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Bottleneck congestion and distribution of work start times: The economics of staggered work hours revisited*

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Abstract

Since the seminal work of Henderson (1981), a number of studies examined the effect of staggered work hours by analyzing models of work start time choice that consider the trade-off between negative congestion externalities and positive production externalities. However, these studies described traffic congestion using flow congestion models. This study develops a model of work start time choice with bottleneck congestion and discloses the intrinsic properties of the model. To this end, this study extends Henderson's model to incorporate bottleneck congestion. By utilizing the properties of a potential game, we characterize equilibrium and optimal distributions of work start times. We also show that Pigouvian tax/subsidy policies generally yield multiple equilibria and that the first-best optimum must be a stable equilibrium under Pigouvian policies, whereas the second-best optimum in which policymakers cannot eliminate queuing congestion can be unstable.

JEL classification: C62; C72; C73; D62; R41; R48

Keywords: staggered work hours; bottleneck congestion; production effects; potential game; stability; Pigouvian policies;

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1 Introduction

Urban traffic congestion is caused by concentrated demand for travel around the start of the workday, because firms in central business districts (CBDs) generally have fixed work schedules and workers start work at the same time. Introducing staggered work hours (SWH) is a transportation demand management (TDM) measure for alleviating peak congestion. It is widely recognized but rarely implemented, because it may reduce intra-firm communication and productivity (Wilson, 1988). That is, SWH reduces positive production externalities (agglomeration economies) alongside the negative congestion externalities (agglomeration diseconomies). Therefore, considering the trade-off between congestion and productivity is essential when we examine the effect of TDM measures for reducing peak congestion.

Since the seminal work of Henderson (1981), a number of studies have developed models of work start time choice that consider traffic congestion and productivity effects; these studies will be discussed in Section 1.1. By examining the equilibrium and optimal distributions of work start times and optimal congestion tolls, these studies provide insights into TDM measures. However, analytical difficulties inevitably arising in models with agglomeration economies and diseconomies (i.e., nonconvexities) limit these studies. Foremost among their limitations is that they describe traffic congestion using flow congestion models, which are inappropriate for dealing with peak congestion. Second, although their models have multiple equilibria, these studies address only a subset—e.g., cases where work starting times are continuously distributed or completely clustered—and do not examine their stability. Therefore, the equilibrium distribution of work start times may be unstable and may never emerge in their models. Third, Akamatsu et al. (2014b) shows that if we consider models with positive and negative externalities, social optima can be unstable equilibria under Pigouvian policies, and a non-optimal stable equilibrium will exist. Therefore, although previous studies (e.g., Arnott, 2007) investigate the properties of optimum congestion tolls, social optimum may not be achieved under their congestion tolls.

This study shows that the *potential function approach*, which utilizes properties of a potential game, overcomes these limitations and clarifies the intrinsic properties of a model of work start time choice with bottleneck congestion. This paper first develops a model with production effects and bottleneck congestion by combining Henderson (1981)'s model and the standard bottleneck model (Vickrey, 1969; Hendrickson and Kocur, 1981; Arnott et al., 1990). Similar to models in Peer and Verhoef (2013) and Gubins and Verhoef (2014), ours assumes that workers make long-run decisions about work start times and short-run decisions about day-specific work arrival times. In the short-run, workers choose arrival times and take work start times as a given; in the long-run, they choose work start times indirectly through their choice of employer. We then show that the short-run equilibrium is uniquely determined, whereas the long-run equilibrium is not unique.

This study examines the local stability of long-run equilibrium by viewing it as a Nash equilibrium of a potential game (Sandholm, 2001). In this case, the model of the long-run choice of work start time admits a potential function, and the set of long-run equilibria coincides exactly with the set of Karush–Kuhn–Tucker points for the maximization problem of the potential function. Further, all local maximizers of the potential function are locally stable long-run equilibria. We can therefore characterize long-run equilibria and their stability by the shape of the potential function.

After characterizing the long-run equilibria and their stability, this study investigates the properties of the first-best and second-best optimal distributions of work start times and their stability under Pigouvian policies. The first-best optimum is defined as the global maximizer of the social welfare function (workers' total utility), and the second-best optimum is that under the condition whereby policymakers cannot control workers' short-run decisions; that is, the queue at the bottleneck cannot be eliminated. Thus, differences between optimum and stable equilibria are clarified by comparing the shapes of the social welfare function and the potential function. Furthermore, stability of the first-best and second-best optima under Pigouvian policies is analyzed by the potential function approach. This analysis discloses that the first-best optimum must be a stable equilibrium under Pigouvian policies, whereas the second-best optimum can be unstable.

1.1 Related Literature

Theoretical studies of SWH and its variants have appeared since the benchmark study by Henderson (1981). Henderson (1981) assumed that all workers in a city commute from a common residential area to a common CBD along a single congestible road and that the productivity of a worker at a point in time depends on the number of workers at work at that time. These two assumptions yield both traffic congestion and productivity effects in his model. He then analyzed the equilibrium and optimal distributions of work start times. Wilson (1992) and

Arnott et al. (2005) extended Henderson (1981) by introducing workers' choices of residential location and firm heterogeneity, respectively. Arnott (2007) generalized Henderson's model and analyzed optimal congestion tolls. Henderson (1981) and these subsequent studies, however, described traffic congestion using a flow congestion model.

Mun and Yonekawa (2006) and Fosgerau and Small (2014) were the most successful in considering both production effects and peak-period traffic congestion.[1] Mun and Yonekawa (2006) formulated a peak-period congestion based on the standard bottleneck model and developed a model that describes firms' and workers' choices to adopt fixed or flextime schedules. They showed that a situation in which all firms adopt flextime never emerges as equilibrium and that multiple equilibria could exist. However, due to analytical difficulties, they examined the stability of equilibria only by numerical examples.

Fosgerau and Small (2014) presented a model that introduces bottleneck congestion and productivity effects of work and leisure. They systematically investigated the properties of equilibrium and optimal tolls. However, their model presupposed that all workers determine their own work start time, which implies that all firms adopt flextime. This leads to the result that workers' work start times are the same as their arrival times at the CBD. Thus, their model describes only a situation wherein work start times are continuously distributed.

It is noteworthy that the framework of Henderson (1981) is the same as that of social interaction models (e.g., Beckmann, 1976; Tabuchi, 1986), which study spatial agglomeration of economic activities. Beckmann (1976) led to numerous extensions and modifications (Fujita and Ogawa, 1982; Fujita, 1988; Berliant et al., 2002; Mossay and Picard, 2011; Akamatsu et al., 2014a) that provide approaches for characterizing equilibrium and social optimum.[2] This study modifies one of these approaches—the potential function approach[3] in Akamatsu et al. (2014a)—and applies it to the model featuring bottleneck congestion. This approach significantly simplifies characterizing equilibrium, its stability, and optimum of our model. By applying the potential function approach, this study then analytically clarifies the intrinsic properties of the model featuring production effects and

[1]Sato and Akamatsu (2006) also extended the standard bottleneck model to incorporate the productivity effect. Although they provided a rigorous framework, their analysis is limited to a particular set of equilibria, such as cases where work start times are completely clustered and staggered.

[2]For comprehensive reviews of these literature, see Fujita and Thisse (2013).

[3]Methods that utilize the potential function are found in a diverse range of applications (for reviews, see, e.g., Sandholm, 2010), which includes transportation science (e.g., Beckmann et al., 1956; Rosenthal, 1973; Sandholm, 2002).

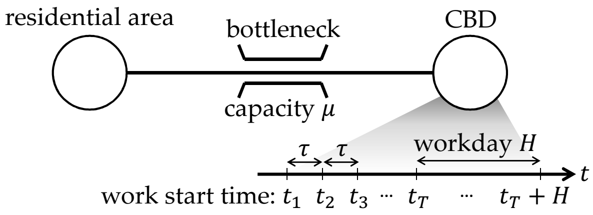


Figure 1: The monocentric city

bottleneck congestion.

This study proceeds as follows. Section 2 formulates a model of work start time choice featuring bottleneck congestion and production effects. Section 3 describes the long-run choice of work start time as a potential game and investigates the uniqueness and stability of the short-run and long-run equilibria by the potential function approach. Section 4 examines the properties of first-best and second-best optima and their stability under Pigouvian policies. Section 5 concludes. Proofs omitted in the text are in the Appendix.

2 The Model

2.1 Basic assumptions

Consider a city that consists of a CBD and a residential area connected by a single road (Figure 1). This road has a single bottleneck with capacity μ. All workers reside in the residential area and commute to the CBD, where all firms are located. If arrival rates of workers at the bottleneck exceed its capacity, a queue develops. To model queuing congestion, we employ first-in-first-out (FIFO) and a point queue in which vehicles have no physical length as in standard bottleneck models (e.g., Vickrey, 1969; Hendrickson and Kocur, 1981; Arnott et al., 1990, 1993).

Each firm chooses its work start time from the feasible set $\mathcal{T} \equiv \{t_1, t_2, \cdots, t_T\}$, where $t_i = t_{i-1} + \tau$ for all $i \in \{2, 3, \cdots, T\}$ and τ is a positive constant. Since the length of a workday is assumed to be identical and fixed at H for all firms, each firm is characterized by its work start time. For convenience, we call the firm that starts work at time t_i "firm i." We further assume there is an interval in the workday when all firms begin work, i.e., $t_T < t_1 + H$.

2.1.1 Behavior of workers

The N workers are ex ante identical. Each chooses his or her work start time t_i indirectly by choosing an employer (i.e., a firm $i \in \mathcal{I} \equiv \{1, 2, \cdots, T\}$ to work for) and the departure time t at the bottleneck to maximize utility $u_i(t)$. The utility of a worker who starts work at t_i, whom we call "worker i," is given by

$$u_i(t) = w_i - c_i(t), \tag{1}$$

where w_i denotes the wage from firm i and $c_i(t)$ denotes commuting cost. The commuting cost $c_i(t)$ of worker i who departs the bottleneck at time t is expressed as the sum of queuing time cost at the bottleneck, $q(t)$, schedule delay cost, $s(t_i - t)$, and fixed travel time cost, c_f:

$$c_i(t) = q(t) + s(t_i - t) + c_f. \tag{2}$$

We assume that $s(x)$ is differentiable, strictly convex, and strictly minimized at $x = 0$, and that $s'(x) \equiv ds(x)/dx < 1$ as in Daganzo (1985), Kuwahara (1990), and Lindsey (2004). Following Arnott et al. (1990, 1993), we set $c_f = 0$ without affecting the results of interest.

