Modeling Systematic Risk and Point-in-Time Probability of Default under the Vasicek Asymptotic Single Risk Factor Model Framework

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MODELING SYSTEMATIC RISK AND POINT-IN-TIME PROBABILITY OF DEFAULT UNDER THE VASICEK ASYMPTOTIC SINGLE RISK FACTOR MODEL FRAMEWORK*

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Abstract

Systematic risk has been a focus for stress testing and risk capital assessment. Under the Vasicek asymptotic single risk factor model framework, entity default risk for a risk homogeneous portfolio divides into two parts: systematic and entity specific. While entity specific risk can be modelled by a probit or logistic model using a relatively short period of portfolio historical data, modeling of systematic risk is more challenging. In practice, most default risk models do not fully or dynamically capture systematic risk. In this paper, we propose an approach to modeling systematic and entity specific risks by parts and then aggregating together analytically. Systematic risk is quantified and modelled by a multifactor Vasicek model with a latent residual, a factor accounting for default contagion and feedback effects. The asymptotic maximum likelihood approach for parameter estimation for this model is equivalent to least squares linear regression. Conditional entity PDs for scenario tests and through-the-cycle entity PD all have analytical solutions. For validation, we model the point-in-time entity PD for a commercial portfolio, and stress the portfolio default risk by shocking the systematic risk factors. Rating migration and portfolio loss are assessed.

Keywords: point-in-time PD, through-the-cycle PD, Vasicek model, systematic risk, entity specific risk, stress testing, rating migration, scenario loss

1. Introduction

Let n denote the size of a portfolio, and k the number of defaults in one-year horizon. Portfolio default rate in horizon is given by $r = k/n$. Assume that the default count k follows a binomial distribution, given the event probability $p(s)$ dictated by a latent factor s in horizon. We call $p(s)$ the portfolio level probability of default (PD) given systematic risk s in horizon. The quantity $p(s)$ contains all information for systematic risk. We can think of $p(s)$ as the asymptotic portfolio default rate, when portfolio size is sufficiently large ([12]).

A risk profile x for an entity consists of a vector of current values for a given list of entity specific risk drivers. Let $p(s, x)$ denote the entity PD in one-year horizon, given systematic risk s and entity risk profile x; and $E(p(s, x) \mid x)$ the expected value of $p(s, x)$ given x. We call $p(s, x)$ the point-in-time (PIT) entity PD, and $E(p(s, x) \mid x)$ the through-time-cycle (TTC) entity PD ([8]).

Let Φ denote the cumulative distribution for a standard normal variable. A random variable y, $0 < y < 1$, is said to follow a Vasicek distribution if $\Phi^{-1}(y)$ is normal ([18, p52]). Under the Vasicek asymptotic single risk factor (ASRF) model framework ([14], [4, p.4-5], [16], [17], [19], [25]), the PIT entity PD for a risk homogenous portfolio (see section 2.1 for definition), as shown in the next section, splits into two parts:

$$\Phi^{-1}(p(s,x)) = w + z,\ \ w \sim N(\mu_w, \sigma_w{}^2),\ \ z \sim N(0, \sigma_z{}^2) \qquad (1.1)$$

where w and z are mutual independent, w represents the systematic risk depending on s, and z represents the entity specific risk depending on entity risk profile x.

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It can be shown (see Proposition 2.3) that, under model (1.1), the systematic risk w and the TTC entity PD are respectively given by:

$$w = [\Phi^{-1}(p(s))]\sqrt{1+\sigma_z^{\ 2}} \qquad (1.2)$$

$$E(p(s,x)\,|\,x) = \Phi[(\mu_w + z)/\sqrt{1+\sigma_w^{\ 2}})] \qquad (1.3)$$

Thus by (1.1) the PIT entity PD is given by:

$$p(s,x) = \Phi(w+z),\ w = [\Phi^{-1}(p(s))]\sqrt{1+\sigma_z^{\ 2}} \qquad (1.4)$$

To model the PIT and TTC entity PDs, it suffices to model the systematic risk $p(s)$ and the entity specific risk z. Each of these risk components can be modelled separately.

The entity specific risk z in model (1.1) can be modelled by a probit or logistic model targeting a default indicator over a relatively short period of portfolio historical data. In practice, such a model uses a list of entity specific risk drivers x, and is calibrated over the current portfolio to a specific level of systematic risk s_0. We can thus assume that $p_m(x)$ is a model for $p(s_0, x)$ with systematic risk s_0. Then entity specific risk is given by:

$$z = \Phi^{-1}(p_m(x)) - E(\Phi^{-1}(p_m(x))) \qquad (1.5)$$

where $E(\Phi^{-1}(p_m(x)))$ denotes the expected value of $\Phi^{-1}(p_m(x))$, estimated by the average of $\Phi^{-1}(p_m(x))$ over the current portfolio.

