Spurious Instrumental Variables

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Abstract

Spurious regression phenomenon has been recognized for a wide range of Data Generating Processes: driftless unit roots, unit roots with drift, long memory, trend and broken-trend stationarity, etc. The usual framework is Ordinary Least Squares. We show that the spurious phenomenon also occurs in Instrumental Variables estimation when using non-stationary variables, whether the non-stationarity component is stochastic or deterministic. Finite sample evidence supports the asymptotic results.

Keywords: IV Estimator, Spurious Regression, Broken-Trend stationarity, Unit Root.

JEL Classification: C12, C13, C22.

1 Introduction

Spurious regression—that is, a statistically significant relationship between two independent random variables—has been uncovered for different forms of non-stationarity in a simple Least Squares (hereinafter, LS) framework. Indeed, related literature has studied the cases where the variables are generated as driftless random walks (Phillips 1986), random walks with drift (Entorf 1997), $I(d)$ processes with d being an integer (Marmol 1995), long memory and fractional integrated processes (Marmol 1998), Trend Stationary (TS) processes,[1] as TS processes with breaks, and, mixed nonstationary DGP's.[2] The approach taken in the study of spurious regression tends to involve the computation of the asymptotics using increasingly complex Data Generating Processes (DGP's), whilst estimation methodology remains the same (LS). This dependence on LS estimators may be considered somewhat limiting, given

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[1]See Hasseler (2000)and Kim, Lee, and Newbold (2004).

[2]See Noriega and Ventosa-Santaulària (2006) and Noriega and Ventosa-Santaularia (2007), respectively.

the variety of estimators commonly used in applied research—Instrumental Variables (IV) and Generalized Method of Moments (GMM), for example. To the best of our knowledge, little consideration has been given to the possibility of a connection between the spurious phenomenon and the IV estimator.[3] This paper focuses in the IV regression estimates under independent nonstationary variables. We prove that, when there is no relationship between the regressand, the regressor and the instrument,[4] IV estimates are statistically significant, that is, IV regression is spurious. We derive the asymptotic behavior of t-statistics in IV-estimated regressions, where the DGP consists of two independent and nonstationary processes with a trending mechanism, be it deterministic with (a possible) structural break or stochastic. Additionally, some Monte Carlo evidence is presented to account for the spurious regression phenomenon in finite samples.

2 IV Estimates using Nonstationary Variables

IV is a classical technique in econometrics; it originated as a proposal to solve the identification problem in the estimation of demand and supply curves (Wright 1928).[5] Typically, in text books, IV is proposed as a solution to the problem of omitted variables and, broadly speaking, when there is no independence between the error term and the regressors. The selection of adequate instruments remains the key issue and little attention has been paid to the problem of the nonstationarity of the series. As mentioned above, Phillips and Hansen (1990) and Hansen and Phillips (1990) studied the asymptotics as well as the finite-sample properties of the IV estimator in the context of a cointegrated relationship, and proved that even "spurious instruments" (i.e. I(1) instruments structurally non-related to the regressors) provide consistent estimates. In this paper, we prove that, when there is no structural relationship between the regressand and a *single* regressor, that is, when there is no cointegration between y and x respectively, the use of spurious instruments does not prevent the

[3]A notable exception is Phillips and Hansen (1990) and Hansen and Phillips (1990) whose results concerning *IV* estimation of cointegrated vectors are discussed in the next section.

[4]We study the case of exact identification.

[5]See Morgan (1992).

phenomenon of spurious regression. We focus on the estimation of the following specification:

$$y_t = \alpha + \delta x_t + u_t \tag{1}$$

Let us suppose that we are dealing with three variables, the dependent, the explanatory and a potential instrument, y, x and z, respectively. The three variables are independent of each other and may be generated by any of the following DGP's:

$$w_t = \mu_w + \beta_w t + \gamma_w DT_{wt} + u_{wt} \tag{2}$$

$$w_t = W_0 + \mu_w t + \sum_{i=1}^{t} u_{wi} \tag{3}$$

where $w = y, x, z$; DGP (2) is referred to as $TS + br$, that is, a Broken-Trend Stationary process, and DGP (3) is referred to as $I(1) + dr$ (Random walk with drift); u_{xt}, u_{yt} and u_{zt} are independent innovations obeying Proposition 17.3 in Hamilton (1994, pp. 505-506), and DT_{wt} is a dummy variable allowing changes in the slope, that is, $DT_{wt} = (t - T_{b_w})\mathbf{1}(t > T_{b_w})$, where $\mathbf{1}(\cdot)$ is the indicator function, and T_{b_w} is the unknown date of the break in w. We denote the break fraction as $\lambda_w = (T_{b_w}/T) \in (0, 1)$, where T is the sample size. W_0 is an initial condition.

