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WHICH ECONOMIC STATES ARE SUSTAINABLE UNDER A SLIGHTLY CONSTRAINED TAX-RATE ADJUSTMENT POLICY

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ABSTRACT. Viability theory is the study of dynamical systems that asks what set of initial conditions will generate evolutions which obey the laws of motion of a system and some state constraints, for the length of the evolution. We apply viability theory to Judd's (JPE, 1987) dynamic tax model to identify which economic states today are sustainable under only slightly constrained tax-rate adjustments in the future, when the dynamic budget constraint and consumers' transversality condition at infinity are satisfied. We call the set of such states the economic viability kernel. In broad terms, knowledge of the viability kernel can tell the planner what economic objectives are achievable and assist in the choice of suitable controls to realise them. We observe, unsurprisingly, that a very high consumption economy lies outside such kernels, at least for annual tax-adjustment levels limited by 20%; higher consumption levels can only be sustained when capital is abundant. Furthermore, we notice that the sizes of the kernel slices for a given taxation level do not diminish as the tax rate rises, hence high taxation economies are not necessarily more prone to explode, or implode, than their low taxation counterparts. In fact, higher tax rates are necessary to keep many consumption choices viable, especially when capital approaches the constraint-set boundaries.

.

Keywords: taxation policy, macroeconomic modeling, dynamic systems, viability theory; VIKAASA
JEL: C61, E61, E62

1. Introduction

This paper uses viability theory (Aubin (1997)) to examine basic problems in dynamic public finance[1]. For specificity, we use the model studied in Judd (1987).

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[1]This paper draws from Krawczyk and Judd (2012).

Viability theory is the study of dynamical systems that asks what set of possible paths obey the system's laws of motion and remain in some state-constraint set. In one example in our paper, we compute the set of possible consumption levels today that remains invariant under only loose restrictions on tax policy and given a fixed level of government expenditure in the future. Another way of putting this is that we perform a kind of robustness analysis to answer the question *what are the sustainable consumption levels today if all we know is that tax policy will satisfy the dynamic budget constraint and that consumers' transversality conditions at infinity will be satisfied?* The usual perfect foresight analysis specifies one future path for taxes. The viability theory approach relaxes this assumption and puts some (loose) restrictions on tax policy. This enables one to ask how much the perfect foresight result depends on having perfect foresight. For example, suppose that we have some debt today and know the future path of tax rates and government expenditure. Then, under the classical approach there would (likely) be only one consumption and capital combination which would be *viable* i.e., one equilibrium path could originate from this combination. In that case, viability reduces to equilibrium. On the other hand, a viability analysis can establish the set of *all* pairs of consumption and capital (c, k) which represent initial conditions such that there is some future tax-rate path which obeys the restrictions we put on the change in tax rate, and is consistent with equilibrium and with initial conditions (c, k). We assert the collection of all such initial conditions, which we call the *viability kernel*, generalizes the notion of equilibrium, which is one theme of viability theory.

We find that if the only tax is a proportional income tax, then uncertainty about future tax policy does not affect consumption much. However, in other tax systems, such as one that taxes labor and capital differently, uncertainty about future tax policy may lead to much greater uncertainty about current consumption.

This paper focuses on some specific questions in a simple dynamic model of expenditure and taxation. However, there is a much more ambitious agenda behind this paper, which is to present viability theory as an important tool for the solution of economic problems.[2] Its main machinery consists of

[2]So far, viability theory has been applied to a handful of economic and financial problems. For applications to *environmental economics* see Martinet and Doyen (2007), De Lara, Doyen, Guilbaud, and Rochet (2006) and Martinet, Thébaud, and Doyen (2007); *finance* – Pujal and Saint-Pierre (2006); *managerial economics* – Krawczyk, Sissons, and Vincent (2012); *macroeconomics* – Krawczyk and Kim (2009), Bonneuil and Saint-Pierre (2008), Bonneuil and

the formulation and solution of differential inclusions. That is, in viability theory the system's dynamics is represented as a *set* of the directions of motion of the system that depend at any moment on the state. The concept of solution is a path of sets instead of a path of points, where the "tube" formed by those sets is the union of all possible paths that stay in the tube but also satisfy the usual terminal constraints and some additional state restriction. Viability theory is therefore part of set-valued analysis.

Solving viability problems is computationally intensive. However, thanks to some specialized software, solving simple models, of 2 – 4 state variables and 1 – 2 controls, is possible. The software we use is VIKAASA (see Krawczyk and Pharo (2011) and Krawczyk and Pharo (2014)).

Here is how the paper is organized. We expound viability theory in Section 2. Following Judd (1987), we introduce a simple model of expenditure and taxation in Section 3. In Section 4, we make an assumption that the only tax charged in this model will be a proportional income tax and calibrate the model according to this assumption. Further, in Section 5, we compute viability kernels and comment on their topology. We also show (in Section 6) a few possible time profiles of the *debt-to-GDP* ratio and observe that a high value of the ratio does not necessarily imply non-viability. The paper ends with concluding remarks.

2. A brief on viability theory and viable solutions

2.1. **An introduction to viability theory.** Viability theory is a relatively new part of mathematics, see e.g., Aubin (1991, 1997, 2001). Viability problems concern systems that evolve over time, where the concern is to identify *viable evolutions* – trajectories that do not violate some set of viability constraints over a given (possibly infinite) time-frame. A *viability domain* is the set of initial states from which viable trajectories originate and the *viability kernel* is the *largest* viability domain. These are the basic tools for analyzing constrained evolutions also known as viability problems.

The basic feature of the viability kernel is that it provides us with the information necessary to determine whether or not a given state-space position has a viable trajectory proceeding from it, i.e., whether starting at that position, the system can be maintained within its constraints, or not. In what

Boucekkine (2008), Krawczyk and Kim (2004), Krawczyk and Sethi (2007), Clément-Pitiot and Saint-Pierre (2006), Clément-Pitiot and Doyen (1999); *microeconomics* – Krawczyk and Serea (2013). However, several of the above publications are working papers of limited circulation.

follows, we give a more technical explanation of viability theory, including a formal definition of the *viability kernel*.

The core ingredients of a viability problem are (compare Krawczyk and Pharo (2011)):

(1) A continuum of time[3] values, $\Theta \equiv [0, T] \subseteq \mathbb{R}_+$, where T can be finite or infinite.
(2) A vector of n real-valued state variables, $x(t) \equiv [x_1(t), x_2(t), \ldots, x_n(t)]' \in \mathbb{R}^n$, $t \in \Theta$ that together represent the dynamic system in which we are interested.
(3) A *constraint set*, $K \subset \mathbb{R}^n$, which is a closed set representing some normative constraints to be imposed on these state variables. Violation of these constraints means that the system has become non-viable. Thus in seeking viable trajectories, we want to ensure that $\forall t (t \in \Theta)$ $x(t) \in K$.
(4) A vector of real-valued controls, $u(t) \equiv [u_1(t), u_2(t), \ldots, u_m(t)]' \in \mathbb{R}^m$, $t \in \Theta$.
(5) Some normative constraints on the controls. In this paper, we assume that $u \in U$ where U is the set of control vectors available at each state. (In general, the set U can depend on x.)
(6) A set of real-valued first-order differential inclusions,

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} \in \left\{ \psi(x,u) = \begin{bmatrix} \psi_1(x,u) \\ \psi_2(x,u) \\ \vdots \\ \psi_n(x,u) \end{bmatrix} \right\}_{u \in U} . \tag{1}$$

Each function $\psi_i : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$, $i = 1, 2 \ldots n$ specifies the range of velocities of the corresponding variable x_i, at the state position $x(t) \in \mathbb{R}^n$ where $u \in U \subset \mathbb{R}^m$ is a control choice available at this position. Some, but not all, inclusions in (1) can be equalities.