We consider utility maximization as a sequence of short-run and long-run optimizations. Specifically, workers in the short-run minimize commuting cost $c_i(t) = q(t) + s(t_i - t)$ by selecting their departure time t taking work start time t_i as given:

$$\min_t c_i(t) = q(t) + s(t_i - t). \tag{3}$$

In the long-run, each worker chooses an employer so as to maximize his/her utility:

$$\max_i u_i = w_i - c_i^*, \tag{4}$$

where c_i^* is the short-run equilibrium commuting cost of worker i, determined by his/her short-run decisions.

2.1.2 Behavior of Firms

All firms produce homogeneous goods under constant returns to scale technology and perfect competition, which requires one unit of labor to produce one unit of output and is chosen as numéraire. For introduction of the production effect, this model assumes that the productivity per worker of a firm at time t is linearly increasing with the total number of workers then on duty. This production effect

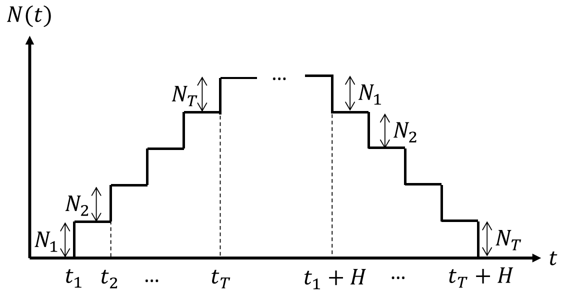


Figure 2: Total number of workers on duty

is represented by the following instantaneous production function:

$$f(t) = \alpha N(t), \tag{5}$$

where coefficient α represents technology of a firm and $N(t)$ is the total number of workers on duty at time t. The daily output F_i per worker of a firm i is simply the sum over the workday of the instantaneous output $f(t)$:

$$F_i = \int_{t_i}^{t_i+H} f(t)dt = \int_{t_i}^{t_i+H} \alpha N(t)dt. \tag{6}$$

Note that because $t_i = t_{i-1} + \tau$, $N(t)$ is represented as follows (Figure 2):

$$N(t) = \begin{cases} \sum_{k=1}^{j} N_k & \text{if } t \in [t_j, t_{j+1}) \quad \forall j \in \{1, 2, \cdots, T-1\}, \\ N & \text{if } t \in [t_T, t_1 + H], \\ \sum_{k=j+1}^{T} N_k & \text{if } t \in (t_j + H, t_{j+1} + H] \quad \forall j \in \{1, 2, \cdots, T-1\}, \end{cases} \tag{7}$$

where N_i denotes total number of workers employed by firm i. Under the production function defined in (6), each firm chooses its work start time to maximize profit per worker:

$$\max_i \pi_i = F_i - w_i. \tag{8}$$

Since a firm cannot change its work start time frequently, its choice of work start time is assumed to be a long-run decision.

2.2 Short-run and long-run equilibrium conditions

2.2.1 Short-run equilibrium conditions

In the short-run, workers decide only the day-specific departure time t at the bottleneck, which implies that the number of workers $\boldsymbol{N} = (N_i)_{i\in\mathcal{I}}$ employed by firm $i \in \mathcal{I}$—which we call the *distributions of work start times*—is assumed to be given. Therefore, short-run equilibrium conditions coincide with those of the standard bottleneck model, given by these three conditions:

$$\begin{cases} c_i^* - \left\{q(t) + s(t_i - t)\right\} = 0 & \text{if} \quad n_i(t) > 0 \\ c_i^* - \left\{q(t) + s(t_i - t)\right\} \geq 0 & \text{if} \quad n_i(t) = 0 \end{cases} \quad \forall t,\ \forall i \in \mathcal{I}, \tag{9a}$$

$$\begin{cases} \mu - \sum_k n_k(t) = 0 & \text{if} \quad q(t) > 0 \\ \mu - \sum_k n_k(t) \geq 0 & \text{if} \quad q(t) = 0 \end{cases} \quad \forall t, \tag{9b}$$

$$\int n_i(t)dt = N_i \quad \forall i \in \mathcal{I}, \tag{9c}$$

where $n_i(t)$ is the number of workers i who arrive at the CBD at time t (i.e., the arrival rate of workers i at the CBD).

Condition (9a) represents the no-arbitrage condition for the choice of departure time. This condition means that at the short-run equilibrium, no worker can reduce commuting cost by changing arrival time at the CBD unilaterally. Condition (9b) is the capacity constraint of the bottleneck, which requires that the total departure rate $\sum_k n_k(t)$ at the bottleneck is equal to the capacity μ if there is a queue; otherwise, the total departure rate is (weakly) lower than μ. The last condition (9c) is flow conservation for commuting demand. These conditions give $n_i(t)$, $q(t)$, and c_i^* at short-run equilibrium as functions of the distribution of work start times, $\boldsymbol{N}$.

2.2.2 Long-run equilibrium conditions

In the long-run, each worker chooses an employer, and each firm chooses its work start time. Thus, the long-run equilibrium conditions are represented as

$$\begin{cases} u^* - \left\{w_i - c_i^*\right\} = 0 & \text{if} \quad N_i > 0 \\ u^* - \left\{w_i - c_i^*\right\} \geq 0 & \text{if} \quad N_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \tag{10a}$$

$$\begin{cases} \pi^* - \left\{F_i - w_i\right\} = 0 & \text{if} \quad N_i > 0 \\ \pi^* - \left\{F_i - w_i\right\} \geq 0 & \text{if} \quad N_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \tag{10b}$$

$$\sum_k N_k = N, \tag{10c}$$

where u^* denotes the equilibrium utility, and π^* is the equilibrium profit which equals zero because firms in the city are perfectly competitive.

Conditions (10a) and (10b) are the equilibrium conditions for workers' choice of firm and firms' choice of work start time, respectively. Condition (10a) implies that at long-run equilibrium, each worker has no incentive to change employer unilaterally. Condition (10b) means that if workers are employed by firm i, the firm i earns the equilibrium profit $\pi^* = 0$; otherwise, the profit must be less than zero. Condition (10b) is the conservation law of the population of workers.

We easily show that conditions (10a) and (10b) are rewritten as the following condition because $\pi^* = 0$.

$$\begin{cases} u^* - \left\{F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N})\right\} = 0 & \text{if} \quad N_i > 0 \\ u^* - \left\{F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N})\right\} \geq 0 & \text{if} \quad N_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \tag{11}$$

where $F_i(\boldsymbol{N})$ and $c_i^*(\boldsymbol{N})$ are determined by (6) and (9) as functions of the distribution $\boldsymbol{N}$ of work start times. Therefore, the long-run equilibrium distribution $\boldsymbol{N}^*$ of work start times and utility u^* are obtained from conditions (10c) and (11).

3 Short-run and Long-run Equilibrium

3.1 Short-run equilibrium

We first characterize short-run equilibrium. Because short-run equilibrium conditions (9) coincide with those of the standard bottleneck model and because the schedule delay cost function $s(x)$ is strictly convex, the following proposition is obtained.

Proposition 1. *The short-run equilibrium is uniquely determined. Furthermore, workers arrive at and leave a bottleneck in the same order as their work start times. That is, the first-in-first-work discipline is valid.*

Proof. See Smith (1984), Daganzo (1985), Kuwahara (1990), and Lindsey (2004). □

In addition, short-run equilibrium commuting cost $c_i^*(\boldsymbol{N})$ has the following desirable properties, which are useful for investigating the properties of long-run equilibrium.

Lemma 1. *The Jacobian matrix* $\nabla \boldsymbol{c}(\boldsymbol{N})$ *of the short-run equilibrium commuting cost* $\boldsymbol{c}(\boldsymbol{N}) = (c_i^*(\boldsymbol{N}))_{i\in\mathcal{I}}$ *is symmetric and positive semidefinite.*

Proof. See Appendix. □

3.2 Long-run equilibrium

3.2.1 Potential game

We next characterize long-run equilibrium. For the analysis, we invoke the properties of a *potential game* introduced by Monderer and Shapley (1996) and Sandholm (2001). Because the long-run equilibrium conditions are represented by (10c) and (11), the model of workers' long-run choice of work start time can be viewed as a population game in which the set of players is $\mathcal{S} \equiv [0, N]$, the common action set is $\mathcal{I}$, and the payoff vector is $\boldsymbol{u}(\boldsymbol{N}) = (F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N}))_{i\in\mathcal{I}}$. As is evident from the definition, a long-run equilibrium is a Nash equilibrium of the game. Thus, let us denote this game by $G = \{\mathcal{S}, \mathcal{I}, \boldsymbol{u}\}$.