It has been proved that the phenomenon of spurious regression occurs when estimating equation (1) using LS when the variables x and y generated by any combination of DGP's (2 and 3). Indeed, the order in probability of $t_{\hat{\delta}_{LS}}$ is $O_p\left(T^{\frac{1}{2}}\right)$ or $O_p\left(T\right)$ [See Noriega and Ventosa-Santaulària (2006) and Noriega and Ventosa-Santaularia (2007)]. In this paper, we are concerned with the estimation of equation (1) by Instrumental Variables (hereinafter, IV). All variables, y, x and a single instrument, z, remain independent of each other. Each may be generated by either of DGP's (2) or (3). For the purposes of clarity, we denote DGP's (2) and (3) as a and b; subsequently, C_{aba} represents the IV estimation using y, x, and z generated by DGP's a, b and a, respectively.

Proposition 1 *The order in probability of $t_{\hat{\delta}_{IV}}$ in model (1) for x, y, and z generated independently by any combination of DGP's (2) and (3) is:*

1. *Combinations C_{bbb} and C_{bba}: $t_{\hat{\delta}_{IV}} = O_p(T)$*
2. *Any other Combination: $t_{\hat{\delta}_{IV}} = O_p\left(T^{1/2}\right)$*

where $\hat{\delta}_{IV}$ denotes the IV estimate of δ in eq. (1).

Proof: see Appendix **A**.

For any combination of DGP's, the t-statistic diverges at a rate of $\sqrt{T}$ or faster, indicating a spurious relationship amongst independent variables. When y and x are $I(1)+dr$ processes–independently of the DGP of z—the IV estimates diverge at rate T. Moreover, when x and z are $I(1)+dr$ processes–independently of the DGP of y–the IV estimates do not differ from their LS counterparts:

Corollary 1 *Let x and z be generated independently by DGP (3) and let y be generated by either DGP (2) or (3). Hence:*

$$\begin{aligned} \hat{\delta}_{IV} &\overset{a}{=} \hat{\delta}_{LS} \\ t_{\hat{\delta}_{IV}} &\overset{a}{=} t_{\hat{\delta}_{LS}} \end{aligned}$$

where $\overset{a}{=}$ stands for asymptotical equivalence and $\hat{\delta}_{LS}$ denotes the LS estimate of δ in eq. (1).

Proof: see Appendix **A**.

Amidst these results it can be questioned whether these hold when the researcher happens to choose a valid instrument, that is, an instrument correlated with the regressor. In order to further investigate this issue, we modify the DGP of x_t. Let z_t be generated by equation

(3); assume further that x_t holds a cointegrated relationship with z_t:

$$x_t \quad = \quad \mu_x + \beta_x z_t + u_{xt} \tag{4}$$

It can be proved that the use of a valid instrument does not preclude the spurious phenomenon previously identified:

Proposition 2 *Let z_t and x_t be generated by DGPs (3) and (4), respectively.*

1. *The order in probability of $t_{\hat{\delta}_{IV}}$ in model (1) for y_t generated independently by DGP (2) is:*
$$t_{\hat{\delta}_{IV}} = O_p\left(T^{\frac{1}{2}}\right)$$

2. *The order in probability of $t_{\hat{\delta}_{IV}}$ in model (1) for y_t generated independently by DGP (3) is:*
$$t_{\hat{\delta}_{IV}} = O_p\left(T\right)$$

Proof: see Appendix **A**.

Proposition (2) shows that, even when the instrument is related to the regressor in an ideal manner, the IV estimate of β does not converge to its true value of zero. In other words, IV yields spurious estimates whether the instruments are spurious or not, at least asymptotically.

3 Finite Sample Evidence

We computed rejection rates for $t_{\hat{\delta}_{IV}}$ in model (1), using a 1.96 Critical Value (5% level for a standard normal distribution). The asymptotic results presented in Proposition (1) were evaluated in finite samples that varied from 50 to 500. The variables y, x and z were simulated according to different combinations of DGP's (2), (3) and (4). The values of the parameters were inspired on real data from Perron and Zhu (2005) and can be found in Appendix **B**. The number of replications is 10,000. Tables (1) and (2) summarize the finite sample findings: the first presents the results when the DGP's include white-noise innovations, whereas the second table uses DGP's where the innovations are first-order autogressive processes, $AR(1)$. Again, for the purposes of clarity, we denote DGP (4) as c.

