Uncertainty and the Zero Lower Bound: A Theoretical Analysis

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Abstract

This paper demonstrates how to analytically characterize the set of rational expectations equilibria in a simple stochastic New Keynesian model with the zero lower bound. In this environment, purely forward-looking (non-history-dependent) monetary policies are not generally consistent with existence of rational expectations equilibria. In particular, equilibria exist only when the volatility of the shocks is below some threshold level. This non-existence result is a consequence of the fact that the expected average policy rate rises with the level of uncertainty in the presence of the zero lower bound under forward-looking policies. History-dependent policies can be designed to eliminate the tendency of the expected average policy rate to rise with uncertainty, thereby potentially mitigating the non-existence problems. The non-existence results are likely quite robust, as the only structural feature of the economy upon which they depend is the Fisher condition.

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1. Introduction

Recent years have witnessed substantial advances in the theory of monetary policy in the presence of a lower bound on the nominal interest rate. In deterministic environments, the literature has developed some interesting analytical results. However, in stochastic environments research has largely focused on numerical analyses. In this paper, I provide theoretical results on the existence of rational expectations equilibria in a standard stochastic New Keynesian model with the zero lower bound (ZLB). In particular, I show that non-history-dependent policy rules are only consistent with existence of a stationary rational expectations equilibrium when the volatility of the exogenous shocks is below some threshold level. I provide an analytical characterization of this threshold, and I discuss its implications for the feasibility of different targeted rates of inflation. In addition, I provide an example of how a history-dependent rule may be able to overcome this non-existence problem.

The degree to which uncertainty about future shocks could affect existence of equilibria is quite varied in the ZLB literature. Some authors, such as Kato and Nishiyama (2005), study effectively backward-looking models in which the volatility of shocks affects optimal policy, but not the behaviour of private agents. In this strand of the literature, forward-looking expectations do not play a role, so existence of rational expectations equilibria is not an issue. Another approach is to focus on deterministic forward-looking models (e.g., Jung, Teranishi and Watanabe, 2005), or models in which there is an absorbing state (e.g., Eggertsson and Woodford, 2003). Since these models are ultimately deterministic, uncertainty does not affect existence. More recently, advances in numerical methods have permitted the quantitative study of ZLB models under standard conditions of uncertainty and rational expectations (e.g., Adam and Billi, 2007). In these models uncertainty can, in principle, affect existence of rational expectations equilibria. However, the fact that they are solved numerically makes it difficult, if not impossible, to distinguish numerical problems from true non-existence.

Another branch of the literature studies the global properties of monetary policy rules in deterministic models with the ZLB. As Benhabib, Schmitt-Grohe and Uribe (2001) show, standard Taylor rules can lead to the existence of multiple deterministic steady-states in the presence of the ZLB. This is a consequence of the fact that both the Fisher condition and the Taylor rule must hold in equilibrium. In the presence of the ZLB, there are two pairs of steady-state interest rates and inflation that satisfy both equations simultaneously (Figure 1). One of the equilibria involves inflation on target and a positive nominal interest rate. The other involves deflation and a zero nominal interest rate.[1]

In this paper, I discuss the set of rational expectations equilibria in a stochastic New Keynesian model with the ZLB. I show that under non-history-dependent rules, it is possible for there to exist zero, one or two equilibria. The possibility of no equilibrium is a direct consequence of uncertainty about future shocks. The intuition for non-existence is quite straightforward. An increase in the volatility of the shocks causes an increase in the volatility of the central bank's desired nominal interest rate. But since the nominal rate is censored at zero, the mean of the actual nominal rate increases when the volatility of the desired rate increases, *ceteris paribus* (e.g., in Figure 2 the mean increases from L to H). All else equal, an increase in the average nominal rate raises the average real interest rate, which puts downward pressure on inflation. If the Taylor principle is satisfied, then the nominal rate falls more than one-for-one with inflation, until the average real rate has fallen to its natural level. The policy rule now associates a higher level of the nominal interest rate with any given level of expected inflation (Figure 3). As volatility increases, the expected nominal interest rate continues to increase, *ceteris paribus*. It is possible that the level of volatility can be high enough that the expected nominal rate implied by the policy rule is always higher than the expected nominal rate implied by the Fisher condition (Figure 4). This is the case of non-existence.

