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Abstract

We characterize the determinacy properties of the intertemporal equilibrium for a continuous-time, pure-exchange, overlapping generations economy with logarithmic preferences. Using recent advances in the theory of functional differential equations, we show that the equilibrium is locally unique and that prices converge to a balanced growth path and are determined.

Keywords: Overlapping generations models · Local dynamics · Existence · Determinacy · Functional differential equations

JEL Classification Numbers: C61 · C62 · D51 · D91

1 Introduction

In this article, we propose a formal proof of the local determinacy of prices in a continuous-time, overlapping generations exchange economy.[1] It has been shown with models in discrete time that the equilibrium may be indeterminate and that the degree of indeterminacy may increase with the dimension of the model.[2] This is because the number of missing initial conditions increases with either the number of goods that are exchanged or the number of cohorts that participate in the market.[3] Demichelis [13] and Demichelis and Polemarchakis [14] however prove that the discretization of the model may artificially produce indeterminacy. This is shown by building a discrete-time model that is parameterized by the frequency of exchanges and that converges to a model with a continuum of agents. They prove that the degree of indeterminacy increases with the frequency of trade but vanishes at the continuous-time limit. This artificial indeterminacy is due to the basic dynamic equations of the model, whose dimension increases with the frequency of exchanges. In this article, we propose an original analysis of the local properties of the equilibrium of an continuous-time overlapping generations model.

In continuous-time, the equilibrium of an overlapping generations economy is the solution of a system of algebraic equations of mixed type. Until now, few studies have proposed an analytical characterization of such equilibria and, in most cases, they proceed indirectly. Demichelis [13] and Demichelis and Polemarchakis [14] study the limit of the discrete-time counterpart of the continuous-time model. Boucekkine et al. [9] and Edmond [15] perform numerical simulations. Mierau and Turnovsky [30], [31] consider an approximation of the solution with a system of ordinary differential equations. D'Albis and Augeraud-Véron

[1] The first models are due to Allais [6], Samuelson [32], Shell [33], Gale [16] and Balasko and Shell [8]. Continuous-time models were first developed by Cass and Yaari [12] and Tobin [34].

[2] See notably Kehoe and Levine [23], [24], Kehoe *et al.* [26], Geanakoplos and Polemarchakis [18], Ghiglino and Tvede [19] as well as the very nice survey in Geneakoplos [17].

[3] Under some conditions, both dimensions are equivalent: see Balasko *et al.* [7]. In short, a model with many cohorts exchanging one good is equivalent to a model with two cohorts of *heterogeneous* agents exchanging many goods.

[2] use the fact that the dynamics on the stable manifold is given by a delay differential equation. It should also be noted that Burke [11] and Mertens and Rubinshick [28], [29] also analyze the existence and the stability of the equilibrium but within the perspective of the general equilibrium theory.

In this article, we use methods recently developed in Hupkes and Augeraud-Véron [21] and d'Albis et al. [4], [5] to solves the system of algebraic equations of mixed type that characterizes our economy. Using the analytical properties of the model, we give a precise characterization of the dynamics lying on the stable manifold. We compute a threshold value such that all roots with real part smaller than this threshold belong to the stable manifold. Moreover, we analytically prove that for a given initial condition, the equation that characterize the dynamics on the stable manifold has a unique solution. Consequently, we find that despite the infinite dimensions of both the stable and the unstable manifolds, the equilibrium is unique. Locally, there exists a unique trajectory of prices that converge to a balanced growth path (or, depending on the parameters of the model, to a steady-state). The level of equilibrium prices is therefore determined. This property of local uniqueness holds in the neighborhood of a classical steady-state, for which the aggregate assets holdings are equal to zero, or of monetary stead-state, for which assets are positive.

The model we consider is more general than most of those cited above as it contains a general survival function. However, it could be improved by e.g. the inclusion of a general utility function or a production process. For instance, d'Albis and Augeraud-Véron [1] show that the degree of indeterminacy of an overlapping generations model may increase by considering a power utility function. We made our choice in order to start working with a linear equation. Obviously, it is possible to generalize the model, linearize it and apply the method we present below in order to characterize the existence and uniqueness of the equilibrium.

2 The model

We consider an overlapping generations model with continuous trading, which builds on Demichelis [13] and Demichelis and Polemarchakis [14] but with a general survival function. The economy is stationary, the distribution of the fundamentals being invariant with calendar time, and there is one commodity available at each date that cannot be stored or produced.

2.1 Agents

The agents behavior closely follows that described by Yaari [35]. Let $(t, s) \in \mathbb{R}_+^2$ respectively denote the time index and the birth date of an agent. Agents are uncertain about the length of their lifetime; let $l\,(t-s)$ be the survival function to age $t-s$, with $l\,(0) = 1$ and $l\,(\omega) \geq 0$, where $\omega \in \mathbb{R}_+$ is the maximum possible lifetime.[4] Let $\theta\,(t-s)$ be the pure discount factor at age $t-s$, with $\theta\,(0) = 1$ and $\theta\,(\omega) \geq 0$. We assume that l and θ are differentiable. Agents derive satisfaction from their consumption, denoted $c\,(s,t)$, and have no bequest motives. The expected utility at date t of an agent who was born on date $s \leq t$, is:

$$\int_t^{s+\omega} \frac{l\,(z-s)\,\theta\,(z-s)}{l\,(t-s)\,\theta\,(t-s)} \ln c\,(s,z)\,dz. \tag{1}$$

During his lifetime an agent receives a given nonnegative stream of endowment, denoted as $e\,(t-s)$, and which is a differentiable function of age. Agents have access to competitive consumption-loans and complete annuities markets. The intertemporal budget constraint of an agent who was born on date $s \leq t$ as of date t is:

$$\int_t^{s+\omega} l\,(z-s)\,p\,(z)\,c\,(s,z)\,dz = l\,(t-s)\,p\,(t)\,a\,(s,t) + \int_t^{s+\omega} p\,(z)\,d\mu\,(z-s)\,, \tag{2}$$

where $a\,(s,t)$ denotes the asset holdings at age $t-s$, $p\,(t)$ the price at date t, and $d\mu\,(z-s) := l\,(z-s)\,e\,(z-s)\,dz$. Moreover, initial and terminal conditions

[4] The case studied by Demichelis and Polemarchakis [14] is obtained by setting $l\,(x) = 1$ when $x \in [0, \omega]$ and $l\,(x) = 0$ otherwise. We are aware that realistic survival functions are such that $l'\,(x) \leq 0$, but this property is not necessary for the purpose of this article.

can be written as:

$$a(s,s)=0 \text{ and } a(s,s+\omega)\geq 0. \tag{3}$$

The optimization problem at date t of the agent born on s is to maximize (1) subject to (2) and (3). As we are interested in asymptotic properties of the equilibrium, we only consider agents who were born on $s\in\mathbb{R}_+$ and ignore those who were born on $s\in(-\omega,0)$ and are still alive at date $t=0$. For $t\geq s\geq 0$, the optimal consumption path of the considered agent satisfies:

$$p(t)\,c(s,t)=\frac{\int_s^{s+\omega}p(z)\,d\mu(z-s)}{\int_0^{\omega}d\nu(z)}\theta(t-s), \tag{4}$$

where $d\nu(z):=l(z)\,\theta(z)\,dz$. Substituting (4) into (2) gives the optimal asset holdings over the life cycle:

$$\begin{aligned} l(t-s)\,p(t)\,a(s,t) &= \frac{\int_t^{s+\omega}d\nu(z-s)}{\int_0^{\omega}d\nu(z)}\int_s^{s+\omega}p(z)\,d\mu(z-s) \\ &\quad -\int_t^{s+\omega}p(z)\,d\mu(z-s). \end{aligned} \tag{5}$$