A potential game is defined as a game G that holds the following condition: there exists a continuously differentiable function P such that

$$\frac{\partial P(\boldsymbol{N})}{\partial N_i} = u_i(\boldsymbol{N}) \quad \forall \boldsymbol{N} \in \Delta \equiv \left\{ \boldsymbol{N} \in \mathbb{R}_+^T \,\middle|\, \sum_k N_k = N \right\},\ \forall i \in \mathcal{I}, \tag{12}$$

where P is defined on an open set containing Δ so that its partial derivative is well-defined on Δ. The function P is the *potential function* of the game G. This condition requires the existence of a function in which gradient $\nabla P(\boldsymbol{N})$ equals the payoff vector $\boldsymbol{u}$. As Sandholm (2001) proves, if payoffs $\boldsymbol{u}(\boldsymbol{N})$ are continuously differentiable, this condition is equivalent to the following condition called *externality symmetry*:

$$\frac{\partial u_i(\boldsymbol{N})}{\partial N_j} = \frac{\partial u_j(\boldsymbol{N})}{\partial N_i} \quad \forall \boldsymbol{N} \in \Delta,\ \forall i, j \in \mathcal{I}. \tag{13}$$

We now show that our game G is a potential game. It follows from (6) that the payoff vector $\boldsymbol{u}(\boldsymbol{N})$ is represented as

$$\boldsymbol{u}(\boldsymbol{N}) = \boldsymbol{F}(\boldsymbol{N}) - \boldsymbol{c}(\boldsymbol{N}) = \alpha \left\{ H\boldsymbol{E} - \tau \boldsymbol{D} \right\} \boldsymbol{N} - \boldsymbol{c}(\boldsymbol{N}), \tag{14}$$

where $\boldsymbol{E}$ is $T \times T$ matrix with all elements equal to 1, and $\boldsymbol{D}$ is the symmetric Toeplitz matrix whose (i, j) element is given by $|i - j|$. From this and Lemma 1, the Jacobian matrix $\nabla \boldsymbol{u}(\boldsymbol{N})$ of the payoff vector $\boldsymbol{u}(\boldsymbol{N})$ is symmetric, which implies that externality symmetry holds in our game. Therefore, we have the following proposition.

Proposition 2. *The game G is a potential game with the potential function*

$$P(\boldsymbol{N}) = P_1(\boldsymbol{N}) - P_2(\boldsymbol{N}), \tag{15a}$$

where $P_1(\boldsymbol{N})$ and $P_2(\boldsymbol{N})$ are convex functions such that

$$\nabla P_1(\boldsymbol{N}) = \boldsymbol{F}(\boldsymbol{N}), \tag{15b}$$

$$\nabla P_2(\boldsymbol{N}) = \boldsymbol{c}(\boldsymbol{N}). \tag{15c}$$

Proof. See Appendix. □

The equilibrium of a potential game is characterized with the maximization problem of the potential function. Let us consider the following problem:

$$\max_{\boldsymbol{N}} P(\boldsymbol{N}) \quad \text{s.t.} \quad \sum_k N_k = N,\ N_i \geq 0\ \forall i \in \mathcal{I}. \tag{16}$$

Let u^* be a Lagrange multiplier for the constraint $\sum_k N_k = N$. We then can readily verify that the Karush–Kuhn–Tucker (KKT) conditions of this problem are equivalent to long-run equilibrium conditions (10c) and (11). Therefore, the equilibrium set of the game G exactly coincides with the set of KKT points for problem (16).

From problem (16), we recognize the trade-off between positive production externalities (agglomeration economies) and negative congestion externalities (agglomeration diseconomies) as the trade-off between the convexity of $P_1(\boldsymbol{N})$ and concavity of $-P_2(\boldsymbol{N})$. If the concavity of $-P_2(\boldsymbol{N})$ dominates such that $P(\boldsymbol{N})$ is strictly concave, a staggered work hours equilibrium is attained as a unique equilibrium. On the other hand, if the convexity of $P_1(\boldsymbol{N})$ dominates, the equilibrium distributions of work start times would be more clustered. Therefore, $P_1(\boldsymbol{N})$ represents positive production externalities, whereas $-P_2(\boldsymbol{N})$ represents negative congestion externalities.

This fact suggests that the capacity expansion of the bottleneck may worsen traffic congestion in our model. The mechanism is as follows. The capacity expansion decreases commuting costs, and thus $-P_2(\boldsymbol{N})$ will be less dominant. This may lead to more clustered distribution of work start times, thereby exacerbating the bottleneck congestion. Although this paradoxical result does not always arise in our model, we can show that such a situation actually exists, as discussed in Section 3.2.4.

3.2.2 Uniqueness

To characterize the long-run equilibrium, we first examine its uniqueness. Since the KKT points of problem (16) are long-run equilibrium, the uniqueness can be investigated by checking the shape of the potential function $P(\boldsymbol{N})$. Specifically, if $P(\boldsymbol{N})$ is unimodal, the long-run equilibrium is unique; otherwise, it is non-unique. It follows from Proposition 1 that $P(\boldsymbol{N})$ is not generally unimodal because of the convexity of $P_1(\boldsymbol{N})$. Thus, we have

Lemma 2. *The long-run equilibrium is generally not unique.*

It is noteworthy that Lemma 2 does not suggest *essential* multiplicity of equilibria because even if all of the equilibrium distributions of work start times are essentially the same (e.g., completely clustered distributions: $(N, 0, \cdots, 0)^\top$, $(0, N, \cdots, 0)^\top$, $(0, 0, \cdots, N)^\top$), the number of equilibria is not one. Hence, we next investigate the essential uniqueness of the long-run equilibrium. For the investigation, we show a property of the support $\operatorname{supp} \boldsymbol{N}^* \equiv \{i \in \mathcal{I} \mid N_i > 0\}$ of the long-run equilibrium.

Lemma 3. *Suppose $\boldsymbol{N}^* \in \Delta$ is a long-run equilibrium. Then,* $\operatorname{supp} \boldsymbol{N}^* \in \mathcal{S}_C$ *where*

$$\mathcal{S}_C = \left\{ \{i_1, i_2, \cdots, i_a\} \subseteq \mathcal{I} \mid a \in \mathcal{I},\ i_{j+1} = i_j + 1\ \forall j \in [1, a-1], \right\}. \tag{17}$$

Proof. See Appendix. □

Lemma 3 means that the set of work start times such that $N_i > 0$ is a convex set. In other words, if we suppose $\tau = 30$ (min) and some employees start work at 8:00 and 9:00, there must be workers who start at 8:30.

Because of the symmetry of our model, Lemma 3 implies that if the long-run equilibrium $\boldsymbol{N}^*$ is not full support (i.e., $\operatorname{supp} \boldsymbol{N}^* \neq \mathcal{I}$) and $N_1 = 0$, there is a long-run equilibrium $\hat{\boldsymbol{N}}^*$ that is essentially the same with $\boldsymbol{N}^*$ such that

$$\hat{\boldsymbol{N}}^* = \boldsymbol{P}\boldsymbol{N}^*, \tag{18}$$

where $\boldsymbol{P} = (P_{ij})_{i,j\in\mathcal{I}}$ is the $T \times T$ permutation matrix given by

$$P_{ij} = \begin{cases} 1 & \text{if} \quad j - i = 1 \text{ or } j - i = 1 - T, \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

that is, $\hat{N}_i^* = N_{i+1}^*$ for all $i \in \mathcal{I}\backslash\{T\}$ and $\hat{N}_T^* = 0$. Furthermore, if we define the schedule delay cost function $s(x)$ such that $s(x) = s(-x)$, there also exists essentially the same long-run equilibrium $\tilde{\boldsymbol{N}}^*$ with $\boldsymbol{N}^*$ such that

$$\tilde{\boldsymbol{N}}^* = \boldsymbol{R}\boldsymbol{N}^*, \tag{20}$$

where $\boldsymbol{R} = (R_{ij})_{i,j\in\mathcal{I}}$ is the $T \times T$ permutation which acts as the upside-down reflection given by

$$R_{ij} = \begin{cases} 1 & \text{if} \quad i + j = T + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

The essentially identical long-run equilibria $\boldsymbol{N}^*, \hat{\boldsymbol{N}}^*, \tilde{\boldsymbol{N}}^*$ satisfy

$$P(\boldsymbol{N}^*) = P(\hat{\boldsymbol{N}}^*) = P(\tilde{\boldsymbol{N}}^*), \tag{22a}$$

$$\det(\nabla^2 P(\boldsymbol{N}^*)) = \det(\nabla^2 P(\hat{\boldsymbol{N}}^*)) = \det(\nabla^2 P(\tilde{\boldsymbol{N}}^*)), \tag{22b}$$

where $\nabla^2 P(\boldsymbol{N}^*)$ is the Hessian matrix of P at $\boldsymbol{N}^*$ and $\det(\boldsymbol{A})$ is the determinant of $\boldsymbol{A}$. Moreover, from the index theorem of Simsek et al. (2007), the set $\mathrm{KKT}(P, \Delta)$ of the KKT points of problem (16) (i.e., the set of the long-run equilibria) satisfies

$$\sum_{\boldsymbol{N}\in\mathrm{KKT}(P,\Delta)} \mathrm{ind}_P(\boldsymbol{N}) = 1, \tag{23a}$$

$$\mathrm{ind}_P(\boldsymbol{N}) \equiv \begin{cases} -1 & \text{if} \quad \det(\nabla^2 P(\boldsymbol{N})) < 0, \\ 0 & \text{if} \quad \det(\nabla^2 P(\boldsymbol{N})) = 0, \\ 1 & \text{if} \quad \det(\nabla^2 P(\boldsymbol{N})) > 0. \end{cases} \tag{23b}$$

However, the total value of indices of essentially the same long-run equilibria cannot be one because of (22b). Therefore, we can obtain the following proposition.

