) is calculated as

$$\frac{1}{Y}\frac{dA}{dt}(n) = (A(n) - A(n-1))/Y(n-1) \qquad \text{(G.4)}$$

List of Explanatory Variables for the following items:

Figure 4

Variables	Description	Source
Labor income share, wL/Y	Ratio of 'Compensation of employees' (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)', line 1.1) to 'Gross domestic product' (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)')	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011

Figure 5

Variables	Description	Source
US GDP changes	$\frac{1}{Y}\frac{dY}{dt}$ is computed here by Equation (G.1) where GDP (Y) is 'Gross domestic product' (A191RC1)	NIPA Table 1.1.5, line1
Calculated US GDP changes	$\frac{1}{Y}\frac{dY}{dt}$ is calculated here by Equation (13) where the relative	NIPA Tables, see sources for Tables F1 and

according to Equation (13)	changes of quantities: $\frac{1}{w}\frac{dw}{dt}, \frac{1}{L}\frac{dL}{dt}, \frac{1}{K}\frac{dK}{dt}$ and $\frac{ds}{dt}$ are computed in the same way as for Tables F1 and F2 (see below). The average value (*K*/*wL*)* is the mean value of the *K*/*wL* ratio during the period 1929–2012, and it is equal to 5.963.	F2 below

Table 1

Variables	Description	Source
Real output growth rate (%)	Calculated using Equation (G.3) and multiplied by 100, where the nominal output is 'Gross domestic product' (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)'), and the corresponding price index is the 'Gross domestic product (expenditure approach)' deflator (table 'Flow/Main Time Series/ Gross Domestic Expenditure/ Deflators/ Calendar Year', line 5)	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Net rate of profit (%)	Hundred times the ratio of 'Saving, net' (table 'Flow/Capital Finance Accounts classified by Institutional Sectors/Non-financial Incorporated Enterprises') to the sum of 'Fixed assets' and 'Stocks' (line 2 plus line 1) up to 1979; or the sum of 'Net fixed assets' and 'Inventories' (line 1/(1)/b plus line 1/(1)/a) since 1980, from table 'Stock/ Accounts by Institutional Sectors/ Non-financial Corporations'	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Investment rate (%)	Hundred times the ratio of the sum of 'Gross fixed capital formation' and 'Increase in stocks' (line 2) up to 1979, or 'Changes in inventories' (line 1.3) since 1980, from table 'Flow/ Capital Finance Accounts classified by Institutional Sectors/Non-financial Incorporated Enterprises' to 'Gross domestic product' (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)'	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Labor income share (%)	Hundred times the ratio of 'Compensation of employees' (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)', line 1.1) to 'Gross domestic product' (see above)	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Capital-to-output ratio	Hundred times the ratio of the sum of 'Fixed assets' and 'Stocks' (line 2 plus line 1 up to 1979); or the sum of 'Net fixed assets' and 'Inventories' (line 1/(1)/b plus line 1/(1)/a) since 1980), from table 'Stock/ Accounts by Institutional Sectors/ Non-financial Corporations' to 'Gross domestic product' (see above).	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011

Table B1 Appendix B All quantities in the table (except "$(K_j/Y)^*$ calculated" column) are the average values of the corresponding variables over the period 1949–2011. Calculations are made for three types of assets *j*: durable goods (*j* = 1), residential fixed capital (*j* = 2), and total capital (*j* = 3).

