A Simple Bargaining Model where Parties Make Errors

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Abstract

In this paper, we develop a bargaining model where parties (or their intermediaries) make errors when reporting their bid. We characterize the Nash equilibria of the game and show that there is a unique equilibrium where trade takes place. This trade equilibrium is shown to converge to the Nash Bargaining Solution of the problem as trembles diminish. Finally, we discuss our results in the context of the previous literature providing a critique of the model and analysis found in Carlsson (1991).

Keywords: Nash Program, Nash Bargaining Solution, Equilibrium Selection

JEL Codes: C7, C72, C78

1 Introduction

The typical exchange between a buyer and seller is no longer face-to-face, but rather takes place in online markets where rules of exchange are pre-established and governed by a computer program. Since the exact details of the computer program (e.g., rounding rules, mistakes in computer code, etc.) are unknown to participants, some shared uncertainty is introduced into the

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exchange process. In this paper, we show this type of shared uncertainty can serve to coordinate behavior in a well known bargaining game. Additionally, as the uncertainty diminishes, the prediction of our bargaining game with errors converges to a unique prediction in the same bargaining game with no errors – the Nash Bargaining Solution.

The main idea of this paper can be traced back to Nash who, in a collection of papers, built a formal game theoretic framework for the study of bargaining problems. In 1950, Nash developed his well known axiomatic bargaining solution. The axiomatic method abstracts away from the specific procedural details involved in bargaining, but Nash (1953) later argued that his bargaining solution should also be supported by the equilibria of a non-cooperative model of the bargaining process.[1] The game studied by Nash, now known as the Nash Demand Game, involves two players who simultaneously announce payoff demands. If these demands are "compatible" according to some pre-established definition, then each player receives his demand. Otherwise the outcome ends in disagreement. It is well known that this game has a continuum of Nash equilibria where the parties reach an agreement and equilibria where the parties disagree.[2] In contrast, the Nash Bargaining Solution suggest a unique payoff pair.

The multiplicity of equilibria in the game therefore forced Nash to consider a refinement.[3] In particular, Nash changed the game to allow for some uncertainty about whether certain pairs of demands would be compatible. Roughly speaking, the game was altered so the probability of agreement was equal to one for all compatible demands and then would go to zero quickly as demand pairs got further from compatibility. The payoff structure in the original game is discontinuous around the agreement equilibria – i.e., small changes in behavior may lead to large changes in a player's payoff. The act of perturbing the game effectively "smooths" the payoffs in the game. Nash argued that the equilibria outcomes of the smoothed game should approach his bargaining solution as the level of smoothing went to zero.[4]

[1] This type of exercise of supporting axiomatic solutions with non-cooperative solutions is called the Nash Program.

[2] See, for example, van Damme (1991) p.146 or Malueg (2010).

[3] Young (1993) develops an evolutionary model to study this game where conventions are derived to deal with the indeterminacy.

[4] Nash's argument is informal. See Binmore (1987a,b), van Damme (1991), or Osborne and Rubinstein (1990) for a formal description and analysis of Nash's perturbed game approach to equilibrium selection. Kaneko (1981) uses a similar technique in a bilateral

This "perturbation" approach taken by Nash generates an attractive outcome, but the plausibility of the perturbation has been criticized since it is tailored to the structure of the Nash Demand Game.[5] In contrast, Carlsson (1991) considers a Nash Demand Game with a more plausible perturbation. In his model, a buyer and seller bargain over the price of an indivisible object by submitting bids, but the participants' bids are subject to trembles. Formally, bids are determined by adding a random error term to each player's action, where the support of the error variables are compact intervals in $\mathbb{R}$.[6] Bids are compatible when the buyer's "random" bid is larger than seller's "random" bid. Similar to Nash's perturbation idea, the addition of the error terms serves to eliminate the discontinuity in the each player's payoff function. Carlsson asserts this game also yields a convergence result where a Nash equilibrium approaches the Nash Bargaining Solution as the error terms go to zero. However, while both Carlsson's model and convergence claim are attractive, Carlsson fails to take into account how his compact support assumption affects the derivation of the expected payoff functions and the derivatives of these functions. This oversight leaves incorrect payoff functions for the players as well as incorrect derivatives of those functions. Moreover, because these flawed expressions are used throughout the analysis, we cannot be certain of the validity of any of the claims found in the paper – including the convergence result.

In this paper we consider a simplified version of Carlsson's model and analyze a Nash Demand Game where the actions taken by players' bids are subject to trembles. We develop this model in detail and completely characterize the set of Nash equilibria. The simplifications in the model allow us to depart from Carlsson's approach to the problem and illustrate our results in a more direct fashion. We show there is a unique equilibrium where the buyer and seller trade and that this equilibrium converges to the Nash Bargaining Solution as the magnitude of the errors diminishes to zero. This provides the intuitive perturbation of the Nash Demand Game envisioned by Carlsson and illustrates the robustness of Nash's Bargaining Solution.

monopoly model.

[5]See, for example, Luce and Raiffa (1957) pg. 141-142.

[6]A similar model of bargaining with errors was first sketched out by Binmore (1987b, p.146). In his model, the errors have unbounded support and Binmore illustrates how any Nash equilibria will likely convergence to the Nash Bargaining Solution – so long as the distribution functions are not "pathological." Carlsson's paper may be seen as an attempt to formalize some of these ideas.

Finally, we provide a detailed critique of Carlsson (1991) and discuss our results in the context of the previous literature.

2 Model

A buyer and a seller are bargaining over the price of an indivisible object. However, rather than bargaining "face-to-face," the players interact through an intermediary. In this interaction, the buyer chooses a bid price b and the seller chooses an asking price s. These choices are given to the intermediary who helps arbitrate the sale of the object on the players' behalf.

The intermediary is known to make a random error when reporting prices. So, if the buyer submits a bid b, the intermediary reports a bid of $\tilde{b} = b + \epsilon_b$, where ϵ_b is a random error. Similarly, if the seller submits an asking price s, the intermediary reports an asking price of $\tilde{s} = s + \epsilon_s$, where ϵ_s is the error term. These error terms are independently and identically distributed according to the uniform distribution on the interval $[-z, z]$, where $z \in \mathbb{R}_+$ is a strictly positive number. Thus, the distribution of the error term is $F(x) = \frac{x+z}{2z}$ on $[-z, z]$ with associated density $f(x) = \frac{1}{2z}$.