In contrast, modeling of systematic risk is more challenging due to the data limitation of portfolio historical default rate time series, and the lack of efficient methodologies in parameter estimation. In practice, most PD models do not fully or dynamically capture systematic risk. Default contagion and feedback effects ([3], [9], [11], [13]) are thus not captured. We will propose in section 2.2 a multifactor Vasicek model with residual for the systematic risk, using the parameter estimation methodology proposed in [27].

As shown in later sections, advantages of the proposed models include:

(a) Systematic and entity specific risk components each is modelled separately, then aggregated together analytically by (1.4).
(b) Conditional entity PDs for scenario tests and entity TTC PDs all have analytical solutions (Propositions 2.3-2.5).
(c) Feedback and default contagion effects are captured and quantified (Propositions 2.4-2.5).
(d) Portfolio scenario loss can be assessed (Section 4).

The proposed approaches extend to a general portfolio where it contains multiple segments with each homogeneous but heterogeneous between segments: train for the portfolio a probit model for entity specific risk and a multifactor Vasicek model (2.3) for systematic risk, calibrate each model over each segment, and follow (1.4) to combine systematic and entity specific risks by segment.

The paper is organized as follows: In section 2, we introduce model (1.1) under the Vasicek ASRF model framework, review the parameter estimation methodologies proposed in [27] for the multifactor Vasicek model (2.3), and show formulations (1.2) - (1.4). Analytical formulas for conditional PDs for stress testing are also shown in this section. We propose in section 3 the steps for scenario tests. An empirical example is given in section 4, where we model dynamically the entity PD for a commercial portfolio. Portfolio scenario loss and rating migration are assessed.

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2. Dynamic Entity PD Models under the Vasicek ASRF Model Framework
2.1. Point-in-Time and Through-the-Cycle Entity PDs

Under the Vasicek ASRF model framework ([14], [4, p.4-5], [16], [17], [19], [25]), default risk in one-year horizon for *i-th* entity in a portfolio is driven by a normalized latent variable r_i at time *t* ($0 < t \le 12$): a default occurs in horizon if r_i falls below a threshold value called default point d_i, and the latent variable r_i splits as:

$$r_i = \sqrt{\rho_i}\, s + \sqrt{1-\rho_i}\,\varepsilon_i, \quad s,\ \varepsilon_i \sim N(0,1), \quad 0 \le \rho_i \le 1$$

where s, ε_i, and ε_j are the independent variables at time *t*, with *s* the systematic risk, and ε_i the entity specific (idiosyncratic) risk. The quantity ρ_i is called the asset correlation. When a risk profile x_i for the *i-th* entity is observed at current time $(t = 0)$, we assume $\varepsilon_i = \mu_i + \omega_i$, where μ_i depends only on x_i, and $\omega_i \sim N(0, \sigma_\omega^2)$, with σ_ω being the same for all entities. Given *s* and x_i, the probability of default in horizon for *i-th* entity is given by

$$\begin{aligned} p(s, x_i) &= P(\sqrt{\rho_i}\, s + \sqrt{1-\rho_i}\,(\mu_i + \omega_i) \le d_i \mid s, x_i) \\ &= P[\omega_i / \sigma_\omega \le -\mu_i / \sigma_\omega + (d_i - s\sqrt{\rho_i}\,) / (\sigma_\omega \sqrt{1-\rho_i}\,) \mid s, x_i] \\ &= \Phi[-\mu_i / \sigma_\omega + (d_i - s\sqrt{\rho_i}\,) / (\sigma_\omega \sqrt{1-\rho_i}\,)] \\ &\Rightarrow \Phi^{-1}(p(s, x_i)) = -\mu_i / \sigma_\omega + (d_i - s\sqrt{\rho_i}\,) / (\sigma_\omega \sqrt{1-\rho_i}\,) \end{aligned}$$

Let $z_i = d_i / (\sigma_\omega \sqrt{1-\rho_i}\,) - \mu_i / \sigma_\omega - \mu$, and $w_i = \mu - s\sqrt{\rho_i}\ / (\sigma_\omega \sqrt{1-\rho_i}\,)$, where μ denotes the average of $d_i / (\sigma_\omega \sqrt{1-\rho_i}\,) - \mu_i / \sigma_\omega$ over the portfolio. We call w_i the systematic risk and z_i the entity specific risk for *i-th* entity. The portfolio is said to be risk homogeneous if w_i is the same for all entities (i.e., asset correlation is the same for all entities), and z_i can be regarded as being sampled independently from the same distribution $N(0, \sigma_z^2)$. Then we have:

$$\Phi^{-1}(p(s, x_i)) = w + z_i, \ w \sim N(\mu_w, \sigma_w^2) \qquad (2.1)$$

Suppressing subscript *i*, we have:

$$\Phi^{-1}(p(s, x)) = w + z, \ w \sim N(\mu_w, \sigma_w^2), \ z \sim N(0, \sigma_z^2) \qquad (2.2)$$

The following lemma is important for subsequent discussions, where statement (b) is implied by statement (a). Statement (c) (see Appendix for a proof) implies that the volatility of default risk $p(s, x) = \Phi(w + z)$, given the systematic risk *w*, is an increasing function of *w* when $w < 0$, a generally desirable property for model (2.2).