Variables	Description	Source
s_j	Investment rate s_j is the ratio of investment in asset *j* to GDP (*Y*), (A191RC1), $s_j = (I_j/Y)$, where:	NIPA Table 1.1.5, line 1
	I_1 is the investment in the durable goods (DDURRC1)	NIPA Table 1.1.5, line 4
	I_2 is the investment in the residential fixed capital (A011RC1)	NIPA Table 1.1.5, line 12
	I_3 is the investment in the total capital (i3ttotl1es000)	NIPA Fixed assets Table 1.5, line 2

i_j	Inflation rate i_j is the relative change of corresponding price index P_j of the asset j which is calculated by Equation (G.1) where:	
	P_1 is the price index of the durable goods (DDURRG3)	NIPA Table 1.1.4, line 4
	P_2 is the price index of the residential fixed capital (B011RG3)	NIPA Table 1.1.4, line 12
	P_3 is the price index of the total capital (B007RG3)	NIPA Table 1.1.4, line 8
i	GDP inflation rate i is the relative change of the GDP price index (B191RG3) calculated by Equation (G.1)	NIPA Table 1.1.4, line 1
δ_j	Depreciation rate δ_j is the ratio of the current-cost depreciation of fixed asset j to the capital stock of the relevant type of assets K_j where	
	Current-cost depreciation of durable goods (m1ctotl1cd000) corresponds to $j = 1$	NIPA Fixed assets Table 1.3, line 13
	Current-cost depreciation of residential fixed capital (m1r53101es000) corresponds to $j=2$	NIPA Fixed assets Table 1.3, line 7
	Current-cost depreciation of total capital (m1ttotl1es000) corresponds to $j = 3$	NIPA Fixed assets Table 1.3, line 2
	K_1 is the stock of durable goods (k1ctotl1cd000)	NIPA Fixed assets Table1.1, line13
	K_2 is the stock of residential fixed capital (k1r53101es000)	NIPA Fixed assets Table1.1, line7
	K_3 is the total stock of capital (k1ttotl1es000)	NIPA Fixed assets Table1.1, line 2
(K_j/Y) fact	Ratio of the capital stock K_j of asset j to GDP, K_j and GDP descriptions see above in this table	
$(K_j/Y)^*$ calculated	Values of "$(K_j/Y)^*$ calculated" are computed by Equation (B2) from Appendix B by using the average values of variables which are already calculated above instead of equilibrium ones	

Figure 6, Appendix C

Variables	Description	Source
Investment-to-GDP ratio, or investment rate	Ratio of non-financial corporation's investment (series code FDCL) to 'Gross domestic product at market prices' (series code YBHA)	ONS
Profits-to-GDP ratio, or capital income share	Ratio of 'total change in liabilities and net worth' of non-financial corporations (series code NRMG) to 'Gross domestic product at market prices' (series code YBHA)	ONS

Figure 7, Appendix C

Variables	Description	Source
Investment-to-GDP ratio or investment rate	Ratio of 'Gross fixed capital formation' (line 1 up to 1979, or line1.1 since 1980, from table 'Flow/ Capital Finance Accounts classified by Institutional Sectors/ Non-financial Incorporated Enterprises') to 'Gross domestic product' (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)')	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Profits-to-GDP ratio, or capital	Ratio of the sum of 'Saving, net' and 'Consumption of fixed capital' (lines 5 plus line 6 up to 1979; or lines 1.6 plus line 1.2 since 1980) from table 'Flow/Capital Finance Accounts classified	Cabinet Office, Annual reports on National

income share	by Institutional Sectors/Non-financial Incorporated Enterprises' to 'Gross domestic product' (see above).	Accounts of 2000 and of 2011

Figure 8, Appendix C

Variables	Description	Source
Investment-to-GDP ratio, or investment rate	Ratio of 'Total capital expenditures' (series code FA105050005.A) to GDP (series code FA086902005.A)	Flow of Funds Accounts of the US (Z.1), Table F102 and Table F6
Profits-to-GDP ratio, or capital income share	Ratio of the sum of 'Undistributed corporate profits excluding IVA and CCAdj' (series code FA106006405.A) and 'Capital consumption allowance' (series code FA106300015.A) to GDP (series code FA086902005.A)	Flow of Funds Accounts of the US (Z.1), Table F7, Table F102, and Table F6

