Housing Dynamics: Theory Behind Empirics

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Abstract: We construct a dynamic general equilibrium model of housing, incorporating some key features that bridge time and space. We model explicitly the evolution of housing structures/household durables and the separate role played by land, fully accounting for households' locational choice decisions. Housing services derive positive utility, but are decayed away from the city center. Our model enables a full characterization of the dynamic paths of housing as well as housing and land prices. The model is particularly designed to be calibrated to fit some important stylized facts, including faster growth of housing structure/household durables than housing, faster growth of land prices than housing prices, a locationally steeper land rent gradient than the housing price gradient, and relatively flatter housing quantity and price gradients in larger cities with flatter population gradients. The calibrated model is then used to quantitatively assess the dynamic and spatial consequences of demand and supply shifts. We find that nonhomotheticity in forms of income-elastic spending on housing/household durables and minimum structure requirement in housing production are essential ingredients.

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1 Introduction

The housing sector is very significant in size. While the value of the American housing stock accounts for more than 30% of national wealth, the housing-related expenditure is about one-fourth of the total household spending. Moreover, housing activity can generate large macroeconomic effects, for example, Case, Quigley and Shiller (2005) find rather large effects of housing wealth on household consumption using a panel of 14 developed countries over the period of 1975-1999 and a panel of U.S. states over the period of 1982-1999. Yet, not until the turn of the century, the housing sector has largely been ignored by macroeconomists.[1] Even in this limited literature, to be reviewed below, the model of housing lacks some key features. Rather, housing is often simply modelled as a type of capital or a form of durable goods. Because a house is tied to a plot of land at a specific location usually close to the occupant's workplace, it is locationally immobile and the consumption set of housing is nonconvex.[2]

In this paper, we intend to model housing with care, particularly in some aspects that bridge time and space. In a recent insightful study, Davis and Heathcote (2007) find that properly decomposing a house into housing structure and land enables better understanding of the time series movements and cross location variation in housing prices. We go one step further: by constructing a dynamic general equilibrium model of housing, we are able to model explicitly the evolution of housing structures and the separate role played by land and to take explicitly the households' locational choice decisions into account. Some crucial ingredients are incorporated into our basic framework so as to capture a minimum set of four stylized facts, both over time and across locations, based on the U.S. observations:

- Measured by housing structures plus household durables, the housing durable out-grows housing.
- Housing prices grow at much lower rates than land rents.
- By putting aside urban ghettos, both housing price and land rent gradients are downward-sloping away from urban centers (or subcenters), though the land rent gradient is much steeper.
- In larger MSAs with flatter population gradients, both housing quantity and price gradients are flatter.

[1] See Leung (2004) for a critical survey, documenting clearly such an ignorance in the literature.

[2] While a house in San Francisco and a house in New York are both in the consumption set, a convex combination of a fraction of a house in San Francisco and a fraction of a house in New York is not.

We believe that this model, specifically calibrated to fit all these facts, would serve as a good basis for future research on related issues where housing is an integral part of the analysis.

Specifically, we construct a two-sector optimal growth model with a composite final good sector and a housing sector. The composite final good can be used for consumption or for capital investment. In addition to composite good consumption, housing services also enters the utility function, with two special features. First, we allow housing to have a different income elasticity, dictated by a nonhomothetic preference, than the composite good consumption, and let the data spell out the difference. Second, we allow housing services to be decayed away from the city center to capture spatial discounting as observed in the market. On the supply side, housing is produced by land and housing structures/durables. Similarly, we also allow housing production function to be nonhomothetic, capturing the possibility that there might be a minimum structure required for a house, which is yet again to be determined by calibration.

Both housing structures/durables and the composite good are produced with the use of physical capital. In equilibrium, both goods and land market clear (no vacant land) and no household has incentive to relocate (locational no-arbitrage). We begin by solving the social planner's problem in a tractable manner and then decentralize it by finding supporting prices with location-dependent redistributions (housing taxes/subsidies and redistribution of nonhousing wealth). Upon obtaining the steady-state competitive spatial equilibrium, we derive a basket of analytical comparative statics and then calibrate the theoretical model to fit the average U.S. data over 1960-2000 to further quantify our analysis.

The main analytic findings of our paper are summarized as follows. First, an increase in the housing production technology or in the supply of land raises housing quantity but reduces the relative price of housing. Second, if housing is more luxury than the composite consumption good, which is shown to be the case by calibration, an increase in the consumption good production technology lowers the cost of producing the consumption good and enables reallocation of resources to housing production, thus raising both the quantity and the relative price of housing.

The model, calibrated to fit the four stylized facts, can also deliver additional results that are consistent with other observations. First, a set of comparative statics regarding the housing related quantities and prices fit the observed spatial patterns. For example, housing exhibits much higher cross-location variations than consumption and housing durable schedules; and, a larger MSA with a flatter population gradient is found to have the quantity of housing rising less rapidly away from the CBD and housing and land prices declining less rapidly away from the CBD. Second, along a dynamic path with accumulation of capital and housing durables, the prices of housing durables exhibit a slight downward trend over time, corroborating with findings in the home production literature. Moreover, the housing expenditure ratio exhibits a moderate increase initially and remains largely unchanged afterward, which is again consistent with empirical findings. Finally, as a by-product of our numerical exercises, the computed wealth share of housing, including household durables, is viewed reasonable as well.

An important take-away message of this paper is that the nonhomothetic specifications in the preferences and in the housing production are both essential. With homothetic preferences, our robustness analysis finds spatial distributions of various housing related quantities and prices to be inconsistent with the observations. Similarly, with homothetic housing production function, the responses to demand and supply shifts turn out to be quantitatively too large to be reasonable.

Related Literature

There are two streams of conventional research: one is the durable housing literature in regional science and urban economics and another is the microfinance literature. Because these studies do not focus on macroeconomic issues, we would not discuss the details but simply refer the reader to the survey by Leung (2004).

More recently, there is a small but growing literature of housing that is macro-based. Kan, Kwong and Leung (2004) study the upward trend of residential and commercial property prices and the relative volatility. Davis and Heathcote (2005) examine the movements in housing construction and other related macro aggregates over the business cycle. Ortalo-Magné and Rady (2006) model the trade-up of houses over a household's life cycle facing borrowing constraints. Bajari, Chan, Krueger and Miller (2008) and Flavin and Nakagawa (2008) study housing demand and asset portfolio in a world with income or asset return uncertainty. While Davis and Martin (2008) investigate whether the home production model of housing can explain equity or value premium puzzles, Davis and Ortalo-Magné (2008) examine cross-MSA variation in housing rentals and household wages.

In these papers, housing is introduced with its service entering the utility function either directly (cf. Leung 2001; Kan, Kwong and Leung 2004; Davis and Heathcote 2005; Ortalo-Magné and Rady 2006; Bajari, Chan, Krueger and Miller 2008; Davis and Ortalo-Magné 2008; Flavin and Nakagawa 2008) or indirectly via a consumption aggregator and home production (cf. Davis and Martin 2008). In most studies, housing is produced by capital or labor or a combination of the two. The only exceptions are Leung (2001) and Davis and Heathcote (2005) in which land is considered as an input of new house production.

To see how our paper is situated in the existing literature, we note that none of these aforementioned models incorporates the location-specific feature of land and housing structures, thereby ignoring endogenous locational choice.[3] To account for the key stylized facts regarding spatial distributions, nonhomotheticity, articulated as above, is a key ingredient. In addition, our paper provides a characterization of the dynamic paths of housing as well as housing and land prices, which is largely unexplored in previous studies.

2 The Model

Let the city (or MSA) be situated in a segment of real line, $[-1, 1]$, with location 0 representing the central business district (CBD). Let the land supply be distributed along the real line according to an exogenous density function $\bar{T}(z)$, for $z \in [-1, 1]$, where z indexes a location. We assume $\bar{T}'(z) > 0$ to capture the fact that land is more abundant away from the city center. Moreover, we assume that the land supply at $z = 0$ is positive ($\bar{T}(0) > 0$).

For convenience, the population of agents is assumed constant over time with mass two. Further assume that each agent supplies labor inelastically at $\frac{1}{2}$. Thus, the aggregate labor supply in the economy is one. We will focus on a symmetric equilibrium in which locational choice yields a negative exponential distribution of households over $[-1, 1]$. More specifically,

$$N(z) = \frac{\omega e^{-\omega|z|}}{1 - e^{-\omega}},$$

which is widely supported by empirical evidence (see the original work by Clark 1951 and a comprehensive survey by McDonald 1989). By changing ω, we can analyze various city-economies such as Chicago, New York and Philadelphia.

Our spatial economy has two theaters of production activities: one produces a composite final good and another accumulates housing durables. Production of both of these mobile goods take place at the CBD to which workers commute.

2.1 The Housing Sector

Housing of a representative household at location z is specified as:

$$H_z = T_z^{\gamma} \left(D_z - \theta\right)^{1-\gamma} \tag{1}$$

[3] Berliant, Peng and Wang (2002), Lucas and Rossi-Hansberg (2002), Xie (2008) and Lin, Mai and Wang (2004) allow for endogenous locational choice. However, the first three papers are static, whereas the last paper only considers a unified household capital without separating residential and nonresidential uses. Moreover, all of these studies focus on the issues of urban land use and internal structure of cities, which are very different from ours.

where T_z is the use of land, D_z is the housing structure and household durable component of the house, and θ is introduced to allow for the possibility that a minimum structure might be needed for producing reasonable quantitative results.[4] The Cobb-Douglas form ensures that land and housing structures/durables are Pareto complement in the sense that an increase in one input raises the marginal product of another. In equilibrium, land demand equals supply at each location z,

$$T_z N(z) = \bar{T}(z).$$

The output of housing durable investment at location z is produced with the use of physical capital:

$$X_z = BK_z^{\beta}$$

where $\frac{\dot{B}}{B} = G(t)$ with $G(t) > 0$, $G' < 0$ and $\lim_{t\to\infty} G(t) = 0$ for any z. Abstracting the labor input from the production of housing durable investment is innocuous, as housing durable investment is more capital intensive relative to the composite final good. Although one may easily allow labor to enter this production process while maintaining the factor intensity ranking, labor allocation across locations $z \in [-1, 1]$ would lead to unnecessary complication in the analysis.

The stock of housing durables evolves according to,

$$\dot{D}_z = X_z - \delta D_z = BK_z^{\beta} - \delta D_z \tag{2}$$

where $\delta > 0$ denotes the demolishment rate of housing structure/household durables and $D_z(0) = d \geq \theta$ for any z.

2.2 The Composite Final Good Sector

The composite final goods sector features the following Cobb-Douglas production function:

$$Y = AK_c^{\alpha} L^{1-\alpha} \tag{3}$$

where labor, L, is inelastically supplied at one and A is a constant.