Finally, the players have agreed to the following rules to govern their interaction. Players simultaneously submit their bids (s, b) to the intermediary who reports $(\tilde{s}, \tilde{b})$. If the buyer's reported price $\tilde{b}$ is higher then the seller's reported asking price $\tilde{s}$, then there is trade. In this case, the buyer receives the item and pays a price equal to $\tilde{b}$. The seller receives $\tilde{s}$. Otherwise there is no trade. Thus, if the realized prices $(\tilde{s}, \tilde{b})$ are such that $\tilde{b} \geq \tilde{s}$, then the seller and buyer payoffs are $u_S = \tilde{s}$ and $u_B = b^* - \tilde{b}$ respectively, where $b^* > 0$ is the buyer's reservation price. Both players attach a zero value to the "no trade" outcome. The Nash Bargaining Solution of the underlying problem is for the seller and buyer to both demand half of the surplus – i.e., $s = b = \frac{b^*}{2}$.

3 Expected Payoff Functions

In the bargaining game, players submit a bid to the intermediary. The error made by the intermediary induces a distribution of potential reported bids for each player. As a consequence, a player i knowing the profile $(s, b) \in \mathbb{R}^2$ must compute an expected payoff function $\pi_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ to evaluate his choices. We now detail the computation of each player's expected payoff function.

The key difficulty in computing the expected payoff function arises because the support of the error distribution is a bounded interval. The players' payoff functions will depend on whether $b \geq s$ and the distance between b and s in relation to the error parameter z. This gives us four cases.

Case 1

In the first case, trade occurs with certainty. The set of profile with this property, C_I, is the collection of profiles (s, b) where $s < b$ and $\tilde{s} \leq \tilde{b}$ for all possible realizations – i.e.,

$$C_I = \{(s, b) : s + z < b - z\}.$$

Therefore, for $(s, b) \in C_I$, the expected payoff functions for the seller and buyer are

$$\begin{aligned}
\pi_S(s, b) &= \frac{1}{4z^2} \int_{-z}^{z} \int_{-z}^{z} (s + \epsilon_s)\, d\epsilon_s d\epsilon_b = s \\
\pi_B(s, b) &= \frac{1}{4z^2} \int_{-z}^{z} \int_{-z}^{z} (b^* - (b + \epsilon_B))\, d\epsilon_s d\epsilon_b = b^* - b
\end{aligned}$$

respectively.

Case 2

In the second case, $s \leq b$, but trade does not occur with certainty. The set of profiles with this property is

$$C_{II} = \{(s, b) : s - z \leq b - z < s + z \leq b + z\}.$$

Alternatively, $(s, b) \in C_{II}$ if $s \in [b - 2z, b]$.

The expected payoff function in C_{II} is more complex than C_I. There is trade when $b + \epsilon_b \geq s + \epsilon_s$. If $\epsilon_b \leq s - b + z$ then there is only trade for small realizations of ϵ_s. In contrast, for $\epsilon_b \geq s - b + z$, there is trade for all realizations of ϵ_s – since $-z \leq \epsilon_s \leq z$. Thus, the expected payoff function for the seller is

$$\begin{aligned}
\pi_S(s, b) &= \frac{1}{4z^2} \int_{-z}^{s-b+z} \int_{-z}^{b-s+\epsilon_b} (s + \epsilon_s)\, d\epsilon_s d\epsilon_b + \frac{1}{4z^2} \int_{s+z-b}^{z} \int_{-z}^{z} (s + \epsilon_s)\, d\epsilon_s d\epsilon_b \\
&= \frac{1}{4z^2} \int_{-z}^{s-b+z} \int_{-z}^{b-s+\epsilon_b} (s + \epsilon_s)\, d\epsilon_s d\epsilon_b + s \left(\frac{b - s}{2z} \right).
\end{aligned}$$

The expected payoff function for the buyer is similarly computed to be

$$\pi_B(s,b) = \frac{1}{4z^2}\int_{-z}^{s-b+z}\int_{-z}^{b-s+\epsilon_b}\left(b^* - (b+\epsilon_b)\right) d\epsilon_s d\epsilon_b + \frac{(b-s)}{4z}\left(2b^* - b - s - 2z\right).$$

Case 3

In the third case, there is again trade with positive probability, but now $b < s$. The set of profiles with this property is

$$C_{III} = \{(s,b) : b - z < s - z < b + z < s + z\}.$$

Alternatively, $(s,b) \in C_{III}$ if $s \in (b, b+2z)$.

Trade occurs whenever $\epsilon_s \leq b - s + \epsilon_b$. Since $s > b$, there are values of ϵ_b where no trade occurs. Specifically, if $\epsilon_b \leq s - b - z$, then $\hat{b}$ is always less than $\hat{s}$. Hence, the expected profit is

$$\pi_S(s,b) = \frac{1}{4z^2}\int_{s-b-z}^{z}\int_{-z}^{b-s+\epsilon_b}\left(s+\epsilon_s\right) d\epsilon_s d\epsilon_b = \frac{1}{24z^2}\left(b-s+2z\right)^2\left(b+2s-z\right)$$

The expected payoff for the buyer is similarly computed to be

$$\begin{aligned}\pi_B(s,b) &= \frac{1}{4z^2}\int_{s-b-z}^{z}\int_{-z}^{b-s+\epsilon_b}\left(b^* - (b+\epsilon_b)\right) d\epsilon_s d\epsilon_b \\ &= -\frac{1}{24z^2}(b-s+2z)^2\left(2b - 3b^* + s + z\right).\end{aligned}$$

Case 4

In the final case there is *never* trade. Specifically, we have $b < s$ such that $\tilde{s} > \tilde{b}$ for all possible realizations – i.e.,

$$C_{IV} = \{(s,b) : b + 2z \leq s\}.$$

The traders profits are both zero for $(s,b) \in C_{IV}$.

Summary: Expected Payoff Functions

We now summarize the different expression for the expected payoff function.[7] The expected payoff function for the seller is

$$\pi_S(s,b) = \begin{cases} s, & \text{if } (s,b) \in C_I \\ \frac{1}{4z^2} \int_{-z}^{s-b+z} \int_{-z}^{b-s+\epsilon_b} (s+\epsilon_s)\, d\epsilon_s d\epsilon_b + s\left(\frac{b-s}{2z}\right), & \text{if } (s,b) \in C_{II} \\ \frac{1}{24z^2}(b-s+2z)^2(b+2s-z), & \text{if } (s,b) \in C_{III} \\ 0, & \text{if } (s,b) \in C_{IV} \end{cases} .$$

The expected payoff function for the buyer is

$$\pi_B(s,b) = \begin{cases} \bar{b}-b, & \text{if } (s,b) \in C_I \\ \begin{array}{l} \frac{1}{4z^2} \int_{-z}^{s-b+z} \int_{-z}^{b-s+\epsilon_b} \left(\bar{b}-(b+\epsilon_b)\right) d\epsilon_s d\epsilon_b \\ +\frac{(b-s)}{4z}\left(2\bar{b}-b-s-2z\right), \end{array} & \text{if } (s,b) \in C_{II} \\ -\frac{1}{24z^2}(b-s+2z)^2\left(2b-3\bar{b}+s+z\right), & \text{if } (s,b) \in C_{III} \\ 0, & \text{if } (s,b) \in C_{IV} \end{cases} .$$

4 The Best Response Correspondences and their Properties

In this section, we present two theorems that detail some properties of the best reply correspondences for the seller and buyer. These properties are used in the next section to establish existence and characterize the set of Nash equilibria for each parameter z.