Proposition 3. *The long-run equilibrium is essentially non-unique.*

3.2.3 Stability

We next consider the local asymptotic stability of long-run equilibria because our model generally includes multiple equilibria as shown above. To investigate the stability of the long-run equilibrium, we need to assume adjustment dynamics $\dot{\boldsymbol{N}} = \boldsymbol{V}(\boldsymbol{N})$ that maps a distribution of work start times $\boldsymbol{N}^0 \in \Delta$ to a set of Lipschitz paths in Δ that starts from $\boldsymbol{N}^0$. Although we usually consider a specific evolutionary dynamic for stability analysis, we see that a more general analysis is possible due to the existence of a potential function. That is, the stability of equilibria can be characterized under a broad class of dynamics. In particular, we consider the class of *admissible dynamics* that satisfies the following conditions:

$$\boldsymbol{V}(\boldsymbol{N}) \cdot \boldsymbol{u}(\boldsymbol{N}) > 0 \text{ whenever } \boldsymbol{V} \neq \boldsymbol{0}, \tag{24}$$

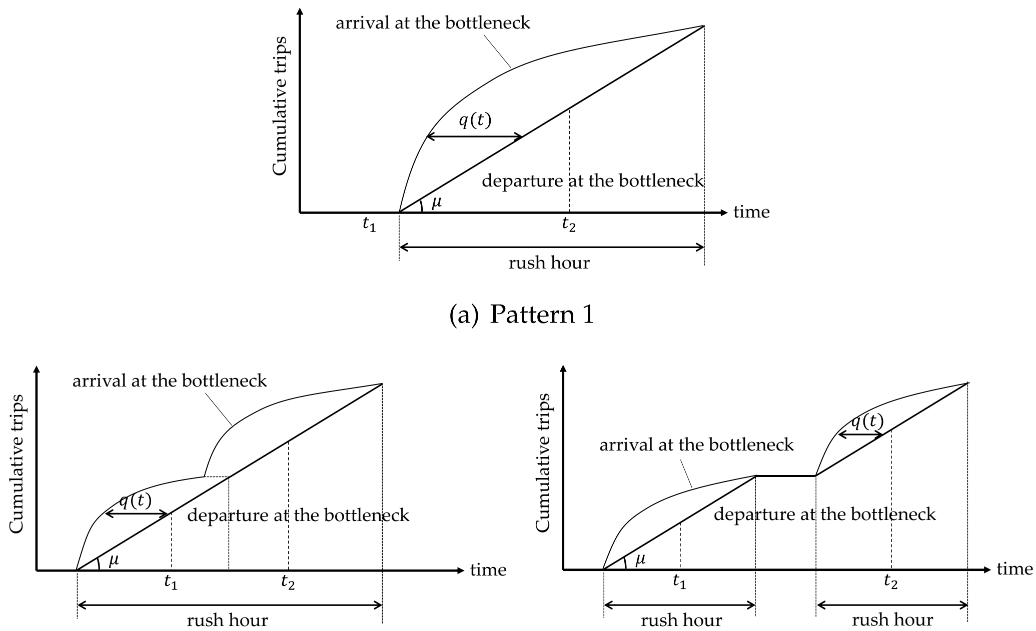
$$\boldsymbol{V}(\boldsymbol{N}) = \boldsymbol{0} \text{ implies that } \boldsymbol{N} \text{ is a Nash equilibrium of the game } G. \tag{25}$$

The former condition (24), called *positive correlation,* requires that out of rest points, there is a positive correlation between the adjustment dynamics $\boldsymbol{V}(\boldsymbol{N})$ and the payoffs $\boldsymbol{u}(\boldsymbol{N})$. This implies that, under this condition, all Nash equilibria of the game G are rest points of the adjustment dynamics $\boldsymbol{V}(\boldsymbol{N})$.[4] The latter condition (25), called *Nash stationarity,* asks that every rest points of the adjustment dynamics $\boldsymbol{V}(\boldsymbol{N})$ be a Nash equilibrium of the game G. Therefore, under the conditions (24) and (25), $\dot{\boldsymbol{N}} = \boldsymbol{V}(\boldsymbol{N}) = \boldsymbol{0}$ if and only if $\boldsymbol{N}$ is a Nash equilibrium of the game G. Specific examples of admissible dynamics include the *best response dynamic* (Gilboa and Matsui, 1991), the *Brown–von Neumann–Nash dynamic* (Brown and von Neumann, 1950), and *projection dynamic* (Dupuis and Nagurney, 1993).[5]

Under the admissible dynamics, we can easily characterize the local asymptotic stability of Nash equilibria of a potential game because Sandholm (2001) proves that *a Nash equilibrium of a potential game is asymptotically stable under any admissible dynamics if and only if it locally maximizes an associated potential function.* This implies that we can examine the stability of long-run equilibria only by checking the shape of the potential function. The following section compares the stable long-run equilibrium and optimal distributions of work start times by utilizing this property.

[4]See Proposition 4.3 of Sandholm (2001).

[5]See Sandholm (2005a) for more examples.



(a) Pattern 1

(b) Pattern 2.1

(c) Pattern 2.2

Figure 3: Distributions of work start times

3.2.4 A simple example

To demonstrate the usefulness of the potential function approach and to show the properties of the long-run equilibrium of our model, we analyze the model under the simple setting such that $T = 2, s(x) = \beta x^2$ where β is a positive constant. In this setting, the FIFO principle is satisfied at the short-run equilibrium if $\beta N \leq \mu$. Thus, we suppose that parameters μ, N, β satisfy this condition in this example.

In the case that $T = 2$, the distribution of work start times can be classified into three patterns (Figure 3):

Pattern 1: work start times are completely clustered.

Pattern 2.1: work start times are staggered, and the rush hour in which queuing congestion occurs is a single interval.

Pattern 2.2: work start times are staggered, and the rush hour is divided into two intervals.

Because of $s(x) = s(-x)$ in this example, Pattern 2.1 arises only if $\tau \leq N/(2\mu)$, and Pattern 2.2 arises only if $\tau > N/(2\mu)$.

The short-run equilibrium commuting costs $c_1^*(N_1)$ and $c_2^*(N_1)$ are obtained as functions of N_1:

$$c_1^*(N_1) = \begin{cases} \beta\left(\dfrac{N}{2\mu} - \tau\dfrac{N-N_1}{N}\right)^2 & \text{if} \quad \tau \leq \dfrac{N}{2\mu}, \\ \dfrac{\beta}{4\mu^2}N_1^2 & \text{if} \quad \tau > \dfrac{N}{2\mu}, \end{cases} \tag{26a}$$

$$c_2^*(N_1) = \begin{cases} \beta\left(\dfrac{N}{2\mu} - \tau\dfrac{N_1}{N}\right)^2 & \text{if} \quad \tau \leq \dfrac{N}{2\mu}, \\ \dfrac{\beta}{4\mu^2}(N-N_1)^2 & \text{if} \quad \tau > \dfrac{N}{2\mu}. \end{cases} \tag{26b}$$

Therefore, the potential function is represented as

$$P(N_1) = P_1(N_1) - P_2(N_1), \tag{27a}$$

$$P_1(N_1) = \alpha\left\{\frac{HN^2}{2} - \tau N_1(N-N_1)\right\}, \tag{27b}$$

$$P_2(N_1) = \begin{cases} \dfrac{\beta N^3}{12\mu^2} + \beta\tau\left(\dfrac{\tau}{N} - \dfrac{1}{\mu}\right)N_1(N-N_1) & \text{if} \quad \tau \leq \dfrac{N}{2\mu}, \\ \dfrac{\beta N}{12\mu^2}\left\{N^2 - 3N_1(N-N_1)\right\} & \text{if} \quad \tau > \dfrac{N}{2\mu}. \end{cases} \tag{27c}$$

Because the potential function is quadratic and the second derivative of the potential function is given by

$$\frac{\partial^2 P(N_1)}{\partial N_1^2} = \begin{cases} 2\tau\left\{\alpha + \beta\left(\dfrac{\tau}{N} - \dfrac{1}{\mu}\right)\right\} & \text{if} \quad \tau \leq \dfrac{N}{2\mu}, \\ 2\left(\alpha\tau - \dfrac{\beta N}{4\mu^2}\right) & \text{if} \quad \tau > \dfrac{N}{2\mu}, \end{cases} \tag{28}$$

the stable and unstable long-run equilibria N_1^s, N_1^u are obtained as follows:

$$\begin{cases} N_1^s = 0, N, \ \ N_1^u = \frac{N}{2} & \text{if} \quad \left\{\tau > \frac{\beta N}{4\alpha\mu^2} \text{ and } \tau > \frac{N}{2\mu}\right\} \text{ or } \left\{\tau > N\left(\frac{1}{\mu} - \frac{\alpha}{\beta}\right) \text{ and } \tau \leq \frac{N}{2\mu}\right\}, \\ N_1^s = \frac{N}{2} & \text{if} \quad \left\{\tau \leq \frac{\beta N}{4\alpha\mu^2} \text{ and } \tau > \frac{N}{2\mu}\right\} \text{ or } \left\{\tau \leq N\left(\frac{1}{\mu} - \frac{\alpha}{\beta}\right) \text{ and } \tau \leq \frac{N}{2\mu}\right\}. \end{cases} \tag{29}$$

Figure 4 illustrates the relation between the stable equilibrium and parameters τ and μ when $\alpha = 0.2, \beta = 1.0$, and $N = 1.0$.

We next show there is a situation in which capacity expansion exacerbates traffic congestion. We consider the case that capacity μ is expanded to 1.5μ

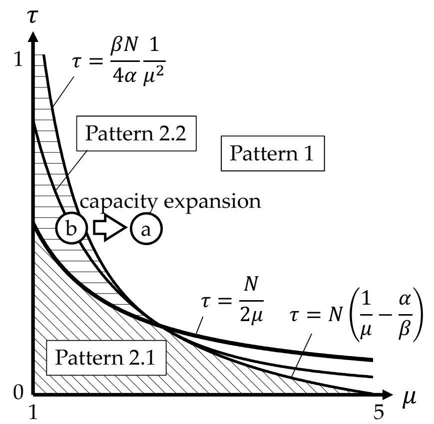


Figure 4: Parameters and the stable equilibrium ($N = 1.0, \alpha = 0.2, \beta = 1.0$)

and the stable equilibrium changes from Pattern 2.2 to Pattern 1 as illustrated in Figure 4. In this case, the total queuing time costs before and after the capacity expansion Q^b, Q^a are given by

$$Q^b = \int_{t_1^f}^{t_1^l} n_1(t)\, q(t)dt + \int_{t_2^f}^{t_2^l} n_2(t)\, q(t)dt = \mu \int_{t_1^f}^{t_1^l} c_1^* - s(t_1 - t)dt + \mu \int_{t_2^f}^{t_2^l} c_2^* - s(t_2 - t)dt$$
$$= \left\{N_1 c_1^* + N_2 c_2^*\right\} - \frac{2\mu\beta}{3}\left\{\left(\frac{N_1}{2\mu}\right)^3 + \left(\frac{N - N_1}{2\mu}\right)^3\right\} = \frac{8\mu\beta}{3}\left(\frac{N}{4\mu}\right)^3, \tag{30a}$$

$$Q^a = \int_{t_i^f}^{t_i^l} n_i(t)\, q(t)dt = 1.5\mu \int_{t_i^f}^{t_i^l} c_i^* - s(t_i - t)dt = Nc_i^* - \mu\beta\left(\frac{N}{3\mu}\right)^3 = 3\mu\beta\left(\frac{N}{3\mu}\right)^3, \tag{30b}$$

where t_i^f and t_i^l are the fastest and latest arrival time at the CBD of worker i. This result clearly indicates that $Q^b < Q^a$. That is, the capacity expansion exacerbates traffic congestion.