Combination \ Sample size	50	100	200	500	1000
C_{aaa}	0.59	1.00	1.00	1.00	1.00
C_{aab}	0.49	0.85	0.93	0.96	0.98
C_{aba}	0.33	0.73	0.86	0.96	0.99
C_{baa}	0.63	0.80	0.90	0.99	0.99
C_{abb}	0.28	0.58	0.75	0.90	0.97
C_{bab}	0.51	0.67	0.80	0.93	0.98
C_{bba}	0.41	0.60	0.78	0.95	0.99
C_{bbb}	0.34	0.48	0.66	0.87	0.96
C_{acb}	0.07	0.41	0.68	0.82	0.92
C_{bcb}	0.11	0.35	0.61	0.81	0.92

Table 1: Rejection rates of $t_{\hat{\delta}}$ under white noise innovations

The results above suggest that the spurious phenomenon in IV estimates is indeed present, even for samples as small as 50, whether the regressor and the instrument are cointegrated or not. When the innovation's structure is more complex, the rejection rates seem to fall slightly, as is illustrated in table (2). Nevertheless, the spurious phenomenon remains strong.

4 Concluding remarks

We have shown that the spurious regression phenomenon (i.e. diverging t-statistics) in the estimation of the linear relationship using IV is present when the variables exhibit nonstationary behaviour (such nonstationarity being deterministic (with a structural break) and/or stochastic). Moreover, when both the explanatory variable and its instrument are random walks with drift, IV and LS produce exactly the same spurious asymptotic estimates. These

Combination \ Sample size	50	100	200	500	1000
C_{aaa}	0.37	0.99	1.00	1.00	1.00
C_{aab}	0.46	0.81	0.91	0.94	0.95
C_{aba}	0.19	0.61	0.72	0.85	0.93
C_{baa}	0.42	0.61	0.73	0.86	0.93
C_{abb}	0.24	0.48	0.61	0.75	0.84
C_{bab}	0.43	0.52	0.62	0.77	0.86
C_{bba}	0.27	0.44	0.56	0.73	0.87
C_{bbb}	0.30	0.37	0.46	0.64	0.77
C_{acb}	0.20	0.45	0.52	0.59	0.65
C_{bcb}	0.28	0.41	0.45	0.55	0.62

Table 2: Rejection rates of $t_{\hat{\delta}}$ under autocorrelated innovations

results may complete those obtained in Phillips and Hansen (1990) and Hansen and Phillips (1990); the latter demonstrated that IV is able to provide consistent estimates in a cointegrated relationship and may actually outperform LS when there is a strong problem of endogeneity, even if the instruments are spurious. Nevertheless, when there is no cointegrated relationship, the reality is that IV provides spurious estimates, just as LS does. The main result indicates the need for caution with regard to the inferences to be drawn from IV regression analysis which may in fact be spurious.

A Proof of Propositions 1 and 2 and Corollary 1

We present a guide as to how to obtain the order in probability of a t-ratio appearing in Proposition (1) in the estimation of regression (1) using IV where the variables y and x are generated by DGP (3) and z by DGP (2) (all other combinations follow the same steps. Proof of such was provided with the aid of *Mathematica 4.1* software[6]) We use the classical IV formulas where the number of instruments matches the number of regressors:

$$\begin{aligned}
\hat{\delta}_{IV} &= \left(Z'X\right)^{-1} Z'y \\
\hat{\sigma}^2_{\hat{\delta}_{IV}} &= \hat{\sigma}^2 \left(Z'X\right)^{-1} \left(Z'Z\right) \left(X'Z\right)^{-1} \\
t_{\hat{\delta}_{IV}} &= \frac{\hat{\delta}_{IV}}{\sqrt{\hat{\sigma}^2_{\hat{\delta}_{IV}}}}
\end{aligned}$$

where,

[6]The corresponding codes are available at **http://www.ventosa-santaularia.com/VSC_07.zip**.

$$Z'X = \begin{bmatrix} T & \sum x_t \\ \sum z_t & \sum x_t z_t \end{bmatrix}; \quad X'Z = \begin{bmatrix} T & \sum z_t \\ \sum x_t & \sum x_t z_t \end{bmatrix}; \quad Z'Z = \begin{bmatrix} T & \sum z_t \\ \sum z_t & \sum z_t^2 \end{bmatrix};$$

$$Z'Y = \begin{bmatrix} \sum y_t \\ \sum y_t z_t \end{bmatrix};$$

and,

$$\hat{\sigma}^2 = \sum y_t^2 + \hat{\alpha}_{IV}^2 T + \hat{\delta}_{IV}^2 \sum x_t^2 - 2\hat{\alpha}_{IV} \sum y_t - 2\hat{\delta}_{IV} \sum x_t y_t + 2\hat{\alpha}_{IV}\hat{\delta}_{IV} \sum x_t$$

