Controlling polluting firms: Nash and Stackelberg strategies

Halkos, George and Papageorgiou, George

Department of Economics, University of Thessaly

September 2014

Controlling polluting firms: Nash and Stackelberg strategies

George E. Halkos and George J. Papageorgiou
Laboratory of Operations Research
Department of Economics
University of Thessaly

Abstract
In this paper we model the conflict between the group of polluting firms of a country and the social planer of the same country which attempts to control the volume of emissions generated during the production process. Both players of the game have their own control policies which are the rate of emissions on behalf the polluting firms and the rate of pollution control (e.g. abatement or taxation) on behalf the home country. The common state variable of the model is the number of the polluting firms, which is better to minimized through the country's control policy, but beneficial to maximized on the polluters' side. From the game theoretic point of view the model setup is very simple and belongs in to the special class of differential games also called state separable differential games. An important property for these games is that the open-loop Nash equilibrium coincides with the Markovian (closed-loop) equilibrium and in the case of hierarchical moves the analytical solutions are easy obtained. The game proposed here is analyzed for both types of equilibrium, i.e. Nash and Stackelberg. In the simultaneous move game (i.e. the Nash game) we find the equilibrium analytical expressions of the controls for both players as well as the steady state stock of the polluting firms. A sensitivity analysis of the crucial variables of the model takes place. In the hierarchical move game (i.e. the Stackelberg game) we find the equilibrium values of the controls as well as of the state variable. As a result a comparison between the two types of equilibrium for the game takes place. The analysis of the comparison reveals that the conflict is more intensive (since both controls have greater values) for the case in which the polluting firms play as the leader of the hierarchical move game.

Keywords: Pollution control; Environmental Economics; Differential games.

JEL Codes: C61; C62; D43; H21; Q50; Q52; Q53.

1. Introduction

The choice of the differential game models, in order to design efficiently conflicting situations between the polluters and the victims of pollution, is rather the rule than the exception. In this paper, we may use the efficiency of the differential game models to study the dynamic interactions of the polluting firms in a country and the social planer of the same country. The strength of the polluting firms as a group changes over time and it is measured by the volume of active polluters, the transactions made among them, by how dangerous for the environmental amenities are the polluting firms as a group and so on. New polluting firms are initiated and encouraged by the existing.

Regarding the polluter's attrition, their decay rate is affected by their own actions and by the counter–pollution actions of the home country as well. The essential targets of the home country are to derive utility from the polluting firms' emissions reduction, but the home country face substantial costs combating the polluters and suffer from disutility stemming from the size of the polluters. Conversely, each polluting firm wants to maximize the size of the group of the polluters as well as its utility stemming from the emissions.

In this study we deal with a special class of differential games called the state–separable game. The state–separable differential games belong into the special class of dynamic games which allow, in the most cases, the derivation of the Nash solutions in explicit form. The advantage of the analytical solutions, according to Dockner et al. (2000), is of great importance because the derived mathematical expressions of the solutions are crucial for the study of the qualitative properties of equilibrium.

Due to the simplicity of the structure the state separable differential games are characterized by the linearity of the objective functional with respect to the state variable(s) and by no interaction between control and state variables (Dockner et al, 1985). An important property of the state separable games is related with the information structure employed. The importance of that property is that the open loop Nash solution coincides with the closed loop (Markovian) Nash solution.

Another important property hinges on the way the game played, i.e. simultaneously (Nash) or hierarchically (Stackelberg). As it is known (e.g. Başar and Olsder 1992, Dockner et al 2000), in the Stackelberg games, the adjoint variable of the leader w.r.t. the adjoint variable of the follower plays a crucial role at the solution process, but due to the state separability the interconnection between these variables vanishes.

In the rest of the paper we determine the Nash and the Stackelberg solutions of the environmental differential game and the state–separability advantage allow to write down some useful propositions and to carry out sensitivity analyses. On the design efficient counter–pollution actions against the polluting firms of a country, the model parameters of the game and the relevance of the two solutions offers useful information as well.