Furthermore, I show that a corollary of the non-existence result is that there exists a minimum feasible inflation target. Attempting to target inflation below this minimum leads to non-existence. In particular, the Friedman rule inflation target is only feasible in deterministic economies. In stochastic economies, the minimum feasible inflation target is always greater than the negative of the real interest rate.

Finally, I conjecture that an appropriately designed history-dependent policy can eliminate the non-existence property. This conjecture is based on the proposition that history-dependent policies can be designed to eliminate the tendency of the expected policy rate to rise with the level of uncertainty. This suggests that history-dependence not only leads to better outcomes, but it can also help to guarantee existence of stationary rational expectations equilibria.

In Section 2, I review the canonical New Keynesian model that I employ in the analysis. I present the main analytical results on existence in Section 3. In

[1] There are several ways of eliminating this "bad" equilibrium, including appropriately history-dependent policy (see, e.g., Sugo and Ueda (2008)).

Section 4, I discuss the potential benefits of history-dependence. I conclude in Section 5.

2. Model

I use a canonical New Keynesian model to conduct the analysis in the next section. The microeconomic foundations of this model have been discussed at length elsewhere, so I will not review them here.[2] The structural equations of the model are:

$$x_t = E_t x_{t+1} - \frac{1}{\gamma}\left(i_t - E_t\pi_{t+1} - \overline{r}_t\right) \tag{1}$$

$$\pi_t = (1-\beta)\,\overline{\pi} + \beta E_t \pi_{t+1} + \kappa x_t \tag{2}$$

where,

$$\pi_t \equiv p_t - p_{t-1} \tag{3}$$

and p_t is the log price level and $\overline{r}_t$ is the exogenous natural real interest rate. I assume that $\overline{r}_t$ follows an exogenous stochastic process:

$$\overline{r}_t \sim N\left(\overline{r}, \sigma^2_{\overline{r}}\right) \tag{4}$$

I consider the case of persistent shocks in Appendix B (unfinished), and I introduce a shock to the Phillips curve in Appendix C (unfinished).

I will analyze the implications of the zero lower bound when monetary policy is conducted according to a simple rule in the spirit of Taylor (1993). The rule specifies the central bank's desired nominal interest rate, i^d_t, as a function of inflation, π_t, and the output gap, x_t (the output gap is defined as the log deviation of output from its efficient level). The monetary policy rule implements the desired rate subject to the constraint imposed by the zero lower bound:

$$i^d_t = \overline{r} + \overline{\pi} + \eta_\pi\left(\pi_t - \overline{\pi}\right) + \eta_x x_t \tag{5}$$

$$i_t = \max\left(0, i^d_t\right) \tag{6}$$

where $\overline{r}$ is the non-stochastic steady-state real interest rate, $\overline{\pi}$ is the central bank's inflation target, and $\eta_\pi > 0$ and $\eta_x > 0$ are parameters. Note that (6) respects the

[2] See Woodford (2003) for the microfoundations of the this model.

ZLB and will therefore lead to the existence of two non-stochastic steady-states as described in the introduction. I assume that η_π and η_x satisfy:

$$\eta_\pi + \left(\frac{1-\beta}{\kappa}\right)\eta_x > 1 \tag{7}$$

This condition ensures that the policy rule respects the Taylor principle – that is, the nominal interest rate responds more than one-for-one to expected inflation in periods in which the ZLB is not binding.

Although the results presented in the next section are based on (5)-(6), they appear to apply much more generally to non-history-dependent rules. In particular, analogous results can be derived for rules with expectations of future variables and for the optimal policy under discretion.