Note that we have formulated viability problems above in terms of *differential inclusions* whereby the evolution of some or all of the system's variables is *set-valued*. That is, for a given $x(t)$ we have an array of possible controls U to choose from and hence have a *set* of velocities $\psi(x(t), u)$, $u \in U$, associated with state $x(t)$. The symbol ψ denotes a point-to-set map, or correspondence,

[3] A similar formulation could be made for a viability problem in discrete time.

from states x to velocities $\psi(x, U)$. We will abbreviate the notation and write $\Psi(x)$ instead of $\psi(x, U)$.[4]

Given problem formulation (1), we can attempt to find one or more *viability domains*, $D \subseteq K$, where each viability domain is a set of initial conditions $x(0)$, for which there exist viable trajectories. That is, for every element $x \in D$, $D \subseteq K \subset \mathbb{R}^n$ there must exist a trajectory that originates at x and is a solution to (1) in D. The problem's *viability kernel*, $\mathcal{V} \subseteq K$ is then the *largest* possible viability domain (or the union of all viability domains), giving all initial conditions in K, for which a viable evolution exists.

We will characterize a viability domain using the Viability Theorem from Cardaliaguet, Quincampoix, and Saint-Pierre (1999) :

Proposition 1. *Assume D is a closed set in $\mathbb{R}^N$. Suppose that $\psi : \mathbb{R}^N \times U \to \mathbb{R}^N$ is a continuous function, Lipschitz in the first variable; furthermore, for every x we define a set valued map $\psi(x, U) = \{\psi(x, u); u \in U\}$, which is supposed to be Lipschitz continuous with convex, compact, nonempty values.*

Then the two following assertions are equivalent[5]*:*

(i)

$$\forall x \in D, \quad \forall p \in \mathcal{NP}_D(x), \quad \min_u \langle \psi(x, u), p \rangle \leq 0 \tag{2}$$

(respectively, $\max_u \langle \psi(x, u), p \rangle \leq 0$*) ;*

(ii) *there exists a function $u : \Theta \mapsto U$ such that (respectively, for all such functions)*

$$\text{the solution of } \begin{cases} \dot{x}(s) = \psi(x(s), u(s)) \text{ for almost every } s \\ x(t) = x \end{cases} \tag{3}$$

remains in D.

To be precise, Proposition 1 merges two results first proved in Veliov (1997) (concerning $\exists\, u$) and in Krastanov (1995) (concerning $\forall\, u$).

Notice that the inequality $\min_u \langle \psi(x, u), p \rangle \leq 0$ in (2) means that there *exists* a control for which the system's velocity $\dot{x}$ "points inside" the set D.

[4]In a numerical algorithm commented on in Section 2.2 we seek controls from U for which the trajectories are *viable* i.e., $x(t) \in K$ for all $t \in \Theta$. For existence and characterisation of feedback controls assuring viability see Veliov (1993).

[5]Here $\mathcal{NP}_D(x)$ denotes the set of *proximal normals* to D at x i.e., the set of $p \in \mathbb{R}^N$ such that the distance of $x + p$ to D is equal to $\|p\|$.

Respectively, $\max_u \langle \psi(x,u), p\rangle \leq 0$ means that the system's velocity $\dot{x}$ "points inside" the set D for *all* controls from U.

When i. (or ii.) holds we say that D is a *viability domain* (or, respectively, D is an *invariance domain*) for the dynamics Ψ.

This introduces the classical notion of the viability (respectively, invariance) domain Aubin (2001), as opposed to viability domains in problems with *targets*, see Quincampoix and Veliov (1998).

Definition 2.1. *Let K be a closed set in $\mathbb{R}^N$. We call the* ***viability kernel*** *in K, for the dynamics Ψ, denoted:*

$$\mathcal{V}_\Psi(K)$$

the largest closed subset of K, which is a viability domain for Ψ.

It was proved (see e.g., Aubin (1991) or Quincampoix and Veliov (1998)) that $\mathcal{V}_\Psi(K)$ is the set of x such that there exists $x(\cdot)$, a solution of

$$\dot{x}(s) \in \Psi(x(s)) \tag{4}$$

starting from x, which is defined on $[0,\infty)$ and $x(s) \in K$ for all $s \geq 0$.

If Ψ is the collective vector of right hand sides like in (1) then the problem that we want to *solve* is

$$\textit{establish viability kernel} \quad \mathcal{V}_\Psi(K) \quad \textit{for the dynamics } \Psi\,. \tag{5}$$

We will approximate $\mathcal{V}_\Psi(K)$ by looking for solutions to (4).

2.2. A method for the determination of viability kernels.

In Gaitsgory and Quincampoix (2009) we can find a base for how to approximate $\mathcal{V}_\Psi(K)$ using the solutions to (4). In broad terms, they say that if a constrained optimal control problem, subjected to the system's dynamics $\Psi(\cdot)$ and the constraint set K, can be solved for $x \in K$ and $x(t) \in K\ \forall t$, then x is viable.

VIKAASA[6], is a computational tool which computes viability kernel approximations (actually, domains) for the class of viability problems introduced in Section 2.1, using a user-selected algorithm. In this paper, we have selected one that solves a truncated optimal stabilization problem, rather than a general optimal control problem, for each $x^h \in K^h \subset K$ where K^h is a suitably discretized K.

For each $x^h \in K^h$, VIKAASA assesses whether a dynamic evolution originating at x^h can be controlled to a (nearly) steady state without leaving the constraint set in finite time. Those points that can be brought close enough

[6]See Krawczyk and Pharo (2011) and Krawczyk and Pharo (2014); also Krawczyk, Pharo, and Simpson (2011), Krawczyk, Pharo, Serea, and Sinclair (2013).

to such a state are included in the kernel by the algorithm, whilst those that are not are excluded.[7]

In Section 5 we present some results from running the algorithm on the taxation problem, introduced in the next section.

3. The tax model

Our goal in this paper is to use viability theory for an analysis of a tax model based on Judd (1987). In that model capital, labor, consumption, debt, marginal utility of consumption and tax rates are all variables of time. To unburden the notation we will drop the time argument on each of them.

The fundamental law of motion for capital k is determined by net output i.e., $y-\delta k$, where y is output and $\delta > 0$ is the rate of depreciation, diminished by consumption $c > 0$ and government expenditure is $g \geq 0$. If so and assuming a Cobb-Douglas type production function for output, we get, in continuous time,

$$\frac{dk}{dt} = Ak^{\alpha}\ell^{1-\alpha} - \delta k - c - g\,. \tag{6}$$

As usual, $\ell > 0$ is labor, $A > 0$ — total factor productivity and α, $0 < \alpha < 1$ — output elasticity of capital. In this model, expenditure g is assumed constant but several values of g will be checked in the computations.

Let the utility of consumption of a representative agent be

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \tag{7}$$

and the disutility of labor

$$v(\ell) = V\frac{\ell^{1+\eta}}{1+\eta} \tag{8}$$

[7]This algorithm (called *inclusion* algorithm, see Krawczyk et al (2013)) employed by VIKAASA will miss any viable points that cannot reach a steady state; e.g., because they form (large) 'orbits'. However, experimenting with the tax model (18) - (20), (26), which consisted of using different discretisation grids and trying various controls, did not lead to discovery of a point like that. In particular, VIKAASA has produced results in Krawczyk et al (2011) that coincide with with those from Krawczyk and Serea (2009), where a method based directly on Gaitsgory and Quincampoix (2009) was applied to the same problem. In turn, the outputs in Krawczyk and Serea (2009) coincide with those published in Krawczyk and Kim (2009).

where V, γ, η are positive. If $\lambda > 0$ is the private marginal value of capital at time t, then it follows from maximization of the utility function $u(c) - v(\ell)$, on an infinite horizon with some discount rate $\rho > 0$, that[8]

$$\frac{d\lambda}{dt} = \lambda(\rho - \bar{r})\,. \tag{9}$$

Here, $\bar{r} = (1-\tau_K)\left(\frac{\partial y}{\partial k} - \delta\right)$ is the after tax marginal product of capital, where τ_K $(0 < \tau_K < 1)$ is capital tax. Expanding $\bar{r}$ in (9) yields

$$\frac{d\lambda}{dt} = \lambda\left(\rho - (1-\tau_K)\left(\alpha A \left(\frac{\ell}{k}\right)^{1-\alpha} - \delta\right)\right). \tag{10}$$