[7] It is straightforward to verify that both of these payoff functions are continuous.

Best Response Behavior of the Seller

Theorem 1 presents some properties of the seller's best response correspondence σ.

Theorem 1: *The seller's best response correspondence $\sigma(b)$ has the following features:*

1. *If $b \leq -z$, then any selection in the set $\{s|s \geq b+2z\}$ is a best response;*
2. *If $-z < b \leq z$, then $\sigma(b) = z$;*
3. *If $b > z$, then the seller has a unique best response $\sigma(b)$ such that $z < \sigma(b) < b$ and $0 < \sigma'(b) < 1$;*
4. *For any b, $\sigma(b) \geq b - 2z$.*
5. *The $\lim_{b\to\infty} \sigma(b) = \infty$.*

Proof: See Appendix.

These properties are derived by examining the partial derivative of the seller's expected payoff function with respect to s:

$$\frac{\partial \pi_S(s,b)}{\partial s} = \begin{cases} 1, & \text{if } (s,b) \in C_I \\ \frac{b-s}{2z} + \frac{1}{4z^2}(s-z)(b-s-2z), & \text{if } (s,b) \in C_{II} \\ -\frac{1}{4z^2}(s-z)(b-s+2z), & \text{if } (s,b) \in C_{III} \\ 0, & \text{if } (s,b) \in C_{IV} \end{cases}.$$

On inspection, several things stand-out. First, the derivative is continuous in s. Second, since the seller's reservation price is zero, the best response behavior depends on the relation of b to the points $-z$ and z. Essentially, these are cutoff values of b where trades go from "unprofitable" to "potentially profitable," and from "potentially profitable" to "always profitable."

If, for example, $b \leq -z$, then any trades that would occur result in a negative profit for the seller. A best response ensures that no-trade occurs.

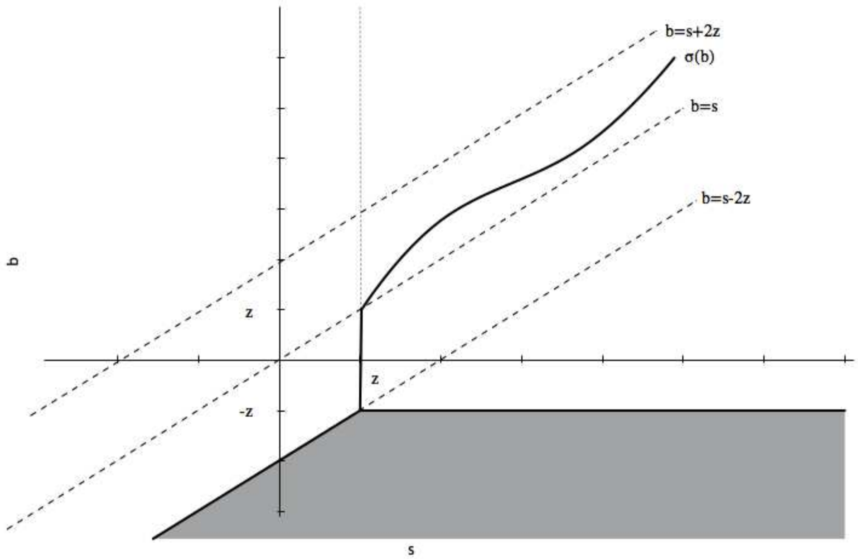

Figure 1: Seller's Best Response Correspondence

This is done by choosing $\sigma(b)$ sufficiently large so that $b + z \leq s - z$ or $b + 2z \leq s$. This is the first part of the theorem.

Alternatively, if $-z < b \leq z$, the second part of the theorem states the seller should set $\sigma(b) = z$. This behavior allows trade to occur with positive probability. Moreover, it ensures that any trades that occur are profitable. The seller does not risk receiving a price below his reservation price.

The interesting behavior occurs when $b > z$ and corresponds to the third part of the theorem. In this range, all trades would be profitable, the seller therefore responds so trade occurs with positive probability. Specifically, if $b > z$, the seller chooses his response $\sigma(b) \in (z, b)$. This response is unique, increasing in b, and $\sigma(b)$ is such that $0 < \sigma'(b) < 1$. Finally, for all b, $\sigma(b)$ is bounded below by $b - 2z$. This implies that if trade occurs with certainty, then the seller is not best responding. Moreover, this implies that as $b \to \infty$, we have $\sigma(b) \to \infty$.

Figure 1 illustrates the seller's best response correspondence.

Best Response Behavior of the Buyer

Theorem 2 presents some properties of the buyer's best response correspondence β.

Theorem 2: *The buyer's best response correspondence $\beta(s)$ has the following features:*

1. *If $s \geq b^* + z$, then any $b \in \{b : b \leq s - 2z\}$ is a best response;*
2. *If $b^* - z \leq s < b^* + z$, then $\beta(s) = b^* - z$;*
3. *If $s < b^* - z$, then the seller has a unique best response $\beta(s)$ such that such that $s < \beta(s) < b^* - z$ and $0 < \beta'(s) < 1$;*
4. *For any s, $\beta(s) \leq s + 2z$.*
5. *The $\lim_{s \to -\infty} \beta(s) = -\infty$.*

Proof. See the Appendix. ■

These properties are found by examining the following partial derivative of the buyer's expected payoff function with respect to b

$$\frac{\partial \pi_B(s,b)}{\partial b} = \begin{cases} -1, & \text{if } (s,b) \in C_I \\ \begin{array}{l} \frac{1}{4z^2}\left(b - (b^* - z)\right)(b - s - 2z) \\ -\frac{1}{2z}(b - s) \end{array}, & \text{if } (s,b) \in C_{II} \\ -\frac{1}{4z^2}\left(b - (b^* - z)\right)(b - s + 2z), & \text{if } (s,b) \in C_{III} \\ 0, & \text{if } (s,b) \in C_{IV} \end{cases}.$$

Not surprisingly, the above derivative is similar to $\frac{\partial \pi_S(s,b)}{\partial s}$. As a result, we get analogous properties of the buyer's best response correspondence in Theorem 2.