The paper is organized as follows. In section 2 we setup the basic model. Section 3 considers the solutions of the Nash equilibrium and performs a simple sensitivity analysis. In section 4 we compute the analytical expressions of the open–loop Stackelberg equilibrium while the polluting firms leads and the social planer of home country follows. Section 5 compares the two solution strategies, while the last section concludes the paper.

2. The model

In the real world scenario, it seems plausible that the mere existence of polluting firms (the polluters) is considered as being an intertemporal threat to any home country's environmental quality. Translating into strategies, the polluting firms on one hand, have to decide about the volume of the emission attacks will carry out, while the home country on the other hand has to defend in the "war of pollution". In the model presented here the state variable of the above clash is the volume of polluting firms, which denoted by x.

Moreover, we make the assumption that the new polluting firms are supported and financed by the existing, thus it is reasonable to face the growth of the polluting firms as in the population models in the absence of controls. Analogously to the models of population a very simple equation that is suitable to describe the evolution of population of the polluting firms at time t, $x(t)$, is the following differential

$$\dot{x} = gx, \qquad x(0) > 0 \tag{1}$$

where g denotes the endogenous growth rate of the polluting firms.

The volume of emissions realizations (denoted by υ) reduces the number of polluting firms due to the compliance costs, i.e. the more (stronger) the emissions the higher the penalties imposed by authorities, consequently the lower the number of the polluting firms that survive from the curse of compliance costs. We assume for simplicity that this fact is proportional to the number of emissions realizations, i.e. $\gamma\upsilon$, and as reduces the volume of the polluters, it is added as an outflow term to equation (1), i.e. it is entered into (1) with the minus sign.

Moreover, we set as the control variable of the home country the intensity u of the counter–emissions effort. The greater the intensity of the counter–emissions effort, the more resources there are that can be devoted to investigating the implications of emissions realization. Moreover, the stronger the home country's counter–pollution effort, the more effective is the reduction of the polluting firms. We assume that this fact is the linear term $f(u) = \beta u$, and the parameter β denotes the percentage looses per emission realization, on behalf the polluters, when the social planer of the home country, abates (or taxes) the pollutants (is counter–offensive). Again, the above term reduces the volume of the polluting firms, and therefore we add a second outflow term to (1) that weights the volume of emissions υ with βu.

Regarding the control variable of the home country, i.e. the intensity of counter–pollution effort, this control certainly reduces the volume of the polluters and therefore a new negative term is entered into the equation (1). This term represents the looses due to the intensity of counter–measures at the initiation phase and is proportional to the control u, i.e. is the term ϕu. Here we note that taking measures against the polluting firms' initiation is very sensitive process as the planer of the home country has to discriminate among the firms. Since the discrimination process lurking risks (e.g. the taxation must be not a blind taxation), we designate this inflow to equation (1), as a quadratic, with respect to the intensity of pollution control measures, cost function (e.g. the square of abatement or taxation).

After all, the volume of polluting firms evolves according to the following equation:

$$\frac{dx}{dt} = \dot{x} = gx - \phi u + \frac{a}{2}u^2 - \gamma \upsilon - \beta u \upsilon$$

where:

$x \geq 0$ the state variable (the volume of polluting firms)

$u \geq 0$ the control variable of the home country i.e. the intensity of the home country's counter pollution effort,

$\upsilon \geq 0$ emissions' rate (control variable of the polluting firms)

$g \geq 0$ endogenous growth rate of the group of polluters

$\phi \geq 0$ rate at which the counter pollution measures would reduces the polluting firms

$\frac{a}{2} \geq 0$ the cost factor which faces the home country due to the unsuccessful discrimination among the overall firms during the abatement (or taxation).

$\beta \geq 0$ percentage looses of the polluters per emission

$\gamma \geq 0$ average number of polluting firms which are not able to face the compliance costs.

In this paper, we assume that the social planer of home country wishes to minimize the following objectives. First, he wants to minimize the volume of emissions υ and second to minimize the volume of the polluting firms x (which is the state variable of the model). An important reason the social planer may wish to minimize the volume of polluters is that the threat of pollutants concentration is costly for the home country, because of costs associated with the uncertainty of business investments which in turn leads to the market shrinkage. As the third objective, the home country has an interest in minimizing the counter–pollution effort (e.g. in lowering the environmental tax factor), by minimizing its control variable u. It is well known that the pollution–control activities cost money, as almost any control policy execution.