2.2 Aggregate variables and equilibrium conditions

It is assumed that new cohorts of identical agents continuously enter the economy and that the age distribution of the population is stationary. Let the constant growth rate of the population be denoted by n. The aggregate per capita counterpart, denoted $x(t)$, of any individual variable $x(s,t)$ is given by:

$$x(t)=\frac{\int_{t-\omega}^{t}e^{-n(t-s)}l(t-s)\,x(s,t)\,ds}{\int_0^{\omega}e^{-nz}l(z)\,dz}. \tag{6}$$

The aggregate per capita endowment is normalized to one, which implies that:

$$\int_0^{\omega}e^{-nz}d\mu(z)=\int_0^{\omega}e^{-nz}l(z)\,dz. \tag{7}$$

At equilibrium, aggregate demand equals the aggregate endowment. Integrating the agents' consumption path given in (4) over birth dates yields the equilibrium condition on the goods market:

$$p(t)=\frac{\int_{t-\omega}^{t}\int_s^{s+\omega}p(z)\,d\mu(z-s)\,e^{-n(t-s)}d\nu(t-s)}{\int_0^{\omega}e^{-nz}d\mu(z)\int_0^{\omega}d\nu(z)} \tag{8}$$

for $t \in [\omega, +\infty)$. Equation (8) is an algebraic equation of mixed type, which means that delays and advances influence the dynamics. By differentiating it with respect to time, one does not obtain ordinary differential equations but mixed-type functional differential equations (MFDE). We notice that limit cases exist when endowments are distributed at the very beginning or end of the life span. By substituting the Dirac measures $d\mu(u) := l(u)\,\delta_0(u)\,du$ or $d\mu(u) := l(u)\,\delta_\omega(u)\,du$ into (8), one can obtain either a delay differential equation (DDE) or an advance differential equation (ADE).[5] In what follows, these two limit cases are omitted.

Aggregate asset holdings, denoted by $a(t)$, can be obtained by integrating individual asset holdings given in (5) over birth dates:

$$\begin{aligned} a(t) &= \int_{t-\omega}^{t} \int_{t}^{s+\omega} d\nu(z-s) \int_{s}^{s+\omega} p(z)\, d\mu(z-s)\, e^{-n(t-s)} ds \\ &\quad - \int_{0}^{\omega} d\nu(z) \int_{t-\omega}^{t} \int_{t}^{s+\omega} p(z)\, d\mu(z-s)\, e^{-n(t-s)} ds \end{aligned} \tag{9}$$

for $t \in [\omega, +\infty)$. We impose a non-negativity condition on aggregate asset holdings, which is written as:

$$a(t) \geq 0. \tag{10}$$

The case $a(t) > 0$ corresponds to a monetary equilibrium whereas the case $a(t) = 0$ is a classical equilibrium such that the money has not value and for which the consumption-loans market is in equilibrium.

3 Prices determinacy at equilibrium

As we are interested in asymptotic properties, we limit our characterization of the equilibrium to $t \in [\omega, +\infty)$. Prices, which are not predetermined, satisfy the following initial condition:

$$p(t) = p_0(t) \text{ for } t \in [0, \omega), \tag{11}$$

[5] Brito and Dilão [10] study similar kinds of endowment distributions. Moreover, Boucekkine *et al.* [9] use a particular set of assumptions in order to obtain a problem characterized by a DDE. In both articles, there is no analytical characterization of the stability.

for $p_0(.) \in \mathcal{C}^b([0,\omega])$ where $\mathcal{C}^b$ denotes the space of continuous functions on $[0,\omega)$ such that $p_0(\omega^-)$ exists. Let us start with a definition.

Definition 1. *An equilibrium is a function $p : C[\omega,+\infty) \to \mathbb{R}_+$ that satisfies the market clearing condition on the goods market (8), the non-negativity condition on aggregate asset holdings (10), and the initial condition (11).*

We first consider the solutions of equation (8) that grow at a constant rate and satisfy condition (10). Those growth rates are the real roots of $\delta(\lambda) = 0$ and $d(\lambda) \geq 0$, where

$$\delta(\lambda) = 1 - \frac{\int_0^\omega e^{-(\lambda+n)z} d\nu(z) \int_0^\omega e^{\lambda z} d\mu(z)}{\int_0^\omega d\nu(z) \int_0^\omega e^{-nz} d\mu(z)} \tag{12}$$

and

$$d(\lambda) = \int_0^\omega e^{-(n+\lambda)s} \left(\frac{\int_s^\omega d\nu(z) \int_0^\omega e^{\lambda z} d\mu(z)}{\int_0^\omega d\nu(z)} - \int_s^\omega e^{\lambda z} d\mu(z) \right) ds. \tag{13}$$

Lemma 1. *There exists a unique real root, denoted $\bar{\lambda}$, that satisfies $\delta(\bar{\lambda}) = d(\bar{\lambda}) = 0$. If $-n < \bar{\lambda}$, there exists a unique real root, namely $-n$, that satisfies $\delta(-n) = 0$ and $d(-n) > 0$.*

<u>Proof.</u> The proof, which is developed in Appendix I, uses the fact that:

$$\frac{\int_0^\omega e^{-nz} d\mu(z)}{(\lambda+n)} \delta(\lambda) = d(\lambda). \ \square \tag{14}$$

Prices that grow asymptotically at rate $\bar{\lambda}$ are compatible with a classical equilibrium with no money. They converge to either a steady-state if $\bar{\lambda} \leq 0$, or to a balanced growth path if $\bar{\lambda} > 0$. In particular, for $\theta(z) := e^{-\rho z}$ and for $e(z)$ constant, one obtains $\bar{\lambda} = -\rho$. The asymptotic growth rate for prices is the opposite of the pure discount rate. Conversely, and provided that $-n < \bar{\lambda}$, prices that grow asymptotically at rate $-n$ are compatible with a monetary equilibrium where aggregate asset holdings are positive. Here, the asymptotic growth rate for prices is the opposite of the growth rate of the population, which always clears the good market but not necessarily the asset market. Notice that we do not consider the particular case given by $\bar{\lambda} = -n$, which corresponds to a coincidental equilibrium.

3.1 Monetary equilibrium

We first study the determinacy of equilibrium prices in the neighborhood of a monetary steady-state where prices grow at rate $-n$. We impose $-n < \bar{\lambda}$ and study equation (8) with Theorem 4 of d'Albis et al. [5]. This theorem provides conditions –based on the localization of characteristic roots– for the existence and uniqueness of a solution to a scalar algebraic equation of mixed type with a non-predetermined variable. In algebraic equations of mixed type and MFDEs, it has been shown[6] that characteristic roots are isolated and that for any line $\text{Re}\,(z) = \eta$ with $\eta \in \mathbb{R}$ in the complex plane, there generically contains infinitely many roots in both half-planes $\{z \in \mathbb{C} : \text{Re}\,(z) < \eta\}$ and $\{z \in \mathbb{C} : \text{Re}\,(z) > \eta\}$. Let us consider all η such that there are no characteristic roots λ that satisfy $\text{Re}(\lambda) = \eta$. Using the theorem mentioned above, we are considering the existence and uniqueness of η-solutions, which are defined as follows:

Definition 2. *An η-solution is an equilibrium that satisfies:*

$$\|p\|_\eta := \sup_{t \in \mathbb{R}_+} e^{-\eta t} \mid p(t) \mid < \infty. \tag{15}$$

Below, we show that a particular η-solution of the problem exists and is uniquely defined. Recall that this characterization is local as is based on the localization of the characteristic roots in the complex plane.