Figure 2 illustrates the buyer's best response correspondence.

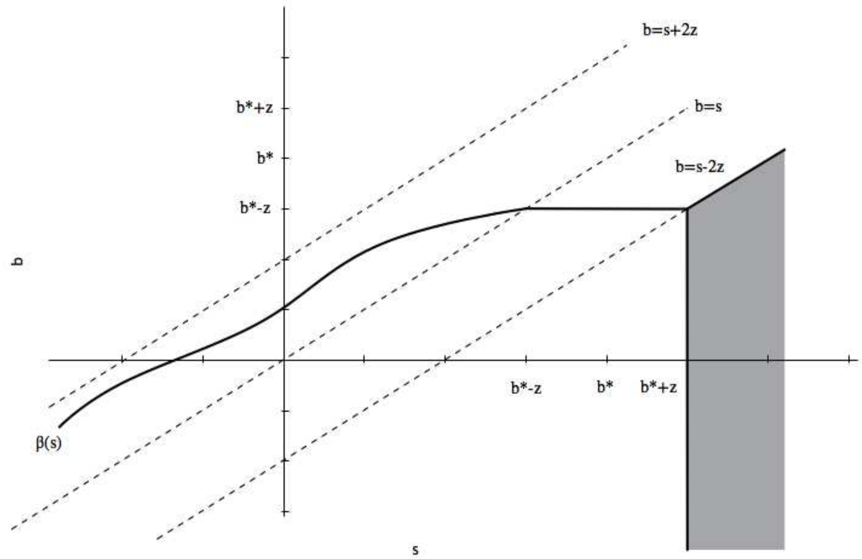

Figure 2: Buyer's Best Response Correspondence β

5 Nash Equilibrium

In this section, we prove the existence and characterize two types of Nash equilibria: Nash Equilibrium with Trade (NEWT); and Nash Equilibria without Trade (NEWOT). A NEWT is an equilibrium profile $(\bar{s}, \bar{b})$ where trade occurs with positive probability. In contrast, a NEWOT is a equilibrium profile $(\bar{s}, \bar{b})$ where there is never trade.

Nash Equilibria without Trade

We start by characterizing the set of NEWOT.

Theorem 3: *The profile* (s, b) *is a NEWOT if and only if* $s \geq b^* + z$ *and* $b \leq -z$.

Proof: If $s \geq b^* + z$, then any $b \in \{b : b \leq s - 2z\}$ is a best response including $b \leq -z$. Similarly, if $b \leq -z$, then any $s \geq b + 2z$ is a best response including $s \geq b^* + z$. Thus, any such (s, b) is a NE and since $s \geq b + 2z$ there is no trade.

Now suppose (s, b) is a NEWOT. First, we have $s \geq b + 2z$. If $b > -z$, then s is not a best response since the seller could achieve a positive payoff by setting $s = z$. Hence, $b \leq -z$. Similarly, if $s < b^* + z$, then the buyer is not best responding since setting $b = b^* - z$ ensures him a positive payoff. Hence, $s \geq b^* + z$. ■

Nash Equilibrium with Trade

The set of NEWOT is large, but there is always a unique NEWT. The characterization of this NEWT, however, depends on the magnitude of the error parameter z in relation to $\frac{b^*}{2}$. First, for z sufficiently small, we show that the unique NEWT occurs in the C_{II} payoff region where $s \leq b$.

Theorem 4: *Suppose $z < b^* - z$, then the a unique NEWT is such that $s \leq b$.*

Proof: Suppose $z < b^* - z$. From Theorem 1, parts 2 and 5, we have $\sigma(z) = z$ and we know there exists a $\mathring{b} > z$ such that $\sigma(\mathring{b}) = b^* - z$. In addition from part 3 of Theorem 1 we have that $0 < \sigma' < 1$ for $b > z$. We conclude $\mathring{b} > b^* - z$.

Analogously, from Theorem 2, parts 2 and 5, we have $\beta(b^* - z) = b^* - z$ and we know there exists a $\mathring{s} < b^* - z$ such that $\beta(\mathring{s}) = z$. From part 3 of Theorem 2, we have that $0 < \beta' < 1$ for $s < b^* - z$, we conclude $\mathring{s} < z$.

Next, for $b > z$ and $s < b^* - z$, the best responses are determined according to the Case 2 first order condition and are clearly continuous. Since the best responses are continuous, they must intersect at some $(\bar{s}, \bar{b})$ such that $\bar{s} < b^* - z$ and $\bar{b} > z$. Moreover, since the reaction functions are increasing in this region each with a slope less than one the intersection must be unique. ■

Figure 3 illustrates the intersection of the two players' best response correspondences when $z < b^* - z$. The set of NEWOT is illustrated by the darkened rectangle at the south-east corner of the figure. The unique NEWT is the north-west most intersection of the best response graphs. The intersection of the NEWT with the line $b + s = b^*$ is not a coincidence, but we defer a proof of this fact until the next section.

Next, if the error parameter z is large so that $z \geq b^* - z$, then in the unique NEWT occurs where both players choose their fail safe bids. The buyer bids $b^* - z$ and the seller chooses z. At these bids, the players guarantee themselves a positive payoff, but the size of z makes both players non-responsive (at least locally) to small changes in their rival's action.

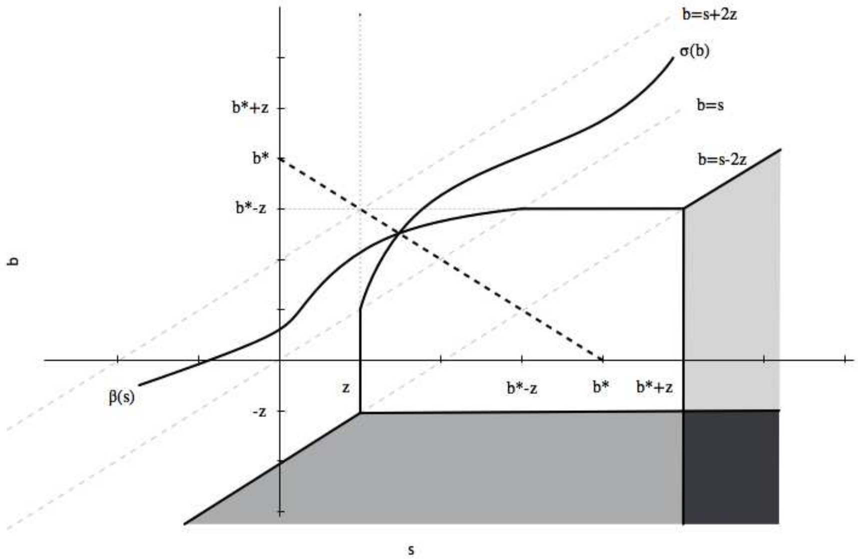

Figure 3: Trade Nash Equilibrium ($z < b^* - z$)