We shall now describe the process involved in establishing the aforementioned proof. $\hat{\delta}_{IV}$, $\hat{\sigma}^2_{\hat{\delta}_{IV}}$ and $t_{\hat{\delta}_{IV}}$ are functions of the following expressions (unless indicated otherwise, all sums run from $t = 1$ to T). Denote $\xi_{w,t} = \sum_{i=1}^{T} u_{w,i}$. Let $w = y, x$:

$$
\begin{aligned}
\sum w_t &= W_0 T + \mu_w \sum t + \underbrace{\sum \xi_{w,t-1}}_{O_p\left(T^{\frac{3}{2}}\right)} \\
\sum z_t &= \mu_z T + \beta_z \sum t + \gamma_z \sum DT_{zt} + \underbrace{\sum u_{z,t}}_{O_p\left(T^{\frac{1}{2}}\right)} \\
\sum w_t^2 &= W_0^2 T + \mu_w^2 \sum t^2 + \underbrace{\sum \xi_{w,t-1}^2}_{O_p(T^2)} + 2W_0\mu_w \sum t \\
&\quad + 2W_0 \sum \xi_{w,t-1} + 2\mu_w \underbrace{\sum \xi_{w,t-1} t}_{O_p\left(T^{\frac{5}{2}}\right)} \\
\sum z_t^2 &= \mu_z^2 T + \beta_z^2 \sum t^2 + \gamma_z^2 \sum DT_{zt}^2 + \underbrace{\sum u_{z,t}^2}_{O_p(T)} + 2\mu_z\beta_z \sum t \\
&\quad + 2\mu_z\gamma_z \sum DT_{zt} + 2\mu_z \sum u_{z,t} + 2\beta_z\gamma_z \sum DT_{zt} t \\
&\quad + 2\beta_z \underbrace{\sum u_{z,t} t}_{O_p\left(T^{\frac{3}{2}}\right)} + 2\gamma_z \underbrace{\sum DT_{zt} u_{z,t}}_{O_p\left(T^{\frac{3}{2}}\right)} \\
\sum w_t z_t &\quad W_0\mu_z T + W_0\beta_z \sum t + W_0\gamma_z \sum DT_{zt} + W_0 \sum u_{zt} + \mu_w\mu_z \sum t \\
&\quad \mu_w\beta_z \sum t^2 + \mu_w\gamma_z \sum DT_{zt} t + \mu_w \sum u_{z,t} t + \mu_z \sum \xi_{w,t-1} \\
&\quad + \beta_z \sum \xi_{w,t-1} t + \gamma_z \underbrace{\sum DT_{zt} \xi_{w,t-1}}_{O_p\left(T^{\frac{5}{2}}\right)} + \underbrace{\sum \xi_{w,t-1} u_{z,t-1}}_{O_p(T)} \\
\sum x_t y_t &\quad X_0 Y_0 T + X_0 \mu_y \sum t + X_0 \sum u_{yt} + \mu_x Y_0 \sum t + \mu_x \mu_y \sum t^2 \\
&\quad + \mu_x \sum u_{y,t} t + Y_0 \sum \xi_{x,t-1} + \mu_y \sum \xi_{x,t-1} t + \underbrace{\sum \xi_{x,t-1} \xi_{y,t-1}}_{O_p(T^2)}
\end{aligned}
$$

where,

$$\begin{aligned}
\sum t &= \frac{1}{2}\left(T^2+T\right) \\
\sum t^2 &= \frac{1}{6}\left(2T^3+3T^2+T\right) \\
\sum DT_{zt} &= \frac{1}{2}\left[T^2\left(1-\lambda_z\right)^2+T\left(1-\lambda_z\right)\right] \\
\sum DT_{zt}^2 &= \frac{1}{6}\left[2T^3\left(1-\lambda_z\right)^3+3T^2\left(1-\lambda_z\right)^2+T\left(1-\lambda_z\right)\right] \\
\sum DT_{zt}t &= \lambda_z T\sum DT_{zt}+\sum DT_{zt}^2 \\
\sum DT_{yt}DT_{zt} &= \sum DT_{yt}^2+\left(\lambda_y-\lambda_z\right)T\sum DT_{yt}
\end{aligned}$$

The orders in convergence of the underbraced expressions can be found in Hamilton (1994) pp. 505-506. and in Noriega and Ventosa-Santaularia (2007). The last sum, $\sum DT_{yt}DT_{zt}$, is not needed in this example, but it appears in other combinations; we assume, for simplicity, that $\lambda_y > \lambda_x > \lambda_z$.