To characterize the economy at hand, we will also use government debt B, which grows in g and diminishes with tax T as follows:

$$\frac{dB}{dt} = \bar{r}B - T + g \tag{11}$$

where, as above, $\bar{r}$ is the net-of-tax interest rate. In this economy, tax rates on capital and labor are τ_K and τ_L $(0 < \tau_L < 1, 0 < \tau_K < 1)$, respectively; if so, the expression for total tax T in (11) at time t becomes

$$T \;=\; \tau_K \alpha A k^\alpha \ell^{1-\alpha} + \tau_L (1-\alpha) A k^\alpha \ell^{1-\alpha} = \Big(\alpha(\tau_K - \tau_L) + \tau_L\Big) A k^\alpha \ell^{1-\alpha}\,.$$

Combining (12) and (11) results in the following debt dynamics

$$\frac{dB}{dt} = \bar{r}B - \Big(\alpha(\tau_K - \tau_L) + \tau_L\Big) A k^\alpha \ell^{1-\alpha} + g\,, \tag{12}$$

where $\bar{r} = (1-\tau_K)(\alpha A k^{-(1-\alpha)} \ell^{1-\alpha} - \delta)$ will be included in this expression later. In simple terms, we see that debt can diminish if output is large or if the tax rates are high (and when output is not too small).

While the private marginal value of capital, λ, can adequately characterize the consumer's behavior, it lacks an easy economic interpretation. We will

[8] Except where stated otherwise, all settings in our model are the same as in Judd (1987), which can also be traced down to Brock and Turnovsky (1981). In particular, the private marginal value of capital λ (or, agent's marginal utility of consumption, see (14)) is the adjoint state in the perfect-foresight household utility $u(c) - v(\ell)$ maximisation problem. Part of its specification is a request for the satisfaction of the consumers' transversality condition at infinity. To obtain optimal consumption, it is sufficient to solve the underlying optimal control problem and use (15). Solving the viability problem will tell us which such optimal consumption decisions are compatible with current capital, labour and a limited-variation (hence only "near-perfect" foresight) tax policy. When we say that the viability kernel is non-empty we imply that the consumers' transversality condition at infinity is fulfilled.

replace the equation for $\frac{d\lambda}{dt}$, (9), by a differential equation for consumption, easily interpretable.

The marginal utility of consumption (see (7)) is

$$\frac{du}{dc} = \frac{1}{c^{\gamma}}; \tag{13}$$

on the other hand, λ is the marginal utility of consumption, so

$$\frac{du}{dc} = \lambda \tag{14}$$

hence,

$$c = \frac{1}{\lambda^{1/\gamma}}, \tag{15}$$

which, after differentiation in the time domain, yields

$$\frac{dc}{dt} = \frac{-1}{\gamma} \cdot \frac{1}{\lambda^{1+1/\gamma}} \cdot \frac{d\lambda}{dt} = \frac{-1}{\gamma}\, c^{1+\gamma}\, \frac{d\lambda}{dt}. \tag{16}$$

Using (10), after some simplifications, we get

$$\frac{dc}{dt} = -\,c \cdot \frac{\rho + \left(\delta - \alpha\, A\, k^{\alpha-1}\, \ell^{1-\alpha}\right)\,(1-\tau_K)}{\gamma} \tag{17}$$

We can see that consumption has one trivial steady state and will grow if ρ (discount rate) and/or δ (depreciation) are "small".

We will now write the three equations of motion (6), (17), (12) together, for a better look at the economy we want to analyze:

$$\frac{dk}{dt} = Ak^{\alpha}\ell^{1-\alpha} - \delta k - c - g \tag{18}$$

$$\frac{dc}{dt} = -\,c \cdot \frac{\rho + \left(\delta - \alpha\, A\, k^{\alpha-1}\, \ell^{1-\alpha}\right)\,(1-\tau_K)}{\gamma} \tag{19}$$

$$\frac{dB}{dt} = \bar{r}B - \left(\alpha(\tau_{_K} - \tau_{_L}) + \tau_{_L}\right)Ak^{\alpha}\ell^{1-\alpha} + g\,. \tag{20}$$

The system of differential equations (18) - (20) is the basic representation of the economy at hand, for which we want to establish the viability kernel i.e., the loci of economic states, from which moderate tax adjustments can guarantee a balanced evolution of the economy.

We recognize that this system is nonlinear with multiple steady states. We can see that, as one would expect, the consumption growth or decline can be moderated by adjusting the capital tax rate while debt will (mainly) depend on the labor tax rate. If the rates were identical ($\tau_{_L} = \tau_{_K}$), then increasing them/it will slow down the consumption rate and diminish debt. With high taxation rate, consumption and debt will naturally diminish and capital will grow (because labor increases, see below). We also notice that debt will grow very fast for large B and non-excessive capital taxation.

We now want to express labour ℓ through capital and consumption and thus "close" the dynamic system (18) - (20).

Let w denote (time-dependent) wages; they equal to the marginal product of labour:

$$w = \frac{dy}{d\ell} = \frac{(1-\alpha)k^{\alpha} A}{\ell^{\alpha}} \tag{21}$$

In equilibrium, the marginal utility of consumption weighted by the after-tax wages must be equal to the marginal disutility from labor:

$$\frac{(1-\tau_{_L})w}{c^{\gamma}} = \ell^{\eta} V\,. \tag{22}$$

Substituting wages and solving for labor yields,

$$\ell = \left(\frac{(1-\tau_{_L})(1-\alpha)Ak^{\alpha}}{c^{\gamma} V}\right)^{\frac{1}{\alpha+\eta}}\,, \tag{23}$$

from which we see that labor can be determined by capital and consumption.

We could now use (23) to substitute labor in (18) - (20), but the resulting formulae would appear more complicated than the original equations, even if they contained one variable less. We will not show them here. We will however use them in the computations, after we have calibrated the equations. Here, we can observe that if $\gamma > \alpha$ then labor decreases in consumption faster than it grows in capital. Allowing for this tells us that the sign of (19) will be negative for large discount and depreciation rates hence high consumption levels will quickly diminish. Large consumption will also contribute to a decline of capital and a rise of debt. However, this multiple downturn may be avoided by an "early" (preemptive) drop of taxes on capital. We will see from which states such an preventive drop can be efficient, after we have computed the viability kernel for this economy, in Section 5 .

To fully describe the tax model dynamics, the equations (18) - (20) (with (23)) need be completed by two differential inclusions for the two tax rates $\tau_{_L}$ and $\tau_{_K}$:

$$\frac{d\tau_{_L}}{dt} = u_{_L} \in [-d_{_L}, d_{_L}] = U_{_L} \tag{24}$$

and

$$\frac{d\tau_{_K}}{dt} = u_{_K} \in [-d_{_K}, d_{_K}] = U_{_K} \tag{25}$$

where d_L, d_K are positive numbers. The inclusions represent bounds on the speed at which tax rates can change. This corresponds to the government policy of "smooth" tax rates adjustments determined by d_L and d_K.

In the current version of the model we will assume that the only tax is a proportional income tax, so the tax rate on labour and capital are equal i.e., $\tau_L = \tau_K = \tau$. Therefore, the above inclusions (24), (25) collapse to

$$\frac{d\tau}{dt} = u \in [-d, d] = U, \, d \geq 0 \, . \tag{26}$$

4. Model calibration

We propose that neglecting depreciation will not greatly affect the economic dynamics and so set δ to zero. Government expenditure g is assumed to be constant. We will construct a couple of different calibrations for the model, each with a different level of government expenditure. First, we set g at 10% of no-tax steady-state output.