Theorem 1. *Assume that $-n < \bar{\lambda}$. There exists a unique $p : C\,[\omega, +\infty) \to \mathbb{R}$ that satisfies the equilibrium condition on the goods market (8), the initial condition (11), and the growth property $\|p\|_{-n+\varepsilon} < \infty$, for $0 < \varepsilon \ll 1$.*

Proof. The proof proceeds in four steps and relies on Lemmas 2 to 6, which can be found in Appendix I. It shows that assumptions of Theorem 4 (recalled in Appendix II) are satisfied for $\eta = -n + \varepsilon$ and use it to conclude.

Step 1. Assumption 4 of Theorem 4 is satisfied if there exists a function Δ that is the characteristic function of a MFDE and satisfies:

$$(\lambda - \gamma)\,\delta\,(\lambda) = \Delta\,(\lambda)\,, \tag{16}$$

[6] See Hupkes [20] for a presentation of those results.

where $\gamma \in \mathbb{R}$ is such that the equation $\delta(\lambda) = 0$ has no complex root with real part equal to γ. We give the expression of $\Delta(\lambda)$ in Lemma 2.

Step 2. Assumption 1 of Theorem 4 imposes that $\Delta(\lambda) = 0$ should have no complex root such that $\mathrm{Re}(\lambda) = -n+\varepsilon$. In Lemma 3, we show that $\delta(\lambda) = 0$ has no complex root such that $\mathrm{Re}(\lambda) \in \left[\min\left(-n, \bar{\lambda}\right), \max\left(-n, \bar{\lambda}\right)\right]$. We conclude by using (16), choosing $\gamma > \max\left(-n, \bar{\lambda}\right)$ and recalling that the roots of MFDEs are isolated. Moreover, Assumption 2 of Theorem 4 is satisfied, as we have assumed that the measure μ is not is a Dirac function on 0 or ω.

Step 3. In order to compute the invariant integer that permits to conclude, the characteristic function $\Delta(\lambda)$ must be factorized. The factorization always exists (Mallet-Paret and Verduyn Lunel [27]) but it is, in general, difficult to obtain directly. We therefore propose a regular perturbation of the MFDE. We introduce $\Delta_-(\lambda)$ and $\Delta_+(\lambda)$, the characteristic functions of a DDE and of an ADE respectively. These functions satisfy:

$$\Delta_-(\lambda)\,\Delta_+(\lambda) = (\lambda - \gamma)\left[\Delta(\lambda) + \varphi(\lambda)\right], \tag{17}$$

where $\varphi(\lambda)$ is the characteristic function of an MFDE. Lemma 4 gives the expressions of $\Delta_-(\lambda)$, $\Delta_+(\lambda)$ and $\varphi(\lambda)$. Then, we build a continuous path $\Gamma(\mu)$ for $\mu \in [0,1]$ that is associated with an MFDE, the characteristic function of which is

$$\Delta_{\Gamma(\mu)}(\lambda) = \Delta(\lambda) + (1-\mu)\,\varphi(\lambda)\,. \tag{18}$$

Note that (18) satisfies:

$$\Delta_{\Gamma(0)}(\lambda) = (\lambda - \gamma)^{-1}\,\Delta_-(\lambda)\,\Delta_+(\lambda) \text{ and } \Delta_{\Gamma(1)}(\lambda) = \Delta(\lambda)\,. \tag{19}$$

Hence, the path $\Gamma(\mu)$ interpolates between an operator for which a Wiener Hopf factorization is available, namely $\Gamma(0)$, and the operator we are interested in, $\Gamma(1)$. Let us now introduce the integer associated with operator $\Gamma(0)$, denoted $n^{\sharp}_{\Gamma(0)}(-n+\varepsilon)$, which satisfies:

$$n^{\sharp}_{\Gamma(0)}(-n+\varepsilon) = n^{-}_{\Delta_+}(-n+\varepsilon) - n^{+}_{\Delta_-}(-n+\varepsilon) + 1, \tag{20}$$

where $n^{-}_{\Delta_+}(\eta)$ is the number of characteristic roots of $\Delta_+(\lambda) = 0$ such that $\mathrm{Re}(\lambda) < \eta$, $n^{+}_{\Delta_-}(\eta)$ is the number of characteristic roots of $\Delta_-(\lambda) = 0$ such that $\mathrm{Re}(\lambda) > \eta$. In Lemma 5, we show that $n^{\sharp}_{\Gamma(0)}(-n+\varepsilon) = 0$. In order to compute the value of the integer of interest, i.e. the one that is associated with operator $\Gamma(1)$, and that is denoted $n^{\sharp}_{\Gamma(1)}(-n+\varepsilon)$, we characterize the roots of $\Delta_{\Gamma(\mu)}$ as μ goes from 0 to 1, and show in Lemma 6 that $n^{\sharp}_{\Gamma(1)}(-n+\varepsilon) = n^{\sharp}_{\Gamma(0)}(-n+\varepsilon)$. In this latter Lemma we also show that Assumption 3 is satisfied. Using Theorem 4 in d'Albis and al. [5], we conclude. □

Theorem 1 proves the existence of a unique trajectory satisfying the market clearing conditions, the initial condition and an asymptotic property. This trajectory is called a $(-n+\varepsilon)$-solution. An equilibrium is a $(-n+\varepsilon)$-solution but, due to the positivity constraint on prices, the reverse is not necessarily true. Lemma 7 gives a sufficient condition for a $(\bar{\lambda}+\varepsilon)$-solution to be an equilibrium.

Lemma 7. If the initial condition (18) is sufficiently close to the balanced growth path, the $(-n+\varepsilon)$-solution is an equilibrium.
Proof. See Appendix.

As a consequence, in the neighborhood of the monetary steady-state, there exists a unique equilibrium. Equilibrium prices are therefore determined. Due to the complex roots in the spectral decomposition, the trajectory is characterized by exponentially decreasing oscillations.

3.2 Classical equilibrium

Now, we study the determinacy of equilibrium prices in the neighborhood of a classical steady-state where prices grow at rate $\bar{\lambda}$. As for the monetary equilibrium, we are going to study the properties of a particular solution, namely the $(\bar{\lambda}+\varepsilon)$-solution.

Theorem 2. *There exists a unique $p : C[\omega, +\infty) \to \mathbb{R}$ that satisfies the equilibrium condition on the good market (8), the equilibrium condition on the asset* *market* $a(t)=0$*, where* $a(t)$ *is given by (9), the initial condition (11), and the growth property* $\|p\|_{\bar{\lambda}+\varepsilon}<\infty$*, for* $0<\varepsilon\ll 1$*.*

Proof. The proof is similar to the one of Theorem 1 and proceeds in four steps.

Step 1. We first show that a trajectory that satisfies condition $a(t)=0$ also satisfies condition (8). As shown in the proof of Lemma 1, this is due to the fact that the roots of $\delta(\lambda)=0$ encompass those of $d(\lambda)=0$. The only root belonging to the former that doesn't belong to the latter is $-n$. Consequently, it is sufficient to study functions $p:C[\omega,+\infty)\rightarrow\mathbb{R}$ that satisfies the equilibrium condition $a(t)=0$, the initial condition (11), and the growth property $\|p\|_{\bar{\lambda}+\varepsilon}<\infty$. As the considered system is now reduced to one equation, we may apply Theorem 4 of d'Albis et al. [5].