Lemma 2.1. Let $s \sim N(\mu_s, \sigma_s^2)$, $\varepsilon \sim N(0, \sigma_e^2)$. Assume that *s* and ε are independent. Then

(a) ([22, p47]) $E(\Phi(s + \varepsilon) \mid s) = \Phi(s / \sqrt{1 + \sigma_e^2}\,)$

(b) $E(\Phi(\varepsilon)) = 1/2$

(c) Given s, the variance of $\Phi(s+\varepsilon)$ is an increasing function of s for $s<0$.

Corollary 2.2. Under model (2.2)**,** the portfolio level PD, given systematic risk s, is:

$$p(s)=E(\Phi(w+z)\,|\,s)=\Phi(s/\sqrt{1+\sigma_z^{\;2}}\,)$$

By model (2.2), we have the following proposition (see Appendix for a proof).

Proposition 2.3. Under model (2.2), we have:

(a) $w=[\Phi^{-1}(p(s))]\sqrt{1+\sigma_z^{\;2}}$, $p(s,x)=\Phi(w+z)$

(b) $E(p(s,x)\,|\,x)=\Phi[(\mu_w+z)/\sqrt{1+\sigma_w^{\;2}}\,)]$

2.2. Multifactor Vasicek Models for Systematic Risk

In this section we propose approaches to modeling $p(s)$, i.e., the portfolio level PD in one-year horizon, given systematic risk s. Restriction to risk homogeneity is not required, and the discussion extends to a general portfolio. Recall that $p(s)$ contains all information for systematic risk.

We propose the following multifactor Vasicek model for $p(s)$:

$$p(s)=\Phi(a+\sum_{j=1}^{k}a_ju_j+\sum_{j=1}^{m}b_js_j+s')\,,\ s'\sim N(0,\sigma^2) \qquad (2.3)$$

where $u_1,u_2,...,u_k$ measure current portfolio credit quality, such as current portfolio default rate, region, and industry sector; while $s_1,s_2,...,s_m$ are external market or macroeconomic variables in horizon, i.e., $s_i=s_i(t)$ for a time in future with $0<t\le 12$ in month. Variables $u_1,u_2,...,u_k$ and $s_1,s_2,...,s_m$ are subjected to a transformation by Φ^{-1} when necessary. The specification for model (2.3) is for stress testing purpose. When the model is used for forecasting, current values for $s_1,s_2,...,s_m$ are then used. The latent factor s' denotes the model residual, a dynamic accounting for default contagion and feedback effects, capturing all the remaining effects in horizon not explained by $u_1,u_2,...,u_k$ and $s_1,s_2,...,s_m$, including the effects after time t and before the end of the horizon.

Let $S=\{(u_{1i},u_{2i},...,u_{ki},\ s_{1i},s_{2i},...,s_{mi},\ r_i)\},\ 1\le i\le N,$ be a given multivariate time series sample with N observations, where r_i is the portfolio default rate at time i. Let p_i be the unobservable portfolio PD at time i. Parameter estimation for model (2.3) will follow the asymptotic maximum likelihood approach proposed in [27]. With this approach, portfolio default rate is equated to portfolio level PD, which in general exaggerates the variance of portfolio level PD, causing a bias to parameter estimates. For this reason, we propose a variance correlation as follows.

Variance correction to portfolio default rates:

(a) Assume a constant size n for the portfolio over time. Let p_0 be the expected value of portfolio level PD over time. Estimate p_0 by the simple average of sample default rates. Estimate as $v(r)$ the sample variance of all r_i. Then the variance v_0 of portfolio level PD can be estimated as ([27], Proposition 2.3 (c)):

$$v_0 = v(r) - [p_0(1-p_0) - v(r)]/(n-1)$$

(b) Let $\bar{r}$ denote the sample average of all r_i, and $w_0 = \sqrt{v_0 / v(r)}$. Replace r_i by rr_i:

$$rr_i = \bar{r} + (r_i - \bar{r}) w_0$$

Note that $p_0(1-p_0) - v(r) = (r_1 + r_2 + ... + r_N)/N - (r_1^2 + r_2^2 + ... + r_N^2)/N > 0$ unless $r_i = 0$ or 1 for all i. We thus have $w_0 = \sqrt{v_0 / v(r)} < 1$ and $0 < rr_i < 1$ unless $r_i = 0$ or 1 for all i. This correction has the advantage of transforming extreme values of 0 and 1 to other regular values between 0 and 1, which would have been an issue for the asymptotic approach with no variance correction. More importantly, the sample variance of rr_i is now adjusted to the sample variance v_0 of portfolio level PD.