Figure 9, Appendix C

Variables	Description	Source
Changes in the capital income share $(r+\delta)\times K/Y$	Calculated using Equation (G.1) where the capital income share is the same as 'profits-to-GDP ratio, or capital income share' for Figure 7 data sources explanation above	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Changes in the capital-to-output ratio" K/Y	Calculated using Equation (G.1) where K/Y is the same as 'Capital-to-output ratio' for Table 1 data sources explanation above	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Changes in the rate of profit" $(r + \delta)$	Rate of profit $(r + \delta)$ is the ratio of the two calculated above values $(r + \delta)= [(r+\delta)\times K/Y]/ (K/Y)$	

Figure 10, Appendix D

Variables	Description	Source
Consumption share	Ratio of 'Personal consumption expenditures' (DPCERC1) to GDP (A191RC1)	NIPA Table 1.1.5, line 2 and line 1
Labor income share	Ratio of the labor income to GDP (row A191RC1) where:	NIPA Table 1.1.5, line1
	Labor income is the sum of 'Wage and salary accruals' (A4102C1) and 'Proprietors' income…' (A041RC1)	NIPA Table 1.10, line 3 and line 15

Figure 11, Appendix D

Variables	Description	Source
The ratio of the rest of the world net lending to GDP	Ratio of the difference between 'Foreign income from U.S.' (series code FA266905005.A) and 'Foreign outlays to U.S.' (series code FA266900005.A) to GDP (series code FA086902005.A)	Flow of Funds Accounts of the US (Z.1), Table F106, Table F106 and Table F6
The ratio of the government social contribution to GDP	Ratio of 'Consolidated governments/social contributions paid' (series code FA366404005.A) to GDP (series code FA086902005.A)	Flow of Funds Accounts of the US (Z.1), Table F105c and Table F6
The difference between the consumption share and the labor	Ratio of the difference between 'Households and nonprofit organizations; personal consumption expenditures' (series code FA156901001.A) and 'Households and nonprofit organizations; wages and	Flow of Funds Accounts of the US (Z.1), Table F6 and Table F7

income share$(C - wL)/Y$	other labor income' (series code FA156020105.A) to GDP (see above)	

Table D1, Appendix D

UK, Japan, and U.S.A. are under consideration. The table shows the correlation coefficients between the changes in households' saving in the UK (net lending in Japan, and saving minus investment in the U.S.A.) and changes in output, consumption, and labor income. All changes of variables values are calculated relative to GDP according to Equation (G.4)

UK

Variables	Description	Source
GDP	'Gross domestic product at market prices' (series code YBHA)	ONS
Households saving	Product of 'Gross disposable income' (series code RPHQ) and 'Household saving ratio' (series code NRJS), divided by 100	ONS
Consumption	'Final consumption by households' (series code ABJQ)	ONS
Labor income	'Wages and salaries' (series code ROYJ)	ONS

Japan

Variables	Description	Source
GDP	'Gross domestic product', time series can be found in table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)'	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Households net lending	'Households, net lending', time series can be found in line 4 up to 1979, and in line 1.5 since 1980 of table 'Flow/Capital Finance Accounts classified by Institutional Sectors/Households (Including Private Unincorporated Enterprises)'	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Consumption	'Private final consumption expenditure', (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)', line 1.7)	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011
Labor income	'Compensation of employees', (table 'Flow/Integrated Accounts/Gross Domestic Product Account (Production and Expenditure Approach)', line 1.1)	Cabinet Office, Annual reports on National Accounts of 2000 and of 2011

U.S.A.

Variables	Description	Source
GDP	'Gross domestic product' (series code FA086902005.A)	Flow of Funds Accounts of the US (Z.1), Table F6
Households saving-investment	Difference between 'Gross saving less net capital transfers…' (series code FA156000105.A) and 'Total capital expenditures including consumer durables' (series code FA155050005.A)	Flow of Funds Accounts of the US (Z.1), Table F100, Table F100
Consumption	'Personal consumption expenditures' (series code FA156901001.A)	Flow of Funds Accounts of the US (Z.1), Table F6
Labor income	'Wages and other labor income' (series code FA156020105.A)	Flow of Funds Accounts of the US (Z.1), Table F7