We can fill the previously-cited matrices and then compute the IV parameter estimates and the t-statistic associated with $\hat{\delta}$. The asymptotics are computed by the program and is represented below. Note that the code provides $\hat{\delta}_{LS}$ and $\hat{\sigma}^2_{\hat{\delta}_{LS}}$ in addition to $\hat{\delta}_{IV}$ and $\hat{\sigma}^2_{\hat{\delta}_{IV}}$. To understand it, a brief glossary is required. Let $w = x, y, z$:

Term	Represents	Term	Represents	Term	Represents	Term	Represents
St	$\sum t$	$St2$	$\sum t^2$	$W0$	W_0	Mw	μ_w
Sw	$\sum w_t$	$Sw2$	$\sum w_t^2$	Uw	$\sum u_{wt}$	Uwt	$\sum u_{wt}t$
Uxy	$\sum u_{xt}u_{yt}$	$DTxy$	$\sum DT_{xt}DT_{yt}$	Ew	$\sum \xi_{w,t-1}$	$Ew2$	$\sum \xi^2_{w,t-1}$
Ewt	$\sum \xi_{w,t-1}t$	Bw	β_w	Gw	γ_w	$Uw2$	$\sum u^2_{wt}$
Sxy	$\sum x_t y_t$	Mxx	$\left(X'X\right)^{-1}$	Exy	$\sum \xi_{x,t-1}\xi_{y,t-1}$	$Exuz$	$\sum \xi_{x,t-1}u_{zt}$
DTw	$\sum DT_{wt}$	$DTw2$	$\sum DT^2_{wt}$	$DTwt$	$\sum DT_{wt}t$	L_w	λ_w
$DTzuz$	$\sum DT_{zt}u_{zt}$	$DTzey$	$\sum DT_{zt}\xi_{y,t-1}$	$Szey$	$\sum z_t\xi_{y,t-1}$	$Szux$	$\sum z_t u_{xt}$
Sxz	$\sum x_t z_t$	Syz	$\sum y_t x_t$	Swt	$\sum w_t t$	$Sdtxy$	$\sum DT_x y_t$

Table 3: Glossary of the Mathematica Code

```
ClearAll; St = 1/2 * (T^2 + T); St2 = 1/6 * (2 * T^3 + 3 * T^2 + T);
DTz = 1/2 * (T^2 * (1 - Lz)^2 + T * (1 - Lz));
DTz2 = 1/6 * (2 * T^3 * (1 - Lz)^3 + 3 * T^2 * (1 - Lz)^2 + T * (1 - Lz));
DTzt = DTz2 + T * Lz * DTz;

Sx = X0 * T + Mx * St + Ex * T^(3/2);
Sy = Y0 * T + My * St + Ey * T^(3/2);
```