Theorem 5: *If* $z \geq b^* - z$*, then the unique NEWT is* $(\bar{s}, \bar{b}) = (z, b^* - z)$*.*

Proof: Suppose $z \geq b^* - z$. Let (s, b) is a NEWT where $b > z$. From Theorem 1, we have that $b^* - z \leq z < \sigma(b) < b$. Since this is NEWT, by Theorem 3, $\beta(\sigma(b)) = b^* - z$ which is a contradiction. Next, suppose (s, b) is a NEWT where $b \leq z$. Since this is NEWT, by Theorem 2 and Theorem 3, $\sigma(b) = z \geq b^* - z$ and $\beta(z) = b^* - z$. The profile $(s, b) = (z, b^* - z)$ is a therefore a NEWT. ■

Figure 4 illustrates the intersection of the two players' best response correspondences when $z > b^* - z$.

6 Relation of the Nash Equilibrium with Trade to the Nash Bargaining Solution

In this section, we show that the unique NEWT of the game converges to the Nash Bargaining Solution of the game without errors as the errors go to zero. The error in the model is parameterized by the support parameter z. Denote the unique NEWT of the game with parameter $z > 0$ by $\left(\bar{s}_z, \bar{b}_z\right)$.

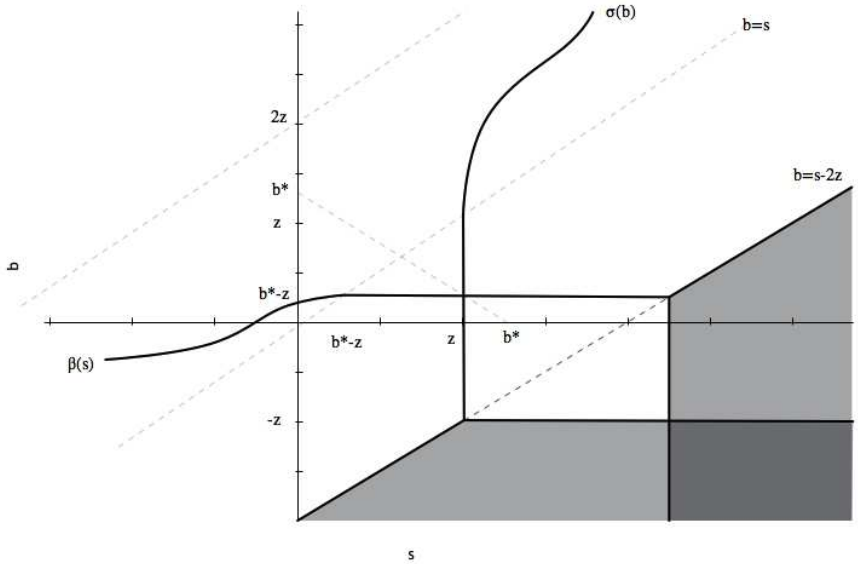

Figure 4: Trade Nash Equilibrium ($z > b^* - z$)

Since we are interested in a convergence result, without loss of generality, we set $z < \frac{b^*}{2}$.

The main theorem is shown in two steps. We first establish that in a NEWT, the seller's bid and the buyer's bid add up to b^*.[8] The lemma allows us to provide a straightforward proof of the main theorem.

Lemma 1: *In every NEWT,* $\bar{s}_z + \bar{b}_z = b^*$.

Proof: From Theorem 4, we know the C_{II} payoff function is the one that applies in equilibrium. The C_{II} first order conditions for the two players that are satisfied in this equilibrium are

$$\begin{aligned}\frac{\partial \pi_S}{\partial s} &= \frac{\bar{b}_z - \bar{s}_z}{2z} + \frac{1}{4z^2}\left(\bar{s}_z - z\right)\left(\bar{b}_z - \bar{s}_z - 2z\right) = 0 \\ \frac{\partial \pi_B}{\partial b} &= -\frac{1}{4z^2}\left(b^* - \bar{b}_z - z\right)\left(\bar{b}_z - \bar{s}_z - 2z\right) - \frac{1}{2z}(\bar{b}_z - \bar{s}_z) = 0\end{aligned}$$

From $\frac{\partial \pi_B}{\partial b}$ we have

$$\frac{1}{2z}(\bar{b}_z - \bar{s}_z) = -\frac{1}{4z^2}\left(b^* - \bar{b}_z - z\right)\left(\bar{b}_z - \bar{s}_z - 2z\right).$$

[8]This lemma is also clearly true when $z \geq b^* - z$ and can be directly verified.

Substituting this expression into $\frac{\partial \pi_S}{\partial s}$ gives us

$$-\frac{1}{4z^2}\left(b^* - \bar{b}_z - z\right)\left(\bar{b}_z - \bar{s}_z - 2z\right) + \frac{1}{4z^2}\left(\bar{s}_z - z\right)\left(\bar{b}_z - \bar{s}_z - 2z\right) = 0$$

which implies

$$\left(\bar{b}_z - \bar{s}_z - 2z\right)\left(-b^* + \bar{b}_z + \bar{s}_z\right) = 0$$

Since $\bar{b}_z < \bar{s}_z + 2z$, we have $\bar{b}_z + \bar{s}_z = b^*$. ■

We now show that the sequence of NEWT converge to the Nash Bargaining Solution.

Theorem 6: *The trade equilibrium allocation* $(\bar{s}_z, \bar{b}_z)$ *converges to the Nash Bargaining Solution as* $z \to 0^+$.

Proof: We have $z < \frac{b^*}{2}$. From Theorem 4, we know the C_{II} payoff function is the one that applies in equilibrium. If we re-arrange the C_{II} first order condition for the seller we have

$$\bar{b}_z - \bar{s}_z = 2z\left(\frac{\bar{s}_z - z}{\bar{s}_z + z}\right).$$

Since $\bar{s}_z = b^* - \bar{b}_z$, from the previous lemma, the above equation can be re-written

$$\bar{b}_z = \frac{b^*}{2} + z\left(\frac{b^* - \bar{b}_z - z}{b^* - \bar{b}_z + z}\right).$$

Note for $z > 0$, we have

$$1 > \frac{b^* - \bar{b}_z - z}{b^* - \bar{b}_z + z} > 0$$

since $b^* - z > \bar{b}_z$. Hence, for all $z > 0$, we have

$$\frac{b^*}{2} \le \bar{b}_z \le \frac{b^*}{2} + z.$$

It follows from the Squeeze Theorem that $\lim_{z \to 0^+} b_z = \frac{b^*}{2}$.