We will assume $\rho = 0.04$, $\alpha = 0.3$, $\eta = 1$ and $\gamma = 0.5$ that, in broad terms, characterize a reasonably industrialized economy composed of rational agents interested in the near future (notably, $exp(-0.04 \cdot 10) = 0.67$ and $exp(-0.04 \cdot 50) = 0.13$), drawing a fair satisfaction from consumption and feeling, quite strongly, the burden of labor.

We will use a stylized steady state $\underline{k} = \underline{\ell} = 1$ with no taxes and no government expenditure to calibrate A and V. Setting the right hand sides of (6) and (9) to zero yields

$$A = \underline{c}, \quad \text{and} \quad A = \frac{\rho}{\alpha} \quad \text{hence} \quad A = \underline{c} = 0.1333 \tag{27}$$

where $\underline{c}$ is the no-tax consumption steady state. Then, we get from (23) that

$$V = (1 - \alpha)\left(\frac{\rho}{\alpha}\right)^{1-\gamma} \quad \text{hence} \quad V = 0.2556 \, . \tag{28}$$

Finally, in our initial calibration, $g = 0.1A = 0.0133$.

As said in Section 2, we also need to set boundaries that the economy should *not* cross. We propose that

(I) **capital** should be between 10% and 200% of no-tax steady state capital stock i.e., $k \in [0.1, 2]$;

(II) **consumption** should range between 1/5 of and 5 times the no-tax steady state consumption $\underline{c}$ i.e., $c \in [0.0267, 0.6667]$;

(III) **debt** may be allowed to grow to 150-200% of the maximum steady-state capital stock and also drop below zero so, in this study, $B \in [-1, 3.5]$;

(IV) **tax rate** $\tau \in [0. 0.8]$;

(V) **tax-rate adjustment speed** i.e., the amount by which the regulator can change the current tax-rate level within a year will be between -20 and 20 percentage points so, $u \in [-0.2, 0.2]$, where u is the tax-adjustment speed.

These constraints have been chosen somewhat arbitrarily. In a "real world" calibration, constraints would come from a combination of positive and normative sources, as well as from the requirement to close K. For instance, the lower bound on capital might be tied to a normative requirement concerning the nation's GDP, whereas the upper bound might be based simply on the observations that capital would never realistically fluctuate that far from its steady state. Bounds on consumption, debt and tax would be similarly determined. In general, normative requirements might be determined through some auxilliary optimisation procedure, or they might be externally given (e.g., politically).

The calibrated system's movements can be learned from Figure 1, which presents vector fields in the *capital-consumption* state space, for no debt, for two different tax levels. The no-tax, no government expenditure steady state is shown as the big dot in the left panel. We observe in each panel that the closer we are to the centre, the slower the system will be moving so, for a large central area of consumption choices, the economy appears stabilizable. We also notice that consumption above 0.2 appears unsustainable in the long-run because it causes capital to quickly diminish or vanish. With this observation, we will reduce the top consumption level to 0.225.

Finally, the constraint set K, for which we will seek the viability kernel, is

$$K = [0.1, 2] \times [0.0267, 0.225] \times [-1, 3.5] \times [0, 0.8]\,. \tag{29}$$

The viability problem is then to determine the kernel $\mathcal{V} \in K \subset \mathbf{R}^4$ for the dynamics $\Psi(\cdot)$ defined through the vector differential inclusion[9] (18) - (20), (26) (with (23)). We will use VIKAASA to compute $\mathcal{V}$.

5. The viability kernel

We will show several viability kernel *slices* for the following two situations:

- $\overline{B} = 3.5$ and $g = 0.0133$, as introduced in Section 4;
- government expenditure doubles to $g = 0.0266$.

[9]Because of (26), system (18) - (20) is now a differential inclusion in $\mathbf{R}^4$.

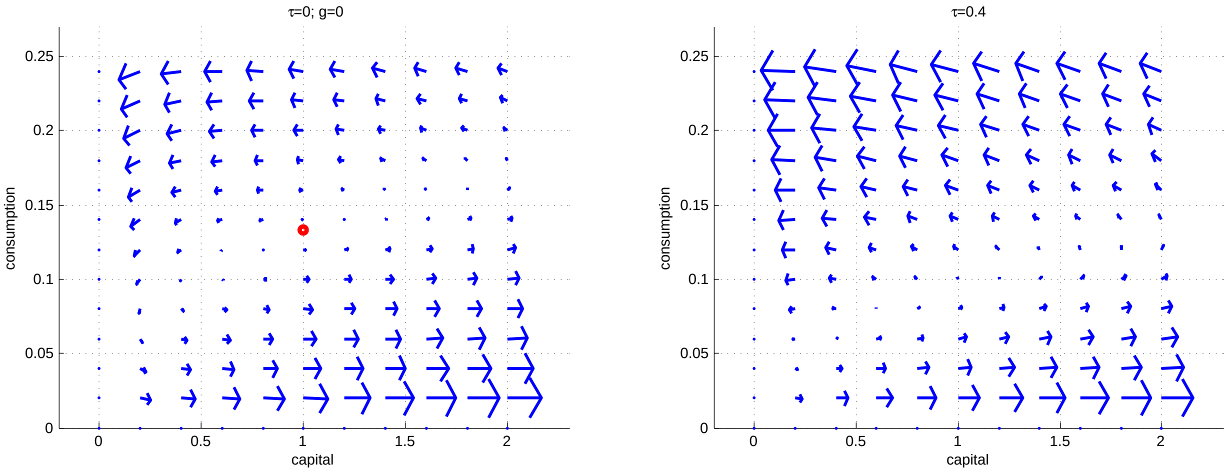

FIGURE 1. k, c-vector fields for $\tau = 0$ $g = 0$, left panel and $\tau = 0.4$, right panel.

5.1. **How to interpret 3D *slices* of the 4D kernel?** Given that $\mathcal{V} \subset K \subset \mathbf{R}^4$ where we cannot display sets, the analysis will be conducted using 3D (sometimes 2D) cross-sections, or "slices" of $\mathcal{V}$.

EXPLANATION BOX 1.

To analyze the tax policy, we will use 3D slices of the 4D space (k, c, B, τ) where evolutions of the economy "live". The first such a slice is shown in Figure 2. The three dimensions, for which the slice is cut, are labelled along the respective axes (here: capital, consumption and tax rate); the fourth dimension is kept constant (here: debt=1.25). The rectangular box in each figure delimits a 3D projection of $K \subset \mathbf{R}^4$ where K is the constraint set, within which the economy is supposed to remain. A 3D body ("boulder") is a snapshot of the viability kernel taken for a particular value of the fourth dimension, written down in the caption or as the figure's title. If there is a line (trajectory) shown in the figure, then each point of this line corresponds to a different value of the fourth dimension; i.e., the 3D line is parametrized in the fourth dimension.

Explanation Box 2.

We remind the reader that by the kernel definition:

• for each economic state represented as a point in the boulder, there exists a smooth tax-rate policy ($u \in [-0.2, 0.2]$), which maintains the economy in the constraint set K;

• the points outside the boulder are the economic states that cannot be controlled by this policy to remain in K.

A smooth tax-rate policy that maintains the economy in K, keeps it also in $\mathcal{V}$. (This is because we deal with infinite-horizon viability problems.) Henceforth, given the restrictions we put on the change in tax rate, we can apprise where the economy will be in the future even if our knowledge about the economy today is only of debt and capital.

5.2. **Maximum allowable debt** $B = 3.5$**.** Figure 2 shows two kernel slices for a medium debt level, $B = 1.25$. We first observe that some low consumption levels (see the far right bottom corner along *capital*) and a lot of high consumption levels ($c \geq 0.14$) are not viable. This is so because the former would lead to overcapitalization of the economy while the latter would de-capitalize the economy.