Next, to estimate the parameters for model (2.3), we equate p_i, the portfolio PD at time i, to rr_i, and set $z_i = \Phi^{-1}(p_i)$. It was shown ([27], Theorem 4.2) that the maximum likelihood approach is equivalent to the least squares linear regression, which minimizes the sum-square of errors:

$$\sum_{i=1}^{N} [z_i - (a + \sum_{j=1}^{k} a_j u_{ji} + \sum_{j=1}^{m} b_j s_{ji})]^2$$

where σ, the standard deviation of s' in model (2.3), is estimated as the standard deviation of the model errors.

To address the serial correlation issue for the time series, we will use the bootstrap technique assuming that the time series of residual s' in (2.3) is stationary. Below are the steps for parameter estimation for model (2.3) proposed in [27].

Steps for parameter estimation for model (2.3):

(i) Do a variance correction to portfolio default rate as proposed, equate p_i, the portfolio PD, to the adjusted default rate rr_i, and set $z_i = \Phi^{-1}(p_i)$.

(ii) Generate B (e.g. B= 200) bootstrap samples each is of the same size as the input sample. For each bootstrap sample, train a model of the form (2.3) using least squares linear regression, and estimate the standard deviation σ of the model residual.

(iii) For each parameter, calculate the average of all its bootstrap estimates. Select from all bootstrap models the one with parameters the closest to their parameter averages.

2.3. Conditional PDs

Plugging the multifactor Vasicek model (2.3) into model (1.4), we have:

$$p(s,x) = \Phi[\sqrt{1+\sigma_z^{\ 2}}(a + \sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j s_j + s') + z] \qquad (2.4)$$

where s' and z are independent, $s' \sim N(0, \sigma^2)$, $z \sim N(0, \sigma_z^{\ 2})$. Note that by (1.4) the scalar $\sqrt{1+\sigma_z^{\ 2}}$ must be multiplied to $(a + \sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{m} b_j s_j + s')$ before adding to the entity specific risk z.

We consider the following conditional PDs:

(a) $p(s_1,...,s_m)$ - Portfolio level PD given a scenario of systematic risk factors $s_1, s_2, ..., s_m$ *in horizon* and *current* portfolio conditions $u_1, u_2, ..., u_k$.

(b) $p(s_1(0),...,s_m(0))$ -Portfolio level PD given *current* systematic risk factors $s_1(0),...,s_m(0)$ and *current* portfolio conditions $u_1, u_2, ..., u_k$.

(c) $p(s_1,...,s_m, x)$ - Entity scenario PD given a scenario of the systematic risk factors $s_1, s_2, ..., s_m$ *in horizon*, c*urrent* portfolio conditions $u_1, u_2, ..., u_k$, and entity *current* risk profile *x*.

Using the notations of model (2.3), we define *u* and *v* as:

$$u = a + \sum_{j=1}^{k} a_j u_j, \quad v = \sum_{j=1}^{m} b_j s_j$$

We assume that *u* and *v* are normal, and the latent factor s' in (2.3) is independent of *u* and *v*. Let $v(0)$ be the current value of *v*. Regress the horizon value v on its current value $v(0)$ over a time series sample by a linear regression to get a model: $v = d + \rho_v v(0) + \Delta v$, where *d* is the intercept, ρ_v the parameter for $v(0)$, and Δv the residual of the regression model. Let $\sigma_{\Delta v}$ denote the standard deviation of Δv.

The proposition below calculates the portfolio level conditional PDs (See Appendix for a proof).

Proposition 2.4. Under model (2.3), where $s' \sim N(0, \sigma^2)$, we have

(a) $p(s_1,...,s_m) = E(\Phi(u + v + s') \mid s_1,...,s_m, u_1,..., u_k) = \Phi((u + v) / \sqrt{1+\sigma^2})$

(b) $p(s_1(0),...,s_m(0)) = E(\Phi(u + v + s') \mid s_1(0),...,s_m(0), u_1,..., u_k)$

$$= \Phi((u + d + \rho_v v(0)) / \sqrt{1 + \sigma_{\Delta v}{}^2 + \sigma^2})$$

Since $\Phi((u + v) / \sqrt{1+\sigma^2}) \geq \Phi(u + v)$ whenever $(u + v) \leq 0$, Proposition 2.4 (a) implies that the latent residual effect s' in model (2.3) contributes to an increase to the portfolio level PD whenever $\Phi(u + v) \leq 1/2$.

The proposition below calculates the scenario entity PD (See Appendix for a proof).