3. Analytical Results on Uncertainty and the ZLB

In order to obtain analytical results, I will first characterize the distribution of the nominal interest rate in the model described in the previous section. Most work on the ZLB in stochastic forward-looking environments has resorted to numerical methods because of the difficulty of analytically characterizing the equilibrium. However, the absence of persistent state variables in the model of the previous section facilitates the derivation of analytical results. In particular, the following observation drastically simplifies the problem.

Observation 1. *The model (1)-(6) contains no lagged variables and $\overline{r}_t$ is an i.i.d. random variable. It follows that, in any stationary rational expectations equilibrium (REE), the conditional expectations in this model are time-invariant and equal to the respective unconditional expectations. That is, for any variable y, we have:*

$$E_t\left[y_{t+j}\right] = E\left[y_{t+j}\right] \qquad \forall j \geq 1$$

Moreover, for the purposes of analyzing stationary REEs, the model (1)-(6) can be written as:

$$i_t^d = \overline{r} + \overline{\pi} + \eta_\pi\left(\pi_t - \overline{\pi}\right) + \eta_x x_t \tag{8}$$

$$i_t = \max\left(0, i_t^d\right) \tag{9}$$

$$x_t = x^e - \frac{1}{\gamma}\left(i_t - \pi^e - \overline{r}_t\right) \tag{10}$$

$$\pi_t = (1-\beta)\,\overline{\pi} + \beta\pi^e + \kappa x_t + \lambda\widehat{\mu}_t \tag{11}$$

where the constants π^e and x^e are defined as $\pi^e \equiv E[\pi_{t+j}]$ and $x^e \equiv E[x_{t+j}]$.[3]

This observation effectively recasts the original system of expectational difference equations as a static system. Of course, this restatement of the problem is only valid in a stochastic steady-state. However, this will suffice for studying the properties of stationary REEs.[4] A necessary condition for the existence of stationary REEs is that the unconditional expectations of endogenous variables exist.

Condition 1. *Existence of stationary rational expectations equilibria requires that equations (9)-(11) hold in unconditional expectation. That is, the following equations must hold:*

$$
\begin{aligned}
i^e &= E\left[\max\left(0, \overline{r} + \overline{\pi} + \eta_\pi\left(\pi_t - \overline{\pi}\right) + \eta_x x_t\right)\right] && (12)\\
i^e &= \pi^e + \overline{r} && (13)\\
x^e &= \left(\frac{1-\beta}{\kappa}\right)\left(\pi^e - \overline{\pi}\right) && (14)
\end{aligned}
$$

where $i^e \equiv E[i_t]$.

Proof. See Appendix A. ■

This necessary condition states that there must exist values of i^e, π^e and x^e that simultaneously solve (12)-(14). Equation (12) is the unconditional expectation of the policy rule (6), equation (13) is the unconditional expectation of the IS curve (1), and (14) is the unconditional expectation of the Phillips curve (2). Note that the unconditional expectation of the IS curve yields the unconditional Fisher relation.

In order to characterize the solution set of (12)-(14), I must first evaluate the expectation in (12). Examination of (12) reveals that the unconditional expectation of the nominal interest rate must satisfy:

$$
\begin{aligned}
i^e &= \Pr\left(i_t^d < 0\right) \cdot 0 + \left[1 - \Pr\left(i_t^d < 0\right)\right] E\left[i_t^d | i_t^d \geq 0\right]\\
&= \Pr\left(i_t^d \geq 0\right) E\left[i_t^d | i_t^d \geq 0\right] && (15)
\end{aligned}
$$

[3]Note that in making this observation, I am implicitly restricting attention to non-oscillatory equilibria. This seems reasonable since any minimum state variable solution of this model (in the sense of McCallum (1983)) will be non-oscillatory.

[4]I am using the terms "stochastic steady-state" and "stationary rational expectations equilibrium" interchangeably.

That is, the mean of i_t is simply a weighted average of zero and the mean of i_t^d conditional on i_t^d being at or above the lower bound. In order to characterize the probability and expectation in (15), note that the time-invariance of the expectations substantially simplifies the task of describing the behaviour of the endogenous variables in a stochastic steady-state. The following lemma exploits this fact.