This is visible from the right panel. Three exemplary evolutions show what can happen to the economy depending on the "initial" state. If the state is $[1.6833, 0.0598, 1.2500, 0.4000] \in \mathcal{V}$ then there are smooth[10] tax-rate strategies, for which the evolution remains contained in $\mathcal{V} \in K \subset \mathbf{R}^4$, see the solid line. (Actually, the evolution stabilizes when $B = 0$, i.e., within a *different* kernel slice, not shown here.)

However, if the evolution starts at $[1.6833, \mathbf{0.0433}, 1.2500, 0.4000] \notin \mathcal{V}$, then even the fastest tax-rate *growth* (i.e., $u = 0.2$) cannot prevent overcapitalization and the economy violates the capital upper bound $k = 2$.

[10] I.e., $u \in [-0.2, 0.2]$.

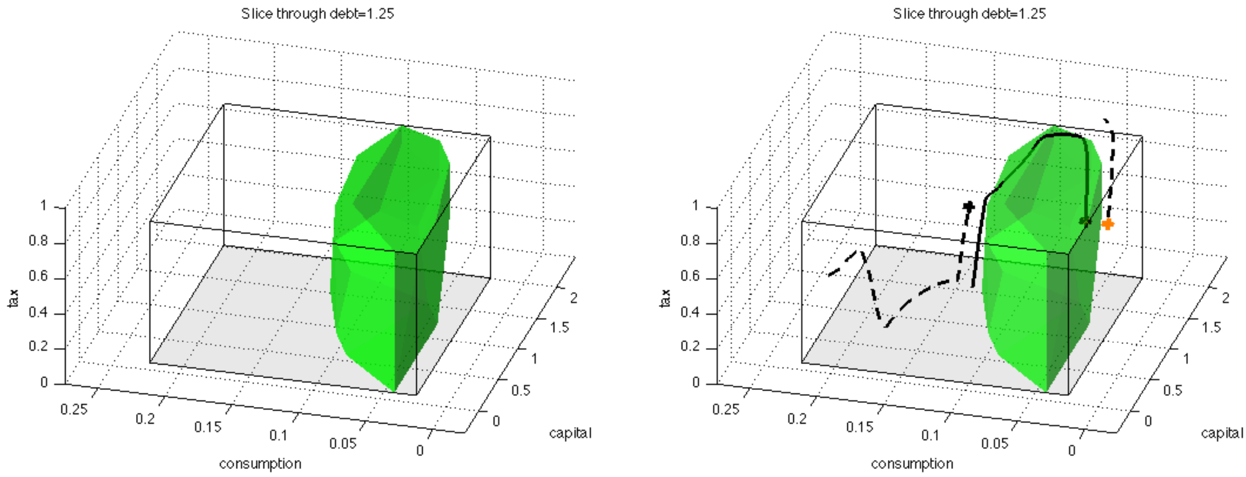

FIGURE 2. Kernel slices for $B = 1.25$.

If the evolution starts at [1.6833, **0.1454**, 1.2500, 0.4000] $\notin \mathcal{V}$ then even the fastest tax-rate *decrease* (i.e., $u = -0.2$) cannot prevent the dramatic capital reduction to below its lower bound $k = 0.2$. In Section 6, page 22, we compute the *debt-to-GDP* ratio for each of these evolutions.

Furthermore, this 3D slice's (i.e., in Figure 2) projections onto the planes: *tax-consumption* and *tax-capital*, not shown but easy to visualize, are almost rectangular. This implies that, for this moderate debt level (i.e., $B = 1.25$), the income tax-rate "initial" conditions are non-essential for the consumption choices.

Figure 3 shows two kernel slices: for an economy with savings, $B = -0.55$ left panel and a high debt economy, right panel $B = 2.6$. Overall, we notice that while the left slice is slanted toward higher consumption, with respect to the position of the slice in Figure 2, the right panel slice (high debt) is slanted toward lower consumption.

Moreover, the kernel slice for an economy without debt (left panel) appears largest among the so far analyzed slices. This implies that when the debt level is low there are more viable consumption choices for a given level of capital and tax, than when debt is high (or higher). We also notice that viable consumption decisions are different for each level of debt. When debt is low (left panel), there are fewer consumption decisions that would de-capitalize the economy, than when debt is high. Also, there are more consumption levels that could lead to overcapitalization in a low tax economy.

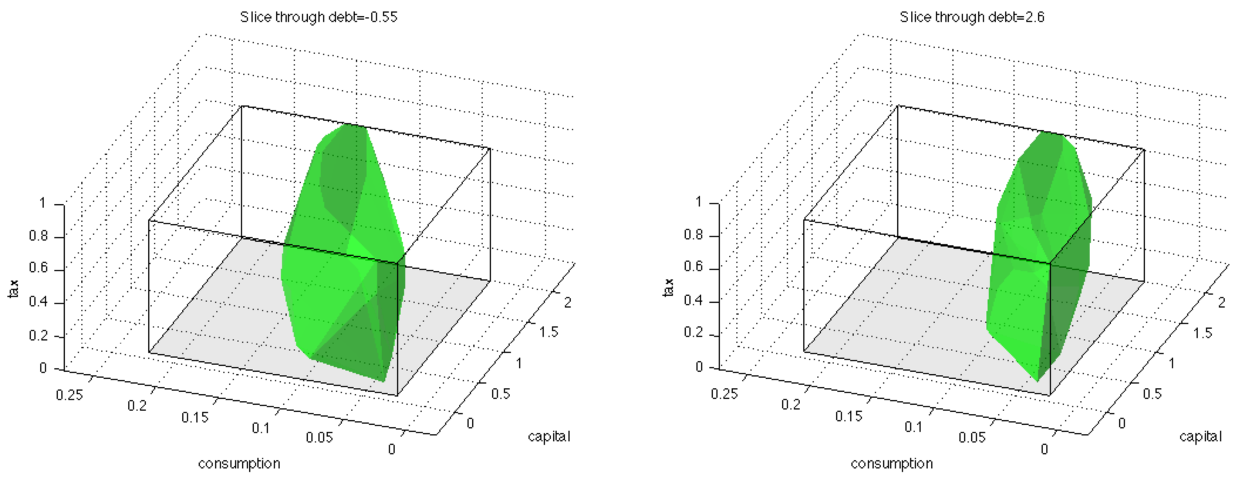

Figure 3. Kernel slices for $B = -0.55$, left panel and $B = 2.6$, right panel.

The slice projections onto the planes of *tax-consumption* and *tax-capital* are less rectangular than for $B = 1.25$. This implies that, for these debt levels (i.e.,

$B = -0.55$ and $B = 2.6$), the income tax-rate "initial" conditions need be taken into account when the consumption choices are made. This is exemplified in Figure 4 where the slices' cuts are shown for capital $k = 1.525$. The left (darker) shape is for the high debt economy, the right one is for the economy with savings. We can see how viable consumption choices depend on debt. When the economy has savings, $B = -0.55$, the right shape, consumption can be "lavish" and reach $c = 0.175$. This is not the case of an economy with debt ($B = 2.6$, the left shape); here, the highest consumption can attain $c = 0.12$. Evidently, with higher debt, consumption must be lower.

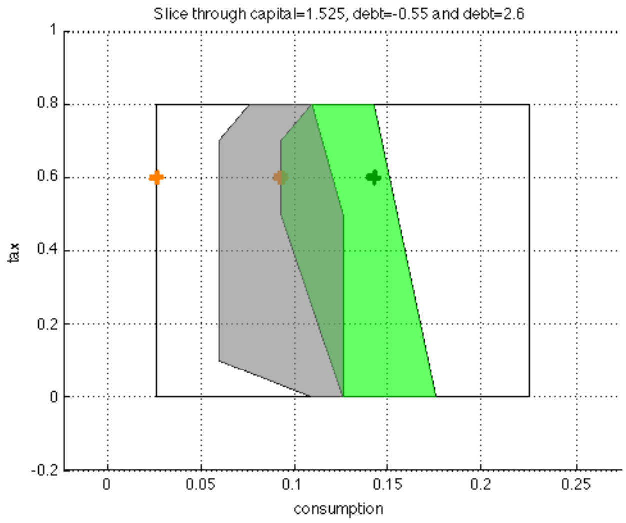

FIGURE 4. Kernel slices for $c = 1.525$ for $B = -0.55$ and $B = 2.6$.