4 Social Optimum

Because of the positive and negative externalities, the long-run equilibrium is not generally efficient. Therefore, this section discusses TDM policies such as SWH and taxation for achieving the optimal distribution of work start times. To address this issue, we first characterize the social (i.e., first-best) optimum and the second-best optimum in which policymakers cannot control workers' short-run decisions. That is, the queue at the bottleneck cannot be eliminated. We then analyze the effectiveness of Pigouvian policies for achieving first-best and second-best optima.

4.1 First-best optimum

We define the first-best optimum as a state wherein total utility is maximized. This means that the first-best optimum coincides with a solution of the following maximization problem:

$$\max_{\{n_i(t)\},\boldsymbol{N}} W = W_1(\boldsymbol{N}) - W_2(\{n_i(t)\}) \tag{31a}$$

$$\text{s.t.} \quad \mu - \sum_{k\in\mathcal{I}} n_k(t) \geq 0 \;\; \forall t, \qquad \int n_i(t)dt = N_i \;\; \forall i \in \mathcal{I}, \quad n_i(t) \geq 0 \;\; \forall t, \; \forall i \in \mathcal{I}, \tag{31b}$$

$$\boldsymbol{N} \in \Delta, \tag{31c}$$

where $W_1(\boldsymbol{N})$ and $W_2(\{n_i(t)\})$ are given by

$$W_1(\boldsymbol{N}) = \sum_{k\in\mathcal{I}} F_k(\boldsymbol{N})N_k = 2P_1(\boldsymbol{N}), \tag{32a}$$

$$W_2(\{n_i(t)\}) = \sum_{k\in\mathcal{I}} \int n_k(t) \; \{q(t) + s(t_k - t)\} \, dt. \tag{32b}$$

As is the case with P_1 and P_2, W_1 and W_2 represent the strength of positive production externalities and negative congestion externalities, respectively. W_1 denotes workers' total wages in the city and W_2 is total commuting cost.

The queue at the bottleneck is completely eliminated at the first-best optimum as proved in studies involving standard bottleneck models (e.g., Vickrey, 1969; Hendrickson and Kocur, 1981; Arnott et al., 1990, 1993, 1994). It follows from this that $W_2(\{n_i(t)\})$ can be rewritten as

$$\tilde{W}_2(\{n_i(t)\}) = \sum_{k\in\mathcal{I}} \int n_k(t) \, s(t_k - t)dt, \tag{33}$$

which denotes the total schedule delay costs in the city. It is noteworthy that $\tilde{W}_2(\{n_i(t)\})$ coincides with the objective function of the optimization problem that is equivalent to the equilibrium conditions of the standard bottleneck model (Iryo and Yoshii, 2007). Specifically, we obtain $\{n_i(t)\}$ at the short-run equilibrium by solving the following minimization problem:

$$\min_{\{n_i(t)\}} \tilde{W}_2(\{n_i(t)\}) \quad \text{s.t.} \quad (31b) \tag{34}$$

Furthermore, this problem has the following useful property:

Lemma 4. $Z(\boldsymbol{N}) = \min_{\{n_i(t)\}} \tilde{W}_2(\{n_i(t)\})$ *s.t.* (31b) *satisfies*

$$\nabla Z(\boldsymbol{N}) = \boldsymbol{c}(\boldsymbol{N}). \tag{35}$$

Proof. See Appendix. □

Lemma 4 indicates that $P_2(\boldsymbol{N})$ is given by $Z(\boldsymbol{N})$. Therefore the distribution $\boldsymbol{N}^o$ of work start times at the first-best optimum is the solution of the following problem.

Lemma 5. *The distribution* $\boldsymbol{N}^o$ *of work start times at the first-best optimum is obtained by solving the following maximization problem.*

$$\max_{\boldsymbol{N}} P(\boldsymbol{N}) + P_1(\boldsymbol{N}) \quad s.t. \quad \boldsymbol{N} \in \Delta. \tag{36}$$

Lemma 5 implies that the positive production externalities should be strengthened to achieve the first-best optimum because the objective function of problem (36) is the sum of the potential function $P(\boldsymbol{N})$ and the convex function $P_1(\boldsymbol{N})$, which represents the production externalities. Therefore, we have the following propositon.

Proposition 4. *The first-best optimal distribution* $\boldsymbol{N}^o$ *of work start times is more clustered than the stable equilibrium* $\boldsymbol{N}^s$*.*

4.2 Second-best optimum

Although there are numerous effective schemes for managing traffic congestion, including dynamic congestion pricing (e.g., Yang and Huang, 2005; Tsekeris and Voß, 2008; de Palma and Lindsey, 2011) and tradable permits schemes (e.g., Verhoef et al., 1997; Yang and Wang, 2011; Wada and Akamatsu, 2013), eliminating queuing congestion has been difficult thus far. Thus, we next consider the second-best optimum wherein policymakers cannot control workers' short-run behaviors. That is, the queue at the bottleneck cannot be eliminated. The distribution $\hat{\boldsymbol{N}}^o$ of work start times at the second-best optimum is defined as the solution of the following problem:

$$\max_{\boldsymbol{N}} \hat{W}(\boldsymbol{N}) = \hat{W}_1(\boldsymbol{N}) - \hat{W}_2(\boldsymbol{N}) \quad \text{s.t.} \quad \boldsymbol{N} \in \Delta, \tag{37}$$

where $\hat{W}_1(\boldsymbol{N})$ and $\hat{W}_2(\boldsymbol{N})$ are total wage and commuting cost, respectively, which are expressed as

$$\hat{W}_1(\boldsymbol{N}) = W_1(\boldsymbol{N}) = \sum_{k \in \mathcal{I}} F_k(\boldsymbol{N}) N_k, \tag{38a}$$

$$\hat{W}_2(\boldsymbol{N}) = \sum_{k \in \mathcal{I}} c_k^*(\boldsymbol{N}) N_k. \tag{38b}$$

To compare the second-best optimum and the stable equilibrium, we examine the shape of $\hat{W}(\boldsymbol{N})$ from its Hessian matrix. Because the Hessian matrix of $\hat{W}(\boldsymbol{N})$ is given by

$$\nabla^2 \hat{W}(\boldsymbol{N}) = 2\nabla^2 P(\boldsymbol{N}) - \sum_{k \in \mathcal{I}} N_k \nabla^2 c_k^*(\boldsymbol{N}), \tag{39}$$

we see that only $-\sum_{k \in \mathcal{I}} N_k \nabla^2 c_k^*(\boldsymbol{N})$ makes a difference in the shape of $\hat{W}(\boldsymbol{N})$ and $P(\boldsymbol{N})$. This yields the following proposition.

Proposition 5. *The second-best optimal distribution of work start times is more clustered than the stable equilibrium if the matrix* $-\sum_{k \in \mathcal{I}} N_k \nabla^2 c_k^*(\boldsymbol{N})$ *is positive definite, and it is more staggered than the stable equilibrium if* $-\sum_{k \in \mathcal{I}} N_k \nabla^2 c_k^*(\boldsymbol{N})$ *is negative definite.*

Note here that in many cases $-\sum_{k \in \mathcal{I}} N_k \nabla^2 c_k^*(\boldsymbol{N})$ is expected to be negative definite because the schedule delay cost function $s(x)$ is assumed to be convex. In fact, if the number of intervals of rush hour at the second-best optimum equals the cardinality of supp $\hat{\boldsymbol{N}}^o$ (i.e., rush hour is completely separated), $c_i^*(\boldsymbol{N})$ must be convex due to the convexity of $s(x)$. Therefore, $\hat{\boldsymbol{N}}^o$ is generally expected to be more staggered than $\boldsymbol{N}^s$, which implies that the TDM policies for staggering work hours are generally effective in the case where queuing congestion cannot be eliminated.

4.3 Pigouvian policies

We next discuss tax/subsidy policies that attain the first-best and second-best optima as a *stable* long-run equilibria. To achieve the optimum, we generally consider Pigouvian policies, such as congestion tolls. We do so because the optimal state is supported as an equilibrium by imposing such policies that workers are responsible for their externalities at the optimum. However, as mentioned in the introduction, Akamatsu et al. (2014b) shows that if we consider a model with positive and negative externalities, social optimum can be an unstable equilibrium under Pigouvian policies and a non-optimal stable equilibrium will exist. This implies the possibility that the social optimum cannot be achieved only by Pigouvian policies in our model. Therefore, this section analyzes the stability of the first-best and second-best optima under Pigouvian policies.