Denote $\delta_k > 0$ as the capital depreciation rate. The output of the composite final goods can then be used for consumption (c_z for those residing in z) or capital investment ($\dot{K} + \delta_k K$), implying:

$$\dot{K} = AK_c^{\alpha} L^{1-\alpha} - \int_{-1}^{1} c_z N(z) dz - \delta_k K, \tag{4}$$

which governs the evolution of capital over time.

[4] Alternatively, we could specify $H_z = (T_z + \theta)^{\gamma} D_z^{1-\gamma}$, with $\theta > 0$, to achieve the same purpose, but this is much less intuitive.

The total stock of capital, K, can be allocated as follows:

$$K = K_c + K_d = K_c + \int_{-1}^{1} K_z N(z) dz \tag{5}$$

where K is equally owned by all the agents and K_d is the aggregate capital stock allocated to the housing sector.

2.3 Preferences

The lifetime utility function of an individual residing at location z is specified as:

$$U_z = \int_0^\infty u(c_z, \phi(z)H_z)e^{-\rho t} dt \tag{6}$$

where $\rho > 0$ is the subjective rate of time preference and $\phi(z)$ is a spatial discounting function capturing the idea that the further away the house is from the CBD, the lower the utility one derives from the house. Part of the reduction in utility may be thought of capturing the detrimental effect from commuting. With spatial discounting, it is not necessary to consider a separate resource cost of commuting, which we assume. Without loss of generality, we normalize $\phi(0) = 1$.

The point-in-time utility function takes the following form:

$$u(c_z, \phi(z)H_z) = c_z^\sigma (\phi(z)H_z + \eta)^{1-\sigma}, \ \sigma \in (0,1) \tag{7}$$

where nonhomotheticity is introduced via parameter η to allow for a different income elasticity of housing than the composite consumption good. If η is positive, which is to be confirmed by calibration, housing is said to be more luxurious in the subsequent discussion than the composite good. Moreover, the Cobb-Douglas form ensures that composite good consumption and housing service ($\phi(z)H_z$) are Pareto complement.[5]

2.4 Locational Choice

Given the ex ante symmetry between all agents, it has to be the case that in equilibrium, $u(c_z, \phi(z)H_z)$ is independent of z. In other words, the following locational no-arbitrage condition holds:

$$u(c_z, \phi(z)H_z) = u(c_0, H_0) \tag{8}$$

Thus, in equilibrium, individual agents feel indifferent in residing in any location.

[5] An alternative to allow housing services to be unnecessary is to use the constant elasticity of substitution form with the elasticity less than one. However, this implies that composite good consumption and housing are Pareto substitutes, which is unrealistic.

3 Equilibrium Analysis

In this section, we solve the optimization problem and then derive the steady-state equilibrium. We begin by solving the central planner's problem instead of solving the competitive equilibrium directly. We then identify a necessary redistribution scheme to support the decentralization of the optimal allocation obtained from the central planner's problem.

3.1 Optimization

For convenience, we define:

$$\Psi_z\left(D_0, D_z\right) \equiv \frac{T_0^{\gamma}\left(D_0-\theta\right)^{1-\gamma}+\eta}{\phi(z) T_z^{\gamma}\left(D_z-\theta\right)^{1-\gamma}+\eta}$$

which is increasing in D_0 but decreasing in D_z, satisfying $\Psi_0\left(D_0, D_0\right)=1$. We can then simplify the central planner's problem by utilizing (7) and (8) to express the locational no-arbitrage condition in forms of final good consumption:

$$c_z=c_0 \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} \tag{9}$$

That is, Ψ_z governs relative composite good consumption across locations.

Using (9), we can write the central planner's problem as:

$$\max \int_0^{\infty} c_0^{\sigma}\left(T_0^{\gamma}\left(D_0-\theta\right)^{1-\gamma}+\eta\right)^{1-\sigma} e^{-\rho t} d t$$

subject to

$$\dot{K}=A\left(K-\int_{-1}^1 K_z N(z) d z\right)^{\alpha} L^{1-\alpha}-\int_{-1}^1 c_0 \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} N(z) d z-\delta_k K \tag{10}$$

$$\dot{D}_z=B K_z^{\beta}-\delta D_z \text{ for all } z \tag{11}$$

This optimization problem can be solved by setting the current-value Hamiltonian,

$$\begin{aligned}
\mathcal{H} = & \max_{c_0, K_z} c_0^{\sigma}\left(T_0^{\gamma}\left(D_0-\theta\right)^{1-\gamma}+\eta\right)^{1-\sigma} \\
& +\lambda\left[A\left(K-\int_{-1}^1 K_z N(z) d z\right)^{\alpha} L^{1-\alpha}-\int_{-1}^1 c_0 \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} N(z) d z-\delta_k K\right] \\
& +\int_{-1}^1 \mu_z\left[B K_z^{\beta}-\delta D_z\right] d z
\end{aligned}$$

where λ and μ_z are co-state variables.

We next define:

$$\Gamma=\int_{-1}^1 \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} N(z) d z \tag{12}$$

which is indeed the endogenous social welfare weight on those residing at location 0.[6] The first-order conditions with respect to c_0 and K_z are:

$$\sigma c_0^{\sigma-1}\left(T_0^{\gamma}\left(D_0-\theta\right)^{1-\gamma}+\eta\right)^{1-\sigma}=\lambda\Gamma \tag{13}$$

$$\beta\mu_z BK_z^{\beta-1}=\alpha\lambda A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha}N(z) \tag{14}$$

While (13) equates the marginal benefit from raising location-0 resident's consumption and the marginal cost from reducing others' consumption, (14) equates the value of marginal product of capital between the two sectors. From (14), we have:

$$K_z=\left(\frac{\mu_zN(0)}{\mu_0N(z)}\right)^{1/(1-\beta)}K_0 \tag{15}$$

That is, the ratio of capital allocated to the housing sector between two locations depends positively on the ratio of the shadow value of housing durables. When the shadow value of housing durables is relatively high at a particular location, it encourages more housing durable investment at that location, thus creating more induced demand for capital input into the production of housing durable investment.

The Euler equations with respect to K and D_z are given by,

$$\begin{aligned}\dot{\lambda} &= (\rho+\delta_k)\lambda-\alpha\lambda A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha}\\ \dot{\mu}_z &= (\rho+\delta)\mu_z-\lambda\left[(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0,D_z\right)^{\frac{1-\sigma}{\sigma}}N(z)\right]\end{aligned}$$

where $\Pi_z\left(D_z\right)\equiv\frac{1}{D_z-\theta}\frac{\phi(z)T_z^{\gamma}(D_z-\theta)^{1-\gamma}}{\phi(z)T_z^{\gamma}(D_z-\theta)^{1-\gamma}+\eta}$ is decreasing in D_z. By rewriting these above expressions using the first-order conditions, (13) and (14), we obtain:

$$\frac{\dot{\lambda}}{\lambda}=(\rho+\delta_k)-\alpha A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha} \tag{16}$$

$$\frac{\dot{\mu}_z}{\mu_z}=(\rho+\delta)-\frac{\beta BK_z^{\beta-1}}{\alpha A\left(K-\int_{-1}^{1}K_zN(z)dz\right)^{\alpha-1}L^{1-\alpha}}(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0,D_z\right)^{\frac{1-\sigma}{\sigma}} \tag{17}$$

The above two expressions govern the shadow price of capital and housing durables, respectively.

[6] This can be easily verified by maximizing the social welfare function given by $\int_{-1}^{1}\Omega_zu(c_z,\phi(z)H_z)dz$, subject to (2) and (4). Applying Negishi (1960), we can compute the social welfare weights consistent with the decentralized equilibrium allocation, yielding: $\Omega_0=\Gamma$.

3.2 Decentralization

We are now ready to find competitive support to the central planner's solution under an appropriate redistribution scheme.

The relative price of housing can be defined as $P_{D_z} = \frac{\mu_z}{\lambda}$. lead to an intertemporal no-arbitrage condition:

$$\frac{\dot{P}_{D_z}}{P_{D_z}} = \alpha A \left(K - \int_{-1}^{1} K_z N(z) dz \right)^{\alpha - 1} L^{1-\alpha} - \left[\frac{\beta B K_z^{\beta-1} (1-\gamma) \frac{1-\sigma}{\sigma} c_0 \Pi_z (D_z) \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}}}{\alpha A \left(K - \int_{-1}^{1} K_z N(z) dz \right)^{\alpha-1} L^{1-\alpha}} - \delta \right] \tag{18}$$

That is, if the net return on capital (first term on the right hand side) exceeds the net return on housing durables, then there must be a capital gain associated with housing durables ($\frac{\dot{P}_{D_z}}{P_{D_z}} > 0$) in order for both sectors to remain operative (see Bond, Wang and Yip 1996). Moreover, since $\Pi_z (D_z)$ and $\Psi_z (D_0, D_z)$ are both decreasing in D_z, it is clear that the rate of capital gain associated with housing durables at a particular location rises with the stock of housing durables but falls with the stock of capital at that location.

From our model, the rental price housing must be equal to the marginal rate of substitution between housing and the composite good,

$$R_{H_z} = \frac{1-\sigma}{\sigma} \frac{\phi(z) c_z}{\phi(z) H_z + \eta}$$

We can then define the price of housing as:

$$P_{H_z} = \frac{R_H}{\rho} = \frac{1}{\rho} \frac{1-\sigma}{\sigma} \frac{\phi(z) c_z}{\phi(z) H_z + \eta} \tag{19}$$

That is, housing price is the capitalization of housing rental. From the specification of housing, the rental price of housing durables is simply its value marginal product given by,

$$R_{D_z} = \frac{(1-\gamma) R_{H_z} H_z}{D_z - \theta}$$

which yields a useful relationship governing the prices of housing durables and housing,

$$R_D D = (1-\gamma) \frac{D}{D-\theta} R_H H$$

The land rent can then be defined based on the bid rent concept,

$$R_{T_z} = \frac{R_{H_z} H_z - R_{D_z} D_z}{T_z}$$

That is, the land rent is the unit surplus of housing rental in excess of housing durable cost.

We claim that these are location-specific supporting prices to the allocation derived from the central planner problem under an appropriate redistribution scheme. Specifically, consider a distribution of the ownership, ν_z, of capital stock, K, together with a housing tax τ_z (subsidy if negative). Let w denote the wage rate and r denote the capital rental rate, which equal the respective marginal products: $w = (1-\alpha)\, AK_c^{\alpha}$ and $r = \alpha AK_c^{\alpha-1}$. Each agent's wealth is measured by,

$$\Omega_z = \nu_z K + P_{H_z} H_z$$

which is the sum of the value of capital and the value of housing per individual. The individual wealth evolves according to,

$$\dot{\Omega}_z = \frac{1}{2}w + (r-\delta_k)\nu_z K - c_z - rK_z - \tau_z P_{H_z} H_z$$

which is equal to wage income (recall that individual labor supply is $\frac{1}{2}$) plus net capital income subtracting consumption expenditure, capital user cost paid for producing housing durable investment and housing tax payment. To satisfy locational no-arbitrage, it must be that $\Omega_z = \Omega_0$ and $\dot{\Omega}_z = \dot{\Omega}_0$ for all z. Using these together with the two redistribution constraints, $\int_{-1}^{1} \tau_z P_{H_z} H_z dz = 0$ and $\int_{-1}^{1} \nu_z N(z) dz = 1$, we can then solve the redistribution pair (τ_z, ν_z) for each location z. This verifies our claim.