Finally, since $\bar{s}_z + \bar{b}_z = b^*$, we have that the $\lim_{z \to 0^*} \bar{s}_z = \frac{b^*}{2}$. ■

7 Discussion of Related Literature and Conclusion

In this section, we justify some of our remarks concerning the analysis of Carlsson (1991) and discuss our model and results in the context of the previous literature.

Discussion of Carlsson (1991)

Carlsson considers a Nash Demand game with two players: a buyer and a seller. The buyer submits a bid b and the seller submits a bid s. The bids (s, b) are transformed into random bids. The seller's random bid is $r = s + \epsilon_s$, where ϵ_s is distributed according to F on $[-x_0, x_1]$ for $x_0, x_1 > 0$. Similarly, let the buyer's random bid be $t = b + \epsilon_b$, where ϵ_b is distributed according to G on $[-y_0, y_1]$ for y_0, $y_1 > 0$. If the buyer's random bid t is higher then the seller's random bid r, then there is trade. Otherwise not.

The players pay/receive a convex combination of the two random bids. For brevity, we restrict attention to expressions involving the seller. If $r \leq t$, then the seller receives a price equal to $\lambda t + (1 - \lambda) r$ where $\lambda \in [0, 1]$ is commonly known to the players. The seller values these trades according to a strictly increasing and concave utility function u_S and attaches a zero value to the "no trade" outcome. The following payoff function and partial derivative for the seller are reported –

$$\pi_S(s, b) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} u_S\left([1 - \lambda] r + \lambda t\right) f(r - s) g(t - b) dr dt.$$

and

$$\frac{\partial}{\partial s} \pi_S(s, b) = - \int_{-\infty}^{\infty} \int_{-\infty}^{t} u_S\left([1 - \lambda] r + \lambda t\right) f'(r - s) g(t - b) dr dt.$$

These expressions, and the analogous ones for the buyer, are incorrect.[9]

In particular, there are three main issues.

First, the notation employed by Carlsson masks the bounds of the integrals in the payoff function. The random variable r has support $[s - x_0, s + x_1]$ and the random variable t has support $[b - y_0, b + y_1]$. While this notation is not incorrect, it hides the fact that the support depends on the choice variables of the players.[10] However, one cannot replace the bounds with their

[9] These are equations (1) - (4) in Carlsson (1991).

[10] This matters when differentiating the function.

correct values and recover the correct expected payoff function – i.e., the function

$$\begin{aligned}\pi_S(s,b) &= \int_{-\infty}^{\infty}\int_{-\infty}^{t} u_S\left([1-\lambda]r+\lambda t\right) f(r-s)g(t-b)drdt \\ &= \int_{b-y_0}^{b+y_1}\int_{s-x_0}^{t} u_S\left([1-\lambda]r+\lambda t\right) f(r-s)g(t-b)drdt\end{aligned}$$

is not always the correct expected payoff function! In particular, the above expression only applies when (s,b) and support parameters x_0, y_0, x_1, and y_1 are such that

$$s-x_0<b-y_0<b+y_1<s+x_1.$$

The second issue, therefore, is that the form of the payoff function varies with b and s and values of the support parameters. Consider the following example. Let (s,b) and support parameters x_0, y_0, x_1, and y_1 be such that

$$s-x_0<b-y_0<s+x_1<b+y_1.$$

We want to determine the expected payoff function. There is trade when $s+\epsilon_s\leq b+\epsilon_b$. If the realization ϵ_b is small (i.e., $\epsilon_b\leq s-b+x_1$), then there is trade when $\epsilon_s\leq b-s+\epsilon_b$. However, for large values of ϵ_b there is trade for all $\epsilon_s\in[-x_0,x_1]$ or equivalently all $r\in[s-x_0,s+x_1]$. Hence, the expected payoff function is

$$\begin{aligned}\dot{\pi}_S(s,b) &= \int_{b-y_0}^{s+x_1}\int_{s-x_0}^{t} u((1-\lambda)r+\lambda t)f(r-s)g(t-b)drdt \\ &\quad+\int_{s+x_1}^{b+y_0}\int_{s-x_0}^{s+x_1} u((1-\lambda)r+\lambda t)f(r-s)g(t-b)drdt.\end{aligned}$$

This is different than the function labeled π_S. Alternatively, if (s,b) and support parameters x_0, y_0, x_1, and y_1 were such that

$$b-y_0<s-x_0<b+y_1<s+x_1,$$

then

$$\ddot{\pi}_S(s,b)=\int_{s-x_0}^{b+y_1}\int_{s-x_0}^{t} u((1-\lambda)r+\lambda t)f(r-s)g(t-b)drdt.$$

The expressions for the expected payoff functions π_S, $\dot{\pi}_S$, and $\ddot{\pi}_S$ are all where $s - x_0 < b + y_1$ (i.e., there is trade with positive probability), but the payoff functions have varied due to the assumption that the supports of the random variables are finite intervals in $\mathbb{R}$.[11] Since Carlsson does not consider the different support cases when deriving his expected payoff functions he ends up with an incomplete representation of the expected payoff function.[12]

Finally, since the player's choice variables b and s appear in the bounds of integration of the expected payoff function one needs to apply *Leibniz's Rule* when taking derivatives. This is an important step missing in Carlsson, and results in incorrect derivatives. It is easily verified that the derivatives $\frac{\partial \pi_S}{\partial s}$, $\frac{\partial \dot{\pi}_S}{\partial s}$, and $\frac{\partial \ddot{\pi}_S}{\partial s}$ lead to different expressions than the single derivative reported by Carlsson. This observation is compounded for the higher order derivatives reported as well.

In summary, the payoff functions and the derivatives found in Carlsson are all either incomplete or incorrect. Since the majority of the analysis is based on these expressions, the assertions made in Carlsson (1991) should be approached with skepticism.

Conclusion

We have provided a careful analysis of a bargaining model where parties make errors. The model and results presented are primarily related to Nash (1953), Binmore (1987b), and Carlsson (1991). Our model is "simpler" than the Carlsson model in several respects. First, our traders had linear utility functions as opposed to generic utility functions. Second, our traders were paid/received their random price as opposed to some convex combination of the random prices.[13] Finally, the error terms in our model were independently drawn from a uniform distribution on $[-z, z]$ as opposed to general distribution functions on arbitrary compact intervals. While less general than Carlsson, the model achieves the desired goal. We have provided a natural

[11] If the supports are identical – i.e., $x_0 = x_1 = y_0 = y_1 = z > 0$, then one cannot have the case

$$s - x_0 < b - y_0 < b + y_1 < s + x_1.$$

[12] While Carlsson's payoff functions mimic the ones reported in Binmore (1987), but the same critique *does not* apply to Binmore's model. In Binmore's model, the expected payoff functions are correct since the support of the error terms is $\mathbb{R}$.