$\mathrm{Sz} = \mathrm{Mz} * T + \mathrm{Bz} * \mathrm{St} + \mathrm{Gz} * \mathrm{DTz} + \mathrm{Uz} * T^{\frac{1}{2}};$
$\mathrm{Sx2} = \mathrm{X0}^2 * T + \mathrm{Mx}^2 * \mathrm{St2} + \mathrm{Ex2} * T^2 + 2 * \mathrm{X0} * \mathrm{Mx} * \mathrm{St} + 2 * \mathrm{X0} * \mathrm{Ex} * T^{\frac{3}{2}} +$
$2 * \mathrm{Mx} * \mathrm{Ext} * T^{\frac{5}{2}};$
$\mathrm{Sy2} = \mathrm{Y0}^2 * T + \mathrm{My}^2 * \mathrm{St2} + \mathrm{Ey2} * T^2 + 2 * \mathrm{Y0} * \mathrm{My} * \mathrm{St} + 2 * \mathrm{Y0} * \mathrm{Ey} * T^{\frac{3}{2}} +$
$2 * \mathrm{My} * \mathrm{Eyt} * T^{\frac{5}{2}};$
$\mathrm{Sz2} = \mathrm{Mz}^2 * T + \mathrm{Bz}^2 * \mathrm{St2} + \mathrm{Gz}^2 * \mathrm{DTz2} + \mathrm{Uz2} * T + 2 * \mathrm{Mz} * \mathrm{Bz} * \mathrm{St} +$
$2 * \mathrm{Mz} * \mathrm{Gz} * \mathrm{DTz} + 2 * \mathrm{Mz} * \mathrm{Uz} * T^{\frac{1}{2}} + 2 * \mathrm{Bz} * \mathrm{Gz} * \mathrm{DTzt} +$
$2 * \mathrm{Bz} * \mathrm{Uzt} * T^{\frac{3}{2}} + 2 * \mathrm{Gz} * \mathrm{DTzuz} * T^{\frac{3}{2}};$
$\mathrm{Sxz} = \mathrm{X0} * \mathrm{Mz} * T + \mathrm{X0} * \mathrm{Bz} * \mathrm{St} + \mathrm{X0} * \mathrm{Gz} * \mathrm{DTz} + \mathrm{X0} * \mathrm{Uz} * T^{\frac{1}{2}} +$
$\mathrm{Mx} * \mathrm{Mz} * \mathrm{St} + \mathrm{Mx} * \mathrm{Bz} * \mathrm{St2} + \mathrm{Mx} * \mathrm{Gz} * \mathrm{DTzt} + \mathrm{Mx} * \mathrm{Uzt} * T^{\frac{3}{2}} +$
$\mathrm{Mz} * \mathrm{Ex} * T^{\frac{3}{2}} + \mathrm{Bz} * \mathrm{Ext} * T^{\frac{5}{2}} + \mathrm{Gz} * \mathrm{DTzex} * T^{\frac{5}{2}} + \mathrm{Exuz} * T;$
$\mathrm{Syz} = \mathrm{Y0} * \mathrm{Mz} * T + \mathrm{Y0} * \mathrm{Bz} * \mathrm{St} + \mathrm{Y0} * \mathrm{Gz} * \mathrm{DTz} + \mathrm{Y0} * \mathrm{Uz} * T^{\frac{1}{2}} +$
$\mathrm{My} * \mathrm{Mz} * \mathrm{St} + \mathrm{My} * \mathrm{Bz} * \mathrm{St2} + \mathrm{My} * \mathrm{Gz} * \mathrm{DTzt} + \mathrm{My} * \mathrm{Uzt} * T^{\frac{3}{2}} +$
$\mathrm{Mz} * \mathrm{Ey} * T^{\frac{3}{2}} + \mathrm{Bz} * \mathrm{Eyt} * T^{\frac{5}{2}} + \mathrm{Gz} * \mathrm{DTzey} * T^{\frac{5}{2}} + \mathrm{Eyuz} * T;$
$\mathrm{Sxy} = \mathrm{X0} * \mathrm{Y0} * T + \mathrm{X0} * \mathrm{My} * \mathrm{St} + \mathrm{X0} * \mathrm{Ey} * T^{\frac{3}{2}} + \mathrm{Y0} * \mathrm{Mx} * \mathrm{St} +$
$\mathrm{Mx} * \mathrm{My} * \mathrm{St2} + \mathrm{Mx} * \mathrm{Eyt} * T^{\frac{5}{2}} + \mathrm{Y0} * \mathrm{Ex} * T^{\frac{3}{2}} + \mathrm{My} * \mathrm{Ext} * T^{\frac{5}{2}} + \mathrm{Exy} * T^2;$

$\mathrm{Mxx} = \begin{pmatrix} T & \mathrm{Sx} \\ \mathrm{Sx} & \mathrm{Sx2} \end{pmatrix}; \mathrm{Vxy} = \begin{pmatrix} \mathrm{Sy} \\ \mathrm{Sxy} \end{pmatrix};$
$\mathrm{iMxx} = \mathrm{Inverse}[\mathrm{Mxx}];$

$\mathrm{Param1} = \mathrm{iMxx.Vxy};$

$\mathrm{P10} = \mathrm{Factor}[\mathrm{Expand}[\mathrm{Extract}[\mathrm{Param1}, \{1, 1\}]]];$
$\mathrm{P11num} = \mathrm{Numerator}[\mathrm{P10}];$
$\mathrm{K1} = \mathrm{Exponent}[\mathrm{P11num}, T];$
$\mathrm{Anum} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P11num}/T^{\mathrm{K1}}], T \to \infty];$
$\mathrm{P12den} = \mathrm{Denominator}[\mathrm{P10}];$
$\mathrm{K2} = \mathrm{Exponent}[\mathrm{P12den}, T];$
$\mathrm{Aden} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P12den}/T^{\mathrm{K2}}], T \to \infty];$
$\mathrm{Apar} = \mathrm{Factor}[\mathrm{Expand}[(\mathrm{Anum}/\mathrm{Aden}) * \frac{T^{\mathrm{K1}}}{T^{\mathrm{K2}}}]];$