In the decision making literature, the social planning, in intertemporal formulations, is described as trying to minimize a weighted sum of the state x and the opponent's control υ, as well as the effort cost stemming from its own control variable u. Therefore after the above simplified assumptions and with a positive discount rate ρ_1, the intertemporal home country's minimized functional will be the following

$$\min_{u(.)} \int_0^\infty e^{-\rho_1 t} \left(c_1 x + c_2 \upsilon + c_3 u\right) dt \qquad (2)$$

The polluting firms as a group, on the other hand, are interested to increasing their number x in order to exert more market power. The emissions' rate υ is their control variable which is maximized. But the emission realizations cost money and this cost is represented in the objective functional by the quadratic cost function $\left(c_4/2\right)\upsilon^2$. Regarding the polluting firms benefits with respect to the counter pollution effort, i.e. the home country's control variable u, the high values of that control may work as an indirect way of stirring up sentiments against the home's environmental policy. Therefore we represent this displeasure as a polluting firms' benefit and we set in their objective functional as the weighted term bu.

Finally, for a positive discount rate ρ_2 the intertemporal objective function of the polluting firm may be the following

$$\max_{\upsilon(.)} \int_0^\infty e^{-\rho_2 t} \left(b_1 x + b_2 \upsilon + b_3 u - \frac{c_4}{2} \upsilon^2\right) dt \qquad (3)$$

with

$$\rho_i > g \quad i = 1,2 \qquad (4)$$

the home country minimizes functional (2) and the polluting firms maximizes (3) subject to (1) and the path constraints

$$x,\ u,\ \upsilon \geq 0$$

In the next sections we proceed with the calculation of both Nash and Stackelberg equilibrium solutions.

3. Nash equilibrium

The Nash equilibrium computation is derived under the assumption that both players play the game at the same time. Then, every player of the game (i.e. the home country and the polluting firms) has to solve their own optimal control problem, taking the opponent's reaction as given. Finally, the two optimal control solutions determine the game optimal controls u^*, υ^*. In the following we denote by λ and μ the shadow prices of the state variable x for the home country and the polluters respectively. Now the current value Hamiltonians of the game described above are given by

$$H_1 = -c_1 x - c_2 \upsilon - c_3 u + \lambda\left(gx - \phi u + \frac{a}{2}u^2 - \gamma\upsilon - \beta u\upsilon\right) \qquad (5)$$

$$H_2 = b_1 x + \left(b_2 - \frac{c_4}{2}\upsilon\right)\upsilon + b_3 u + \mu\left(gx - \phi u + \frac{a}{2}u^2 - \gamma\upsilon - \beta u\upsilon\right) \qquad (6)$$

Proposition 1
Along the optimal path, the shadow price of the home country's state variable is always negative, since one additional polluting firm is always harmful for the home country's environmental quality. Conversely, since one more polluting firm increases the benefits of the group of polluters, the shadow price of the state variable of the polluters is positive along the optimal path.

Proof

The result is obtained through the Pontryagin's maximum principle optimality conditions, i.e.,

$$\dot{\lambda} = (\rho_1 - g)\lambda + c_1 \qquad (7)$$

with the equilibrium
$$\dot{\lambda} = 0 \Rightarrow \hat{\lambda} = -\frac{c_1}{\rho_1 - g} < 0 \qquad (8a)$$

and the polluting firms' shadow price evolves according to the following equation

$$\dot{\mu} = (\rho_2 - g)\mu - b_1$$

with equilibrium
$$\hat{\mu} = \frac{b_1}{\rho_2 - g} > 0 \qquad (8b)$$

According to $(8a)$ the long–run damage to the home country, implied by having one more polluting firm ($\hat{\lambda}$), increases. This is the result of an increasing cost associated with the existence of a polluting firm (i.e. the factor c_1 in the home country's objective functional). Note that according to basic theorems of the optimal control theory the transversality conditions hold for all admissible state trajectories (e.g. Grass et al, 2008).