One might ask why it is not "viable" to have even lower consumption than c=0.0598, which is on the left boundary of the high debt economy slice. In broad terms, the reason is that lower consumption *now*, combined with the restrictions that must be satisfied along the *future* path, which include the rate at which future taxes can change, would put the capital accumulation process on an explosive path, which would violate the capital upper bound and TVC-infinity[11] (i.e., transversality condition when the optimization horizon tends to infinity). We illustrate this in Appendix A, page 26.

Here, we show the impact of tax-rate levels on viable consumption choices. Figure 5 shows two kernel slices for low ($\tau = 0$) and high ($\tau = 0.8$) tax rates. (Notice, we have chosen a different "elevation" for these slices.) As in Figure 4, we see that higher consumption decisions can be made for larger capital values. Furthermore, for a given capital level, the consumption decisions' *ranges* are wider and the consumption *values* are higher when the tax rate is lower. In addition, we can observe that the boulder bases are wider than their tops, which indicates that higher consumption levels are viable when debt is low, for both taxation levels.

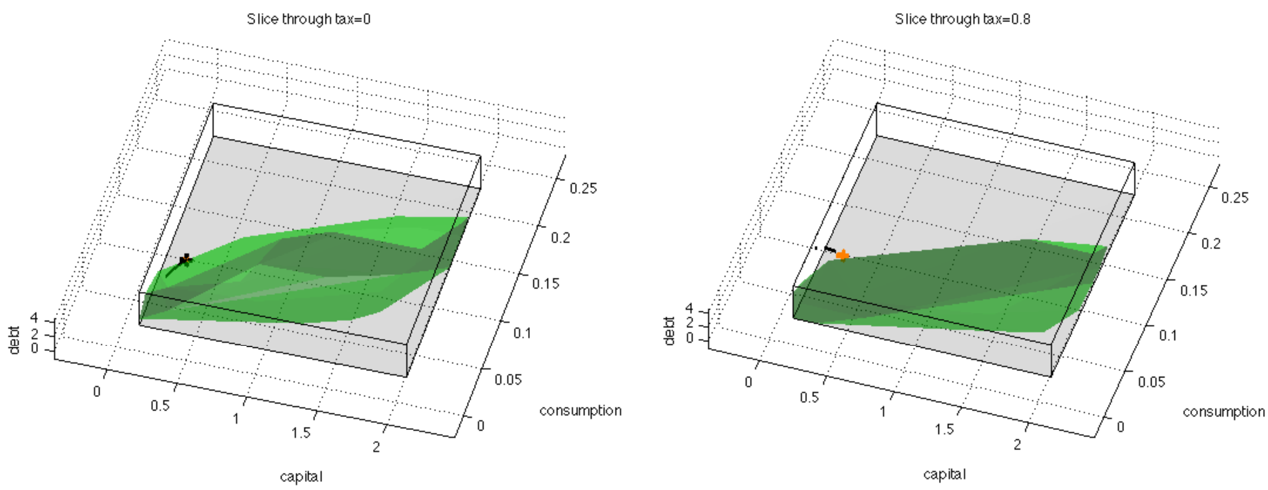

Figure 5. Kernel slices for $\tau = 0$ and $\tau = 0.8$.

We also show some economic evolutions in this figure. In the left panel we start a viable evolution from $[0.2583, 0.0928, -0.1, \mathbf{0}] \in \mathcal{V}$. The evolution in the right panel begins at $[0.2583, 0.0928, -0.1, \mathbf{0.8}] \notin \mathcal{V}$ and the fastest tax drop ($u = -0.2$) is applied. We can see that the latter, which illustrates what

[11]Unless *crisis control* was undertaken, see Cardaliaguet et al (1999).

can happen in a highly taxed economy, crashes through the capital lower boundary. This is because the tax could not drop sufficiently fast to prevent de-capitalization. The former stabilizes at low capital and consumption values and is therefore viable.

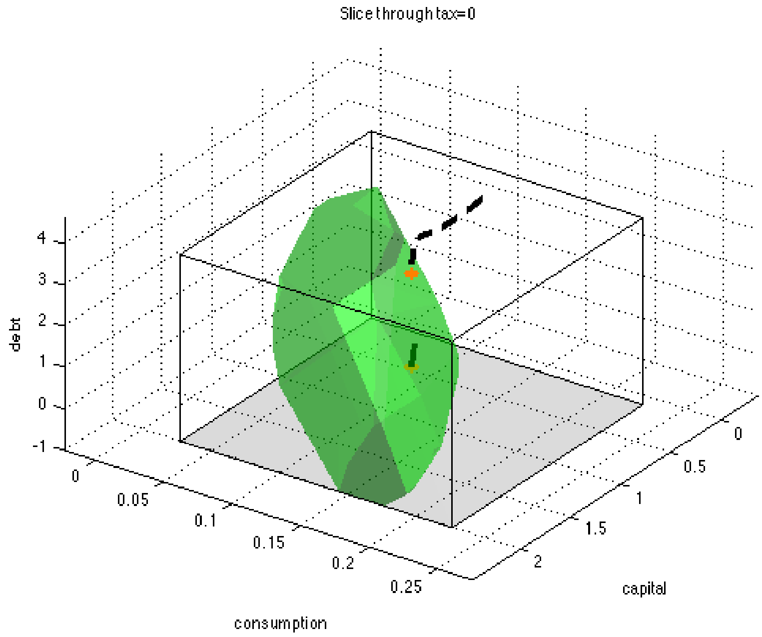

FIGURE 6. Kernel slice for $\tau = 0$.

Figure 6 shows that high debt levels are incompatible with low tax. Here again, we see the slice through $\tau = 0$ but graph "elevation" is different. Notice two evolutions starting at $[1.05, 0.1259, \mathbf{0.35}, 0] \in \mathcal{V}$ slice) and $[1.05, 0.1259, \mathbf{2.6}, 0] \notin \mathcal{V}$. So, the evolutions start from low debt, inside slice, and high debt, outside slice, respectively. We see that the high-debt trajectory rises fast in debt and crashes through its upper boundary. This is because the smooth taxation policy cannot generate enough tax to curb the increasing debt. On the other hand, the initially low-debt economy remains almost stationary.

5.3. **A higher government expenditure.** Here we have computed the kernel when the government expenditure is doubled, so $g = 0.0266$. The other parameters are as in Section 5.2.

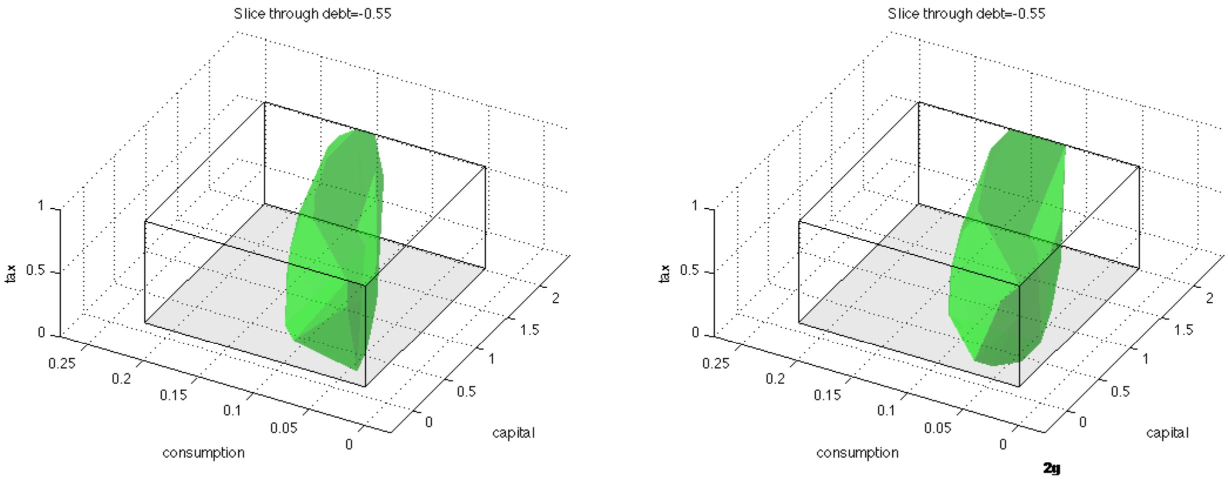

FIGURE 7. Kernel slices for $B = -0.55$. The left panel is as in Figure 3, the right-panel kernel slice is computed for the doubled g.