3.3 Steady-State Equilibrium

From (16), (12), as well as (10) and (11), we obtain the following three steady-state relationships:

$$K_c = K - \int_{-1}^{1} K_z N(z) dz = \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} \tag{20}$$

$$K_z = \left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} \tag{21}$$

$$K = \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} + \int_{-1}^{1} \left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} N(z) dz \tag{22}$$

$$c_0 = \frac{A\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{\alpha}{1-\alpha}} - \delta_k \left[\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} + \int_{-1}^{1} \left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} N(z) dz\right]}{\int_{-1}^{1} \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} N(z) dz} \tag{23}$$

Clearly, a higher composite good technology or a lower time preference rate raises consumption as well as capital allocated to the composite good sector. Moreover, a higher demolishment rate requires more capital to be allocated to the housing sector to maintain the need for housing services.

The above equations can then be combined with (17) to yield,

$$\frac{\beta B}{\rho+\delta_k}\left(\frac{\delta D_z}{B}\right)^{\frac{\beta-1}{\beta}}\left[(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0,D_z\right)^{\frac{1-\sigma}{\sigma}}\right]=\rho+\delta \tag{24}$$

Notice that, at $z=0$, (24) reduces to an expression for solving uniquely $D_0(c_0)$ which turns out to be an increasing function. This can then be substituted into (24) to derive all housing durables $D_z(c_0)$, which are all increasing in c_0 as well. Next, substituting $D_z(c_0)$ into (23) yields a fixed point mapping in c_0. Once the fixed point of c_0 is obtained, it can then be plugged into $D_z(c_0)$ to solve for D_z for all z, and then into (21), (22) and (9) to solve for K_z, K and c_z. Using (1) and (3), we obtain the steady-state value of housing and the composite good output, H and Y. Finally, we can solve all the supporting prices. In particular, the steady-state capital rental rate is: $r=\rho+\delta_k$. One may also compute the price of housing durables as:

$$P_{D_z}=\frac{\mu}{\lambda}=\frac{\rho+\delta_k}{\beta BK_z^{\beta-1}}$$

It can then be verified that in the steady state the housing durable price satisfies $R_{D_z}=(\rho+\delta)\,P_{D_z}$. Recall that the housing price satisfies $R_H=\rho P_H$. Thus, the capitalization of housing durables and housing differs by the demolishment factor δ. Since both ρ and δ are constant over time and across locations, we can examine the dynamic and spatial patterns of housing and housing durable prices by using their corresponding rental price measures (R_{H_z} and R_{D_z}), which are in comparable units to the land rent.

It may be noted that the involvement of c_0 in all the location-specific variables makes the steady-state equilibrium too complicated to be characterized analytically. In particular, all the preference and technology parameters of interest, (A,B,η,ρ,θ,T), will affect the fixed point of c_0 ambiguously due to their opposing effects on $\Psi_z\left(D_0,D_z\right)$ via $D_0(c_0)$ and $D_z(c_0)$. Thus, we will instead perform comparative-static exercises only under the baseline one-location setup, while conducting the equilibrium characterization of the general model only numerically.

3.4 Characterization of the Steady-State Equilibrium

In order to perform comparative statics in the baseline one-location case, we utilize the "hat calculus" that has been frequently adopted by general equilibrium trade theorists. Denoting $\hat{X}=\frac{\dot{X}}{X}$, we can totally differentiate the key relationships in the baseline one-location setup and manipulate the expressions to derive the fundamental equation governing the changes in the housing quantity in response to changes model parameters (A,B,η,ρ,θ,T):

$$\hat{H}=\xi_A\hat{A}+\xi_B\hat{B}+\xi_\theta\hat{\theta}+\xi_\eta\hat{\eta}+\xi_T\hat{T}+\xi_\rho\hat{\rho}, \tag{25}$$

where the elasticities ξ_i, $i = A, B, \eta, \rho, \theta, T$, can be found in Appendix A. Similarly, we can then obtain the fundamental equation governing the changes in the housing price in response to changes in $(A, B, \eta, \rho, \theta, T)$:

$$\hat{P}_H = \varepsilon_A \hat{A} + \varepsilon_B \hat{B} + \varepsilon_\theta \hat{\theta} + \varepsilon_\eta \hat{\eta} + \varepsilon_T \hat{T} + \varepsilon_\rho \hat{\rho}, \tag{26}$$

where the elasticities ε_i, $i = A, B, \eta, \rho, \theta, T$, are also reported in Appendix A.

Based on these two fundamental equations, we can summarize the comparative static results in the following table:

	A	B	η	ρ	θ	T
Housing Quantity (H)	$+$	$+$	$-$	$-$	$-$	$+$
Housing Price (P_H)	$+$	$-^*$	$-$	?	$+^*$	$-$

Note: * if δ_k small

Among these six parameters, B, θ, and T can be characterized as affecting the supply side, η the demand side, A both the demand and supply side (to be elaborated below), and ρ the intertemporal choice.

Intuitively, an increase in the housing production technology (B) lowers the cost of producing housing, thus raising housing quantity but reducing housing price. The responses of housing quantity and price to an increase in the supply of land are similar. We next examine what happens to an increase in the minimum structure requirement for housing (higher θ). Since such a requirement raises the cost of producing a house, housing price rises while housing supply decreases in response. In response to an increase in the luxury good nature of housing relative to the consumption good (higher η), individual preferences shift away from housing and as a result both housing quantity and housing price are lower. Notably, while an increase in B or T or a decrease in θ capture a prototypical outward shift in housing supply, a decrease in η indicate a prototypical outward shift in housing demand.

Turning now to time discounting (ρ), we can see that more impatience discourages allocation of resources for the future. Since housing requires continual inflows to maintain its adequate service, it falls in response to an increase in time discounting. While such a reduction in housing production tends to raise housing price, the resulting increase in the real interest rate tends to lower housing price. The net effect of impatience on housing price is therefore ambiguous. Notice that in partial equilibrium setups adopted by conventional housing models, rising time discounting would reduce housing price unambiguously.

Finally, an increase in the consumption good production technology (A), in addition to a positive wealth effect (demand effect), lowers the cost of producing the consumption good and increases the relative price of housing. As a consequence, it enables reallocation of resources to housing production and raises the quantity of housing (supply effect). Such an effect only arises in multi-sectoral setups within the general equilibrium framework.

It is noted that equations (25) and (26) are useful not only for deriving comparative statics but also for numerically decomposing changes in the quantity and the price of housing once we have calibrated the model economy, to which we now turn.

4 Quantitative Analysis

We now calibrate the model to fit with the average U.S. data over 1960-2000. We then use the calibrated model to perform various numerical analyses. Additionally, we check the robustness of our main quantitative findings using a gammaville.

4.1 Calibration

Under our theoretical framework, the total population is two. Denote c as the per capita flow of non-housing related consumption good, D as the per capita stock of housing structure plus household durables (called housing durable), X as the per capita output of the housing durables sector and H as housing per capita (all without the location subscript z). We specify the land supply as a simple quadratic function: $\bar{T}(z) = (b + q\,|z|)^2$, where b measures the land supply at the CBD and $q > 0$ reflects increasing land supply away from the CBD. We further specify the spatial discounting function in a linear form given by: $\phi(z) = 1 - a\,|z|$, where a measures the locational discount rate. We normalize $b = 1$ so that the amount of land at the CBD is $\bar{T}(0) = 1$. We then select $a = 0.3$ and $q = 0.1$, under which those at city border discount housing consumption by 30% compared to a resident at the CBD and land supply at city border is 21% more than at the CBD. In computing aggregate variables, the per capita land supply is set as: $T = \int_0^1 (1 + 0.1z)^2 dz = 1.1033$. In the benchmark case, we use Chicago configuration where the negative exponential distribution parameter is given by $\omega = 0.3$ using the estimate in McDonald (1989).

In the macroeconomics literature, the time preference rate is taken to be between 2% and 5%; we thus set $\rho = 0.035$. Also in compliance with the literature, we choose the capital income share as one-third (implying $\alpha = 1/3$). We set the rate of capital depreciation, $\delta_k = 5\%$, a number widely used in the literature. The overall depreciation of housing structure and household durables considered herein includes both demolishment of housing structure and depreciation of household durables. While Greenwood and Hercowitz (1991) uses 7.8% as the depreciation rate for the household structures and equipment, Davis and Heathcote (2005) computes the housing demolishment rate as 1.57%. It is reasonable to assume that the latter accounts for 75% of the overall depreciation, which yields $\delta = 0.0313$.

The calibration analysis is conducted using a simpler version of the model in which there is one location, namely all households are situated in location $z = 0$. By choosing units, we normalize one of the two technological scaling factors by setting $A = 1$. Let $\zeta = \rho D/c$ measure the housing durable flow to non-housing consumption ratio. The capital share of housing sector is denoted by s_K. Further denote the capital-output ratio in the housing durable sector as $\chi = K_d/(2X)$, where $2X$ measures the aggregate output of housing durables. In the steady state, $X = \delta D$, which implies: $K_d = 2\delta\chi\zeta c/\rho$. In the home production literature (e.g. Benhabib, Rogerson and Wright 1991; Greenwood and Hercowitz 1991), the housing consumption flow is regarded as large as non-housing consumption; our ρD is only part of the housing consumption flow, we thus set $\zeta = 0.5$. Since the economy-wide capital-output ratio in the U.S. usually falls in the range from 2 to 3, we set $\chi = 2.25$ as the benchmark. Based on our steady-steady relationships, we can then obtain:

$$\begin{aligned} K_c &= \left(\frac{\alpha A}{\rho + \delta_k}\right)^{\frac{1}{1-\alpha}} = 7.7659 \\ c &= \frac{1}{2}AK_c^{\alpha} - \delta_k K = \frac{AK_c^{\alpha} - \delta_k K_c}{2\left(1 + \frac{\delta_k \delta \chi \zeta}{\rho}\right)} = 0.7579 \end{aligned}$$

Subsequently, the capital stock devoted to the housing durable sector, the housing capital share and the steady-state value of housing durables can be computed as:

$$\begin{aligned} K_d &= \frac{2\delta\chi\zeta c}{\rho} = 1.5250 \\ s_K &= \frac{K_d}{K_d + K_c} = 0.1641 \\ D &= \frac{\zeta c}{\rho} = 10.8268 \end{aligned}$$

That is, about 16.5% of the aggregate capital stock is allocated to producing housing durables.