For the following analysis presented here it is assumed that only interior solutions exist and they are positive, i.e. $u,\ \upsilon > 0$. According to Pontryagin's maximum principle, the maximizing condition of the Hamiltonian for the intensity of the home country's pollution–control effort (the home country's control variable) is given by

$$\frac{\partial H_1}{\partial u} = 0 \ \Leftrightarrow\ -c_3 + \lambda\phi - \lambda\beta\upsilon + \lambda a u = 0 \ \Leftrightarrow\ \ u^* = \frac{1}{a}\left(\frac{c_3}{\lambda} + \gamma + \beta\upsilon\right) \qquad (9)$$

The result (9) is recorded in proposition 2.

Proposition 2
The optimal counter–pollution effort u^* increases with: *a) a rising volume of emissions,* *b) an increasing percentage lost of polluting firms per emissions (β),* *c) an increasing rate at which pollution–control reduce the polluting firms (γ).* *The cost factor which faces the home country due to the unsuccessful discrimination among the firms of the country during the exercise of the counter pollution measures $(a/2)$ has a decreasing influence on the home country's intensity of conducting the above effort.*

Looking at the control variable analytical expression (9), it is worth noting that if the cost of control (c_3) is large relative the home's shadow price λ (which is negative along the optimal path), the home country's optimal control u^* becomes low and possibly meets the boundary at $u^* = 0$. Conversely, if the cost of the control is negligible with respect to the shadow price λ, the home's optimal control collapses into a linear function of emissions υ, since the term c_3/λ in (9) vanishes. Therefore it is optimal, in the former case, for the home country to not exert any counter–pollution control.

Turning in the polluters' problem and regarding their emissions, the Hamiltonian maximizing condition is determined by

$$\frac{\partial H_2}{\partial \upsilon} = 0 \quad \Leftrightarrow \quad b_2 - c_4\upsilon - \mu(\gamma - \beta\upsilon) = 0 \quad \Leftrightarrow \quad \upsilon^* = \frac{b_2}{c_4} - \frac{\mu}{c_4}(\gamma + \beta u) \qquad (10)$$

We record the result (10), as

Proposition 3
The optimal rate of the polluters' emissions υ^* *decreases with:* *a) an increasing average number of the polluting firms abandonment* (γ)*,* *b) an increasing* percentage looses per emission (βu)*, and* *c) an increasing state's variable shadow price* μ *of the group of polluters.*

According to (10) if the shadow price of the polluting firms is raised, then it is optimal for the polluters to curb the emissions' rate. Conversely, along the polluters' optimal path, the rate of emissions increases as the emissions' benefits(b_2) increases relative to the costs(c_4).

A useful corollary according to the optimality conditions (9) and (10) it must be the following: "Along the home country's optimal path the intensity of pollution–control measures raises while the rate of emissions increases, and the rate of emissions declines while the intensity of the counter-pollution measures is increasing".

The stationary values of the controls in the Nash equilibrium are the following

$$\begin{aligned}\hat{u}_N &= \frac{\beta\left(b_2 - \hat{\mu}\gamma\right) + c_4\left(\phi + c_3/\hat{\lambda}\right)}{c_4 a + \hat{\mu}\beta^2} \\ \hat{\upsilon}_N &= \frac{a\left(b_2 - \hat{\mu}\gamma\right) - \hat{\mu}\beta\left(\phi + c_3/\hat{\lambda}\right)}{c_4 a + \hat{\mu}\beta^2}\end{aligned} \qquad (11)$$

with $\hat{\lambda}$, $\hat{\mu}$ given by $(8a)$ and $(8b)$, where N in (11) means the Nash solution. The Nash equilibrium value for the polluting firms is given by

$$\hat{x}_N = \frac{1}{g}\left[\left(\phi - \frac{a}{2}\hat{u}_N\right)\hat{u}_N + \left(\gamma + \beta\hat{u}_N\right)\hat{\upsilon}_N\right] \qquad (12)$$

and $\hat{u}_N$, $\hat{\upsilon}_N$ as in (11).