[13] This is the case consided by Nash (1953) and Binmore (1987a,b).

perturbation of the Nash Demand Game and precisely demonstrate that the Nash Bargaining Solution is approximated by the trade equilibrium of this game as the perturbation becomes small.

8 Appendix

8.1 Proof of Theorem 1

Proof of Part 1: If $b \leq -z$, then the best the seller can do is enforce a no trade outcome. If $s - z \geq b + z$, then there is no trade.

Proof of Part 2: We claim that if $b = z$, then $\sigma(b) = z$.

First, if $s < b - 2z$, then there is trade for sure and Case 1 applies. In this region, $\frac{\partial \pi_S}{\partial s} = 1 > 0$ and s cannot be a best response. Second, if $s = z$, then Case 2 applies and $\frac{\partial \pi_S}{\partial s} = 0$. We also have that if $s < z$, then $\frac{\partial \pi_S}{\partial s} > 0$. Third, if $b + 2z > s > z$, then Case 3 applies and

$$\frac{\partial \pi_S}{\partial s} = -\frac{1}{4z^2}(s - z)(b - s + 2z) < 0.$$

Finally, if $b + 2z < s$, then $\pi_S = 0$. Since setting $s = z$ results in a strictly positive expected payoff $s > b + 2z$ cannot be a best response. Hence, $s = z$ is the best response to $b = z$.

Next, we claim that if $-z < b < z$, then $\sigma(b) = z$.

First, if $s < b - 2z$, then there is trade for sure and Case 1 applies. In this region, $\frac{\partial \pi_S}{\partial s} = 1 > 0$ and s cannot be a best response. Second, if $s \in [b - 2z, b]$, then Case 2 applies and

$$\frac{\partial \pi_S}{\partial s} = \frac{b - s}{2z} + \frac{1}{4z^2}(s - z)(b - s - 2z).$$

Since $b - s > 0$, $s - z < 0$, and $b - s - 2z < 0$ we have that $\frac{\partial \pi_S}{\partial s} > 0$ for $s \in [b - 2z, b]$. Third, if $s \in (b, b + 2z]$, then Case 3 applies and

$$\frac{\partial \pi_S}{\partial s} = -\frac{1}{4z^2}(s - z)(b - s + 2z).$$

Since $-z < b < z$, we have that $z < b + 2z$ and therefore that $z \in (b, b + 2z)$. Setting $s = z$ gives us $\frac{\partial \pi_S}{\partial s} = 0$. If $s < z$, then $\frac{\partial \pi_S}{\partial s} > 0$. If $s > z$, then for s such that Case 3 is satisfied we have that $\frac{\partial \pi_S}{\partial s} < 0$. Finally, if s is such that

Case 4 applies then $\pi_S = 0$. However, such s cannot be a best response since setting $s = z$ results in profitable trades with positive probability. Hence, if $-z < b < z$, then $\sigma(b) = z$ is the unique best response.

Proof of Part 3: Suppose that $b > z$ and that $s = \sigma(b)$ is a best response to b.

We first demonstrate that $\sigma(b)$ is unique and occurs in C_{II} payoff region. First, if $s < b - 2z$, then the C_I payoff applies so $\frac{\partial \pi_S}{\partial s} = 1 > 0$ and s cannot be a best response. Second, if $s \in [b - 2z, b]$, then

$$\frac{\partial \pi_S(s, b)}{\partial s} = \frac{b - s}{2z} + \frac{1}{4z^2}(s - z)(b - s - 2z).$$

If $s = b - 2z$, then we have a marginal payoff of $\frac{\partial \pi_S}{\partial s}(b - 2z, b) = 1 > 0$. In addition, setting $s = b$ yields $\frac{\partial \pi_S}{\partial s}(b, b) = \frac{1}{4z^2}(b - z)(-2z) < 0$ because $b > z$ by assumption. Since $\frac{\partial \pi_S}{\partial s}$ is a continuous function in s, by the Intermediate Value Theorem, there is a $\sigma(b) \in (b - 2z, b)$ such that $\frac{\partial \pi_S}{\partial s}(\sigma(b), b) = 0$. Moreover, this value is unique in this region since

$$\frac{\partial^2 \pi_S}{\partial s^2} = -\frac{1}{4z^2}(2s - b + 3z) < 0.$$

The sign follows since the smallest $2s - b + 3z$ can be is when $s = b - 2z$. Thus, sufficiency for $\frac{\partial^2 \pi_S}{\partial s^2} < 0$ is $b > z$ which is true by assumption. Third, if $s > b > z$ and $s < b + 2z$, then C_{III} applies and

$$\frac{\partial \pi_S}{\partial s} = -\frac{1}{4z^2}(s - z)(b - s + 2z) < 0$$

so s cannot be a best response. Last, if $s > b + 2z$, then there is never trade so $\pi_S = 0$. Trades at $\sigma(b)$ never yield a negative profit and sometimes yield a strictly positive profit. Hence, choosing a s that always results in no trade is not a best response when $b > z$. Thus, $\sigma(b)$ is the unique best response if $b > z$.

Next, we must also have $\sigma(b) > z$. If $z < b - 2z$, then we have already shown that $\sigma(b) > z$. If $z \in [b - 2z, b]$, then

$$\frac{\partial \pi_S}{\partial s}(z, b) = \frac{b - z}{2z} > 0$$

since $b - z$. Using a similar argument with the Intermediate Value Theorem we conclude that $\sigma(b) \in (\max\{b - 2z, z\}, b)$.