Proposition 2.5. Under model (2.4), where $s' \sim N(0, \sigma^2)$, $z \sim N(0, \sigma_z{}^2)$, we have

$$\begin{aligned} p(s_1,...,s_m, x) &= E(\Phi(\sqrt{1+\sigma_z{}^2}(u + v + s') + z) \mid s_1,...,s_m, u_1,..., u_k, x) \\ &= \Phi([\sqrt{1+\sigma_z{}^2}(u + v) + z] / \sqrt{1 + \sigma^2(1+\sigma_z{}^2)}) \end{aligned} \tag{2.5}$$

3. Stress Testing for Portfolio Default Risk

Stress testing is widely used by financial institutions to assess the vulnerability to exceptional but plausible events. It is a tool complementing the existing internal models for capital allocation ([2], [5], [8], [10], [13], [15], [23]). In practice, stress testing focuses on systematic risk, with shocks from the external market or macroeconomic factors ([7], [13], [24]). With the dynamic model (1.4) and model (2.3), stress testing can be conducted through shocking the systematic risk factors in the model (2.3), then propagate to entity default risk by model (1.4). We focus on scenario tests.

3.1. Scenario Generation

Scenarios for stress testing can either be historical or hypothetical. Hypothetical scenarios are assumed to capture the interdependence of different risk factors between each other and across time ([13, p.67].

For historical scenario tests, market factors are extracted from historical scenarios; while for hypothetical scenario tests, market factors s_1, s_2, ..., s_m are to be generated appropriately. We propose the following steps for generating hypothetical scenarios:

(a) Assume that $s_1, s_2, ..., s_m$ are multivariate normal. Estimate the covariance matrix and denote it by *R*. Decompose *R* by the Cholesky algorithm ([20, pp.51-54]) as: $R = G^T G$, where G^T is the transpose of the matrix G

(c) Generate $w_i \sim N(0,1)$ independently, and deliver a scenario as:

$$(s_1, s_2, ..., s_m)^T = G(w_1, w_2, ..., w_m)^T$$

3.2. Scenario Tests and Loss Assessments

We propose the following steps for a scenario test:

(a) Model systematic risk by a multifactor Vasicek model (2.3), following the steps proposed in section 2.2. This includes estimating the model parameters, and the standard deviation σ for the latent effect s'.

(b) Model entity specific risk by a probit or a logistic model $p_m(x)$, using a list of entity specific risk sensitive drivers *x*, calibrate the model $p_m(x)$ over the current portfolio, and set $z = \Phi^{-1}(p_m(x)) - E(\Phi^{-1}(p_m(x)))$ by (1.5).

(c) Given a scenario for the systematic risk factors, calculate entity scenario PD by expression (2.5), and portfolio scenario loss (SL) by:

$$Loss = \sum_i \sum_j P_i EAD_{ij} LGD_{ij} \tag{3.1}$$

where P_i denotes the scenario PD for entity *i*, EAD_{ij} the exposure at default for facility *j* of entity *i*, and LGD_{ij} the loss given default for facility *j*.

4. An Empirical Example – A Dynamic PD Model for a Commercial Portfolio

4.1. Modeling Systematic Default Risk

In this section, we model the portfolio level PD, i.e., $p(s)$, for a US commercial portfolio, where historical 1-year default rates are available for each quarter between 2006 and 2012.

The delinquency rate for commercial and industry loans (no seasonal adjustment), posted by US Federal Reserve, is available since 1987. This is the macro variable we use for systematic risk modeling. Based on portfolio historical default data, internal portfolio default rate responds to US delinquency rate by a lag of two quarters.

We follow the steps proposed in section 2.2, do a variance correction to the default rates, and bootstrap 200 times. Each time we train a model of the form (4.1) below over the bootstrap sample:

$$p(s) = \Phi(a + b_1 u_1 + b_2 s_1 + cs''), \ \ s'' \sim N(0, 1) \tag{4.1}$$

where $u_1 = \Phi^{-1}(r_1)$ and r_1 is the current portfolio default rate (original default rate, not the adjusted one by the variance correction), while

$$s_1 = \Phi^{-1}(US\ delinquent\ rate\ in\ 6\ months)$$

We then calculate for each of the parameters a, b_1, b_2, and c, the average its bootstrap estimates, and select from the bootstrap models the one with parameters the closest to their bootstrap averages. This is the final model we will use for systematic risk $p(s)$.

4.2. Modeling Entity Specific Default Risk

For entity specific risk, we train a logistic model over a sample of portfolio historical data, targeting entity default indicator. The sample contains 1161 entities, including all defaults for years 2006-2011, but non-defaults are sampled randomly and proportionally by year for each year in 2006-2012. The model includes six entity specific risk drivers:

1. Debt Service Coverage Ratio
2. Annual Revenue
3. Ratio of Debt to Tangible Net Worth
4. Ratio of Debt to EBITDA
5. Ratio of Cash and Security to Current Liability
6. Years in Business

For assessment of regulatory capital (RC) ([1, pp.59-60]) and expected loss (EL), the model is calibrated at a long-run portfolio PD of 3.1% over the current portfolio (as of September 2012). Denote this model by $p_m(x)$, given entity specific risk profile x. Note that model $p_m(x)$ is not a PIT model yet at the moment.