Lemma 1. *Given the time-invariance of expectations, the aggregate law of motion for* $\mathbf{z}_t = (i_t, x_t, \pi_t)'$ *is a piecewise linear function of the form:*

$$\mathbf{z}_t = \begin{cases} \mathbf{c}_b + \mathbf{d}_b\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t < \overline{r}^* \\ \mathbf{c}_{nb} + \mathbf{d}_{nb}\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t \geq \overline{r}^* \end{cases}$$

where $\mathbf{c}_b$, $\mathbf{d}_b$, $\mathbf{c}_{nb}$ *and* $\mathbf{d}_{nb}$ *are* 3×1 *vectors, and* $\mathbf{c}_b$, $\mathbf{c}_{nb}$ *and* $\overline{r}^*$ *are functions of* π^e. *Moreover, the two linear pieces of the aggregate law of motion are equal at* $\overline{r}_t = \overline{r}^*$.

Proof. See Appendix A. ■

As shown in Appendix A, the piecewise linear law of motion for the nominal interest rate is:

$$i_t = \begin{cases} 0 & \forall \overline{r}_t < \overline{r}^* \\ \overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right) + \alpha_1\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t \geq \overline{r}^* \end{cases} \tag{16}$$

where,

$$\begin{aligned} \alpha_0 &\equiv 1 + \left(\frac{\alpha_1 \gamma}{\eta_\pi}\right)\left[\eta_\pi + \eta_x\left(\frac{1-\beta}{\kappa}\right) - 1\right] \\ \alpha_1 &\equiv \left(\frac{\eta_\pi}{\gamma + \eta_\pi \kappa + \eta_x}\right)\left(\kappa + \frac{\eta_x}{\eta_\pi}\right) \end{aligned}$$

and,

$$\overline{r}^* = \overline{r} - \frac{1}{\alpha_1}\left[\overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right)\right] \tag{17}$$

The only unknown in the laws of motion is π^e ($\overline{r}^*$ is known up to a given π^e). So, in order to characterize the set of steady-state equilibria, I must characterize the set of solutions for π^e consistent with Condition 1. To this end, the next proposition uses the results on the laws of motion to evaluate the expectation in (12).

Proposition 1. *Let $\Phi(\bullet)$ be the standard normal c.d.f. and let $\phi(\bullet)$ be the standard normal p.d.f. Then, in any stationary rational expectations equilibrium of the model (5)-(2), the expectation of the policy rule is given by:*

$$\begin{aligned} i^e &= E\left[\max\left(0, \overline{r} + \overline{\pi} + \eta_\pi\left(\pi_t - \overline{\pi}\right) + \eta_x x_t\right)\right] \\ &= E\left[\max\left(0, \overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right) + \alpha_1\left(\overline{r}_t - \overline{r}\right)\right)\right] \\ &= \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\left[\overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right) + \alpha_1 \sigma_{\overline{r}} \underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right] \end{aligned} \quad (18)$$

where,

$$\underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right) \equiv \frac{\phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)}{1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)}$$

Proof. The first equality is simply (12). The second equality follows from the equilibrium law of motion for i_t, (16). To understand the third equality, recall that equation (16) makes i_t a linear function of normal random variables when $\overline{r}_t \geq \overline{r}^*$, and implies a non-zero probability mass on $i_t = 0$ when $\overline{r}_t < \overline{r}^*$. This is the definition of a censored normal random variable. The expression for the mean of i_t is simply the standard formula for the mean of a censored normal variable (see, e.g., Greene, 2000). ■