According to Davis and Ortalo-Magné (2008), the expenditure share of housing is about 24% ($s_H = 0.24$). Over the four decades between 1960 and 2000, we can use the data provided by Davis and Heathcote (2007) to compute housing growth rate at 1.8% ($g_H = 0.018$), the housing structure growth rate at 2.4% ($g_D = 0.024$), the housing structure price growth rate at 0.68% ($g_{R_D} = 0.0068$) and the land price growth rate at 4.33% ($g_{R_T} = 0.0433$). Moreover, the average land value to housing value share is about 36% ($s_T = 0.36$). Using non-durable consumption as a proxy, we compute the non-housing consumption good growth rate as 3% ($g_c = 0.03$).

These ratios and growth rates can then be used to calibrate some key parameters in our model. Recall that, from our model, $R_H = \frac{1-\sigma}{\sigma}\frac{c}{H+\eta}$, $R_D = \frac{(1-\gamma)R_H H}{D-\theta}$ and $R_T = \frac{R_H H - R_D D}{T}$. Assuming fixed land supply over time, we totally differentiate the above three price relationships around the steady state to obtain:

$$\hat{R}_D = \hat{R}_H + \hat{H} - \frac{D}{D-\theta}\hat{D} \tag{27}$$

$$\hat{R}_H = \hat{c} - \frac{H}{H+\eta}\hat{H} \tag{28}$$

$$\hat{R}_T = \frac{R_H H}{R_H H - R_D D}\left(\hat{R}_H + \hat{H}\right) - \frac{R_D D}{R_H H - R_D D}\left(\hat{R}_D + \hat{D}\right)$$

Denote the land value to housing value share as: $s_T = \frac{R_T T}{R_H H}$. Straightforward manipulations lead to,

$$\hat{R}_T = \left(\hat{R}_H + \hat{H}\right) + \frac{(1-\gamma)\frac{D}{D-\theta}\frac{\theta}{D-\theta}}{1-(1-\gamma)\frac{D}{D-\theta}}\hat{D} \tag{29}$$

$$s_T = \frac{R_T T}{R_H H} = 1 - (1-\gamma)\frac{D}{D-\theta} \tag{30}$$

Let the rates of changes of all price and quantity variables capture their respective transitional growth rates, $(g_{R_D}, g_{R_H}, g_{R_T}, g_D, g_H, g_c)$.[7] From (27) and (28), we have:

$$\frac{\theta}{D} = 1 - \frac{g_D}{g_H + g_{R_H} - g_{R_D}} \tag{31}$$

$$\frac{\eta}{H} = \frac{g_H}{g_c - g_{R_H}} - 1 \tag{32}$$

We utilize (30) to write $(1-\gamma)\frac{D}{D-\theta} = 1 - s_T$, which, together with (29) and (31), gives:

$$g_{R_H} = s_T g_{R_T} + (1-s_T)(g_{R_D} + g_D) - g_H = 0.0173$$

We can now use (30) and (31) to compute:

$$\theta = \left(1 - \frac{g_D}{g_H + g_{R_H} - g_{R_D}}\right) D = 1.7095$$

$$\gamma = 1 - \frac{1-s_T}{\frac{D}{D-\theta}} = 0.4611$$

[7] These transitional changes are consequences of transitional changes in $G(t)$. We do not model these changes as permanent because we must otherwise construct specific unbalanced endogenous growth models which often require adding a third sector with two of the three sectors growing at different rates but balancing each other in aggregation (see Kongsamut, Rebelo and Xie 2001, Bond, Trask and Wang 2003 and Acemoglu and Guerrieri 2008). Adding such a sector would make the analysis more difficult without generating further insight over our simple optimal growth structure.

Thus, the minimum structure requirement for housing is indeed present in the data, which is about one-sixth of the amount of housing durables. Applying the functional form of housing given by $H = T^{\gamma}(D-\theta)^{1-\gamma} = 3.4436$ and the land supply schedule, we can then utilize (32) to calibrate:

$$\eta = \left(\frac{g_H}{g_c - g_{R_H}} - 1\right) H = 1.4371$$

which confirms that housing is indeed more luxurious than the composite good.

Finally, from the first-order condition governing consumption and housing demand, we have:

$$s_H = \frac{R_H H}{c + R_H H} = \frac{1}{1 + \frac{\sigma}{1-\sigma}\frac{H+\eta}{H}}$$

which yields,

$$\sigma = \frac{\left(\frac{1}{s_H} - 1\right)\frac{H}{H+\eta}}{1 + \left(\frac{1}{s_H} - 1\right)\frac{H}{H+\eta}} = 0.6908$$

Furthermore, from the steady-state relationship $B(K_d/2)^{\beta} = \delta D$, we can write:

$$B = \frac{\delta D}{(K_d/2)^{\beta}}$$

Substituting this expression into another steady-state relationship,

$$\frac{\beta B}{\rho + \delta_k}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}} (1-\gamma)\frac{1-\sigma}{\sigma} c \frac{1}{D-\theta}\frac{H}{H+\eta} = \rho + \delta,$$

leads to a single equation in β. This gives the calibrated value $\beta = 0.8963$, which can be plugged back into the previous expression to calibrate $B = 0.4321$.

4.2 Numerical Results

We begin by identifying the redistribution scheme (τ_z, ν_z) that is required for equilibrium support. In our benchmark case, such a scheme features imposing taxes on those in inner city $[-0.517, 0.517]$ and providing subsidies to those in outskirts $[-1, -0.517] \cup [0.517, 1]$. The redistributive tax/subsidy schedules over the right half of the city, $[0, 1]$, are plotted in Figure 1 (dashed line). Intuitively, the consideration of locational discounting $\phi(z)$ can be thought of regarding the CBD as a public good whose services decay with distance. Thus, one would expect that those enjoying more of such public good services (in the inner city) would be taxed. Similarly, those who reside in inner city $[-0.563, 0.563]$ would be allocated a share of capital stock lower than average whereas those who in ourskirts $[-1, -0.563] \cup [0.563, 1]$ a share of capital stock higher than average (solid line). More specifically, the tax rate at the center is 0.17% and the subsidy at the fringe is 0.27%. Those at the center holds 49.87% of capital stock per capita and those at the fringe holds 50.28% of capital stock per capita; all very close to the average of 50%. As a by-product of this decentralization exercise, we can compute the wealth share of housing as 58.33%. Based on the 2000 Census, such a share without including household durables is 32.3%. Since our calculation includes the household durables, it is viewed as reasonably consistent with the data.

Using calibrated parameter values, we can further compute 3 quantity and 3 price ratios across locations in the city, plus 3 aggregate shares/ratios, the housing expenditure share (s_H), the housing capital share (s_K) and the ratio of aggregate housing durables to housing (D/H). The results are reported below:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0402	1.2503	0.9956	0.7952	0.6077	0.9995	0.24	0.1641	3.1441

Thus, the quantity of housing at the city fringe is about 25% more than at the CBD (the amount of land is by construction 21% more). While the land rent is about 39% lower, the housing price is only about 20% less at the border compared to the center. In Figure 2, we plot the schedule of each endogenous quantity or price over the right half of the city, $[0, 1]$. As one can see clearly, while housing schedule shows significant cross-location variations, consumption and housing durable schedules are rather flat. Moreover, the land rent schedule is much steeper than the housing rental price schedule, whereas the housing durable rental price schedule is essentially flat. Intuitively, land is entirely immobile while housing durables are fully mobile. It is expected that the greater the degree of mobility is, the less the cross-location variation will be, thereby explaining our results.

We can also compute the housing quantity and price elasticities with respect to various parameter changes, reported in the table below:

	A	B	η	ρ	θ	T
Housing Quantity (H)	0.6213	0.6405	−0.1536	−0.4654	−0.0188	0.5319
Housing Price (P_H)	0.6439	−0.4520	−0.1592	−0.5426	−0.0012	−0.3877

This table coincides well with our theoretical predictions in Section 3 except the housing price elasticity of θ, the parameter of minimum housing structure. This is because with our calibrated δ_k, an increase in θ raises the need for housing structure D, which in turn raises the demand for K_d, and reduces consumption (higher δ_k implies a more significant reduction), with the tendency of lowering housing price: $P_H = (1-\sigma)c/(\sigma\rho(H+\eta))$.

We next turn to conducting comparative-static exercises quantitatively. We are particularly interested in the responses of the above cross-location ratios and the three aggregate shares/ratios to a 10% increase in each of four key preference and technology parameters, η, θ, a and B. Such responses in percentage are reported as follows.

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
η	−0.08	−0.26	−0.39	−0.26	−0.67	−0.05	−3.06	−2.67	−1.02
θ	0.00	−0.01	0.00	0.00	−0.03	0.00	−0.04	1.33	1.57
a	1.31	−0.18	−0.27	−0.18	−0.44	−0.03	−0.57	−0.51	−0.19
B	0.05	0.13	0.20	0.14	0.36	0.02	1.57	0.19	3.57

Thus, when housing becomes more luxurious (higher η), the out-skirt to inner city ratios of consumption, the quantity of housing and housing durables, and the rental prices of land, housing and housing durables are all lower. Intuitively, when housing becomes less necessary, housing demand must fall. In terms of the production of housing, the derived demand for housing durables will also fall, though normally by not as much.[8] Our quantitative results suggest that while housing expenditure and housing capital shares fall sharply, the ratio of aggregate housing durables to housing falls. Among all the cross-location ratios, housing, housing durables, housing rental prices and land rents are more responsive.

An increase in the minimum housing structure requirement (higher θ) has little influence on any of the cross-location ratios (with many of such changes less than 0.005%). In response to this increased minimum requirement, it is necessary to allocate more capital to housing capital to produce the required housing durables (i.e., the housing capital share must increase). As a result, both housing durable prices and housing prices rise, while the land rent falls. The former changes discourage housing demand, thereby lowering the housing expenditure share and raising the housing durables to housing ratio. Our quantitative results suggest that while the housing expenditure share drops negligibly, both the housing capital share and the aggregate housing durables to housing ratio rise sharply.

Except for the effect on the cross-location consumption ratio, the change in spatial discounting generates qualitatively identical effects to the change in the luxury good nature of housing. Intuitively, in response to higher spatial discounting (higher a in the spatial discounting function, $\phi(z)$), agents are less willing to reside at outskirts, thereby reducing housing demand and housing durables demand as well as their prices and the land rent in the outer city. That is, both the ratios of housing and housing durables at the fringe to the center must fall. Our quantitative results suggest that

[8] In trade theory, the finding that changes in output are larger than changes in inputs is usually referred to as the magnification effect in quantity.

the economy-wide housing durables to housing ratio decreases marginally. It is interesting to note that almost all the cross-location ratios (except housing durable prices) are most responsive to this spatial discounting perturbation.

Concerning an increase in the housing durable technology (higher B), all the responses are exactly reverse to an increase in the luxury good nature of housing. Such reversed effects are not surprising as one may view the luxury good nature of housing as a barrier to housing development, thereby having opposite impact to the productivity of housing durables. Because housing durable productivity has a direct positive impact on housing durables, it tends to increase the aggregate housing durables to housing ratio. Our quantitative results show a sharp rise in both the housing expenditure share and the aggregate housing durables to housing ratio in response to an increase in the housing durable technology.