Therefore we have shown that for $b > z$, the seller's best response $\sigma(b)$ is uniquely defined and $z < \sigma(b) < b$. The final part of the claim is that the slope of the σ in this region satisfies $0 < \sigma'(b) < 1$. This follows since

$$\sigma'(b) = -\frac{\frac{\partial^2 \pi_S}{\partial b \partial s}}{\frac{\partial^2 \pi_S}{\partial s^2}},$$

where the higher order derivatives are of the C_{II} payoff function. The slope $\sigma'(b) > 0$ because $\frac{\partial^2 \pi_S}{\partial b \partial s} = \frac{1}{4z^2}(s+z) > 0$ and $\frac{\partial^2 \pi_S}{\partial s^2} = -\frac{1}{4z^2}(2s - b + 3z) < 0$. In addition,

$$-\frac{\partial^2 \pi_S}{\partial b \partial s} = \frac{\partial^2 \pi_S}{\partial s^2} + \frac{1}{4z^2}(s+z) - \frac{1}{4z^2}(b-z).$$

So,

$$\sigma'(b) = 1 + \frac{\frac{1}{4z^2}(s - b + 2z)}{\frac{\partial^2 \pi_S}{\partial s^2}} < 1,$$

since $s > b - 2z$. ■

Proof of Part 4 and Part 5: A strategy s cannot be a best response to b if $s < b - 2z$ (i.e., if C_I applies). In C_I, trade occurs with certainty and $\frac{\partial \pi_S(s,b)}{\partial s} > 0$. The seller's marginal payoff is increasing in own action so s cannot be optimal. Hence, $\sigma(b) \geq b - 2z$. It follows that $\lim_{b \to \infty} \sigma(b) = \infty$. ■

8.2 Proof of Theorem 2

Proof of Part 1: Since $s - z \geq b^*$ any trade that occurs results in negative profit, the set of best responses are those reports that guarantee no trade will occur. Any report $b \leq s - 2z$ is sufficient. ■

Proof of Part 2: Suppose Case 3 applies so $b \in (s - 2z, s]$. Since $b^* - z \in (s - 2z, s]$ and

$$\frac{\partial \pi_B}{\partial b} = -\frac{1}{4z^2}(b - (b^* - z))(b - s + 2z)$$

setting $\bar{b} = b^* - z$ results in $\frac{\partial \pi_B}{\partial b} = 0$.

In Case 3, if $b < \bar{b}$ and $b > s - 2z$, then $\frac{\partial \pi_B}{\partial b} > 0$; and if $b > \bar{b}$ and $b \leq s$, then $\frac{\partial \pi_B}{\partial b} < 0$.

If $b < \bar{b}$ and $b < s-2z$, then Case 4 applies and $\pi_B = 0$. Setting $\bar{b} = b^* - z$ achieves a strictly positive expected profit since only profitable trades occur with positive probability. Hence, $b < s - 2z$ cannot be a best response.

If $b > \bar{b}$ and $b > s$, then either Case 2 or Case 1 apply. If Case 2 applies then $s < b < s + 2z$

$$\frac{\partial \pi_B}{\partial b} = -\frac{1}{4z^2}\left(b^* - z - b\right)\left(b - s - 2z\right) - \frac{1}{2z}(b - s).$$

Since $b > s$, $b < s + 2z$, and $b > \bar{b} = b^* - z$, we have that $\frac{\partial \pi_B}{\partial b} < 0$. Hence, there are no critical points in Case 2.

Finally, if Case 1 applies, then $b \geq s + 2z$ and $\frac{\partial \pi_B}{\partial b} = -1$. ■

Proof of Part 3: First, if $b \leq s - 2z$, then Case 4 applies and $\pi_B = 0$. Second, if $s - 2z < b < s$, then Case 3 applies and

$$\frac{\partial \pi_B}{\partial b} = -\frac{1}{4z^2}\left(b - (b^* - z)\right)\left(b - s + 2z\right).$$

Since $b < s < b^* - z$, we have $b - (b^* - z) < 0$. In addition, $(b - s + 2z) > 0$ because $s - 2z < b$. Therefore $\frac{\partial \pi_B}{\partial b} > 0$ for b in Case 3.

Third, if $s \leq b \leq s + 2z$, then Case 2 applies and

$$\frac{\partial \pi_B}{\partial b} = -\frac{1}{4z^2}\left(b^* - b - z\right)\left(b - s - 2z\right) - \frac{1}{2z}(b - s).$$

If $b = s$, then $\frac{\partial \pi_B}{\partial b} = \frac{1}{4z^2}\left(b^* - z - b\right)(2z) > 0$ since $s = b < b^* - z$. In contrast, if $b = s + 2z$, then $\frac{\partial \pi_B}{\partial b} = -1$. From the Intermediate Value Theorem, there is a $\bar{b} \in (s, s + 2z)$ such that $\frac{\partial \pi_B}{\partial b} = 0$. Moreover, $\frac{\partial^2 \pi_B}{\partial b^2} = -\frac{1}{4z^2}\left(b^* - 2b + s + 3z\right) < 0$ so long as $\left(b^* - 2b + s + 3z\right) > 0$. The term $b^* - 2b + s + 3z$ is smallest in $b \in (s, s+2z)$ when $b = s+2z$. In this case, we have $b^* - 2(s+2z) + s + 3z = b^* - s - z > 0$ by assumption. Hence, $\frac{\partial^2 \pi_B}{\partial b^2} < 0$ for $b \in (s, s + 2z)$ so $\bar{b}$ is unique.

If $b^* - z \in (s, s + 2z)$, then $b = b^* - z$ results in

$$\frac{\partial \pi_B}{\partial b} = -\frac{1}{2z}(b^* - z - s).$$

The term $b^* - z - s > 0$ so $\frac{\partial \pi_B}{\partial b} < 0$. It then follows if $b^* - z \in (s, s + 2z)$, then $\bar{b} \in (s, b^* - z)$.

Finally, if $b > s + 2z$, then $\frac{\partial \pi_B}{\partial b} = -1 < 0$.

We have therefore shown that $\bar{b}$ is the unique global maximum. Thus, for $s < b^* - z$, the best response mapping $\beta(s)$ is a function whose slope is

$$\beta'(s) = -\frac{\frac{\partial^2 \pi_B}{\partial b \partial s}}{\frac{\partial^2 \pi_B}{\partial b^2}}.$$

The cross partial is $\frac{\partial^2 \pi_B}{\partial s \partial b} = \frac{1}{4z^2}(b^* - b - z) + \frac{1}{2z}$. Since, $\bar{b} \in (s, \min\{b^* - z, s + 2z\})$, it follows $\frac{\partial^2 \pi_B}{\partial b \partial s} > 0$ and therefore $\beta'(s) > 0$.

Next, $\beta' < 1$ if

$$\frac{1}{4z^2}(b^* - b + z) < \frac{1}{4z^2}(b^* - 2b + s + 3z)$$

or

$$b < s + 2z$$

Since, $s + 2z > b$, the above inequality is true. Thus, for $s < b^* - z$, the mapping β is a non-expansive function. ■

Proof of Part 4 and Part 5: The action b cannot be a best response to s if $s + 2z < b$ (i.e., if C_I applies). In C_I, trade occurs with certainty and $\frac{\partial \pi_B(s,b)}{\partial b} < 0$ – i.e., the buyer can increase his payoff by decreasing b. Hence, $\beta(s) \leq s + 2z$. It follows that $\lim_{s \to -\infty} \beta(s) = -\infty$. ■

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