4.3.1 First-best optimum

Since the model of workers' short-run decisions involves no positive externalities, we assume the queue is completely eliminated by some schemes and examine whether the first-best optimal distribution of work start times is a stable long-run equilibrium under Pigouvian policies. We consider a Pigouvian policy that introduces tax/subsidy $\boldsymbol{p} = (p_i)_{i \in \mathcal{I}}$ to workers in order to attain the first-best optimum, which we call the Pigouvian first-best policy. It follows from Proposition 2 and Lemma 5 that $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = \boldsymbol{F}(\boldsymbol{N}^o) = \{H\boldsymbol{E} - \tau\boldsymbol{D}\}\,\boldsymbol{N}^o. \tag{40}$$

Under the Pigouvian first-best policy, our model is viewed as a potential game $G^P = \{\mathcal{S}, \mathcal{I}, \boldsymbol{u}^P\}$, where $\boldsymbol{u}^P(\boldsymbol{N}) = \boldsymbol{u}(\boldsymbol{N}) + \boldsymbol{p}$, because there exists the following potential function:

$$P^P(\boldsymbol{N}) = P(\boldsymbol{N}) + \boldsymbol{p} \cdot \boldsymbol{N}. \tag{41}$$

The KKT conditions of the maximization problem of the potential function $P^P(\boldsymbol{N})$ subject to $\boldsymbol{N} \in \Delta$ is given by

$$\begin{cases} u^* - \left\{F_i(\boldsymbol{N}^o) + F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N})\right\} = 0 & \text{if} \quad N_i > 0 \\ u^* - \left\{F_i(\boldsymbol{N}^o) + F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N})\right\} \geq 0 & \text{if} \quad N_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \tag{42a}$$

$$\sum_{k \in \mathcal{I}} N_k = N. \tag{42b}$$

This implies that the first-best optimum $\boldsymbol{N}^o$ must be a Nash equilibrium of the game G^P because the first-order conditions (i.e., optimality conditions) of problem (36) is represented as

$$\begin{cases} w^* - \left\{2F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N})\right\} = 0 & \text{if} \quad N_i > 0 \\ w^* - \left\{2F_i(\boldsymbol{N}) - c_i^*(\boldsymbol{N})\right\} \geq 0 & \text{if} \quad N_i = 0 \end{cases} \quad \forall i \in \mathcal{I}, \tag{43a}$$

$$\sum_{k \in \mathcal{I}} N_k = N. \tag{43b}$$

However, this policy does not work for stabilizing the first-best optimum because introduction of the Pigouvian first-best policy cannot change the Hessian matrix of the potential function. That is, $\nabla^2 P(\boldsymbol{N})$ equals $\nabla^2 P^P(\boldsymbol{N})$. Note that because $\nabla^2 P(\boldsymbol{N}) = \nabla^2 P^P(\boldsymbol{N})$, $\boldsymbol{N}^o$ is stable under the Pigouvian first-best policy when $\boldsymbol{N}^o = \boldsymbol{N}^s$, and $\boldsymbol{N}^o$ is unstable if $\boldsymbol{N}^o = \boldsymbol{N}^u$.

Since P is not generally unimodal, the equilibrium of the game G^P is generally non-unique. Thus, we examine the stability of the first-best optimum $\boldsymbol{N}^o$ by looking at the shape of P^P at the neighborhood of $\boldsymbol{N}^o$. For this examination, let Λ^o be the set of the neighborhood of $\boldsymbol{N}^o$ in Δ. Then, $\boldsymbol{N}^o$ is locally and asymptotically stable (i.e., a local maximizer of P^P) if and only if

$$(\boldsymbol{N} - \boldsymbol{N}^o)^\top \left\{\nabla^2 P^P(\boldsymbol{N}^o)\right\} (\boldsymbol{N} - \boldsymbol{N}^o) < 0 \qquad \forall \boldsymbol{N} \in \Lambda^o; \tag{44}$$

otherwise, $\boldsymbol{N}^o$ is unstable. Because $\boldsymbol{N}^o$ is the global maximizer of W and $\nabla^2 P_1(\boldsymbol{N})$ is positive definite, we have

$$\begin{aligned} 0 &> (\boldsymbol{N} - \boldsymbol{N}^o)^\top \nabla^2 W(\boldsymbol{N}^o)(\boldsymbol{N} - \boldsymbol{N}^o) = (\boldsymbol{N} - \boldsymbol{N}^o)^\top \left\{\nabla^2 P(\boldsymbol{N}^o) + \nabla^2 P_1(\boldsymbol{N}^o)\right\} (\boldsymbol{N} - \boldsymbol{N}^o) \\ &> (\boldsymbol{N} - \boldsymbol{N}^o)^\top \nabla^2 P(\boldsymbol{N}^o)(\boldsymbol{N} - \boldsymbol{N}^o) = (\boldsymbol{N} - \boldsymbol{N}^o)^\top \nabla^2 P^P(\boldsymbol{N}^o)(\boldsymbol{N} - \boldsymbol{N}^o) \qquad \forall \boldsymbol{N} \in \Lambda^o. \end{aligned} \tag{45}$$

This yields the following proposition.

Proposition 6. *The first-best optimal distribution of work start times is stable under the Pigouvian first-best policy.*

4.3.2 Second-best optimum

We next consider Pigouvian policy to attain the second-best optimum, which we call the Pigouvian second-best policy.

The Pigouvian second-best policy is to introduce tax/subsidy $\hat{\boldsymbol{p}}$ so that $\hat{\boldsymbol{N}}^o$ is a Nash equilibrium of the game $\hat{G}^P = \{\mathcal{S}, \mathcal{I}, \hat{\boldsymbol{u}}^P = \boldsymbol{u} + \hat{\boldsymbol{p}}\}$, where

$$\hat{\boldsymbol{p}} = \nabla^2 P(\hat{\boldsymbol{N}}^o)\hat{\boldsymbol{N}}^o. \tag{46}$$

As stated above, this policy makes $\hat{\boldsymbol{N}}^o$ a long-run equilibrium but cannot stabilize it. Thus, we check its stability.

For this, we consider a potential game $\hat{G}^P$ with the potential function

$$\hat{P}^P(\boldsymbol{N}) = P(\boldsymbol{N}) + \hat{\boldsymbol{p}} \cdot \boldsymbol{N}. \tag{47}$$

Because the model of workers' long-run decisions is viewed as the game $\hat{G}^P$, the second-best optimum $\hat{\boldsymbol{N}}^o$ is stable if and only if

$$\begin{aligned}
&\left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right)^\top \left\{\nabla^2 \hat{P}^P(\hat{\boldsymbol{N}}^o)\right\} \left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right) \\
&= \frac{1}{2}\left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right)^\top \left\{\nabla^2 \hat{W}(\hat{\boldsymbol{N}}^o) + \sum_{k \in \mathcal{I}} \hat{N}_k^o \nabla^2 c_k^*(\hat{\boldsymbol{N}}^o)\right\} \left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right) < 0 \qquad \forall \hat{\boldsymbol{N}} \in \hat{\Lambda}^o,
\end{aligned} \tag{48}$$

where $\hat{\Lambda}^o$ is the set of the neighborhood of $\hat{\boldsymbol{N}}^o$ in Δ. Therefore, we have the following proposition.

Proposition 7. *The second-best optimum $\hat{\boldsymbol{N}}^o$ is a stable equilibrium under the Pigouvian second-best policy if and only if*

$$\begin{aligned}
&\left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right)^\top \left\{\nabla^2 \hat{W}(\hat{\boldsymbol{N}}^o)\right\} \left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right) \\
&\qquad < \left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right)^\top \left\{-\sum_{k \in \mathcal{I}} \hat{N}_k^o \nabla^2 c_k^*(\hat{\boldsymbol{N}}^o)\right\} \left(\hat{\boldsymbol{N}} - \hat{\boldsymbol{N}}^o\right) \qquad \forall \hat{\boldsymbol{N}} \in \hat{\Lambda}^o;
\end{aligned} \tag{49}$$

otherwise, the second-best optimum $\hat{\boldsymbol{N}}^o$ is unstable.

Note that the condition (49) can be violated as shown in Section 4.3.3, and thus the Pigouvian second-best policy can be ineffective. This means that policymakers need to implement other policies for stabilizing optimal distribution of work start times. One of the effective policy is *evolutionary implementation of Pigouvian policies* introduced by Sandholm (2002, 2005b). This policy is to impose the values of externalities evaluated at the current state, rather than the optimal state. We briefly show the effectiveness of this policy. If the current state is $\boldsymbol{N} \in \Delta$, the tax/subsidy $\tilde{\boldsymbol{p}}(\boldsymbol{N})$ to workers is

$$\tilde{\boldsymbol{p}}(\boldsymbol{N}) = \nabla^2 P(\boldsymbol{N})\boldsymbol{N}. \tag{50}$$

Let $\tilde{G}^P = \{\mathcal{S}, \mathcal{I}, \tilde{\boldsymbol{u}} = \boldsymbol{u} + \tilde{\boldsymbol{p}}\}$ be a population game under this policy. We then can show that the game $\tilde{G}^P$ is a potential game for which $\hat{W}(\boldsymbol{N})$ is the potential function. This implies that the second-best optimal distribution of work start times $\hat{\boldsymbol{N}}^o$ must be a stable equilibrium under the policy (50) because $\hat{\boldsymbol{N}}^o$ globally maximizes $\hat{W}(\boldsymbol{N})$.