It is noted that in response to any of these parameter changes, land rents are always much more responsive than other rental prices, while housing is relatively less responsive than housing durables.

Finally, we shift our attention to city configurations. Based on the estimates provided by McDonald (1989), we have used the case of Chicago as the benchmark where the negative exponential distribution parameter is $\omega = 0.3$. We now consider two alternative configurations: New York with a flatter population gradient ($\omega = 0.2$) and Philadelphia with a steeper population gradient ($\omega = 0.4$). For comparison purposes, we normalize both cases with population equal to two and landscape over the same unit interval $[-1, 1]$. The results of the key gradients are reported below and illustrated in Figure 3:

ω	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$
0.2	1.0548	1.1927	0.9940	0.8318	0.6698	0.9993
0.4	1.0258	1.3106	0.9972	0.7602	0.5514	0.9997

While both the quantities and prices of mobile goods do not alter much, those of immobile goods vary substantially. In a larger MSA like New York where the population gradient is flatter compared to a smaller MSA like Philadelphia, the housing quantity gradient as well as housing and land price gradients are all flatter, with land prices much more responsive than housing prices.[9] Thus, a larger MSA with a flatter population gradient will have the quantity of housing rising less rapidly away

[9] Due to our normalization of population and city boundaries, the reader is advised not to pay attention to the absolute level but the gradient of these variables depicted in Figure 3. Should New York be allowed to have 4 times as populated as Philadelphia and twice as big in areas, its population density would be uniformly higher than Philadelphia.

from the CBD and housing and land prices declining less rapidly away from the CBD, conforming with real world observations.

5 Transitional Dynamics

We now turn to examining the property of housing related quantities and prices along a dynamic equilibrium path, especially those highlighted in the introduction. Because migration dynamics and the resulting changes in spatial distribution along the transition is beyond the scope of the present paper, it is sufficient to focus on the aggregate measures. As such, we shall move to a simpler version of the model in which there is one location, with all households residing in location $z = 0$. Moreover, we can also afford to assume away the variability of housing productivity by setting B constant ($G(t) \equiv 0$), as the variability is mainly needed in the calibration exercise above.

The dynamics can be captured by the following equations (see derivation in Appendix B):

$$\dot{K} = A(K - F(K, \lambda, \mu))^{\alpha} - 2C(\lambda, D) - \delta_k K \tag{33}$$

$$\dot{\lambda} = (\rho + \delta_k)\lambda - \alpha\lambda A\left(K - F(K, \lambda, \mu)\right)^{\alpha-1} \tag{34}$$

$$\dot{D} = B\left(F(K, \lambda, \mu)/2\right)^{\beta} - \delta D \tag{35}$$

$$\dot{\mu} = (\rho + \delta)\mu - (1 - \gamma)\frac{1 - \sigma}{\sigma}\frac{2\lambda C(\lambda, D)}{D - \theta}\frac{T^{\gamma}\left(D - \theta\right)^{1-\gamma}}{T^{\gamma}\left(D - \theta\right)^{1-\gamma} + \eta} \tag{36}$$

where

$$C(\lambda, D) = \left(T^{\gamma}\left(D - \theta\right)^{1-\gamma} + \eta\right)\left(\frac{2\lambda}{\sigma}\right)^{1/(\sigma-1)}$$

and $K_d = F(K, \lambda, \mu)$ solves

$$K_d = 2\left(\frac{\beta\mu B}{2\alpha\lambda A}\right)^{\frac{1}{1-\beta}}(K - K_d)^{\frac{1-\alpha}{1-\beta}}$$

While $C(\lambda, D)$ is decreasing in λ and increasing in D, $F(K, \lambda, \mu)$ is decreasing in λ and increasing K and μ. The computation of the steady state values of K, λ, D, and μ can also be found in Appendix B.

Based on our calibrated economy, we can apply backward shooting method to this one-location setup to examine the transitional dynamics. Our numerical computations suggest that as the trajectory approaches the steady state, it oscillates in the space of (K, D). The intuition for oscillation can be illustrated using Figure 4 (a close-up near the steady state). Starting at point Q, $D = D^*$ but $K < K^*$, hence it is intuitive that a large fraction of capital would be allocated to the goods sector, implying $K_D < K_D^*$. As a result, $\dot{D} < 0$ at point Q. Since at point Q, the wealth of the representative agent is below that at the steady state, we must have $C_Q < C^*$ and the consumption is small enough to allow for capital accumulation, namely $\dot{K} > 0$ (see equation (33)). Hence, the trajectory from point Q is south-east. At point Q', $K = K^*$ but $D < D^*$, hence it is intuitive that a large fraction of capital would be allocated to durable structure production, namely, $K_D > K_D^*$, which implies that $\dot{D} > 0$ (see equation (35)). Although this means that $K_C < K_C^*$, but $C_{Q'}$ remains below C^*, making it possible for $\dot{K}$ to remain positive.

Of our particular interest, we can identify a transition path along which both K and D increase monotonically until they are close to the steady state (see Figure 5). Specifically, starting from $(K_0, D_0) = (3.2705, 1.7317)$, both K and D increase toward the steady state. As they approach the steady state (indicated by the big dot), an oscillation occurs as depicted in the three graphs in the lower panel of Figure 5: (i) K overshoots and then starts to fall while D continues to rise, (ii) both fall, and (iii) K then rises while D continues to fall. A repetition of such an oscillation continues until the steady state is reached (the close-up figure is not shown as it has already been illustrated in Figure 4). This path is mimicking the transition dynamics in an economy continuing to evolve by accumulating more capital and housing durables.

In addition to capital and housing durables, it is crucial to understand the transitional dynamics of the rental prices of housing, land and housing durables. One can clearly see from Figure 6 that along the transition, land rents (solid line) grow much more sharply (from 0.027 to 0.08) than housing rental prices (long-dashed line, which rises initially from 0.071 from 0.076 and then falls back to 0.07), whereas the rental price schedule of housing durables (short-dashed line) exhibits slight decline over time (from 0.022 to 0.014). This latter finding is consistent with the home production literature, where cheaper household durables enable house wives to substitute out their time for participating in market activities.

Finally, we note that the presence of the luxury good nature of housing results in changes in the housing expenditure ratio over time. In our calibrated economy, this ratio increases moderately from 20.9% to 24% over the first 25 years and remain largely unchanged afterward (see Figure 7). The moderate increase in the ratio is basically consistent with the evidence in the U.S. For example, Rogers (1988) documents that the ratio increased by 2.7% in urban areas and by 1.9% in rural areas from 1972/73 to 1985, whereas Davis and Martin (2008) finds that the ratio increased by 2.3% from 1975 to 1982 and then becomes relatively stable with a slight downward trend through 2007.

6 Alternative Parametrization and Model Specification

In this section, we will perform sensitivity analysis with regard to some parameter selections that are not entirely based on observations. We will also provide further discussion concerning particularly some key ingredients of our model specification.

6.1 Alternative Parametrization

In our calibration analysis, two parameter selections are not entirely based on observations: one is the ratio of housing durables to consumption (ζ, set as 0.5) and another is the housing-sector capital-output ratio (χ, set as 2.25). To check the robustness of our results, we change ζ up and down by 10% from its benchmark value (0.5) and χ from 2 to 2.5 (reasonable range used in the literature when calibrating the model to fit the U.S. data). We find that our main results are robust to all such changes. More specifically, both the dynamic patterns and the cross-locational patterns of our key variables are essentially unchanged. As reported in Appendix C, the only noticeable changes are the economy-wide capital share and housing durables to housing ratio in the steady state. Such changes are expected. When the model is calibrated with a higher housing durables to consumption ratio, both the housing capital share and the housing durable to housing ratio must rise. When the model is calibrated with a higher housing-sector capital-output ratio, the housing capital share must increase.

Our calibrated economy features increasing land supply away from the CBD where the relative supply at the fringe is about 21% more than at the center. In reality, such relative land supplies vary across different MSAs. We thus perform sensitivity analysis with respect to the land expansion rate away from the CBD (q in the land supply schedule, $T(z)$), changing it to 0.25 and 0.35 (deviating from its benchmark value of 0.30). We find that the dynamic patterns of our key variables are largely unchanged. In response to a steeper land expansion rate, all of the aggregate variables are essentially unchanged. Concerning the cross-locational patterns of our key variables, the most noticeable changes are steeper housing schedule and flatter housing price and land rent gradients away from the CBD (see Appendix C), which are not surprising given the increased supply of land toward fringes.

6.2 Alternative Model Specifications

There are three key factors driving some of the main results in the paper. The obvious one is the spatial structure captured by both spatial discounting and increasing land supply away from the CBD. These ensure reasonable housing ratios at the fringe relative to the center as well as a reasonable downward land rent gradient.

In addition, there are two ingredients worth highlighting. One is the luxury good nature of housing relative to the composite good captured by $\eta > 0$; another is the minimum housing structure requirement captured by $\theta > 0$. Although the calibration confirms the presence of the nonhomotheticity in these specifications, it is of interest to check how quantitatively important they are if each of them is assumed away.

6.2.1 Housing Is Not More Luxurious than Consumption

We abandon the luxury good nature of housing relative to the composite good (i.e., set $\eta = 0$), which does not affect any of the calibrated parameters except σ (whose recalibrated value becomes 0.76). The steady-state values of some key ratios are now recalculated below:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0363	1.2760	1.0278	0.8122	0.6403	1.0032	0.24	0.1641	3.1509

The most significant changes are that both the housing durables ratios and the housing durable price ratios at the fringe compared to at the center are now exceeding one. That is, agents residing in outskirts demand for more housing durables at higher prices. In terms of the dynamics, the non-housing consumption growth rate is now given by $g_c = 1.73\%$, much lower than the observed rate of 3%.

We also redo comparative statics, obtain the following results:

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
θ	0.00	0.00	**−0.05**	−0.01	**0.12**	−0.01	0.00	1.28	1.51
a	1.21	**0.59**	**0.92**	**0.60**	**1.51**	**0.10**	**0.00**	**0.00**	−0.01
B	0.00	0.00	0.03	0.00	**−0.11**	0.00	0.00	**−1.16**	3.06

The most significant changes compared to the benchmark case are three folds. First, and perhaps the most undesirable outcome, the responses of housing-related quantity and price variables to a all have wrong signs. Specifically, greater spatial discounting away from the CBD should cause agents to be less willing to reside at outskirts, thereby reducing housing demand and housing durables demand as well as their prices and the land rent. With $\eta = 0$, agents turn out to be more willing to reside away from the CBD despite they have a stronger preference to be closer to the center.[10] This is because that, with $\eta = 0$, $\phi(z)$ becomes a common multiplier to both composite consumption and housing. In this case, adjustments in consumption may dominate the required adjustments in housing. leading to counter-intuitive results in the relative price of housing and the relative demands for housing. Second, the relative technological changes in the housing sector now have essentially no effect on any of the key ratios except the allocation of capital, which is unlikely in the real world. Indeed, the land rent gradient and the housing capital share respond negatively to a positive technology change in the housing sector, apparently counter-intuitive. Finally, although not reported in the table above, the housing expenditure share is entirely flat, not only over time but across locations within the city. The latter result is inconsistent with the U.S. data, where within the MSA variations are observed as documented by Davis and Ortalo-Magne (2008).