Here it is worth noting the advantage regarding the structure of the state separable games, due to which we have the opportunity to find the analytical expressions of the controls as well as of the state variable. The solution (11) is a unique closed loop Nash equilibrium. This advantage is rather unusual, since the multiple solutions in differential games is the rule. Due the analytical expressions (11) and (12) it is easy to proceed with sensitivity analysis with respect to the model parameters.

Table 1 represents the results of sensitivity analysis. Taking the partial derivatives $\partial(.)/\partial(\text{parameter})$, the symbol "+" means that the partial derivative is greater than zero, the symbol "–" means the opposite case, 0 indicates that the result of the partial derivative is zero (the parameter is not a part of the control), and ? denotes that the result is unknown. The results in Table 1 make some economic sense. Taking into account (8*b*) the polluters' shadow price $\hat{\mu}$ decreases with the discount factor ρ_2, but increases with the factor b_1 and with the endogenous growth rate g.Taking into account (11) the stationary value of the polluting firms $\hat{x}_N$ decreases with increasing endogenous rate g (as the control factor c_3 is equal to zero).

Table 1: A summary of the sensitivity analysis results

	ϕ	α	β	γ	c_1	c_2	b_1	b_2	c_4	ρ_1	ρ_2
$\hat{u}_N$	+	–	?	–	0	0	0	+	0	0	0
$\hat{\upsilon}_N$	–	?	?	0	0	0	0	+	+	0	0
$\hat{x}_N$	+	–	+	+	0	0	0	0	0	0	0

4. The Leader–Follower game (polluting firms as a leader)

In the Nash equilibrium solution, as illustrated above, it is assumed that the two player game played simultaneously. i.e. the moves of the rivals are made at the same time. As it is mentioned above, in this paper we explore and the other class of games in which one player, the leader, moves first, and the opponent, the follower, makes his/her decision at the second time. As it is known, this hierarchical or sequential mode of playing the game is the leader–follower or Stackelberg mode. In the game theoretic literature, e.g. Olsder and Başar, 1999, it has been developed at least one stepwise procedure to derive the equilibrium solution. In order to describe (for completeness) the solution procedure we assume, without any loos of generality, that the first player is the leader and the second is the follower. The control and adjoint variables of the leader are denoted with u, λ respectively, and with υ, μ we denote the same variables for the follower. We assume moreover that the cost of pollution control vanishes, i.e. $c_3 = 0$.

The three step procedure for the (open–loop) Stackelberg solution (e.g. Grass et al, 2008, Dockner et al., 2000, Basar T., Olsder G.,1999):

Step 1: The polluting firms, as group, announce their common strategy, υ

Step 2: For the given strategy υ, the home country (the follower) solves the same Nash optimal control problem. As it is mentioned in the Nash case (see (9)), the home's optimal response to the polluters' strategy υ, will be

$$u^* = u^*(\upsilon) = \frac{1}{a}(\gamma + \beta\upsilon) \qquad (13)$$

since it is assumed that $c_3 = 0$.

the adjoint λ variable for the follower is given by equation (7).

Step 3: Now, in the last step, the leader has to solve the same as in the Nash case optimal control problem, but for the known reaction function (13) of the follower:

$$\max_{\upsilon(.)} \int_0^\infty e^{-\rho_2 t} \left(b_1 x + \left(b_2 - \frac{c_4}{2} \upsilon \right) \upsilon + b_3 u^*(\upsilon) \right) dt$$

subject to the following state equations

$$\dot{x} = gx - \left(\phi - \frac{a}{2} u^*(\upsilon) \right) u^*(\upsilon) - \gamma \upsilon - \beta u^*(\upsilon) \upsilon \qquad (14)$$

$$\dot{\lambda} = (\rho_1 - g)\lambda + c_1 \qquad (15)$$

with $u^*(\upsilon)$ given by (13).