Tables F1 and F2, Appendix F

Variables	Description	Source
$\frac{1}{Y}\frac{dY}{dt}$	Calculated using Equation (G.1) where the GDP (Y) is 'Gross domestic product' (A191RC1)	NIPA Table 1.1.5, line 1
$\frac{1}{w}\frac{dw}{dt}$	Calculated using Equation (G.1) where the wage (w) is 'Wage and salary accruals per full-time equivalent employee' (A4401C0)	NIPA Table 6.6, line 1

$\frac{1}{K}\frac{dK}{dt}$	Calculated using Equation (G.1) where the capital stock K is 'Fixed assets' (k1ttotl1es000)	NIPA Fixed assets Table 1.1, line 2
$\frac{1}{L}\frac{dL}{dt}$	Calculated using Equation (G.1) where the labor L is 'Full-time equivalent employees' (A4301C0)	NIPA Table 6.5, line 1
$\frac{ds}{dt}$	Calculated using Equation (G.2) where the investment rate s is the ratio of 'Gross private domestic investment' (A006RC1) to GDP (Y) (A191RC1)	NIPA Table 1.1.5, line7 and line 1
$\frac{1}{P}\frac{dP}{dt}$	Calculated using Equation (G.1) where P is the GDP price index (B191RG3)	NIPA Table 1.1.4, line 1
$\frac{1}{y}\frac{dy}{dt}$	Calculated using Equation (G.1) where y is the GDP per labor unit, $y = Y/L$, the ratio of GDP (A191RC1) to 'Full-time equivalent employees' (A4301C0)	NIPA Table 1.1.5, line 1and Table 6.5, line 1
$\frac{1}{k}\frac{dk}{dt}$	Calculated using Equation (G.1) where k is the capital intensity $k = K/L$, the ratio of 'Fixed assets' (k1ttotl1es000) to 'Full-time equivalent employees' (A4301C0)	NIPA Fixed assets Table 1.1, line 2 and NIPA Table 6.5, line 1
$\frac{1}{E}\frac{dE}{dt}$	Calculated using Equation (G.1) where E is 'Nonfinancial corporate business; corporate equities; liability' (series code FL103164103.A), the E time series are shifted forward in time by one year.	Flow of Funds Accounts of the US (Z.1), Table L213
$\frac{1}{(Y/P)}\frac{d(Y/P)}{dt}$	Calculated using Equation (G.3) where Y/P is the ratio of GDP (A191RC1) to GDP price index (B191RG3)	NIPA Table 1.1.5, line 1 and Table 1.1.4, line 1
$\frac{1}{(y/P)}\frac{d(y/P)}{dt}$	Calculated using Equation (G.3) where $y/P=Y/(PL)$ is the ratio of GDP (A191RC1) to the product of GDP price index (B191RG3) and labor (A4301C0)	NIPA Table 1.1.5, line 1, Table 1.1.4, line 1 and Table 6.5, line 1
$\frac{1}{(w/P_w)}\frac{d(w/P_w)}{dt}$	Calculated using Equation (G.3) where w/P_w is the ratio of wage (A4401C0) to the personal consumption price index P_w (DPCERG3)	NIPA Table 6.6, line 1 and Table 1.1.4, line 2
$\frac{1}{(K/P_K)}\frac{d(K/P_K)}{dt}$	Calculated using Equation (G.3) where K/P_K is the ratio of capital stock K (k1ttotl1es000) to the price index for fixed investment (B007RG3)	NIPA Fixed assets Table 1.1, line 2, NIPA Table 1.1.4 line 8
$\frac{1}{(k/P_K)}\frac{d(k/P_K)}{dt}$	Calculated using Equation (G.3) where $k/P_K = K/(P_K L)$ is the ratio of capital stock K (k1ttotl1es000) to the product of the price index for fixed investment (B007RG3) and labor (A4301C0)	NIPA Fixed assets Table 1.1, line 2, NIPA Table 1.1.4, line 8, Table 6.5, line 1

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