In summary, the consideration of the luxury good nature of housing is crucial for producing sensible comparative statics, particularly with respect to changes in locational preferences. It is also useful for obtaining a sharp upward trend in the land rent to housing durable price ratio and for the housing-related variables at different locations to respond differently to sector-specific technological changes.

6.2.2 Housing Requires No Minimum Structure

If we recalibrate the model by removing the minimum housing structure requirement (i.e., set $\theta = 0$), three calibrated parameters would change: $\gamma = 0.36$, $\eta = 13.47$ and $\sigma = 0.4526$. The steady-state values of some key ratios become:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0676	1.0999	0.8806	0.7889	0.5312	0.9854	0.24	0.1641	2.2753

Most significant changes are the large drops in the housing and housing durables ratios as well as the land rent gradient and the housing capital share. Although there is no obvious problem associated with any of these changes, we shall point out the calibrated value of the preference bias parameter η appears unusually large relative to housing services $\phi(z)H_z$: the ratio $\frac{\eta}{\phi(z)H_z}$ ranges from 3.4 to 4.4 (much larger than the benchmark counterparts, 0.48 to 0.55). In terms of the dynamics, the housing durables growth rate is now given by $g_D = 3.65\%$, much higher than the observed rate of 2.4%.

[10] A by-product of this result is that the redistribution scheme for decentralization must now feature a housing tax on suburban residents and a housing subsidy to central-city residents. This redistribution scheme is also unlikely in the real world.

We also redo comparative statics, obtain the following results:

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
η	−0.55	−1.35	−2.10	−1.00	−2.33	−0.24	−10.49	−10.88	−4.15
a	1.45	−2.33	−3.62	−1.74	−4.01	−0.43	−2.02	−2.10	−0.74
B	0.35	0.85	1.33	0.63	1.49	0.15	6.79	7.11	6.18

The outcomes are mixed. On the positive side, there are no wrong signs contradicting to the theory. On the negative side, several changes in response to a 10% increase in relative demand in the inner city (captured by higher a), a 10% decrease in city-wide demand for housing services (captured by an increase in the luxury good nature of housing η) and a 10% increase in city-wide supply (captured by higher B) seem too large quantitatively. For example, the more-than-proportional impacts of a 10% decrease in city-wide demand for housing services on the housing expenditure share and the housing capital share are unlikely to arise in the real world. Moreover, a 10% increase in housing durables production technology results in almost 7% increase in the housing expenditure share and the housing capital share, both very excessive to the reality. Moreover, since housing durables are mobile across locations, one would expect their cross-location ratios in quantities and prices not too responsive to locationally uniform changes (η and B). It is not the case under this model specification: a 10% decrease in city-wide demand for housing services leads to a 2.1% drop in the cross-location housing durables ratio, whereas a 10% increase in city-wide supply generates a 1.3% increase in the cross-location housing durables ratio.

In summary, the consideration of the minimum structure requirement for housing is most useful for creating a buffer that produces more plausible responses with respect to changes in city-wide parameters.

7 Concluding Remarks

We have developed a two-sector dynamic general equilibrium model explicitly accounting for locational choice and several special features of housing. We have shown how housing quantities and prices respond to changes in goods and housing production technologies, the supply of land as well as other preference and technology parameters. The model has been calibrated to fit some important stylized facts, not only over time, but also across locations within an MSA and across various MSAs with different population gradients. In particular, the quantitative results have conformed with the four key observations delineated in the introduction, namely, (i) faster growth of housing structure/household durables than housing, (ii) faster growth of land prices than housing prices, (iii) a locationally steeper land rent gradient than the housing price gradient, and (iv) relatively flatter housing quantity and price gradients in larger cities with flatter population gradients.

We have verified the importance of decomposing the housing structure and the land components as well as of the spatial discounting of housing services. Moreover, we have established the crucial role played by nonhomothetic specifications in household preferences and housing production in generating realistic spatial distributions of various housing related quantities and prices and reasonable responses to autonomous demand and supply shifts. It is thereby our recommendation that the above-mentioned features be incorporated into the model framework, in order to properly account for the aspects of time and space of housing.

Along these lines, perhaps the most important future work is to study the housing sector and its interplays with the non-housing sector over the business cycle. This may be done by introducing stochastic shocks to sector-specific technologies (A and B in our model). Another useful venue of future research is to conduct normative analysis, studying the short-run and long-run effects of housing-related policy on the performance of the housing sector and the macroeconomy as a whole. Such policy may include property taxes and provision of public infrastructure that may affect housing development across different locations (such as highways, public transportation, and public utility).

Appendix

(A major portion of the appendix is not intended for publication)

A. Comparative-Static Analysis

The key relationships in the baseline one-location setup are summarized as follows:

$$
\begin{aligned}
K_c &= \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} \\
K_d &= 2\left(\frac{\delta D}{B}\right)^{\frac{1}{\beta}} \\
c &= \frac{1}{2}AK_c^{\alpha}-\delta_k K=\frac{1}{2}AK_c^{\alpha}-\delta_k\left(K_c+K_d\right) \\
H &= T^{\gamma}\left(D-\theta\right)^{1-\gamma} \\
P_H &= \frac{R_H}{\rho}=\frac{1-\sigma}{\sigma}\frac{1}{\rho}\frac{c}{H+\eta}
\end{aligned}
$$

$$
\frac{\beta B}{\rho+\delta_k}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}\left(1-\gamma\right)\frac{1-\sigma}{\sigma}c\frac{1}{D-\theta}\frac{H}{H+\eta}=\rho+\delta
$$

$$
c=\frac{1}{2}AK_c^{\alpha}-\delta_k\left(K_c+K_d\right)
$$

$$
\begin{aligned}
\hat{c} &= \frac{AK_c^{\alpha}}{2c}\left(\hat{A}+\alpha\hat{K}_c\right)-\frac{\delta_k}{c}\left(K_c\hat{K}_c+K_d\hat{K}_d\right) \\
&= \frac{AK_c^{\alpha}}{2c}\hat{A}+\frac{1}{c}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\frac{1}{1-\alpha}\left(\hat{A}-\frac{\rho}{\rho+\delta_k}\hat{\rho}\right)-\frac{\delta_k K_d}{c}\frac{1}{\beta}\left(\hat{D}-\hat{B}\right) \\
&= \frac{1}{\left(1-\alpha\right)c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}-\frac{1}{\left(1-\alpha\right)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{\rho}-\frac{\delta_k K_d}{\beta c}\left(\hat{D}-\hat{B}\right)
\end{aligned}
$$

Utilizing the hat calculus, we first totally differentiate the above expressions to obtain:

$$
\hat{K}_c = \frac{1}{1-\alpha}\left(\hat{A}-\frac{\rho}{\rho+\delta_k}\hat{\rho}\right) \tag{37}
$$

$$
\hat{K}_d = \frac{1}{\beta}\left(\hat{D}-\hat{B}\right) \tag{38}
$$

$$
\hat{c} = \frac{1}{(1\text{-}\alpha)\,c}\left(\frac{AK_c^{\alpha}}{2}\text{-}\delta_k K_c\right)\hat{A}-\frac{1}{(1\text{-}\alpha)\,c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^{\alpha}}{2}\text{-}\delta_k K_c\right)\hat{\rho}-\frac{\delta_k K_d}{\beta c}\left(\hat{D}\text{-}\hat{B}\right) \tag{39}
$$

$$
\hat{H} = \gamma\hat{T}+\left(1-\gamma\right)\left(\frac{D}{D-\theta}\hat{D}-\frac{\theta}{D-\theta}\hat{\theta}\right)\text{, or,} \tag{40}
$$

$$
\hat{D} = \frac{1}{1-\gamma}\frac{D-\theta}{D}\hat{H}-\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T}+\frac{\theta}{D}\hat{\theta} \tag{41}
$$

$$
\begin{aligned}
\hat{P}_H &= \hat{c}-\frac{H}{H+\eta}\hat{H}-\hat{\rho}-\frac{\eta}{H+\eta}\hat{\eta} \\
&= \frac{1}{(1-\alpha)\,c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}-\left[1+\frac{1}{(1-\alpha)\,c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\right]\hat{\rho} \qquad (42) \\
&\quad +\frac{\delta_k K_d}{\beta c}\left(\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T}-\frac{\theta}{D}\hat{\theta}+\hat{B}\right)-\left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c}+\frac{H}{H+\eta}\right]\hat{H}-\frac{\eta}{H+\eta}\hat{\eta}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(1\text{-}\alpha)\,c}\left(\frac{AK_c^{\alpha}}{2}\text{-}\delta_k K_c\right)\hat{A}+\left(\frac{1}{\beta}+\frac{\delta_k K_d}{\beta c}\right)\hat{B}+\frac{\theta}{D\text{-}\theta}\hat{\theta}+\frac{\eta}{H+\eta}\hat{H}-\frac{\eta}{H+\eta}\hat{\eta} \\
&= \left(\frac{1\text{-}\beta}{\beta}+\frac{D}{D\text{-}\theta}+\frac{\delta_k K_d}{\beta c}\right)\hat{D}+\left[\frac{\rho}{\rho+\delta}+\frac{\rho}{\rho+\delta_k}\left[1+\frac{1}{(1\text{-}\alpha)\,c}\left(\frac{\alpha AK_c^{\alpha}}{2}\text{-}\delta_k K_c\right)\right]\right]\hat{\rho} \qquad (43)
\end{aligned}
$$