Note that (15) and (18) have the same structure. In particular, the first term in square brackets in (18) is equal to $\Pr\left(i_t^d \geq 0\right)$ and the second term is equal to $E\left[i_t^d | i_t^d \geq 0\right]$. Using (18), I can evaluate the expectation in (12) and write the the system (12)-(14) as:

$$\begin{aligned} i^e &= \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\left[\overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right) + \alpha_1 \sigma_{\overline{r}} \underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right] \\ &\equiv M\left(\pi^e; \sigma_{\overline{r}}\right) \end{aligned} \quad (19)$$

$$\begin{aligned} i^e &= \pi^e + \overline{r} \\ &\equiv F\left(\pi^e\right) \end{aligned} \quad (20)$$

$$x^e = \left(\frac{1-\beta}{\kappa}\right)\left(\pi^e - \overline{\pi}\right) \quad (21)$$

where, for future notational convenience, I have defined M as the expected monetary policy rule and F as the expected Fisher condition. The unconditional expectations (i^e, π^e and x^e) in any stationary REE are solutions to (19)-(21). The number of solutions to this system is equal to the number of stationary REEs. Since (17) makes $\overline{r}^*$ a function of π^e, this system is nonlinear in π^e. This makes it impossible to solve the system analytically, but it is possible to characterize the number of stationary REEs that exist and some of their properties. In order to do this, note that x^e only appears in (21). So, given π^e, this equation can be used to residually determine x^e. Thus, solutions for i^e and π^e are characterized by the system of two equations, (19) and (20).

Before turning to the formal results, consider the economic meaning of (19) and (20). Recall that the IS curve, (1), is derived from a consumption Euler equation. Taking the unconditional expectation of the IS curve yields the unconditional expectation of the Fisher condition, (20), just as one would expect from an Euler equation. The Fisher condition specifies a linear relationship between the expected nominal interest rate and any given level of expected inflation. This relationship is invariant to the shock variances, policy parameters and structural parameters (with the exception of the discount factor which determines $\overline{r}$).

Equation (19) specifies the expected nominal interest rate consistent with the policy rule. Like the Fisher condition it provides a relationship between the expected nominal interest rate and any given level of expected inflation. However, it differs from the Fisher condition in two important respects: (1) the implied relationship between i^e and π^e is nonlinear, and (2) the implied relationship is dependent upon the policy parameters, structural parameters and variances. Since (19) is the expected monetary policy rule, the reason for the dependence on the policy parameters is obvious. Less obvious is that the relationship between i^e and π^e implied by (19) must depend on the structural parameters and variances. This is a consequence of the fact that the policy rule depends on π_t and x_t. So anything that affects the distributions of π_t and x_t will necessarily affect the expectation of the policy rule implied interest rate.

With these results in hand, I can turn to the task of characterizing the conditions for existence. In order to prove the results regarding existence, the following lemmas will be used. The proofs for these lemmas are provided in Appendix A.

Lemma 2. *$M(\pi^e; \sigma_{\overline{r}})$ is increasing and convex in π^e, $\forall \sigma_{\overline{r}} > 0$.*

Lemma 3. *The limits of the partial derivative of M with respect to π^e are:*

$$\begin{aligned}\lim_{\pi^e \longrightarrow -\infty} \frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \pi^e} &= 0 \\ \lim_{\pi^e \longrightarrow \infty} \frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \pi^e} &= \alpha_0\end{aligned}$$

Lemma 4. *There exists a π^{e*} that is defined such that:*

$$\left.\frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \pi^e}\right|_{\pi^e=\pi^{e*}} \equiv \frac{\partial F\left(\pi^e\right)}{\partial \pi^e} = 1$$

and,

$$\pi^{e*} \equiv \overline{\pi} - \left(\frac{\overline{r}+\overline{\pi}}{\alpha_0}\right) - \left(\frac{\alpha_1}{\alpha_0}\right) \sigma_{\overline{r}} \Phi^{-1}\left(1 - \frac{1}{\alpha_0}\right) \tag{22}$$

Lemma 5. *For a given level of expected inflation, π^e, the expected nominal interest rate implied by the policy rule, $M\left(\pi^e; \sigma_{\overline{r}}\right)$, is increasing in the level of volatility, $\sigma_{\overline{r}}$:*

$$\frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}} = \alpha_1 \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right] \underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right) > 0$$

Evaluating this derivative at $\pi^e = \pi^{e*}$,

$$\left.\frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}}\right|_{\pi^e=\pi^{e*}} = \alpha_1 \phi\left(\Phi^{-1}\left(1 - \frac{1}{\alpha_0}\right)\right) > 0$$