$\mathrm{P20} = \mathrm{Factor}[\mathrm{Expand}[\mathrm{Extract}[\mathrm{Param1}, \{2, 1\}]]];$
$\mathrm{P21num} = \mathrm{Numerator}[\mathrm{P20}];$
$\mathrm{K3} = \mathrm{Exponent}[\mathrm{P21num}, T];$
$\mathrm{Bnum} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P21num}/T^{\mathrm{K3}}], T \to \infty];$
$\mathrm{P22den} = \mathrm{Denominator}[\mathrm{P20}];$
$\mathrm{K4} = \mathrm{Exponent}[\mathrm{P22den}, T];$
$\mathrm{Bden} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P22den}/T^{\mathrm{K4}}], T \to \infty];$
$\mathrm{Bpar} = \mathrm{Factor}[\mathrm{Expand}[(\mathrm{Bnum}/\mathrm{Bden}) * \frac{T^{\mathrm{K3}}}{T^{\mathrm{K4}}}]]$

$\mathrm{P30} =$
$\mathrm{Factor}[$
$\mathrm{Expand}[\mathrm{Sy2} + \mathrm{P10}^2 * T + \mathrm{P20}^2 * \mathrm{Sx2} - 2 * \mathrm{P10} * \mathrm{Sy} - 2 * \mathrm{P20} * \mathrm{Sxy} +$
$2 * \mathrm{P10} * \mathrm{P20} * \mathrm{Sx}]];$

$\mathrm{P31num} = \mathrm{Numerator}[\mathrm{P30}];$
$\mathrm{K5} = \mathrm{Exponent}[\mathrm{P31num}, T];$
$\mathrm{Vnum} = \mathrm{Factor}[\mathrm{Limit}[\mathrm{Expand}[\mathrm{P31num}/T^{\mathrm{K5}}], T \to \infty]];$
$\mathrm{P32den} = \mathrm{Denominator}[\mathrm{P30}];$
$\mathrm{K6} = \mathrm{Exponent}[\mathrm{P32den}, T];$
$\mathrm{Vden} = \mathrm{Factor}[\mathrm{Limit}[\mathrm{Expand}[\mathrm{P32den}/T^{\mathrm{K6}}], T \to \infty]];$
$\mathrm{Vpar} = \mathrm{Factor}[\mathrm{Expand}[T^{-1} * (\mathrm{Vnum}/\mathrm{Vden}) * \frac{T^{\mathrm{K5}}}{T^{\mathrm{K6}}}]];$

$\mathrm{Varianzas1} = \mathrm{Extract}[\mathrm{iMxx}, \{2, 2\}];$

$\mathrm{P40} = \mathrm{Factor}[\mathrm{Expand}[T^{-1} * \mathrm{P30} * \mathrm{Varianzas1}]];$
$\mathrm{P41num} = \mathrm{Numerator}[\mathrm{P40}];$
$\mathrm{K7} = \mathrm{Exponent}[\mathrm{P41num}, T];$
$\mathrm{Bvarnum} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P41num}/T^{\mathrm{K7}}], T \to \infty];$
$\mathrm{P42den} = \mathrm{Denominator}[\mathrm{P40}];$
$\mathrm{K8} = \mathrm{Exponent}[\mathrm{P42den}, T];$
$\mathrm{Bvarden} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P42den}/T^{\mathrm{K8}}], T \to \infty];$
$\mathrm{Bvar} = \mathrm{Factor}[\mathrm{Expand}[(\mathrm{Bvarnum}/\mathrm{Bvarden}) * \frac{T^{\mathrm{K7}}}{T^{\mathrm{K8}}}]]$

$\mathrm{Mzx} = \begin{pmatrix} T & \mathrm{Sx} \\ \mathrm{Sz} & \mathrm{Sxz} \end{pmatrix}; \mathrm{Mxz} = \begin{pmatrix} T & \mathrm{Sz} \\ \mathrm{Sx} & \mathrm{Sxz} \end{pmatrix}; \mathrm{Mzz} = \begin{pmatrix} T & \mathrm{Sz} \\ \mathrm{Sz} & \mathrm{Sz2} \end{pmatrix};$
$\mathrm{Vzy} = \begin{pmatrix} \mathrm{Sy} \\ \mathrm{Syz} \end{pmatrix};$
$\mathrm{iMzx} = \mathrm{Inverse}[\mathrm{Mzx}]; \mathrm{iMxz} = \mathrm{Inverse}[\mathrm{Mxz}];$