Next, substituting (41) into (43) yields,

$$
\begin{aligned}
&\frac{1}{(1-\alpha)\,c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)\hat{A}+\left(\frac{1}{\beta}+\frac{\delta_k K_d}{\beta c}\right)\hat{B}+\frac{\theta}{D-\theta}\hat{\theta}+\frac{\eta}{H+\eta}\hat{H}-\frac{\eta}{H+\eta}\hat{\eta} \\
&= \left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\hat{H}-\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T}+\frac{\theta}{D}\hat{\theta}\right] \\
&\quad +\left[\frac{\rho}{\rho+\delta}+\frac{\rho}{\rho+\delta_k}\left[1+\frac{1}{(1-\alpha)\,c}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\right]\right]\hat{\rho}
\end{aligned}
$$

or, by rearranging terms, we obtain the fundamental equation governing the changes in the housing quantity (25):

$$
\hat{H}=\xi_A\hat{A}+\xi_B\hat{B}+\xi_\theta\hat{\theta}+\xi_\eta\hat{\eta}+\xi_T\hat{T}+\xi_\rho\hat{\rho}
$$

where the elasticities are given by,

$$
\xi_A=\frac{\frac{1}{(1-\alpha)c}\left(\frac{AK_c^{\alpha}}{2}-\delta_k K_c\right)}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}>0
$$

$$
\xi_B=\frac{\frac{1}{\beta}+\frac{\delta_k K_d}{\beta c}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}>0
$$

$$
\xi_\theta=-\frac{\left(\frac{1-\beta}{\beta}+\frac{\delta_k K_d}{\beta c}\right)\frac{\theta}{D}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}<0
$$

$$
\xi_\eta=-\frac{\frac{\eta}{H+\eta}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}<0
$$

$$
\xi_T=\frac{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}>\gamma>0
$$

$$
\xi_\rho=-\frac{\frac{\rho}{\rho+\delta}+\frac{\rho}{\rho+\delta_k}\left[1+\frac{1}{(1-\alpha)c}\left(\frac{\alpha AK_c^{\alpha}}{2}-\delta_k K_c\right)\right]}{\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}+\frac{\delta_k K_d}{\beta c}\right)\frac{1}{1-\gamma}\frac{D-\theta}{D}-\frac{\eta}{H+\eta}}<0
$$

Finally, this latter fundamental equation can then be substituted into (42) to yield the fundamental equation governing the changes in the housing price (26):

$$\begin{aligned}
\hat{P}_H &= \frac{1}{(1-\alpha)c}\left(\frac{AK_c^\alpha}{2} - \delta_k K_c\right)\hat{A} - \left[1 + \frac{1}{(1-\alpha)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^\alpha}{2} - \delta_k K_c\right)\right]\hat{\rho} \\
&\quad -\frac{\eta}{H+\eta}\hat{\eta} + \frac{\delta_k K_d}{\beta c}\left(\frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T} - \frac{\theta}{D}\hat{\theta} + \hat{B}\right) \\
&\quad -\left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\left(\xi_A\hat{A} + \xi_B\hat{B} + \xi_\theta\hat{\theta} + \xi_\eta\hat{\eta} + \xi_T\hat{T} + \xi_\rho\hat{\rho}\right) \\
&= \varepsilon_A\hat{A} + \varepsilon_B\hat{B} + \varepsilon_\theta\hat{\theta} + \varepsilon_\eta\hat{\eta} + \varepsilon_T\hat{T} + \varepsilon_\rho\hat{\rho}
\end{aligned}$$

where the elasticities are given by,

$$\varepsilon_A = \frac{1}{(1-\alpha)c}\left(\frac{AK_c^\alpha}{2} - \delta_k K_c\right) - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_A > 0$$

$$\varepsilon_B = \frac{\delta_k K_d}{\beta c} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_B < 0 \text{ if } \delta_k \text{ small}$$

$$\varepsilon_\theta = -\frac{\delta_k K_d}{\beta c}\frac{\theta}{D} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_\theta > 0 \text{ if } \delta_k \text{ small}$$

$$\varepsilon_\eta = -\frac{\eta}{H+\eta} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_\eta < 0$$

$$\varepsilon_T = \frac{\delta_k K_d}{\beta c}\frac{\gamma}{1-\gamma}\frac{D-\theta}{D} - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_T < 0$$

$$\varepsilon_\rho = -\left[1 + \frac{1}{(1-\alpha)c}\frac{\rho}{\rho+\delta_k}\left(\frac{\alpha AK_c^\alpha}{2} - \delta_k K_c\right)\right] - \left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\frac{\delta_k K_d}{\beta c} + \frac{H}{H+\eta}\right]\xi_\rho$$

B. The Dynamic System with One Location

To make the equilibrium properties consistent on average between this one location model and the multi-location model in the main text, we continue to assume that the population size equals 2 and the land per individual, T, stays the same, which requires:

$$T = \int_0^1 T(z)dz$$

While housing in this one location case is simply $H = T^\gamma (D-\theta)^{1-\gamma}$, the housing durable evolves according to $\dot{D} = B(K_d/2)^\beta - \delta D$ (with $D(0) \geq \theta$). The total labor supply L is assumed to be 1 (i.e., each individual supplies 1/2 unit of labor), so the aggregate capital stock evolves according to

$$\dot{K} = AK_c^\alpha L^{1-\alpha} - 2c - \delta_k K$$

where $K = K_c + K_d$.

Thus, the competitive equilibrium can be derived from solving the central planner's problem as follows:

$$\max \int_0^\infty c^\sigma \left(T^\gamma (D-\theta)^{1-\gamma} + \eta\right)^{1-\sigma} e^{-\rho t} dt$$

$$\text{subject to: } \dot{K} = A(K-K_d)^\alpha L^{1-\alpha} - 2c - \delta_k K \tag{44}$$

$$\dot{D} = B(K_d/2)^\beta - \delta D \tag{45}$$

$$D(0) > \theta$$

The first-order conditions with respect to c and K_d are:

$$\sigma c^{\sigma-1} \left(T^\gamma (D-\theta)^{1-\gamma} + \eta\right)^{1-\sigma} = 2\lambda \tag{46}$$

$$\frac{\beta}{2}\mu B(K_d/2)^{\beta-1} = \alpha\lambda A(K-K_d)^{\alpha-1} L^{1-\alpha} \tag{47}$$

Euler equations with respect to K and D are given by,

$$\dot{\lambda} = (\rho+\delta_k)\lambda - \alpha\lambda A(K-K_d)^{\alpha-1} L^{1-\alpha}$$

$$\dot{\mu} = (\rho+\delta)\mu - (1-\gamma)\frac{1-\sigma}{\sigma}\frac{2\lambda c}{D-\theta}\frac{T^\gamma (D-\theta)^{1-\gamma}}{T^\gamma (D-\theta)^{1-\gamma}+\eta}$$

which can be rewritten using the first-order conditions as:

$$\frac{\dot{\lambda}}{\lambda} = (\rho+\delta_k) - \alpha A(K-K_d)^{\alpha-1} L^{1-\alpha} \tag{48}$$

$$\frac{\dot{\mu}}{\mu} = (\rho+\delta) - (1-\gamma)\frac{1-\sigma}{\sigma}\frac{c}{D-\theta}\frac{\beta B(K_d/2)^{\beta-1}}{\alpha A(K-K_d)^{\alpha-1}L^{1-\alpha}}\frac{T^\gamma (D-\theta)^{1-\gamma}}{T^\gamma (D-\theta)^{1-\gamma}+\eta} \tag{49}$$

From (48) as well as (44) and (45), we obtain:

$$K_c = K - K_d = \left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}} \tag{50}$$

$$K_d = 2\left(\frac{\delta D}{B}\right)^{\frac{1}{\beta}} \tag{51}$$

$$c = \frac{1}{2}AK_c^\alpha - \delta_k K = \frac{AK_c^\alpha - \delta_k K_c}{2\left(1+\frac{\delta_k \delta \chi \zeta}{\rho}\right)} \tag{52}$$

These can then be used together with (49) to yield,

$$\frac{\beta B}{\rho+\delta_k}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1-\gamma)\frac{1-\sigma}{\sigma}\frac{A\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{\alpha}{1-\alpha}} - \delta_k\left(\frac{\alpha A}{\rho+\delta_k}\right)^{\frac{1}{1-\alpha}}}{2\left(1+\frac{\delta_k\delta\chi\zeta}{\rho}\right)}\frac{1}{D-\theta}\frac{T^\gamma (D-\theta)^{1-\gamma}}{T^\gamma (D-\theta)^{1-\gamma}+\eta} = \rho+\delta \tag{53}$$

which solves uniquely D, which can then be plugged into (51) and (50) to solve for K_d and K.

Using (46) and (47), we can write in a recursive manner c as a function of (λ, D) and K_d as a function of (K, λ, μ):

$$c = \left(T^{\gamma}(D-\theta)^{1-\gamma} + \eta\right)\left(\frac{2\lambda}{\sigma}\right)^{1/(\sigma-1)} \equiv C(\lambda, D)$$

$$K_d = 2\left(\frac{\beta\mu B}{2\alpha\lambda A}\right)^{\frac{1}{1-\beta}}(K-K_d)^{\frac{1-\alpha}{1-\beta}}$$

where the latter yields a unique fixed point $K_d = F(K, \lambda, \mu)$. Once we obtain the steady state, we can then solve by backward shooting of the following system of four differential equations given by (33)-(36).

C. Sensitivity Analysis

We consider four sensitivity cases with respect to ζ (housing durable flow to consumption ratio) and χ (housing-sector capital-output ratio), adjusting one parameter each time while keeping another at its benchmark value. We then consider two more cases, adjusting q (land expansion rate away from the CBD) above and below its benchmark value.

	Benchmark	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
ζ	0.5	0.45	0.55	0.5	0.5	0.5	0.5
χ	2.25	2.25	2.25	2.0	2.5	2.25	2.25
q	0.1	0.1	0.1	0.1	0.1	0.05	0.15
η	1.4371	1.3613	1.5089	1.4412	1.4330	1.4051	1.4692
θ	1.7095	1.5460	1.8715	1.7186	1.7004	1.7095	1.7095
B	0.4321	0.4136	0.4433	0.4625	0.3997	0.4321	0.4321
β	0.8963	0.8066	0.9859	0.7967	0.9959	0.8963	0.8963
$\frac{c_1}{c_0}$	1.0402	1.0402	1.0403	1.0402	1.0403	1.0538	1.0274
$\frac{H_1}{H_0}$	1.2503	1.2506	1.2499	1.2507	1.2498	1.1967	1.3037
$\frac{D_1}{D_0}$	0.9956	0.9961	0.9951	0.9961	0.9951	0.9942	0.9969
$\frac{R_{H_1}}{R_{H_0}}$	0.7952	0.7951	0.7953	0.7951	0.7954	0.8293	0.7640
$\frac{R_{T_1}}{R_{T_0}}$	0.6077	0.6079	0.6075	0.6079	0.6075	0.6654	0.5573
$\frac{R_{D_1}}{R_{D_0}}$	0.9995	0.9991	0.9999	0.9990	1.0000	0.9993	0.9996
s_H	0.24	0.24	0.24	0.24	0.24	0.24	0.24
s_K	0.1641	0.1508	0.1769	0.1493	0.1783	0.1641	0.1641
$\frac{D}{H}$	3.1441	3.0016	3.2781	3.1518	3.1364	3.2155	3.0754