The Hamiltonian of player 2 (the follower) becomes

$$H_2 = b_1 x + \left(b_2 - \frac{c_4}{2} \upsilon \right) \upsilon + b_3 u^*(\upsilon) + \mu \dot{x} + \psi \dot{\lambda} \qquad (16)$$

The adjoint variables are the shadow values of the states x, λ for which the equations of motion are given by (14) and (15) respectively. Taking the first order condition for the Hamiltonian (16), i.e., $\partial H_2 / \partial \upsilon = 0$ we found the optimal strategy υ^*. The calculations of the stationary strategies are made through the substitutions in (13) the player's 2 optimal strategy. After the rearrangement the final expressions are:

$$\begin{aligned} \hat{u}_S &= \frac{\beta(b_2 - \hat{\mu}\gamma) + c_4\phi + b_3(\beta^2/a)}{c_4 a + \hat{\mu}\beta^2} \\ \hat{\upsilon}_S &= \frac{a(b_2 - \hat{\mu}\gamma) - \beta(\hat{\mu}\phi - b_3)}{c_4 a + \hat{\mu}\beta^2} \end{aligned} \qquad (17)$$

with *S* to denote the Stackelberg strategy. The number of polluting firms is given by

$$\hat{x}_S = \frac{1}{g}\left(\left(\phi - \frac{a}{2}\hat{u}_S \right)\hat{u}_S + (\gamma + \beta \hat{u}_S)\hat{\upsilon}_S \right) \qquad (18)$$

and the optimal controls are given by (17). Since the analytical expressions of the optimal strategies are computed for both types of the game, in the next section we compare these values.

Note that, In the reverse case at which the home country moves first as a leader and the polluting firms follow the following controls are optimal[1]

$$\hat{u}_{S,L} = \frac{\beta(b_2 - \hat{\mu}\gamma) + c_4\phi - \beta\hat{\mu}\left(\frac{c_2}{\hat{\lambda}} + \gamma\right)}{c_4 a + 2\beta^2\hat{\mu}}$$
$$\hat{\upsilon}_{S,F} = \frac{b_2 a - \hat{\mu}(a\gamma + \beta\phi) + b_3\beta + \frac{\beta^2\hat{\mu}^2}{c_4}\left(\frac{b_2}{\hat{\mu}} + \frac{c_2}{\hat{\lambda}}\right)}{c_4 a + 2\beta^2\hat{\mu}}$$

5. Comparison of the two solutions

Taking the Nash solutions (11) and the Stackelberg solutions (17) the optimal controls can be expressed as

$$\hat{u}_S = \hat{u}_N + \frac{\beta}{a}\Delta$$

$$\hat{\upsilon}_S = \hat{\upsilon}_N + \Delta$$

while
$$\Delta = \frac{b_3\beta}{c_4 a + \beta^2\hat{\mu}} > 0 \qquad (19)$$

the difference between the optimal stationary strategies is given by (19). Some remarks can be drawn about the difference of the two solutions of the same game. These observations could be:

i) The fewer the polluting firms losses per emission (β), the smaller the difference Δ. If the losses rate β vanishes $(\beta = 0)$, the Nash and Stackelberg equilibrium solutions become equal.

---

[1] The analysis of the latter Stackelberg equilibrium case is left for future research.

ii) if the polluting firms have no objective which is related with the unsuccessful discrimination on behalf the social planer ($b_3 = 0$), the Nash and Stackelberg equilibrium solutions are equal. If the same factor b_3 is positive, the group of polluting firms announces a volume of emissions, υ_S, such that the home country reacts with a higher of counter–pollution effort, u_S. As a result the number of polluting firms x increases which in turn increase the volume of emissions.

As follows from the comparison of (11) and (17)

$$\hat{u}_S > \hat{u}_N \qquad \text{and} \qquad \hat{\upsilon}_S > \hat{\upsilon}_N$$

the conflict will be more intensive if the group of polluting firms has the first mover advantage and announces the volume of emissions to be carried out (compared to the simultaneous move game). Consequently, the next result becomes obvious.

Proposition 4
The pollution control hierarchical game in which the group of polluting firms being the leader and the home country the follower, gives a higher volume of emissions and a more intensive counter–pollution effort, i.e. the conflict between the players is more intensive.

The difference between the equilibrium values (12) and (18) is positive, that is $D = \hat{x}_S - \hat{x}_N > 0$, and therefore we can conclude that the polluting firms being the leader verifies its better position due to the increase D in its size.