$\mathrm{Param2} = \mathrm{iMzx}.\mathrm{Vzy};$

$\mathrm{P50} = \mathrm{Factor}[\mathrm{Expand}[\mathrm{Extract}[\mathrm{Param2}, \{1, 1\}]]];$
$\mathrm{P51num} = \mathrm{Numerator}[\mathrm{P50}];$
$\mathrm{K9} = \mathrm{Exponent}[\mathrm{P51num}, T];$
$\mathrm{Fnum} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P51num}/T^{\mathrm{K9}}], T \to \infty];$
$\mathrm{P52den} = \mathrm{Denominator}[\mathrm{P50}];$
$\mathrm{K10} = \mathrm{Exponent}[\mathrm{P52den}, T];$
$\mathrm{Fden} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P52den}/T^{\mathrm{K10}}], T \to \infty];$
$\mathrm{Fpar} = \mathrm{Factor}[\mathrm{Expand}[(\mathrm{Fnum}/\mathrm{Fden}) * \frac{T^{\mathrm{K9}}}{T^{\mathrm{K10}}}]];$

$\mathrm{P60} = \mathrm{Factor}[\mathrm{Expand}[\mathrm{Extract}[\mathrm{Param2}, \{2, 1\}]]];$
$\mathrm{P61num} = \mathrm{Numerator}[\mathrm{P60}];$
$\mathrm{K11} = \mathrm{Exponent}[\mathrm{P61num}, T];$
$\mathrm{Dnum} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P61num}/T^{\mathrm{K11}}], T \to \infty];$
$\mathrm{P62den} = \mathrm{Denominator}[\mathrm{P60}];$
$\mathrm{K12} = \mathrm{Exponent}[\mathrm{P62den}, T];$
$\mathrm{Dden} = \mathrm{Limit}[\mathrm{Expand}[\mathrm{P62den}/T^{\mathrm{K12}}], T \to \infty];$
$\mathrm{Dpar} = \mathrm{Factor}[\mathrm{Expand}[(\mathrm{Dnum}/\mathrm{Dden}) * \frac{T^{\mathrm{K11}}}{T^{\mathrm{K12}}}]]$

```
P70 =
Factor[
Expand[Sy2 + P50^2 * T + P60^2 * Sx2 - 2 * P50 * Sy - 2 * P60 * Sxy+
2 * P50 * P60 * Sx]];
P71num = Numerator[P70];
K13 = Exponent[P71num, T];
Wnum = Factor[Limit[Expand[P71num/T^K13], T -> ∞]];
P72den = Denominator[P70];
K14 = Exponent[P72den, T];
Wden = Factor[Limit[Expand[P72den/T^K14], T -> ∞]];
Wpar = Factor[Expand[T^-1 * (Wnum/Wden) * T^K13/T^K14]];

Varianzas20 = (iMzx.Mzz.iMxz);
Varianzas2 = Extract[Varianzas20, {2, 2}];

P80 = Factor[Expand[T^-1 * P70 * Varianzas2]];
P81num = Numerator[P80];
K15 = Exponent[P81num, T];
Dvarnum = Limit[Expand[P81num/T^K15], T -> ∞];
P82den = Denominator[P80];
K16 = Exponent[P82den, T];
Dvarden = Limit[Expand[P82den/T^K16], T -> ∞];
Dvar = Factor[Expand[(Dvarnum/Dvarden) * T^K15/T^K16]]
```

B Parameter values of simulations

1. **Rejection rates of $t_{\hat{\delta}_{IV}}$ under white noise innovations**
   The values of the parameters in the DGP's are as follows: all DGP's: $\sigma_w = 1$ and no-autocorrelation in u_{wt}. DGP's with one break: $\lambda_y = 0.5$, $\lambda_x = 0.3$, and $\lambda_z = 0.6$; $\gamma_y = -0.015$, $\gamma_x = 0.035$, and $\gamma_z = 0.02$. Constants (or drifts): $\mu_y = 0.11$, $\mu_x = 0.09$, and $\mu_z = 0.05$. Trends: $\beta_y = 0.04$, $\beta_x = 0.07$, and $\beta_z = -0.07$.

2. **Rejection rates of $t_{\hat{\delta}_{IV}}$ under autocorrelated innovations**
   all DGP's are generated as in Table (1) except for the properties of u_{wt}; the innovation processes are generated as $AR(1)$. The values of the parameters in the $AR(1)$ specification are: $\rho_y = 0.5$, $\rho_x = 0.4$, and $\rho_z = 0.7$

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