Step 2. The characteristic function of $a(t)=0$ is

$$\frac{d(\lambda)}{\int_0^\omega e^{-nz}d\mu(z)}=0, \tag{21}$$

where $d(\lambda)$ is given in 13. Given equations (14) and (16), one obtains

$$(\lambda-\gamma)\frac{d(\lambda)}{\int_0^\omega e^{-nz}d\mu(z)}=\Delta(\lambda)\frac{1}{(\lambda+n)}. \tag{22}$$

According to Lemma 5.7 of Mallet-Paret and Verduyn Lunel [27], for any γ_0, there exists D, the characteristic function of an MFDE, that satisfies:

$$\Delta(\lambda)\frac{(\lambda-\gamma_0)}{(\lambda+n)}=D(\lambda). \tag{23}$$

We therefore conclude that:

$$(\lambda-\gamma)(\lambda-\gamma_0)\frac{d(\lambda)}{\int_0^\omega e^{-nz}d\mu(z)}=D(\lambda), \tag{24}$$

which is sufficient to prove that Assumption 4 of Theorem 4 is satisfied. We notice that the roots of $D(\lambda)=0$ are those of $\delta(\lambda)=0$ except γ and γ_0 that are roots of $D=0$ only and $-n$ that is the root of $\delta=0$ only. By choosing $\gamma_0>\bar{\lambda}+\varepsilon$ and using Lemma 3, we conclude that Assumption 1 is satisfied. Following the same argument as the one proposed in the proof of Theorem 1, Assumption 2 is also satisfied.

Step 3. Using (23), the characteristic function $D(\lambda)$ can be factorized as in (17). We first compute $n^{\sharp}_{\Gamma(1)}\left(\bar{\lambda}+\varepsilon\right)$, the invariant integer of $\Delta(\lambda)$ for $\eta = \bar{\lambda}+\varepsilon$. In Lemma 8, we show that $n^{\sharp}_{\Gamma(1)}\left(\bar{\lambda}+\varepsilon\right)$ is equal to 0 if $-n > \bar{\lambda}$ and to 1 if $-n < \bar{\lambda}$. Then, we link $n^{\sharp}_{\Gamma(1)}\left(\bar{\lambda}+\varepsilon\right)$ to the invariant integer of $D(\lambda)$, which will be denoted $n^{\sharp}\left(\bar{\lambda}+\varepsilon\right)$. In Lemma 9, we show that $n^{\sharp}\left(\bar{\lambda}+\varepsilon\right) = 0$. Assumption 3 can be show following the argument developed in the proof if Theorem 1. Using Theorem 4 in d'Albis and al. [5], we conclude. □

Theorem 2 is the counterpart of Theorem 1 for a trajectory that converges to the classical steady-state. It is completed with:

Lemma 10. If the initial condition (18) is sufficiently close to the balanced growth path, the $\left(\bar{\lambda}+\varepsilon\right)$-solution is an equilibrium.
Proof. See Appendix.

As a consequence, in the neighborhood of the classical steady-state, there exists a unique equilibrium. As there is no complex roots between the two real roots, the trajectory is characterized by damped oscillations if $-n > \bar{\lambda}$ while it is monotonic if $-n < \bar{\lambda}$.

Theorems 1 and 2 complete the works of Demichelis [13], and Demichelis and Polemarchakis [13] by directly analyzing the continuous-time model rather than the limit of its discrete-time counterpart. In addition to the equilibrium equation of the goods market, we also consider that of the asset market, which explains why we do not obtain an indeterminacy of degree one. Our result further supports that obtained by Kehoe et al. [25] in a discrete-time economy with many goods and two-period agents.

4 Conclusion

In this article, we use methods recently developed in Hupkes and Augeraud-Véron [21] and d'Albis et al. [4], [5] on MFDEs to analyze the local determinacy of a pure exchange overlapping generations economy. We find that the equilibrium is locally unique in the neighborhoods of the monetary or classical steady-states.

It is possible to extend our results using a more general utility function or by considering a production economy by linearizing the equation characterizing the equilibrium. A first step has been made in d'Albis and Augeraud-Véron [1] who studied the spectral decomposition of an exchange economy with instantaneous utility characterized by a power function.

Further work, which is beyond the scope of this paper, might be to characterize the global dynamics of a continuous-time, pure-exchange, overlapping generations economy, accounting for the behavior of agents present at the initial date of the economy as well as the constraint on the non-negativity of prices.

APPENDIX I

Proof of Lemma 1. We first consider the real roots of $\delta(\lambda) = 0$, where $\delta(\lambda)$ is given by (12). As $\delta''(\lambda) < 0$, $\delta(-n) = 0$ and $\lim_{\lambda \to \pm\infty} \delta(\lambda) = -\infty$, we conclude that there exist two real roots: $-n$ and another one that we shall denote $\bar{\lambda}$ that solve $\delta(\lambda) = 0$.

Second, we aim at proving that the roots of $\delta(\lambda) = 0$ encompass those of $d(\lambda) = 0$ and that the only root of the former that doesn't belong to the latter is $-n$. This can be proved by showing that

$$\frac{\int_0^\omega e^{-nz} d\mu(z)}{(\lambda + n)} \delta(\lambda) = d(\lambda). \tag{25}$$

To do so, we make use of (12) in order to rewrite the LHS of (25) as

$$-\int_0^\omega \int_0^u e^{-(n+\lambda)z} dz e^{\lambda u} d\mu(u) + \frac{\int_0^\omega \int_0^s e^{-(n+\lambda)z} dz d\nu(s) \int_0^\omega e^{\lambda z} d\mu(z)}{\int_0^\omega d\nu(z)}, \tag{26}$$

which, by changing the order of integration, is equal to $d(\lambda)$. Then, by applying l'Hospital Rule on the LHS of (25), we obtain that

$$d(-n) = \int_0^\omega e^{-nz} d\mu(z) \delta'(-n), \tag{27}$$

which implies that $d(-n) \geq 0$ if and only if $-n \leq \bar{\lambda}$.

As a consequence, $\bar{\lambda}$ is the only real root that solves $\delta(\lambda) = 0$ and $d(\lambda) = 0$. Moreover, $-n$ solves $\delta(\lambda) = 0$ and satisfies $d(\lambda) > 0$ if $-n < \bar{\lambda}$. □

Lemma 2. *The characteristic function* $\Delta(\lambda)$ *in (16) is:*

$$\begin{aligned} \Delta(\lambda) &= \lambda - \gamma - \frac{\left[e^{-(n+\lambda)\omega} \nu'(\omega) - 1\right] \int_0^\omega e^{\lambda z} d\mu(z)}{\int_0^\omega e^{-nz} d\mu(z) \int_0^\omega d\nu(z)} \\ &\quad - \frac{\int_{-\omega}^0 \int_s^{s+\omega} e^{\lambda z} d\mu(z-s) e^{ns} d\nu'(-s)}{\int_0^\omega e^{-nz} d\mu(z) \int_0^\omega d\nu(z)}. \end{aligned} \tag{28}$$

$\Delta(\lambda) = 0$ *is the characteristic equation of the following MFDE of variable* $y_1(t)$*:*

$$\begin{aligned} y_1'(t) &= \gamma y_1(t) - (\gamma + n) \frac{\int_{t-\omega}^t \int_s^{s+\omega} y_1(z) d\mu(z-s) e^{-n(t-s)} d\nu(t-s)}{\int_0^\omega e^{-nz} d\mu(z) \int_0^\omega d\nu(z)} \\ &+ \frac{\int_{t-\omega}^t \int_s^{s+\omega} y_1(z) d\mu(z-s) e^{-n(t-s)} d\nu'(t-s)}{\int_0^\omega e^{-nz} d\mu(z) \int_0^\omega d\nu(z)} \\ &+ \frac{\int_t^{t+\omega} y_1(z) d\mu(z-t) - \nu'(\omega) \int_{t-\omega}^t y_1(z) d\mu(z-t+\omega)}{\int_0^\omega e^{-nz} d\mu(z) \int_0^\omega d\nu(z)}. \end{aligned} \tag{29}$$

Proof. We start with $\lambda\delta(\lambda)$ where $\delta(\lambda)$ is given in (12). Integrating by parts $\lambda\int_0^\omega e^{-(\lambda+n)z}d\nu(z)\int_0^\omega e^{\lambda z}d\mu(z)$ and rearranging terms gives (28). It corresponds to the MFDE given by (29). □