4.3.3 A simple example revisited

To show concretely the properties of the first-best optimum, second-best optimum, and Pigouvian policies, we revisit the simple example presented in Section 3.2.4. In this simple case, W and $\hat{W}$ are represented as functions of N_1.

$$W(N_1) = 2P_1(N_1) - P_2(N_1), \tag{51}$$

$$\hat{W}(N_1) = 2P_1(N_1) - \hat{P}_2(N_1) \tag{52}$$

$$\hat{P}_2(N_1) = \begin{cases} \dfrac{\beta N}{4\mu^2}\{N^2 - 3N_1(N - N_1)\} & \text{if} \quad \tau > \dfrac{N}{2\mu}, \\ \dfrac{\beta N^3}{4\mu^2} - \dfrac{2\beta\tau}{N}\left(\dfrac{N}{\mu} - \dfrac{\tau}{2}\right)N_1(N - N_1) & \text{if} \quad \tau \leq \dfrac{N}{2\mu}. \end{cases} \tag{53}$$

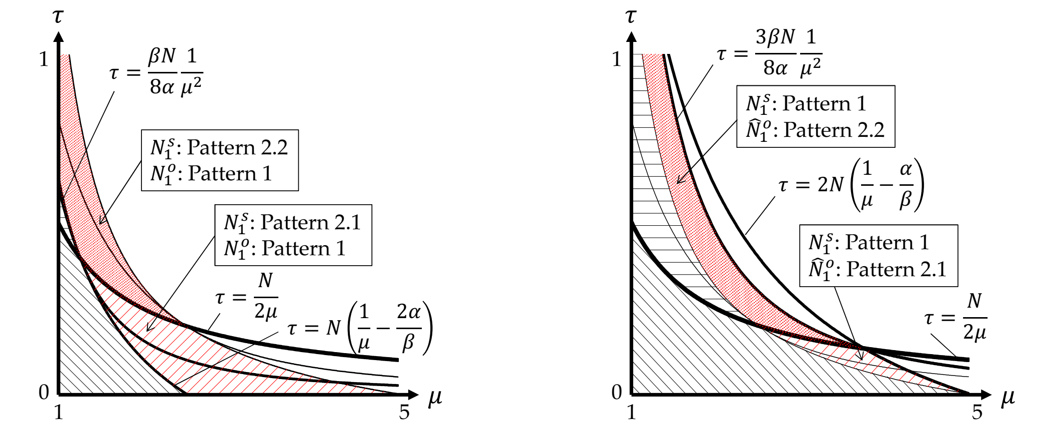
Because W and $\hat{W}$ are quadratic, the first-best and second-best optima $N_1^o, \hat{N}_1^o$ are easily obtained.

$$\begin{cases} N_1^o = 0, N, & \text{if} \quad \left\{\tau > \frac{\beta N}{8\alpha\mu^2} \quad \text{and} \quad \tau > \frac{N}{2\mu}\right\} \quad \text{or} \quad \left\{\tau > N\left(\frac{1}{\mu} - \frac{2\alpha}{\beta}\right) \quad \text{and} \quad \tau \leq \frac{N}{2\mu}\right\}, \\ N_1^o = \frac{N}{2} & \text{if} \quad \left\{\tau \leq \frac{\beta N}{8\alpha\mu^2} \quad \text{and} \quad \tau > \frac{N}{2\mu}\right\} \quad \text{or} \quad \left\{\tau \leq N\left(\frac{1}{\mu} - \frac{2\alpha}{\beta}\right) \quad \text{and} \quad \tau \leq \frac{N}{2\mu}\right\}, \end{cases} \tag{54a}$$

$$\begin{cases} \hat{N}_1^o = 0, N, & \text{if} \quad \left\{\tau > \frac{3\beta N}{8\alpha\mu^2} \quad \text{and} \quad \tau > \frac{N}{2\mu}\right\} \quad \text{or} \quad \left\{\tau > 2N\left(\frac{1}{\mu} - \frac{\alpha}{\beta}\right) \quad \text{and} \quad \tau \leq \frac{N}{2\mu}\right\}, \\ \hat{N}_1^o = \frac{N}{2} & \text{if} \quad \left\{\tau \leq \frac{3\beta N}{8\alpha\mu^2} \quad \text{and} \quad \tau > \frac{N}{2\mu}\right\} \quad \text{or} \quad \left\{\tau \leq 2N\left(\frac{1}{\mu} - \frac{\alpha}{\beta}\right) \quad \text{and} \quad \tau \leq \frac{N}{2\mu}\right\}. \end{cases} \tag{54b}$$

It follows from (29) and (54) that the first-best optimal distribution of work start times is more clustered than the stable equilibrium, and that the second-best optimum is more staggered, which is illustrated by the red areas in Figure 5. Both are consistent with Propositions 4 and 5.

These results also indicate that $\hat{N}_1^o$ can be equal to N_1^u. That is, the second-best optimum can be unstable under the Pigouvian second-best policy. Therefore, we have to carefully implement Pigouvian policies for alleviating traffic congestion, such as a congestion toll, if policymakers cannot control workers' short-run decisions.



(a) Pigouvian first-best optimum policy (b) Pigouvian second-best optimum policy

Figure 5: Differences between the stable equilibrium and the optimum (N = 1.0, α = 0.2, β = 1.0)

5 Conclusions

This study presented a model of work start time choice with bottleneck congestion and an analytical approach utilizing the properties of a potential game. By using this approach, we showed that equilibrium distribution of work start times is essentially non-unique and that stability of equilibria can be examined by checking the shape of the potential function. Furthermore, by comparing the social welfare function and the potential function, we clarified that if policymakers can eliminate the queue at the bottleneck, distribution of work start times should be more clustered than the stable equilibrium; otherwise, it should be more staggered. After characterizing the equilibrium and optimal distribution of work start times, we investigated the effectiveness of tax/subsidy policies and pointed out that if the queue cannot eliminated, Pigouvian tax/subsidy policies can be ineffective for achieving a optimum.

The analytical approach presented herein can be used not only for a model of work start time choice but also for a wide class of models considering bottleneck congestion. For instance, this approach is applicable to models of location choice with bottleneck congestion, such as Arnott (1998). Therefore, it would be valuable for future research to investigate the intrinsic properties of other models by applying the approach.

A Proofs

A.1 Proof of Lemma 1

Let t_i^f and t_i^l be the fastest and latest arrival time at CBD of worker i. It follows from Proposition 1 (i.e., uniqueness of the short-run equilibrium and the first-in-first-work discipline) that t_i^f and t_i^l satisfy

$$t_{i-1}^l \leq t_i^f \quad \forall i \in \mathcal{I} \backslash \{1\}. \tag{55}$$

Let $\mathcal{I}_i \equiv \{j \mid q(t) > 0 \ \forall t \in [t_i, t_j] \text{ or } [t_j, t_i]\}$. Then, we can say that the short-run equilibrium commuting cost of worker i is affected by behavior of worker $j \in \mathcal{I}_i$. That is,

$$\frac{\partial c_i^*(\boldsymbol{N})}{\partial N_j} \neq 0 \quad \forall j \in \mathcal{I}_i. \tag{56}$$

Note that if there exists $t \in [t_i, t_j]$ or $[t_j, t_i]$ such that $q(t) = 0$, Lindsey (2004) proves that

$$\frac{\partial c_i^*(\boldsymbol{N})}{\partial N_j} = \frac{\partial c_j^*(\boldsymbol{N})}{\partial N_i} = 0 \quad \forall j \notin \mathcal{I}_i. \tag{57}$$

Thus, for the proof of the symmetry of $\nabla \boldsymbol{c}(\boldsymbol{N})$, we will show here that

$$\frac{\partial c_i^*(\boldsymbol{N})}{\partial N_j} = \frac{\partial c_j^*(\boldsymbol{N})}{\partial N_i} \quad \forall j \in \mathcal{I}_i. \tag{58}$$

The short-run equilibrium commuting cost $c_i^*(\boldsymbol{N})$ of worker i is expressed as

$$c_i^*(\boldsymbol{N}) = \begin{cases} s(\Delta t_i^f) = q(t_i^l) + s(\Delta t_i^l) & \text{if} \quad i = a_i \equiv \min \mathcal{I}_i, \\ q(t_i^f) + s(\Delta t_i^f) = s(\Delta t_i^l) & \text{if} \quad i = b_i \equiv \max \mathcal{I}_i, \\ q(t_i^f) + s(\Delta t_i^f) = q(t_i^l) + s(\Delta t_i^l) & \text{otherwise,} \end{cases} \tag{59}$$

where $\Delta t_i^f \equiv t_i - t_i^f$ and $\Delta t_i^l \equiv t_i - t_i^l$, because $q(t_{a_i}^f) = q(t_{b_i}^l) = 0$. This is rewritten as

$$\begin{aligned} c_i^*(\boldsymbol{N}) &= q(t_i^f) + s(\Delta t_i^f) = \left\{ q(t_{i-1}^f) + s(\Delta t_{i-1}^f) - s(\Delta t_{i-1}^l) \right\} + s(\Delta t_i^f) = \cdots \\ &= \sum_{k=a_i}^{i-1} \left\{ s(\Delta t_k^f) - s(\Delta t_k^l) \right\} + s(\Delta t_i^f). \end{aligned} \tag{60}$$

Furthermore, t_i^f and t_i^l are represented as functions of $t_{a_i}^f$ and $\boldsymbol{N}$:

$$t_i^f = t_{a_i}^f + \frac{\sum_{k=a_i}^{j-1} N_k}{\mu}, \qquad t_i^l = t_{a_i}^f + \frac{\sum_{k=a_i}^{j} N_k}{\mu}. \tag{61}$$

Therefore, differentiating $c_i^*(\boldsymbol{N})$ with respect to N_i, we have

$$\frac{\partial c_i^*(\boldsymbol{N})}{\partial N_j} = \begin{cases} -\dfrac{\partial t_{a_i}^f}{\partial N_j}\left[\displaystyle\sum_{k=a_i}^{i-1}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\} + s'(\Delta t_i^f)\right] - \dfrac{1}{\mu}\left[-s'(\Delta t_j^l) + \displaystyle\sum_{k=j+1}^{i-1}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\} + s'(\Delta t_i^f)\right] \\ -\dfrac{\partial t_{a_i}^f}{\partial N_j}\left[\displaystyle\sum_{k=a_i}^{i-1}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\} + s'(\Delta t_i^f)\right] \end{cases} \tag{62}$$

where the prime denotes differentiation. In addition, it follows from $q(t_{b_i}) = 0$ that

$$\sum_{i \in \mathcal{I}_i}\left\{s(\Delta t_k^f) - s(\Delta t_k^l)\right\} = 0. \tag{63}$$

Differentiating both side of (63) with respect to N_j, we obtain

$$-\frac{\partial t_{a_i}^f}{\partial N_j}\left[\sum_{k \in \mathcal{I}_i}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\}\right] - \frac{1}{\mu}\left[-s'(\Delta t_j^l) + \sum_{k=j+1}^{b_i}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\}\right] = 0. \tag{64}$$

Substituting this into (62) yields

$$\frac{\partial c_i^*(\boldsymbol{N})}{\partial N_j} = \begin{cases} -\dfrac{\partial t_{a_i}^f}{\partial N_i}\left[\displaystyle\sum_{k=a_i}^{j-1}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\} + s'(\Delta t_j^f)\right] & \text{if} \quad i > j, \\ -\dfrac{\partial t_{a_i}^f}{\partial N_j}\left[\displaystyle\sum_{k=a_i}^{i-1}\left\{s'(\Delta t_k^f) - s'(\Delta t_k^l)\right\} + s'(\Delta t_i^f)\right] & \text{if} \quad i \leq j, \end{cases} \tag{65}$$

which shows the symmetry of $\nabla \boldsymbol{c}(\boldsymbol{N})$.