4.3. Scenario Tests

The portfolio is assumed to be risk homogeneous, e.g., entities have the same systematic risk (i.e. the same asset correlation), and each entity specific risk z can be regarded as being sampled independently from the same distribution $N(0, \sigma_z^{\ 2})$. Then the idiosyncratic risk component z in (1.4) can be derived by (1.5) using the entity specific risk model $p_m(x)$ developed in section 4.2:

$$z = \Phi^{-1}(p_m(x)) - E(\Phi^{-1}(p_m(x)))$$

where $E(\Phi^{-1}(p_m(x)))$ is estimated by the average of $\Phi^{-1}(p_m(x))$ over current portfolio (as of September 2012). Estimate the standard deviation σ_z of z over the portfolio. Then the systematic risk component w in (1.4) is given by

$$w = \sqrt{1+\sigma_z^{\ 2}}(a + b_1 u_1 + b_2 s_1 + cs")$$

where $p(s) = \Phi(a + b_1 u_1 + b_2 s_1 + cs")$, $s" \sim N(0, 1)$, is the model developed in section 4.1 for the systematic risk $p(s)$.

Combine w and z together by (1.4). Scenario tests follow the steps (a)-(c) proposed in section 3.2, using the existing portfolio EAD and LGD models for the portfolio. We assume that each of these two models dynamically captures the exposure at default or the loss rate for a facility in the portfolio.

Results are shown in Table 1 below. Entities in the portfolio are grouped into investment (Inv), sub investment (Sub), and problematic (Prblm) grades, based on entity scenario PD given by expression (2.5).

The columns 2-6 in the table are respectively the current US delinquent rate, US delinquent rate in six months, current portfolio default rate, realized portfolio default rate in one year, predicted portfolio-level PD in one year given by Proposition 2.4(b), and scenario portfolio-level PD given by Proposition 2.4(a). Portfolio scenario loss (SL) is calculated as proposed in Section 3.2 (c), as a percentage of total portfolio exposure (the sum of all facility EAD_{ij} in the portfolio).

Recall that the US delinquency rate in six months is the only macro variable used in the model (4.1) for the systematic risk. We are interested in two scenarios as highlighted in Table 1: both assume the current time $(t = 0)$ as of 2nd quarter of 2009.The historical scenario uses the macro variable value of fourth quarter of 2009 (in $t = 6$ months), which is 4.4%, and the hypothetical scenario uses the macro variable value of 3rd quarter of 1987, which is 6.6%.

The results show:

(a) With the hypothetical scenario, most (58.43%) entities migrate to problematic grade (including defaults).

(b) Among all the historical scenarios, portfolio scenario loss (SL) peaks in 2nd quarter of 2010 (the end of one-year horizon), with a loss of 2.9% of the total portfolio exposure; while for the hypothetical scenario, the scenario portfolio loss reaches 5.3% of total portfolio exposure.

Table 1. Scenario assessments for a commercial portfolio

	US Delinq Rate		Portfolio Level Default Risk				Scenario Rating Distribution			
Year /Quarter	Current	In Six Mths	Current	Realized	Pred PD	Scenario PD	Inv	Sub	Prblm	SL
2006Q2	1.3%	1.2%	0.6%	2.4%	1.3%	1.4%	55.8%	37.2%	7.0%	0.5%
2006Q4	1.2%	1.1%	1.5%	1.9%	1.6%	1.9%	55.8%	37.2%	7.0%	0.7%
2007Q2	1.1%	1.3%	2.4%	1.3%	1.8%	2.6%	41.6%	51.4%	7.0%	0.9%
2007Q4	1.3%	1.7%	1.9%	1.3%	1.9%	3.1%	25.7%	58.9%	15.4%	1.1%
2008Q2	1.7%	2.6%	1.3%	2.7%	2.1%	4.3%	25.7%	58.9%	15.4%	1.4%
2008Q4	2.6%	3.7%	1.3%	4.9%	3.1%	6.2%	11.1%	62.0%	26.9%	2.1%
2009Q2	3.7%	4.4%	2.7%	5.3%	5.0%	9.0%	5.2%	50.5%	44.2%	2.9%
2009Q4	4.4%	3.5%	4.9%	4.6%	6.8%	8.7%	7.0%	48.8%	44.2%	2.8%
2010Q2	3.5%	3.0%	5.3%	4.7%	5.9%	7.6%	7.0%	66.1%	26.9%	2.5%
2010Q4	3.0%	2.1%	4.6%	4.5%	4.9%	5.1%	11.1%	73.5%	15.4%	1.7%
2011Q2	2.1%	1.7%	4.7%	3.7%	3.7%	4.1%	25.7%	58.9%	15.4%	1.4%
2011Q4	1.7%	1.4%	4.5%	2.9%	3.0%	3.3%	25.7%	58.9%	15.4%	1.1%
1987Q3	6.6%	6.6%	5.7%	5.7%	9.9%	16.1%	0.8%	40.8%	58.4%	5.3%

Conclusion. In practice, most entity PD models do not fully or dynamically capture systematic risk. The approaches proposed in this paper allow systematic and entity specific risks to be modelled separately and then aggregated together analytically. Systematic risk is quantified and modelled by a multifactor Vasicek model with a latent residual, a factor accounting for default contagion and feedback effects. The asymptotic maximum likelihood approach for parameter estimation for this model is equivalent to least squares linear regression. Conditional entity PDs for scenario tests and TTC entity PD all have analytical solutions. Stress testing can be conducted by shocking the risk factors in the system risk component model.