Lemma 3. *$\delta(\lambda)$ has no complex root such that*

$$\mathrm{Re}(\lambda)\in\left[\min\left(-n,\bar{\lambda}\right),\max\left(-n,\bar{\lambda}\right)\right]. \tag{30}$$

Proof. Let us first show that $\delta(\lambda)$ has no root such that $\min\left(-n,\bar{\lambda}\right)<\mathrm{Re}(\lambda)<\max\left(-n,\bar{\lambda}\right)$. Let $\lambda=p+iq$ be a root of $\delta(\lambda)=0$. Using (12), such a root satisfies:

$$1=\frac{\int_0^\omega e^{-(n+p+iq)z}d\nu(z)\int_0^\omega e^{(p+iq)z}d\mu(z)}{\int_0^\omega d\nu(z)\int_0^\omega e^{-nz}d\mu(z)}. \tag{31}$$

Taking the modulus of both sides of the equality and introducing a majorant on the RHS, we obtain:

$$1<\frac{\int_0^\omega e^{-(n+p)z}d\nu(z)\int_0^\omega e^{pz}d\mu(z)}{\int_0^\omega d\nu(z)\int_0^\omega e^{-nz}d\mu(z)}, \tag{32}$$

which implies that for $p\in\left(\min\left(-n,\bar{\lambda}\right),\max\left(-n,\bar{\lambda}\right)\right)$, $\delta(p)<0$, which is not possible.

Let us now show that there is no root that satisfies $\delta(-n+iq)=0$. We proceed by contradiction and suppose that $-n+iq$ is a root. Then (31) can be rewritten as

$$1=\frac{\int_0^\omega\int_0^\omega e^{-nz}e^{iq(z-u)}d\mu(z)\,d\nu(u)}{\int_0^\omega d\nu\int_0^\omega e^{-nz}d\mu}. \tag{33}$$

Splitting the real and the imaginary parts of (33) and expanding the trigonometric formula gives

$$\begin{cases}\int_0^\omega\cos(qz)\,e^{-nz}d\mu(z)\int_0^\omega\cos(qu)\,d\nu(u)\\ +\int_0^\omega\sin(qu)\,d\nu(u)\int_0^\omega\sin(qz)\,e^{-nz}d\mu(z)=0\\ \\ \int_0^\omega\int_0^\omega(\sin(qz)\cos(qu)-\sin(qu)\cos(qz))\,e^{-nz}d\mu(z)\,d\nu(u)\\ -\int_0^\omega e^{-nz}d\nu\int_0^\omega d\mu(z)=0.\end{cases} \tag{34}$$

Summing the square of left and right members of both expressions gives

$$\begin{aligned}\left(\int_0^\omega d\nu\int_0^\omega e^{-nz}d\mu\right)^2&=\left[\left(\int_0^\omega\sin(qu)\,d\nu(u)\right)^2+\left(\int_0^\omega\cos(qu)\,d\nu(u)\right)^2\right]\\ &\times\left[\left(\int_0^\omega\sin(qz)\,e^{-nz}d\mu(z)\right)^2+\left(\int_0^\omega\cos(qz)\,e^{-nz}d\mu(z)\right)^2\right].\end{aligned} \tag{35}$$

Applying the Cauchy Schwartz inequality to both factors, one has

$$\left(\int_0^\omega d\nu\left(z\right)\int_0^\omega e^{-nz}d\mu\left(z\right)\right)^2 < \int_0^\omega e^{-2nz}d\mu^2\left(z\right)\int_0^\omega d\nu^2\left(u\right), \qquad (36)$$

which is not possible.

The same reasoning can be used to show there is no root that satisfies $\delta\left(\bar{\lambda}+iq\right)=0$. □

Lemma 4. *The characteristic functions* $\Delta_+\left(\lambda\right)$, $\Delta_-\left(\lambda\right)$ *and* $\varphi\left(\lambda\right)$ *in (17) are*

$$\begin{aligned}
\Delta_+\left(\lambda\right) &= \left(\lambda-\gamma\right)\delta_+\left(\lambda\right) \text{ where } \delta_+\left(\lambda\right)=1+\frac{\int_0^\omega e^{\lambda z}d\mu\left(z\right)}{\int_0^\omega e^{-nz}d\mu\left(z\right)}, &(37)\\
\Delta_-\left(\lambda\right) &= \left(\lambda-\gamma\right)\delta_-\left(\lambda\right) \text{ where } \delta_-\left(\lambda\right)=1-\frac{\int_0^\omega e^{-(\lambda+n)z}d\nu\left(z\right)}{\int_0^\omega d\nu\left(z\right)}, &(38)\\
\varphi\left(\lambda\right) &= e^{\lambda\omega}\mu'\left(\omega\right)-\mu'\left(0\right)-\gamma\int_0^\omega e^{\lambda z}d\mu\left(z\right)-\gamma\int_0^\omega e^{\lambda z}d\mu'\left(z\right)\\
&\quad +e^{-(\lambda+n)\omega}\nu'\left(\omega\right)-1-\int_0^\omega e^{-(\lambda+n)z}d\nu'\left(z\right). &(39)
\end{aligned}$$

Proof. Let us consider the two characteristic functions:

$$\Delta_+\left(\lambda\right) = \lambda-\gamma+\frac{e^{\lambda\omega}\mu'\left(\omega\right)-\mu'\left(0\right)-\gamma\int_0^\omega e^{\lambda z}d\mu\left(z\right)-\int_0^\omega e^{\lambda z}d\mu'\left(z\right)}{\int_0^\omega e^{-nz}d\mu\left(z\right)} \qquad (40)$$

and

$$\Delta_-\left(\lambda\right) = \lambda-\gamma-\frac{1-e^{-(n+\lambda)\omega}\nu'\left(\omega\right)+\int_{-\omega}^0 e^{(n+\lambda)z}d\nu'\left(-z\right)}{\int_0^\omega d\nu\left(z\right)}, \qquad (41)$$

which are respectively linked to an ADE of variable y_2 and a DDE of variable y_3, given by:

$$\begin{aligned}
y_2'\left(t\right) &= \gamma y_2\left(t\right)+\frac{y_2\left(t\right)\mu'\left(0\right)-y_2\left(t+\omega\right)\mu'\left(\omega\right)}{\int_0^\omega e^{-nz}d\mu\left(z\right)}\\
&\quad +\frac{\int_t^{t+\omega} y_2\left(z\right)d\mu'\left(z\right)+\gamma\int_t^{t+\omega} y_2\left(z\right)d\mu\left(z\right)}{\int_0^\omega e^{-nz}d\mu\left(z\right)}, &(42)\\
y_3'\left(t\right) &= \gamma y_3\left(t\right)+\frac{y_3\left(t\right)-e^{-n\omega}\nu'\left(\omega\right)y_3\left(t-\omega\right)}{\int_0^\omega d\nu\left(z\right)}\\
&\quad +\frac{\int_{t-\omega}^t e^{-n(t-s)}y_3\left(s\right)d\nu'\left(t-s\right)}{\int_0^\omega d\nu\left(z\right)}. &(43)
\end{aligned}$$

We deduce from (17) that:

$$\begin{aligned}
\varphi\left(\lambda\right) &= \frac{\Delta_{L_{\gamma_-}}\left(\lambda\right)\Delta_{L_{\gamma+}}\left(\lambda\right)}{\left(\lambda-\gamma\right)}-\Delta_{L_\gamma}\left(\lambda\right)\\
&= \left(\lambda-\gamma\right)\left[\frac{\int_0^\omega e^{\lambda z}d\mu\left(z\right)}{\int_0^\omega e^{-nz}d\mu\left(z\right)}-\frac{\int_0^\omega e^{-(\lambda+n)z}d\nu\left(z\right)}{\int_0^\omega d\nu\left(z\right)}\right]. &(44)
\end{aligned}$$