Lemmas 2 and 3 together state that the M is increasing and convex and that the slope of M eventually exceeds the slope of F as π^e gets large (since $\alpha_0 > 1$). This will be useful because it implies that if M lies below F for any π^e, then it must cut F from below for some π^e. Lemma 4 states that M and F are parallel at $\pi^e = \pi^{e*}$, and Lemma 5 characterizes how the expected interest rate consistent with M changes with the shock volatility, $\sigma_{\overline{r}}$. Lemma 5 is important because the tendency of the M curve to shift up in (π^e, i^e)-space is at the root of the non-existence problem.

I can now state the main existence result.

Proposition 2. *There exists a* $\sigma_{\overline{r}}^{*} > 0$ *such that (i)* $M(\pi^{e};\sigma_{\overline{r}}^{*})$ *is tangent to* $F(\pi^{e})$ *at* $\pi^{e}=\pi^{e*}$ *and the unique closed-form solution for* $\sigma_{\overline{r}}^{*}$ *is:*

$$\sigma_{\overline{r}}^{*} = \frac{\left(1-\frac{1}{\alpha_0}\right)(\overline{r}+\overline{\pi})}{\alpha_1 \phi\left(\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right)\right)} \tag{23}$$

(ii) for $\sigma_{\overline{r}} = \sigma_{\overline{r}}^{*}$ *there exists one stationary REE, (iii) for* $\sigma_{\overline{r}} < \sigma_{\overline{r}}^{*}$ *there exist two stationary REEs with* $\pi^{e}=\pi^{eL}<\pi^{e*}$ *and* $\pi^{e}=\pi^{eH}>\pi^{e*}$*, and (iv) for* $\sigma_{\overline{r}} > \sigma_{\overline{r}}^{*}$ *there exist no stationary rational expectations equilibria.*

Proof. See Appendix A. ∎

Thus, the existence of stationary REEs under a non-history-dependent policy rule of the form (5)-(5) is sensitive to the degree of uncertainty in the economy, as measured by the volatility of the exogenous shocks. Figure 3 shows M and F for the case of $\sigma_{\overline{r}}$ less than $\sigma_{\overline{r}}^{*}$, and Figure 4 shows M and F for the case of $\sigma_{\overline{r}}$ greater than $\sigma_{\overline{r}}^{*}$. Economically, in the case of $\sigma_{\overline{r}} > \sigma_{\overline{r}}^{*}$ the expected real interest rate consistent with the policy rule is too high (for all π^{e}) to bring desired saving to zero as it must be in any equilibrium in this economy.

Note that Proposition 2 implies that the Friedman (1969) rule inflation target is infeasible in a stochastic economy with the ZLB and a non-history-dependent policy rule of the form (5)-(6). The Friedman rule inflation target, $\overline{\pi} = -\overline{r}$, would cause $\sigma_{\overline{r}}^{*} = 0$ according to (23). Hence, under the assumed policy rule, the Friedman rule inflation target is infeasible for any positive shock variance. More generally, as the following proposition states, (23) imposes a lower bound on the inflation target.

Proposition 3. *There exists a Minimum Feasible Inflation Target (MFIT),* $\overline{\pi}^{*}$*, such that there exist no stationary REEs if* $\overline{\pi} < \overline{\pi}^{*}$*. The MFIT is given by:*

$$\overline{\pi}^{*} = -\overline{r} + \sigma_{\overline{r}} \frac{\alpha_0 \alpha_1 \phi\left(\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right)\right)}{(\alpha_0 - 1)} \tag{24}$$

Proof. Replace $\sigma_{\overline{r}}^{*}$ with an arbitrary $\sigma_{\overline{r}}$ in (23). Denote the value of $\overline{\pi}$ that makes (23) hold for arbitrary $\sigma_{\overline{r}}$ by $\overline{\pi}^{*}$. This yields (24). The fact that $\partial\sigma_{\overline{r}}^{*}/\partial\overline{\pi} < 0$ implies that $\sigma_{\overline{r}}^{*} < \sigma_{\overline{r}}$ for any $\overline{\pi} < \overline{\pi}^{*}$. This proves that no equilibria exist for $\overline{\pi} < \overline{\pi}^{*}$. ∎

Figure 5 graphically depicts the MFIT. The relationship between $\sigma_{\overline{r}}$ and the MFIT is clearly linear. Moreover, the figure makes clear the infeasibility of the Friedman rule in stochastic economies.