REFERENCES

[1] Basel Committee on Banking Supervision (2004). International Convergence of Capital Measurement and Capital Standards

http://www.bis.org/publ/bcbs118.pdf
[2] Basel Committee on Banking Supervision (2009). Principles for Sound Stress Testing Practices and Supervision
http://www.bis.org/publ/bcbs155.pdf
[3] Basel Committee on Banking Supervision (2009). The Application of Basel II to Trading Activities and the Treatment of Double default Effects
http://www.bis.org/publ/bcbs116.pdf
[4] Basel Committee on Banking Supervision (2005). An Explanatory Note on the Basel II IRB Risk Weight Functions, July 2005.
http://www.bis.org/bcbs/irbriskweight.pdf
[5] Blaschke, W., Jones , M. T., Majnoni, G., and Peria, S. M.(2001). Stress Testing of Financial Systems: An Overview of Issues, Methodologies, and FSAP Experiences, IMF Working Paper, WP/01/88
http://www.imf.org/external/pubs/ft/wp/2001/wp0188.pdf
[6] Breiman, L. (1996). Bagging Predictors. Machine Learning 24: 123-140
http://www.cs.utsa.edu/~bylander/cs6243/breiman96bagging.pdf
[7] Bunn, P. (2005). Stress testing as a tool for assessing systematic risks, Financial Stability Review, 2005:6, pp.116-126
http://www.bankofengland.co.uk/publications/Documents/fsr/2005/fsr18art8.pdf
[8] Carlehed, M., Petrov, A. (2012). A methodology for point-in-time-through-the-cycle probability of default in risk classification systems, Journal of Risk Model Validation, Fall 2012, pp.3-25
[9] Cihak, M. (2007). Introduction to Applied Stress Testing, IMF Working Paper, WP/07/59
http://www.imf.org/external/pubs/ft/wp/2007/wp0759.pdf
[10] Committee on the Global Financial System (2001). A Survey of Stress Tests and Current Practice at Major Financial Institutions
http://www.mirkin.ru/_docs/articles02-055.pdf
[11] Das, S., Duffie, D.,Kapadia, N., Saita, L. (2007) Common Failings: How Corporate Defaults are Correlated. Journal of Finance 62 (1), 93-117
http://www.q-group.org/archives_folder/pdf/Paper-Das.pdf
[12] Demey, P., Jouanin, J., Roget, C, and Roncalli, T. (2004). Maximum likelihood estimate of default correlations, Risk, November 2004
http://thierry-roncalli.com/download/risk-mledc.pdf
[13] Drehmann, M. (2008). Stress tests: Objectives, challenges and modelling choices, Economic Review, 2008:Vol 60 (2), pp. 60-92
http://www.riksbank.se/Upload/Dokument_riksbank/Kat_publicerat
/Artiklar_PV/2008/drehmann2008_2_eng.pdf
[14] Gordy, M. B. (2003). A risk-factor model foundation for ratings-based bank capital rule. Journal of Financial Intermediation12, pp.199-232.
[15] Kupiec P. H. (2000). Stress tests and risk capital, Journal of Risk, Volume 2 (4), pp.27-39
[16] Jacobson, T., Linde, J., Roszbach, K. (2011) Firm Default and Aggregate Fluctuations, Board of Governors of the Federal Reserve System, August 2011
[17] Merton, R. (1974). On the pricing of corporate debt: the risk structure of interest rates. Journal of Finance, Volume 29 (2), 449-470
http://www.jstor.org/discover/10.2307
/2978814?uid=2129&uid=2&uid=70&uid=4&sid=21102236625247
[18] Meyer, C. (2009). Estimation of intra-sector asset correlation. The Journal of Risk Model validation, Volume 3 (3), Fall
http://www.risk.net/digital_assets/5021/jrm_v3n3a3.pdf
[19] Miu, P., Ozdemir, B. (2008). Estimating and validating long-run probability of default with respect to Basel II requirements. The Journal of Risk Model validation, Volume 2/Number 2, 3-41
[20] Nocedal, J., Wright, S. J. (2006), Numerical Optimization, 2nd Edition, Springer Series in Operations Research and Financial Engineering