In Figure 7, we observe that the kernel slice in the right panel (slightly fatter) appears "turned" clockwise, with respect to that in the left panel, computed in Section 5.2 for the lower g. This means that (even) if the economy is in credit i.e., $B = -0.55$, increasing the government expenditure reduces maximum achievable consumption. This is visible from the top consumption in the right panel reaching only 0.174; the top consumption in the left panel attains 0.19 .

The same phenomenon is visible in Figure 8, which shows the kernel slices for a high-debt economy, B=2.6, where the right panel is for the doubled government expenditure The right-panel's empty space between the maximum consumption "wall" is larger, even if the slice may be fatter (for higher taxes) than the one in the left panel. So, again, a higher government expenditure results in that only lower consumption choices are feasible, given the adopted tax policy.

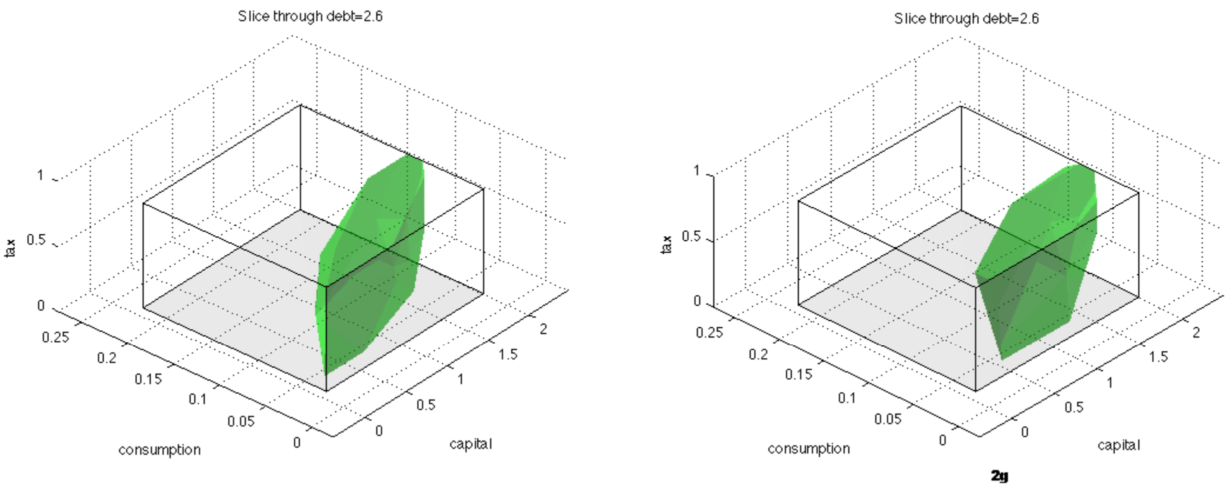

FIGURE 8. Kernel slices for $B = 2.6$. The left panel is as in Figure 3, the right-panel kernel slice is computed for the doubled g.

However, there is a feature of the kernel slice in the right panel i.e., when the government-expenditure is higher, which is absent from the left panel.

Here, the kernel-slice is clearly not rectangular. This means that, for low capital ($k < 0.6$) , consumption choices can be viable only if high tax rates are applied.

6. Debt-to-GDP ratio

An economic evolution could also be characterized by the *debt-to-GDP* ratio (see e.g., Baker, Kotlikoff, and Leibfritz (1999)). We have computed such ratio time-profiles for the three evolutions pictured in Figure 2, right panel (see page 15). As explained in the description of this figure, the evolution from $c_0 = 0.0598$ is viable while the two others: from $c_0 = 0.1454$ and $c_0 = 0.0433$ are not. We show the corresponding *debt-to-GDP* ratios in Figure 9.

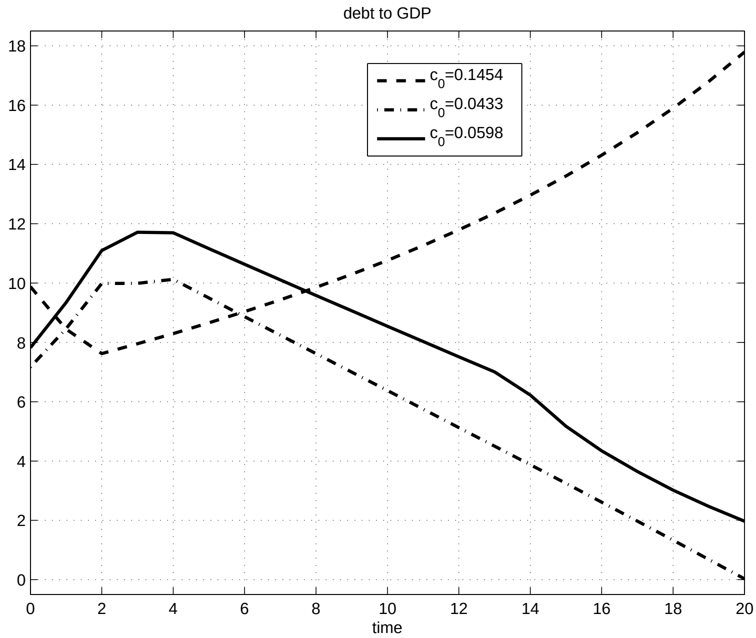

Figure 9. *Debt-to-GDP* ratio time profiles.

The solid line represents an interesting case, which corresponds to the viable trajectory starting at $c_0 = 0.0598$. We see that the *debt-to-GDP* ratio eventually diminishes; however, before diminishing its values rise. (The values are numerically high because of a low value of the stylized output steady-state, which is equal to $A = 0.1333$.) Under an increasing tax, debt diminishes and, eventually, the steady state is such that capital is large enough to assure a growing output and a medium-level consumption.

The similar looking dash-dotted line originating from a lower consumption level $c_0 = 0.0433$ is non-viable because of overcapitalization of the economy, see Figure 2. Here, the *debt-to-GDP* radio (also eventually) diminishes because capital grows fast and so does the output but, as said, the economy becomes over-capitalized.

The third (dash) line displays the (eventually) growing *debt-to-GDP* ratio and corresponds to an evolution from $c_0 = 0.1454$ (highest between the three). This evolution is clearly non-viable. As seen in Figure 2 right panel, this is so because even the fastest tax-drop cannot prevent the capital reduction below its lower bound.

We conjecture that *debt-to-GDP* ratio cannot be used as a proxy for viability; on the other hand, a viable evolution can imply a diminishing *debt-to-GDP* ratio.

7. Concluding remarks

We have presented a computational method based on viability theory for a discovery of consumption choices that are compatible with the state variables of the economy at hand. The compatibility means that viable consumption and capital choices will generate a nearly steady-state path for a smooth tax-rate adjustment policy.

Among other findings we report that increasing government expenditure implies that higher tax rates will be needed to preserve the viability of many consumptions choices, when capital levels approach the constraint set boundaries.

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Appendix A. Why some choices can be non-viable

To help understand why some economic states can be non-viable we will consider evolutions from three *capital-consumption-tax* combinations for two different levels of debt $B = -0.55$ and $B = 2.6$. The evolution starting points are represented by the dots shown in Figure 4.