Integrating by parts gives (39), the characteristic equation of the following MFDE on y_4:

$$\begin{aligned} y_4'(t) &= [1+\mu'(0)]\, y_4(t) - \left[\nu'(\omega)\, e^{-n\omega} + \mu'(\omega)\right] y_4(t-\omega) \\ &\quad + \int_0^{\omega} y_4(-s)\, e^{-ns} d\nu'(s) + \gamma \int_t^{t+\omega} e^{\lambda z} d\mu(z-t) \\ &\quad + \int_t^{t+\omega} e^{\lambda z} d\mu'(z-t)\,. \ \square \end{aligned} \tag{45}$$

Lemma 5. *$n^{\sharp}_{\Gamma(0)}(-n-\varepsilon) = -1$ if $-n > \bar{\lambda}$ and $n^{\sharp}_{\Gamma(0)}(-n+\varepsilon) = 0$ if $-n < \bar{\lambda}$.*

Proof. Using definition (70) and given our assumption on γ, we have

$$n^{\sharp}_{\Gamma(0)}(-n \pm \varepsilon) = n^{-}_{\Delta_+}(-n \pm \varepsilon) - n^{+}_{\Delta_-}(-n \pm \varepsilon) + 1. \tag{46}$$

In order to compute $n^{-}_{\Delta_+}$ and $n^{+}_{\Delta_-}$, we analyze the roots of the characteristic equations $\delta_+(\lambda) = 0$ and $\delta_-(\lambda) = 0$ given in (37) and (38).

Let us first analyze the roots of δ_+. As $\delta_+'(\lambda) > 0$ and $\lim_{\lambda \to -\infty} \delta_+(\lambda) = 1$, $\delta_+(\lambda) = 0$ has no real root. Moreover, $\delta_+(\lambda) = 0$ has no complex root with real part less than or equal to $-n$. Let us suppose the contrary: that there exists a $p + iq$ such that $\delta_+(p+iq) = 0$ and $p < -n$. Then the roots satisfy:

$$\int_0^{\omega} e^{-nz} d\mu(z) = -\int_0^{\omega} e^{\lambda z} d\mu(z)\,. \tag{47}$$

Taking the modulus of each expression gives:

$$\int_0^{\omega} e^{-nz} d\mu(z) = \left| \int_0^{\omega} e^{-nz} d\mu(z) \right| < \int_0^{\omega} e^{pz} d\mu(z)\,. \tag{48}$$

As the RHS of the above inequality increases with p and is equal to $\int_0^{\omega} e^{-nz} d\mu(z)$ for $p = -n$, we conclude there is no root such that $p < -n$. Moreover, $\lambda = -n + iq$ is not a root of $\delta_+(\lambda) = 0$. If it were, one would have

$$\int_0^{\omega} e^{-nz} d\mu(z) + \int_0^{\omega} e^{-nz} e^{iqz} d\mu(z) = 0. \tag{49}$$

Taking the real parts of the above equality gives

$$\int_0^{\omega} e^{-nz} (1 + \cos(qz))\, d\mu(z) = 0, \tag{50}$$

which is not possible as the integrand is positive and not null everywhere.

Let us now consider the roots of $\delta_-(\lambda) = 0$. As $\delta'_-(\lambda) > 0$ and $\delta_-(-n) = 0$, $\delta_-(\lambda) = 0$ has a unique real root, namely $-n$. Moreover, $\delta_-(\lambda) = 0$ has no complex root with real part greater than or equal to $-n$. Let us suppose, on the contrary, that there exists a $p + iq$ such that $\delta_-(p + iq) = 0$ and $p > -n$. Then:

$$\left| \frac{\int_0^\omega \int_0^\omega e^{-(p+n)z} \cos(qz)\, d\nu(z)}{\int_0^\omega d\nu(z)} \right| < \frac{\int_0^\omega e^{-(p+n)z} d\nu(z)}{\int_0^\omega d\nu(z)} < 1, \tag{51}$$

which is not possible. Moreover, $\lambda = -n + iq$ is not a root of $\delta_-(\lambda) = 0$. If it were, we would have

$$\int_0^\omega (1 - \cos(qz))\, dv(z) = 0, \tag{52}$$

which is not possible.

If $\bar{\lambda} < -n$, $n^-_{\Delta_+}(-n-\varepsilon) = 0$, and $n^+_{\Delta_-}(-n-\varepsilon) = 2$, (the two misplaced roots of $\Delta_+(\lambda)$ are $\lambda = -n$ and $\lambda = \gamma$). This implies (using formula 20) that $n^\sharp_{\Gamma(0)}(-n-\varepsilon) = -1$. Alternatively, if $\bar{\lambda} > -n$, $n^-_{\Delta_+}(-n+\varepsilon) = 0$, and $n^+_{\Delta_-}(-n+\varepsilon) = 1$ (which is due to root $\lambda = \gamma$), which is given by $-n$. This implies that $n^\sharp_{\Gamma(0)}(-n+\varepsilon) = 0$. □

Lemma 6. $n^\sharp_{\Gamma(1)}(-n-\varepsilon) = n^\sharp_{\Gamma(0)}(-n-\varepsilon)+1$ *if* $-n > \bar{\lambda}$ *and* $n^\sharp_{\Gamma(1)}(-n+\varepsilon) = n^\sharp_{\Gamma(0)}(-n+\varepsilon)$ *if* $-n < \bar{\lambda}$.

We relate $n^\sharp_{\Gamma(0)}(-n \pm \varepsilon)$ to $n^\sharp_{\Gamma(1)}(-n \pm \varepsilon)$ by characterizing the roots of $\Delta_{\Gamma(\mu)}$ as μ goes from 0 to 1. The objective is to show how roots cross the line $\{z \in \mathbb{C}, \mathrm{Re}(z) = -n\}$ as μ changes. Let us consider:

$$\delta(\lambda, \mu) = \delta(\lambda) - (1-\mu)\left[\frac{\int_0^\omega e^{-(\lambda+n)z} d\nu(z)}{\int_0^\omega d\nu(z)} - \frac{\int_0^\omega e^{\lambda z} d\mu(z)}{\int_0^\omega e^{-nz} d\mu(z)}\right], \tag{53}$$

where $\delta(\lambda)$ is given in (12).

Step 1. We prove that roots cannot cross the set $\{\lambda \in \mathbb{C} \diagdown \{-n\}, \mathrm{Re}(\lambda) = -n\}$. To do so, we show that $-n + iq$, with $q > 0$, is not a root of $\delta(\lambda, \mu) = 0$ $\forall \mu \in (0, 1)$. We have

$$\begin{aligned} \delta(-n+iq, \mu) &= 1 - \frac{\int_0^\omega e^{-iqz} d\nu(z) \int_0^\omega e^{(-n+iq)z} d\mu(z)}{\int_0^\omega d\nu(z) \int_0^\omega e^{-nz} d\mu(z)} - (1-\mu) \\ &\quad \times \left[\frac{\int_0^\omega e^{-iqz} d\nu(z)}{\int_0^\omega d\nu(z)} - \frac{\int_0^\omega e^{(-n+iq)z} d\mu(z)}{\int_0^\omega e^{-nz} d\mu(z)}\right]. \end{aligned} \tag{54}$$

Suppose that $\delta(-n+iq,\mu)=0$. The following equality should hold:

$$
\begin{aligned}
& \int_0^\omega d\nu(z) \int_0^\omega e^{-nz} d\mu(z)\, \delta(-n+iq,\mu) \\
= & \int_0^\omega e^{-nz} d\mu(z) \int_0^\omega \left(1-(1-\mu)\, e^{-iqz}\right) d\nu(z) \\
& - \int_0^\omega \left(e^{-iqz}-(1-\mu)\right) d\nu(z) \int_0^\omega e^{(-n+iq)z} d\mu(z)\,. \qquad (55)
\end{aligned}
$$