[21] Pindyck, R. S., Rubinfeld, D. L. (1998). Econometric Models and Economic Forecasts, 4th Edition, Irwin/McGraw-Hill
[22] Rosen, D., Saunders, D. (2009). Analytical methods for hedging systematic credit risk with linear factor portfolios. Journal of Economic Dynamics & Control, 33 (2009), 37-52
http://www.r2-financial.com/wp-content/uploads/2010/07/LinearFactor.pdf
[23] Segoviano, M. A. and Goodhart, C. (2009). Banking Stability Measures, *IMF working papers* 4 (2009).
http://www.imf.org/external/pubs/ft/wp/2009/wp0904.pdf
[24] Tarashev, N., Borio, C., and Tsatsaronis, K. (2010). Attributing systematic risk to individual institutions," Technical Report Working Papers No 308, BIS, May 2010.
http://www.bis.org/publ/work308.pdf
[25] Vasicek, O. (2002). Loan portfolio value. RISK, December 2002, 160 - 162.
[26] Wolfinger, R. (2008). Fitting Nonlinear Mixed Models with the New NLMIXED Procedure. SAS Institute Inc.
http://www.ats.ucla.edu/stat/sas/library/nlmixedsugi.pdf
[27] Yang, B. H. (2013). Estimating long-run probability of default, asset correlation and portfolio-level probability of default using Vasicek models, Journal of Risk Model Validation, Winter 2013

APPENDIX A

Proof of Lemma 2.1 (c). Given constants a and $b\,(\geq 0)$, it can be shown ([22], Lemma 6, p.48) that

$$E[(\Phi(a+bz))^2] = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\alpha}^{c}\int_{-\alpha}^{c}\exp[-(x^2+y^2-2\rho xy)/(2-2\rho^2)]dxdy$$

where $c = a/\sqrt{1+b^2}$, $\rho = b^2/\sqrt{1+b^2}$, and $z \sim N(0,1)$. By Lemma 2.1 (a), we have $E[\Phi(a+bz)] = \Phi(a/\sqrt{1+b^2})$. Let Δ_1 denote the derivative of $E[(\Phi(a+bz))^2]$ with respect to c, and Δ_2 the derivative of $(E[\Phi(a+bz)])^2$ with respect to c. It suffices to show $\Delta_1 > \Delta_2$ when $c < 0$ and $\rho > 0$. Let $\phi(-)$ be the standard normal distribution. Then $\Delta_2 = 2\phi(c)\Phi(c)$, and we have

$$\begin{aligned}
\Delta_1 &= \frac{2}{2\pi\sqrt{1-\rho^2}}\int_{-\alpha}^{c}\exp[-(c^2+y^2-2\rho cy)/(2-2\rho^2)]dy \\
&= \frac{2}{2\pi\sqrt{1-\rho^2}}\int_{-\alpha}^{c}\exp(-[c^2(1-\rho^2)+(y-\rho c)^2]/(2-2\rho^2))dy \\
&= \frac{2\phi(c)}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\alpha}^{c}\exp[-(y-\rho c)^2/(2-2\rho^2)]dy \\
&= 2\phi(c)\Phi[c\sqrt{(1-\rho)/(1+\rho)}] > 2\phi(c)\Phi(c)
\end{aligned}$$

This is because $c < 0$ and $\rho > 0 \Rightarrow c < c\sqrt{(1-\rho)/(1+\rho)}$ □

Proof of Proposition 2.3. Statement (b) is a corollary of Lemma 2.1 (a). For statement (a), we have by Corollary 2.2

$$w/\sqrt{1+\sigma_z^{\,2}} = \Phi^{-1}(p(s)) \Rightarrow w = \Phi^{-1}(p(s))\sqrt{1+\sigma_z^{\,2}} \quad □$$

Proof of Proposition 2.4. We have:

$$p(s_1,...,s_m) = E(\Phi(u+v+s') \,|\, s_1,...,s_m,u_1,...,u_k)$$
$$p(s_1(0),...,s_m(0)) = E(\Phi(u+v+s') \,|\, s_1(0),...,s_m(0),u_1,...,u_k)$$

Statement (a) is a corollary of Lemma 2.1 (a). For statement (b), we have

$$p(s_1(0),...,s_m(0)) = E(\Phi(u+d+\rho_v v(0)+(\Delta v+s')) \,|\, s_1(0),...,s_m(0),u_1,...,u_k)$$
$$\Rightarrow p(s_1(0),...,s_m(0)) = \Phi((u+d+\rho_v v(0))/\sqrt{1+\sigma_{\Delta v}{}^2+\sigma^2}) \,\square$$

Proof of Proposition 2.5. By (1.4), we have:

$$p(s_1, s_2,..., s_m, x) = E(\Phi(\sqrt{1+\sigma_z{}^2}(u+v+s')+z) \,|\, s_1, s_2,..., s_m,\, u_1, u_2,...,u_k, x)$$

As the variance for the term $\sqrt{1+\sigma_z{}^2}\,s'$ is $\sigma^2(1+\sigma_z{}^2)$, the proposition follows from Lemma 2.1 (a). □