It is also worth noting that the average natural rate of interest, $\overline{r}$, is also an important determinant of $\sigma^*_{\overline{r}}$ in (23). Adam and Billi (2007) find that under discretionary policy, their algorithm fails to converge for low values of the steady-state real interest rate. The role of $\overline{r}$ in (23) suggests that the problem Adam and Billi encountered was not merely numerical. By reasoning analogous to that in Proposition 3, there is a minimum level of the steady-state natural rate that is consistent with equilibrium, holding all else equal. Thus, it is not surprising that numerical techniques would fail to find an equilibrium for sufficiently low values of the $\overline{r}$.

Proposition 4. *When there are two distinct stationary REEs, average inflation will be increasing in $\sigma_{\overline{r}}$ in the low-inflation equilibrium and decreasing in $\sigma_{\overline{r}}$ in the high-inflation equilibrium. That is:*

$$\left.\frac{\partial \pi^{eL}}{\partial \sigma_{\overline{r}}}\right|_{M(\pi^{eL};\sigma_{\overline{r}})=F(\pi^{eL})} > 0$$

$$\left.\frac{\partial \pi^{eH}}{\partial \sigma_{\overline{r}}}\right|_{M(\pi^{eH};\sigma_{\overline{r}})=F(\pi^{eH})} < 0$$

where $\pi^{eL} < \pi^{e*}$ and $\pi^{eH} > \pi^{e*}$ are the levels of expected inflation in the low- and high-inflation equilibria, respectively.

Proof. See Appendix A. ■

4. A Conjecture Regarding the Benefits of History Dependence

The monetary policy rules considered in the preceding sections were all non-history-dependent. That is, they specified the central bank's actions as a function of the current and expected future evolution of the economy. In contrast, in this section I show that history-dependent policies can eliminate the non-existence problem identified earlier.

A history-dependent policy is a policy that makes the central bank's current actions dependent not only upon current and expected future variables, but also upon past economic conditions. In economies with forward-looking agents, the optimal monetary policy commitment will generally be history dependent. History dependence is optimal in such environments because it allows the central bank to influence private sector expectations in a stabilizing manner. In the context of the zero lower bound, history dependence can be particularly beneficial. A commitment to reflate after a ZLB episode can diminish both the length and severity of such episodes. Such a policy is history-dependent because the extent of the post-ZLB reflation is dependent upon the cumulative disinflation that occurred during the ZLB episode.

In the previous section, I showed that the interest rate implied by the non-history-dependent policy rule is increasing in the degree of volatility, *ceteris paribus*. This property of the non-history-dependent rule made it possible for the average interest rate implied by the policy rule to be too high relative to the average natural rate of interest. This misalignment of the average actual and natural rates of interest led to the non-existence result in Proposition 2. In contrast, appropriately designed history-dependent rules can eliminate the positive relationship between the average interest rate implied by the policy rule and the level of volatility. To demonstrate this benefit of history dependence, I employ as an example a "catch-up" rule that adjusts the desired rate based on past ZLB episodes.

Under the catch-up rule, the desired rate evolves according to:

$$i_t^d = \overline{r}_t + \overline{\pi} + \eta_\pi \left(E_t \pi_{t+1} - \overline{\pi}\right) - \left(i_{t-1} - i_{t-1}^d\right) \tag{25}$$

This policy rule ratchets down the desired rate each time the lagged desired rate is below the lagged actual rate (i.e., each time the zero bound is binding). In this way, the rule can ensure that the policy rate is kept lower than it otherwise would be after a ZLB episode.