But the LHS of (55) is greater than:

$$
\begin{aligned}
& \left|\left|\int_0^\omega e^{-nz} d\mu(z)\right| \left|\int_0^\omega e^{-iqz}\left(e^{iqz}-(1-\mu)\right) d\nu(z)\right|\right| \qquad (56) \\
& - \left|\left|\int_0^\omega \left(e^{-iqz}-(1-\mu)\right) d\nu(z)\right|\right|. \qquad (57)
\end{aligned}
$$

To conclude, we have to show that:

$$
\left|\int_0^\omega e^{-iqz}\left(e^{iqz}-(1-\mu)\right) d\nu(z)\right| \neq \left|\int_0^\omega \left(e^{-iqz}-(1-\mu)\right) d\nu(z)\right|, \qquad (58)
$$

for all $\mu \in (0,1)$. Suppose this is not true and that the equality holds. This implies

$$
\begin{aligned}
& \left|\int_0^\omega (1-(1-\mu)\cos(qz))\, d\nu(z) + i(1-\mu)\int_0^\omega \sin(qz)\, d\nu(z)\right| \\
= & \left|\int_0^\omega (\cos(qz)-(1-\mu))\, d\nu(z) - i\int_0^\omega \sin(qz)\, d\nu(z)\right|,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
& \left(\left(\int_0^\omega (1-(1-\mu)\cos(qz))\, d\nu(z)\right)^2 + (1-\mu)\left(\int_0^\omega \sin(qz)\, d\nu(z)\right)^2\right)^{1/2} \\
= & \left(\left(\int_0^\omega (\cos(qz)-(1-\mu))\, d\nu(z)\right)^2 + \left(\int_0^\omega \sin(qz)\, d\nu(z)\right)^2\right)^{1/2}, \qquad (59)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
& (2-\mu)\int_0^\omega (1-\cos(qz))\, d\nu(z) \int_0^\omega (1+\cos(qz))\, d\nu(z) \\
= & \left(\int_0^\omega \sin(qz)\, d\nu(z)\right)^2. \qquad (60)
\end{aligned}
$$

Since the LHS is smaller than $\int_0^\omega \sin(qz)\, d\nu(z)$, rearranging the RHS implies that $(2-\mu)^{1/2} < 1$, which is not possible since $\mu \in (0,1)$.

Step 2. We show how roots cross at point $\lambda = -n$. Let us compute:

$$\begin{aligned}\frac{\delta(\lambda,\mu)}{\lambda+n} &= \frac{\delta(\lambda)}{\lambda+n} + (1-\mu)\left[\frac{\int_0^\omega \int_0^z e^{-(\lambda+n)u} du d\nu(z)}{\int_0^\omega d\nu(z)}\right. \\ &\quad \left. + \frac{\int_0^\omega \int_0^z e^{(\lambda+n)u} du e^{-nz} d\mu(z)}{\int_0^\omega e^{-nz} d\mu(z)}\right]. \qquad (61)\end{aligned}$$

First consider the case where $\bar{\lambda} > -n$. As $\delta'(-n) > 0$, $\lim_{\lambda\to -n} \delta(\lambda,\mu)/(\lambda+n) > 0$, which implies that $-n$ cannot be a double root. Now consider the case where $\bar{\lambda} < -n$. As $\delta'(-n) < 0$, the sign of $\lim_{\lambda\to -n} \delta(\lambda,\mu)/(\lambda+n)$ is ambiguous. But, using the results established in Lemma 5, one has $\lim_{\lambda\to -n} \delta(\lambda,0)/(\lambda+n) > 0$ and, using the fact that $\delta'(-n) < 0$, one has $\lim_{\lambda\to -n} \delta(\lambda,1)/(\lambda+n) < 0$. As $\delta(\lambda,0)/(\lambda+n)$ is linear in μ, we conclude that there exists a $\mu_1 \in (0,1)$ such that $\lim_{\lambda\to -n} \delta(\lambda,\mu_1)/(\lambda+n) = 0$.

Step 3. Finally, we prove that the root is crossing from right to left at $\lambda = -n$, which increase the value of the invariant integer. This is done by showing

$$\lim_{\lambda\to -n} \frac{\partial\left(\frac{\delta(\lambda,\mu)}{\lambda+n}\right)}{\partial\lambda} < 0 \text{ for } \mu > \mu_1, \qquad (62)$$

which can be done by computing:

$$\lim_{\lambda\to -n} \frac{\partial}{\partial\lambda}\delta(\lambda,\mu) = -(\mu-\mu_1)\left[\frac{\int_0^\omega z d\nu(z)}{\int_0^\omega d\nu(z)} + \frac{\int_0^\omega z e^{-nz} d\mu(z)}{\int_0^\omega e^{-nz} d\mu(z)}\right]. \square \qquad (63)$$

<u>Proof of Lemma 7.</u> We proceed in two steps. First, we claim that there exist no η-solutions such that $\eta < -n$. This claim is an immediate implication of Lemma 4.1 in Hupkes and Augeraud-Véron [21]. Second, we claim that η-solutions with $\eta > -n + \varepsilon$ imply $p(t) < 0$ for some t. This is due to the presence of complex roots with real parts greater than $-n$ in the solution. □

Lemma 8. $n^\sharp_{\Gamma(1)}(\bar{\lambda}+\varepsilon) = 0$ *if* $-n > \bar{\lambda}$ and $n^\sharp_{\Gamma(1)}(\bar{\lambda}+\varepsilon) = 1$ *if* $-n < \bar{\lambda}$.

<u>Proof.</u> The result is an immediate consequence of Lemma 3 that says that $\delta(\lambda)$ has no root in $\left[\min(-n,\bar{\lambda}), \max(-n,\bar{\lambda})\right]$ except at $-n$ and $\bar{\lambda}$, and which permit to conclude that $n^\sharp_{\Gamma(1)}(\bar{\lambda}+\varepsilon) = n^\sharp_{\Gamma(1)}(-n-\varepsilon)$ if $-n > \bar{\lambda}$ and that

$n^{\sharp}_{\Gamma(1)}\left(\bar{\lambda}+\varepsilon\right) = n^{\sharp}_{\Gamma(1)}\left(-n+\varepsilon\right)+1$ if $-n<\bar{\lambda}$. Moreover, using Lemmas 5 and 6 we have that $n^{\sharp}_{\Gamma(1)}\left(-n-\varepsilon\right)=0$ if $-n>\bar{\lambda}$ and $n^{\sharp}_{\Gamma(1)}\left(-n+\varepsilon\right)=1$ if $-n<\bar{\lambda}$, the conclusion follows. □

Lemma 9. $n^{\sharp}\left(\bar{\lambda}+\varepsilon\right)=0$.

Proof. Let us denote the characteristic function $D\left(\lambda\right)$ as

$$D\left(\lambda;\gamma_0,\gamma\right) := \frac{\lambda-\gamma_0}{\lambda+n}\Delta\left(\lambda\right), \tag{64}$$

and the invariant integer as $n^{\sharp}\left(\bar{\lambda}+\varepsilon;\gamma_0,\gamma\right)$. We aim at computing $n^{\sharp}\left(\bar{\lambda}+\varepsilon;\gamma_0,\gamma\right)$ for $\gamma_0>\bar{\lambda}+\varepsilon$. Let us first consider the case $-n<\bar{\lambda}$ and then the case $-n>\bar{\lambda}$.