We need to remark that viable evolutions, represented by the solid lines in the following figures, are constructive in that we have found tax-rate adjustments that generate them and lead to a (numerically) steady state. On the other hand, the dash and dash-dotted lines cross a boundary of K in finite time, hence represent nonviable evolutions; they are computed by VIKAASA as "best" in that the sum of their velocities is minimal, but too big to be deemed steady.

Consider the following points (from left to right in Figure 4):

(1) $[1.525, 0.0267, 2.6, 0.6] \notin \mathcal{V}_\Psi(K)$
(2) $[1.525, 0.0267, -0.55, 0.6] \notin \mathcal{V}_\Psi(K)$
(3) $[1.525, 0.0928, 2.6, 0.6] \in \mathcal{V}_\Psi(K)$
(4) $[1.525, 0.0928, -0.55, 0.6] \in \mathcal{V}_\Psi(K)$
(5) $[1.525, 0.1424, 2.6, 0.6] \notin \mathcal{V}_\Psi(K)$
(6) $[1.525, 0.1424, -0.55, 0.6] \in \mathcal{V}_\Psi(K)$

From Figure 4, we know that the point, numbered "1" (high-debt economy) is nonviable. Here we analyze the evolution from this point.

(1) We can see in the left panels in Figure 10 that even with the application of the maximum tax rate, the evolution crashes through the capital upper bound, albeit consumption increases.
(2) Very similarly to what we have seen in the left panels, we notice in the right panels in Figure 10 that with the application of the maximum tax rate, the evolution also crashes through the capital upper bound (consumption increases too).

The evolution that starts at the (low-debt) point number "2" is also nonviable. However, the evolution that starts at the (high-debt) point numbered "3" is viable.

(3) Here, we notice (see the left panels in Figure 10) that with the application of the maximum tax rate, capital decreases and consumption increases faster than from point "1.", especially, after the intermediate tax-rate drop. After the tax-rate hike to 40%, the economy stabilizes.

The evolution that starts at the (low-debt) point number "4" is also viable.

(4) Here, we notice (see the right panels in Figure 10) that with a medium size tax-rate hike, capital decreases and consumption increases albeit both processes are slower than under "3". The economy stabilizes with the tax rate below 10%.

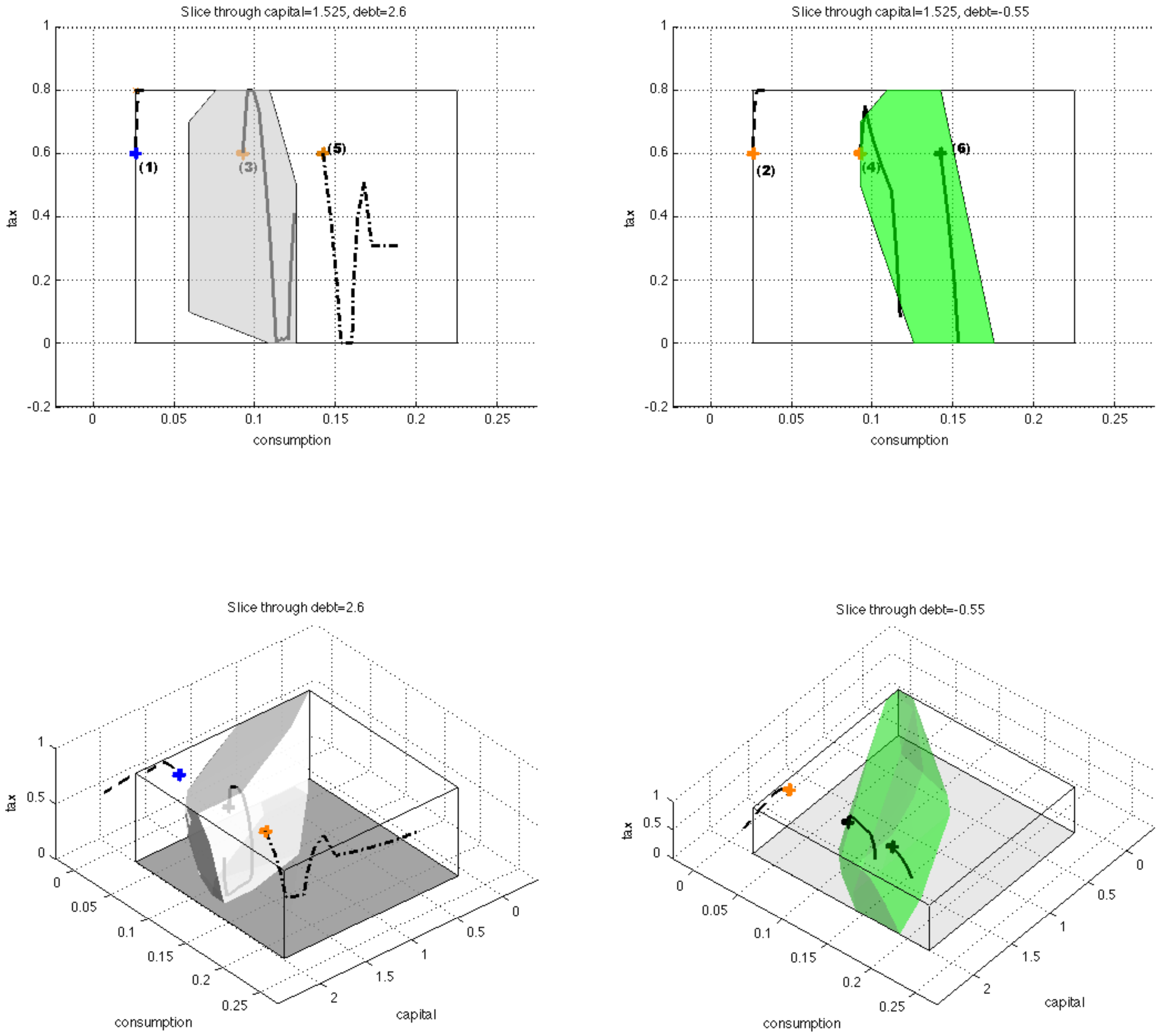

FIGURE 10. Viable trajectories (solid lines) and non-viable trajectories (dash and dash-dotted lines).

The evolution that starts at the (high-debt) point numbered "5" is non-viable.

(5) Here, we see in the left panels in Figure 10 that with a medium size tax-rate drop, capital decreases and consumption increases however both processes are faster than from point "3". After increasing the tax-rate and then decreasing it, capital still diminishes very fast and almost crashes through the lower boundary. However, in this case, debt also grows rapidly and violates the upper limit before capital reaches its border. This is visible from Figure 11.

The evolution that starts at the (low-debt) point number "6" is viable.

(6) Here, we notice (see the right panels in Figure 10) that with a big tax-rate drop, capital decreases and consumption increases however both processes are rather slow and the economy stabilizes with zero tax rate.

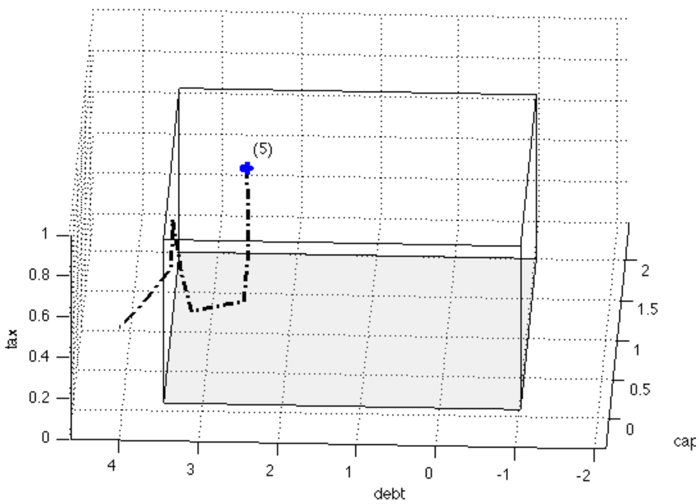

FIGURE 11. Kernel slice for $c = 0.1424$.