We next prove positive definiteness of $\nabla \boldsymbol{c}(\boldsymbol{N})$. Substituting (64) into (65), we obtain

$$\frac{\partial c_i^*(\boldsymbol{N})}{\partial N_j} = \frac{\partial c_j^*(\boldsymbol{N})}{\partial N_i} = \frac{S_{ij}^f S_{ji}^l}{\mu S_i}, \tag{66a}$$

$$S_{ij}^f \equiv s'(\Delta t_j^f) + \sum_{k=a_i}^{j-1} \left\{ s'(\Delta t_k^f) - s'(\Delta t_k^l) \right\} > 0, \tag{66b}$$

$$S_{ij}^l \equiv -s'(\Delta t_j^l) + \sum_{k=j+1}^{b_i} \left\{ s'(\Delta t_k^f) - s'(\Delta t_k^l) \right\} > 0, \tag{66c}$$

$$S_i \equiv \sum_{k \in \mathbb{S}_i} \left\{ s'(\Delta t_k^f) - s'(\Delta t_k^l) \right\} > 0. \tag{66d}$$

Note that since $-s'(\Delta t_i^l) + s'(\Delta t_{i+1}^f) > 0$ for all $i, i+1 \in \mathcal{I}_i$, $s'(\Delta t_{a_i}^f) > 0$, and $s'(\Delta t_{b_i}^l) < 0$, S_i, S_{ij}^f, S_{ij}^l are all positive. Thus, (66a) can be rewritten as follows:

$$\nabla \boldsymbol{c}(\boldsymbol{N}) = \boldsymbol{L}\boldsymbol{L}^\top, \tag{67}$$

where $\boldsymbol{L} = (L_{ij})_{i,j \in \mathcal{I}}$ is a lower triangular matrix, the (i, j) entries of which are given by

$$L_{ij} = \begin{cases} S_{ii}^l \sqrt{\dfrac{1}{S_i} \left\{ \dfrac{S_{ij}^f}{S_{ij}^l} - \dfrac{S_{i(j-1)}^f}{S_{i(j-1)}^l} \right\}} > 0 & \text{if} \quad i \geq j, \ j \in \mathcal{I}_i, \\ 0 & \text{otherwise}, \end{cases} \tag{68}$$

where $S_{i0}^f = 0, S_{i0}^l \neq 0$. This implies that $\nabla \boldsymbol{c}(\boldsymbol{N})$ has a Cholesky decomposition, and thus $\nabla \boldsymbol{c}(\boldsymbol{N})$ is positive semidefinite[6].

A.2 Proof of Proposition 2

Since it is apparent that $P(\boldsymbol{N})$ is the potential function of the game G, we prove here that the convexity of $P_1(\boldsymbol{N})$ and $P_2(\boldsymbol{N})$.

We first show that $P_1(\boldsymbol{N})$ is convex. The Hessian matrix of $P_1(\boldsymbol{N})$ is given by

$$\nabla^2 P_1(\boldsymbol{N}) = \alpha \left\{ H\boldsymbol{E} - \tau \boldsymbol{D} \right\}. \tag{69}$$

[6] For the proof, see, e.g., Corollary 7.2.9 of Horn and Johnson (2013).

Its inverse can be directly computed as

$$\left\{\nabla^2 P_1(\boldsymbol{N})\right\}^{-1} = \frac{1}{\tau}\begin{bmatrix} \gamma & -0.5 & & & & \epsilon \\ -0.5 & 1 & -0.5 & & \boldsymbol{O} & \\ & -0.5 & 1 & -0.5 & & \\ & & \ddots & \ddots & \ddots & \\ & \boldsymbol{O} & & -0.5 & 1 & -0.5 \\ \epsilon & & & & -0.5 & \gamma \end{bmatrix}, \tag{70}$$

$$\gamma \equiv \epsilon + 0.5, \tag{71}$$

$$\epsilon \equiv \frac{\tau}{2\{2H - (T-1)\tau\}}. \tag{72}$$

Then, by Gershgorin circle theorem,[7] every eigenvalue $\lambda_{\{\nabla^2 P_1(\boldsymbol{N})\}^{-1}}$ of $\{\nabla^2 P_1(\boldsymbol{N})\}^{-1}$ lies in

$$0 \leq \lambda_{\{\nabla^2 P_1(\boldsymbol{N})\}^{-1}} \leq \max\{1 + 2\epsilon, 2\}. \tag{73}$$

It follows from this that every eigenvalue of $\nabla^2 P_1(\boldsymbol{N})$ is also nonnegative. This shows the convexity of $P_1(\boldsymbol{N})$.

$P_2(\boldsymbol{N})$ is also convex, because $\nabla^2 P_2(\boldsymbol{N}) = \nabla \boldsymbol{c}(\boldsymbol{N})$ is positive semidefinite as shown in Lemma 1.

A.3 Proof of Lemma 3

Suppose to the contrary that there exists an equilibrium $\boldsymbol{N}^*$ in which, for some $i, j \in \operatorname{supp} \boldsymbol{N}^*$ with $j - i \geq 2$, $N_k^* = 0$ (i.e., $k \notin \operatorname{supp} \boldsymbol{N}^*$) for all $i < k < j$, and let $d_{ij} \equiv |i - j|$, $\tilde{N}_k^- \equiv \sum_{m=1}^{k} N_m$, and $\tilde{N}_k^+ \equiv \sum_{m=k}^{T} N_m$. Then, for $k \in (i, j)$,

$$u_i = u_j \geq u_k, \tag{74a}$$

$$u_i - u_j = \tau d_{ij}\left\{\tilde{N}_k^- - \tilde{N}_k^+\right\} - c_i^*(\boldsymbol{N}) + c_j^*(\boldsymbol{N}) = 0, \tag{74b}$$

$$u_i - u_k = \tau d_{ik}\left\{\tilde{N}_k^- - \tilde{N}_k^+\right\} - c_i^*(\boldsymbol{N}) + c_k^*(\boldsymbol{N}) \geq 0, \tag{74c}$$

$$u_j - u_k = \tau d_{kj}\left\{\tilde{N}_k^+ - \tilde{N}_k^-\right\} - c_j^*(\boldsymbol{N}) + c_k^*(\boldsymbol{N}) \geq 0. \tag{74d}$$

If $\tilde{N}_k^- \geq \tilde{N}_k^+$, we have $c_i^*(\boldsymbol{N}) \geq c_j^*(\boldsymbol{N}) > 0$ and $c_k^*(\boldsymbol{N}) \geq c_j^*(\boldsymbol{N}) > 0$ from (74b) and (74d). This implies that $q(t_k) > 0$, i.e., $t_i^l > t_k$. Furthermore, substituting (74b) into

[7] For the details of this theorem, see, e.g., Strang (2006) and Horn and Johnson (2013).

(74d) yields

$$u_j - u_k = -\frac{d_{ik}}{d_{ij}} c_j^*(\boldsymbol{N}) - \frac{d_{kj}}{d_{ij}} c_i^*(\boldsymbol{N}) + c_k^*(\boldsymbol{N}) = -\frac{d_{ik}}{d_{ij}} s(\Delta t_j^f) - \frac{d_{kj}}{d_{ij}} s(\Delta t_i^l) + s(t_k - t_i^l). \quad (75)$$

Since $s(x)$ is convex, (75) is rewritten as

$$u_j - u_k \leq \begin{cases} \frac{d_{ik}}{d_{ij}} \left\{ -s(\Delta t_j^f) + s(t_k - t_i^l) + \tau d_{kj} s'(t_k - t_i^l) \right\} < 0 & \text{if} \quad t_i^l < t_j, \\ \frac{d_{ik}}{d_{ij}} \left\{ s(t_k - t_i^l) - s(\Delta t_j^f) \right\} + \frac{d_{kj}}{d_{ij}} \left\{ s(t_k - t_i^l) - s(\Delta t_i^l) \right\} < 0 & \text{if} \quad t_i^l \geq t_j. \end{cases} \quad (76)$$

If $\tilde{N}_k^- < \tilde{N}_k^+$, we can easily show that $u_i - u_k < 0$ by the same procedure. But this contradicts $i, j \in \operatorname{supp} \boldsymbol{N}^*$ (i.e., $u_i = u_j \geq u_k$).

A.4 Proof of Lemma 4

It follows from Proposition 1 (uniqueness of the short-run equilibrium and the first-in-first-work discipline) that $Z(\boldsymbol{N})$ is represented as

$$Z(\boldsymbol{N}) = \sum_{k \in \mathcal{I}} \int_{t_k^f}^{t_k^l} \mu s(t_k - t) dt. \quad (77)$$

Since t_i^f and t_i^l is given by (61), differentiation of $Z(\boldsymbol{N})$ with respect to N_i yields

$$\frac{\partial Z(\boldsymbol{N})}{\partial N_i} = s(\Delta t_i^l) + \sum_{k=i+1}^{b_i} \left\{ -s(\Delta t_k^f) + s(\Delta t_k^l) \right\} + \mu \frac{\partial t_{a_i}^f}{\partial N_i} \sum_{k \in \mathcal{I}_i} \left\{ -s(\Delta t_k^f) + s(\Delta t_k^l) \right\}. \quad (78)$$

Substituting (62), we have

$$\frac{\partial Z(\boldsymbol{N})}{\partial N_i} = \sum_{k=a_i}^{i-1} \left\{ s(\Delta t_k^f) - s(\Delta t_k^l) \right\} + s(\Delta t_i^f) = c_i^*(\boldsymbol{N}). \quad (79)$$

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