Case $-n>\bar{\lambda}$. γ_0 can be chosen such that $\gamma_0=-n$. Thus, $D\left(\lambda;\gamma_0,\gamma\right)=\Delta\left(\lambda\right)$ and $n^{\sharp}\left(\bar{\lambda}+\varepsilon\right)=n^{\sharp}_{\Gamma(1)}\left(\bar{\lambda}+\varepsilon\right)$. We use Lemma 6 to conclude.

Case $-n<\bar{\lambda}$. The definitions of $\Delta\left(\lambda;\gamma_0,\gamma\right)$, $d\left(\lambda\right)$ and $\Delta\left(\lambda\right)$ imply that

$$D\left(\lambda;\gamma_0,\gamma\right)=\left(\lambda-\gamma_0\right)\left(\lambda-\gamma\right)\frac{d\left(\lambda\right)}{\int_0^{\omega}e^{-nz}d\mu\left(z\right)}. \tag{65}$$

We have already assumed that $\gamma>\max\left(\bar{\lambda},-n\right)$. Let us now consider the case where $\gamma>\bar{\lambda}+2\varepsilon$ and $\varepsilon>0$. Our objective is to compute $n^{\sharp}\left(\bar{\lambda}+\varepsilon;\gamma_0,\gamma\right)$ for $\gamma_0\in\left[-n,\bar{\lambda}+2\varepsilon\right]$. First, we know that $n^{\sharp}\left(\bar{\lambda}+\varepsilon;-n,\gamma\right)=n^{\sharp}_{\Gamma(1)}\left(\bar{\lambda}+\varepsilon\right)$. Moreover, we can use the relationship 2.20 in Theorem 2.5 of Hupkes and E. Augeraud-Véron [21] to state that

$$n^{\sharp}\left(\bar{\lambda}+\varepsilon;\bar{\lambda}+2\varepsilon,\gamma\right)-n^{\sharp}\left(\bar{\lambda}+\varepsilon;-n,\gamma\right)=-\mathrm{cross}\left(\Gamma\left(0\right),\bar{\lambda}+\varepsilon\right), \tag{66}$$

where the crossing number $\mathrm{cross}\left(\Gamma\left(0\right),\bar{\lambda}+\varepsilon\right)$ denotes the number of roots of $D\left(\lambda;\gamma_0,\gamma\right)=0$, counted with multiplicity, that cross the line $\mathrm{Re}\left(z\right)=\bar{\lambda}+\varepsilon$ from left to right as γ_0 increases from $-n$ to $\bar{\lambda}+2\varepsilon$. Using (65), we know that $\lambda=\bar{\lambda}-\varepsilon$ is the only root of $D\left(\lambda;\gamma_0,\gamma\right)=0$ that crosses $\mathrm{Re}\left(z\right)=\bar{\lambda}+\varepsilon$ for the considered values of γ_0. As a consequence, $\mathrm{cross}\left(\Gamma\left(0\right),\bar{\lambda}+\varepsilon\right)=1$, which implies that $n^{\sharp}\left(\bar{\lambda}+\varepsilon;\bar{\lambda}+2\varepsilon,\gamma\right)=0$. □

Proof of Lemma 10. The same argument as the one proposed in the proof of Lemma 7 can be used, except that we compare η to $\bar{\lambda}+\varepsilon$. □

APPENDIX II

Presentation of Theorem 4 (d'Albis et al. [5])

Let $t \in \mathbb{R}_+$ be the time index. The problem is

$$\begin{cases} \int_{-a}^{b} x(u+t)\, d\nu(u) = \int_{-a}^{b} x(u+t)\, d\mu(u), \\ x(\xi) = x_0(\xi) \text{ for } \xi \in [-a, 0). \end{cases} \tag{67}$$

where $(a, b) \in \mathbb{R}_+^2$ and where μ and ν are measures on $[-a, b]$. Algebraic equations that are considered satisfy the following assumption.

Assumption 4. *There exists $n \in \mathbb{N}_*$ such that for all $\gamma \in \mathbb{R}$, the characteristic function of (67), denoted $\delta(\lambda)$, satisfies $\delta(\lambda)(\lambda - \gamma)^n = \Delta_{L_\gamma}(\lambda)$ where $\Delta_{L_\gamma}(\lambda)$ is the characteristic function of a scalar MFDE whose operator is denoted L_γ.*

Let $\eta \in \mathbb{R}$. One has to assume:

Assumption 1. $\Delta_{L_\gamma}(\lambda) = 0$ *has no roots that satisfy* $\mathrm{Re}(\lambda) = \eta$.

Assumption 2. *There exist $s_\pm \in \mathbb{R}_+$ and $J_\pm \in \mathbb{R}_*$ such that the following asymptotic expansions hold true:*

$$\Delta_{L_\gamma}(\lambda) = \begin{cases} \lambda^{-s_+} e^{\lambda b}(J_+ + o(1)) \text{ as } \lambda \to +\infty, \\ \lambda^{-s_-} e^{-\lambda a}(J_- + o(1)) \text{ as } \lambda \to -\infty. \end{cases} \tag{68}$$

The characteristic function $\Delta_{L_\gamma}(\lambda)$ can be factorized as

$$(\lambda - \lambda_0)\Delta_{L_\gamma}(\lambda) = \Delta_{L_-}(\lambda)\Delta_{L_+}(\lambda), \tag{69}$$

where $\lambda_0 \in \mathbb{R}$, $\Delta_{L_-}(\lambda)$ is the characteristic equation of a DDE, and $\Delta_{L_+}(\lambda)$ is the characteristic equation of an ADE. If the factorization of $\Delta_{L_\gamma}(\lambda)$ is not explicit, one can build a continuous path $\Gamma(\mu)$ for $\mu \in [0, 1]$, which allows a family of operators associated with MFDEs to be defined. Such a path is built so that $\Gamma(1)$ is operator L_γ while $\Gamma(0)$ is an operator for which the characteristic equation can be explicitly factorized. One has to assume:

Assumption 3. $\Delta_{\Gamma(\mu)}(\lambda) = 0$ *has roots with* $\mathrm{Re}(\lambda) = \eta$ *for only a finite number of values of* $\mu \in (0,1)$*, while* $\Delta_{\Gamma(0)}(\lambda) = 0$ *and* $\Delta_{\Gamma(1)}(\lambda) = 0$ *have no roots with* $\mathrm{Re}(\lambda) = \eta$*.*

Let us define an integer, denoted $n^{\sharp}_{L_\gamma}(\eta)$, as

$$n^{\sharp}_{L_\gamma}(\eta) = n^{-}_{L_+}(\eta) - n^{+}_{L_-}(\eta) + n_0(\eta), \tag{70}$$

where $n^{-}_{L_+}(\eta)$ is the number of characteristic roots of $\Delta_{L_+}(\lambda) = 0$ such that $\mathrm{Re}(\lambda) < \eta$, $n^{+}_{L_-}(\eta)$ is the number of characteristic roots of $\Delta_{L_-}(\lambda) = 0$ such that $\mathrm{Re}(\lambda) > \eta$, and $n_0(\eta)$ is equal to 1 if $\lambda_0 > \eta$ and to 0 if $\lambda_0 < \eta$.

Theorem 4. *Let Assumption 4 prevail. Let us consider* γ *such that* $\gamma > \eta$*, no root of* $\delta(\lambda) = 0$ *satisfies* $\mathrm{Re}(\lambda) \in [\eta, \gamma)$*, and Assumptions 1, 2 and 3 prevail for* L_γ*. If* $n^{\sharp}_{L_\gamma}(\eta) \geq 0$*, the degree of indeterminacy of problem (67) is equal to* $n^{\sharp}_{L_\gamma}(\eta)$ *and the steady state, or the Balanced Growth Path, is stable. If* $n^{\sharp}_{L_\gamma}(\eta) < 0$*, problem (67) has no solution and the steady state, or the Balanced Growth Path, is unstable.*

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