The linear state equations (12) and (18) can explicitly solved with respect to the state $x(t)$, yielding:

$$x_N(t) = x_{N0}e^{gt} + \hat{x}_N\left(1 - e^{gt}\right)$$

$$x_S(t) = x_{S0}e^{gt} + \hat{x}_S\left(1 - e^{gt}\right)$$

And the value functions for the Nash and Stackelberg equilibrium is easy computed as:

$$V_{2,N} = \int_0^\infty e^{-\rho_2 t}\left(b_1 x + b_2 \hat{\upsilon}_N + b_3 \hat{u}_N - \frac{c_4}{2}\hat{\upsilon}_N^2\right)dt =$$
$$= b_1 \frac{\rho_2 x_{N0} - g\hat{x}_N}{\rho_2\left(\rho_2 - g\right)} + \frac{2b_2\hat{\upsilon}_N - c_4\hat{\upsilon}_N^2 + 2b_3\hat{u}_N}{2\rho_2}$$

and

$$V_{2,S} = \int_0^\infty e^{-\rho_2 t}\left(b_1 x + b_2 \hat{\upsilon}_S + b_3 \hat{u}_N - \frac{c_4}{2}\hat{\upsilon}_S^2\right)dt =$$
$$= b_1 \frac{\rho_2 x_{S0} - g\hat{x}_S}{\rho_2\left(\rho_2 - g\right)} + \frac{2b_2\hat{\upsilon}_S - c_4\hat{\upsilon}_S^2 + 2b_3\hat{u}_S}{2\rho_2}$$

Moreover, the difference of the two value functions

$$V_{2,S} - V_{2,N} = \frac{b_3\beta\Delta}{2\rho_2 a} > 0$$

is positive, and therefore becomes better for the group of the pollutin firms to lead playing the Stackelberg strategy than playing the Nash strategy. This result is recorded as Proposition 5.

Proposition 5.
In the environmental pollution game between the polluting firms of a country and the social planer of the same country the more beneficial strategy, on behalf of the polluters, is the strategy in which they lead (and the home country follows) in a Stackelberg setting.

6. Conclusions

In this paper we setup a differential game model between the polluting firms of a country and the social planer of the same country. The model belongs into the special tractable class of the state separable games. This class of games has the special feature, in the Nash equilibrium, for which the open-loop equilibrium coincides with the closed-loop (Markovian) equilibrium. During the solution process, of the simultaneous move game, we found the analytical expressions of both players' controls as well as the steady state of the stock variable (which is the volume of the polluting firms). A sensitivity analysis, which is an analysis between the controls and the crucial variables of the model, makes economic sense.

Moreover a number of propositions are stated from the same Nash equilibrium game. As an extension of the model, we setup the game in the case of hierarchical move, i.e. we transform the Nash game into a Stackelberg game. With the above transformation the computed equilibrium values become different. The analysis of the control values comparison, for both equilibrium concepts, reveals that the conflict between the players of the game becomes more intensive in the case of the Stackelberg game. Moreover we found, comparing the payoffs of the polluting firms for both equilibrium concepts, that is better to play as leaders in a Stackelberg game. Finally, some results, recorded as propositions, are stated as well as in the case of the Stackelberg equilibrium.

References

Başar T., Olsder G., (1999). Dynamic Noncooperative Game Theory, Academic Press New York, 2nd Edition

Dockner, J., Feichtinger, G., (1991). On the Optimality of Limit Cycles in Dynamic Economic Systems, Journal of Economics, 53, 31 – 50.

Dockner, J., Feichtinger, G., Jorgensen, S., (1985). Tractable classes of Non Zero Sum open Loop Nash Differential Games: Theory and Examples, Journal of Optimization Theory and Applications, 45(2), 179–197.

Dockner, E., Jorgensen, S., Long, N.V., Sorger, G., (2000), Differential games in economics and management science, Cambridge University Press.

Grass, D., Gaulkins, J., Feichtinger, G., Tragler, G., Behrens, D., (2008), Optimal Control of Nonlinear Processes. With Applications in Drugs, Corruption and Terror, Springer, Berlin.