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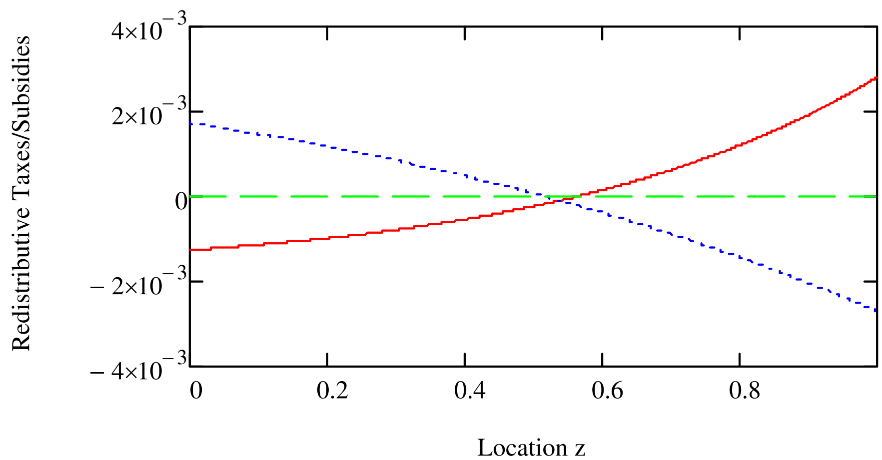

Figure 1. Small Redistributive Measures Needed for Decentralization

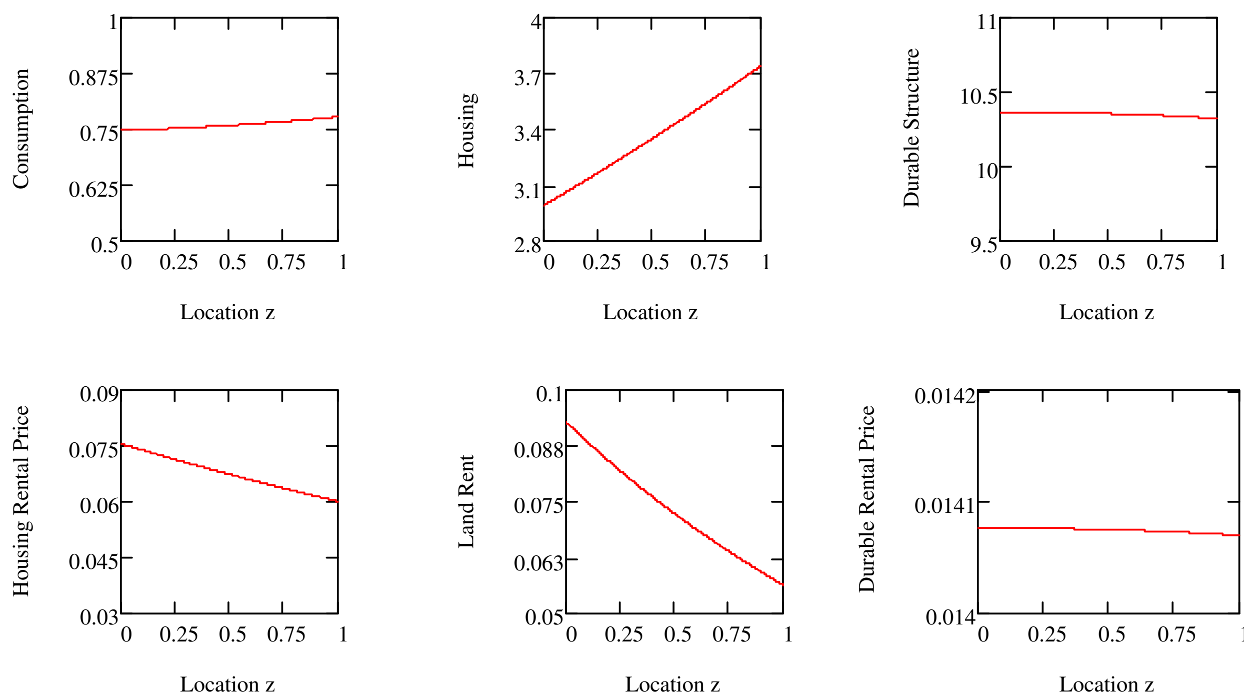

Figure 2. Housing and Land Rent Most Sensitive to Location

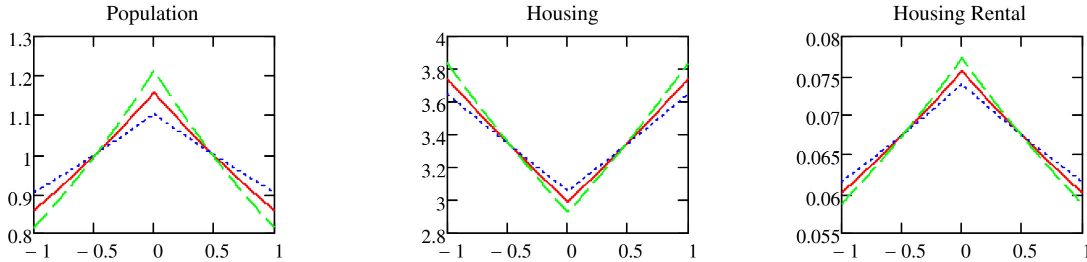

Figure 3. Chicago-solid, NYC-dash, Philly-long dash

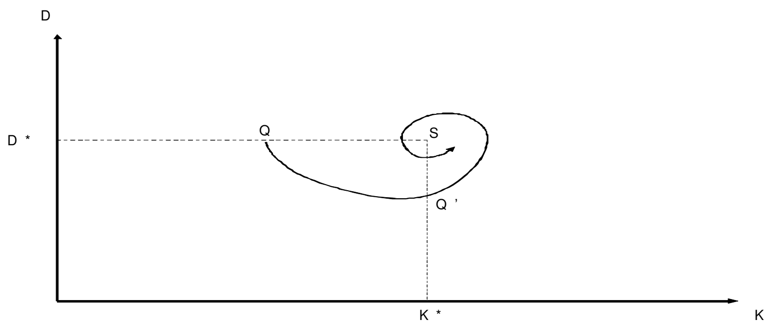

Figure 4. Oscillation Near the Steady State

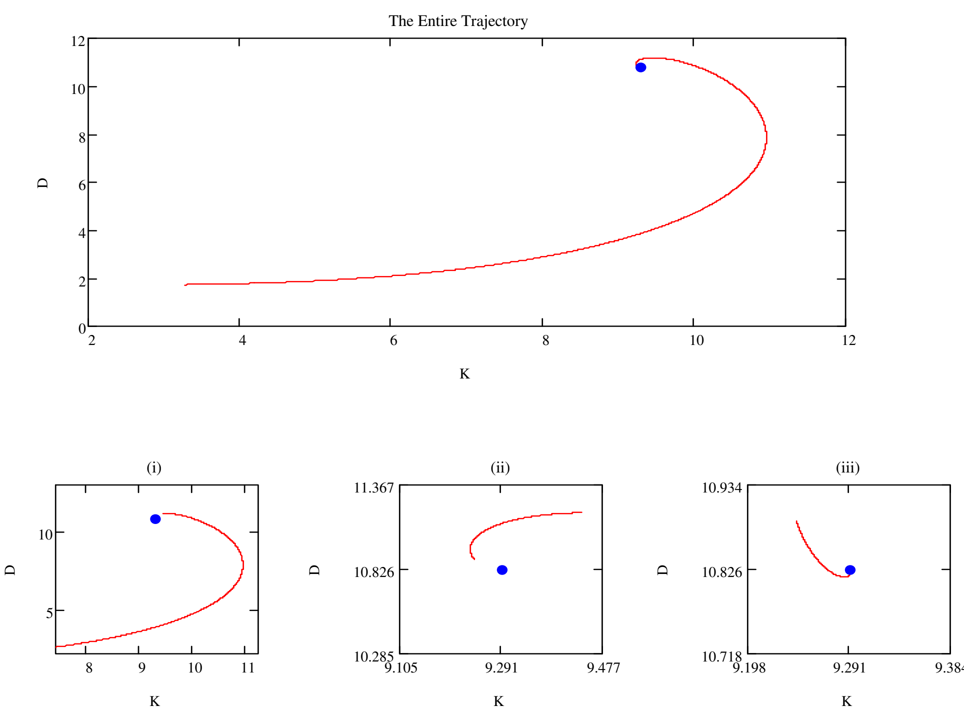

Figure 5. Equilibrium Trajectory

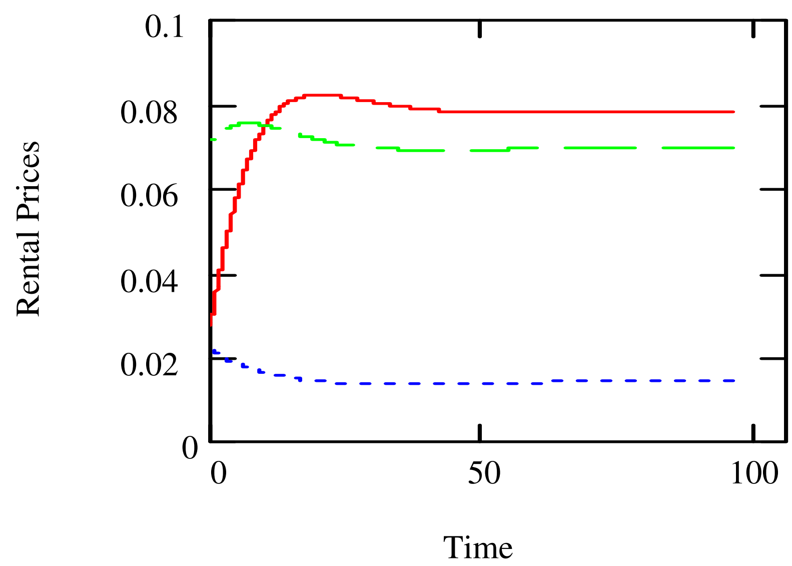

Figure 6. Rental Prices of Housing, Land (solid), and Durables (dash)

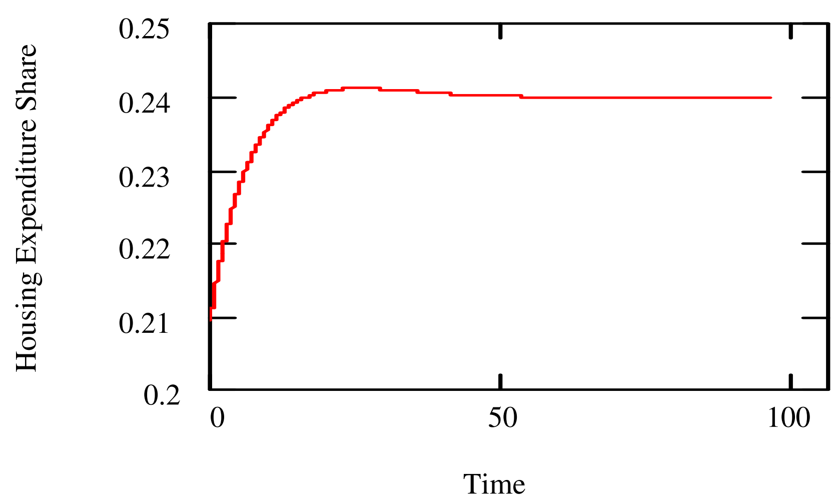

Figure 7. Housing Expenditure Share