As before, the ZLB implies that the actual policy rate is related to the desired rate by:

$$i_t = \max\left(0, i_t^d\right) \tag{26}$$

Define a function $\widehat{M}$ that is the analog of M used in the previous section:

$$i^e = E\left[\max\left(0, i_t^d\right)\right] \equiv \widehat{M}\left(\pi^e; \sigma_{\overline{r}}\right)$$

For the non-history-dependent rule, Lemma 5 showed that $\partial M / \partial \sigma_{\overline{r}} > 0$. That is, the M curve shifts up in (π^e, i^e)-space as the volatility increases. The next proposition shows that this is not true for the history-dependent rules proposed in this section.

Proposition 5. *In any stationary REE, under a history-dependent policy rule of the form (25) or (**??**), the expected policy rate,* $\widehat{M}$*, is invariant to changes in* $\sigma_{\overline{r}}$*, ceteris paribus. In particular,*

$$\frac{\partial \widehat{M}\left(\pi^e;\sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}} = 0$$

Proof. Note that we can write (26) as:

$$\begin{aligned} i_t &= \max\left(0, i_t^d\right) \\ &= i_t^d + \max\left(0, i_t - i_t^d\right) \end{aligned} \tag{27}$$

Substituting (25) into (27) for the first i_t^d yields:

$$i_t = \overline{r}_t + \overline{\pi} + \eta\left(E_t\pi_{t+1} - \overline{\pi}\right) - \left(i_{t-1} - i_{t-1}^d\right) + \max\left(0, i_t - i_t^d\right) \tag{28}$$

Using the fact that $\left(i_{t-1} - i_{t-1}^d\right) = \max\left(0, i_{t-1} - i_{t-1}^d\right)$:

$$i_t = \overline{r}_t + \overline{\pi} + \eta\left(E_t\pi_{t+1} - \overline{\pi}\right) - \max\left(0, i_{t-1} - i_{t-1}^d\right) + \max\left(0, i_t - i_t^d\right) \tag{29}$$

In any stationary REE, the unconditional expectation must exist:

$$i^e = \overline{r} + \overline{\pi} + \eta\left(\pi^e - \overline{\pi}\right) \equiv \widehat{M}\left(\pi^e;\sigma_{\overline{r}}\right) \tag{30}$$

Notice that the last two terms in (29) cancel in unconditional expectation. This is important because it is precisely the expectations of max terms that would be directly affected by $\sigma_{\overline{r}}$. With the max terms gone, it is immediately apparent from (30) that $\partial\widehat{M}/\partial\sigma_{\overline{r}} = 0$ for a rule of the form (25). ■

Of course, this result does not guarantee existence of a stationary REE – it merely rules out the possibility that the $\widehat{M}$ curve shifts up in (π^e, i^e)-space as the volatility increases. However, given that the tendency of M to shift up was at the root of the non-existence result in the previous section, I conjecture that a broad class of history-dependent rules can indeed guarantee existence. I would expect this class to include price-level targeting and the optimal policy under commitment.

5. Conclusion

In this paper, I have shown how to analytically characterize the equilibrium of the canonical New Keynesian model with non-history-dependent policy and the ZLB. The analytical results prove that non-history-dependent rules guarantee existence of stationary REEs only for sufficiently low levels of uncertainty. Moreover, the level of volatility places a lower bound on the level of inflation that can be targeted in a manner that is consistent with a stationary REE. I have also shown that there is reason to expect history-dependent policies to be able to avoid the non-existence problem.

The role of other policy levers at the ZLB remains to be explored. For example, if government spending is used asymmetrically to offset the effect of shocks to the natural rate when the lower bound is binding, it could potentially guarantee existence.

Finally, the results on non-existence are likely to be fairly robust (in the absence of other asymmetric policies). Non-existence arose solely from the interaction of the expected Fisher condition and the expected policy rule. The Fisher condition holds in almost all well-defined macroeconomic models. Thus, the specification of the policy rule is the key factor determining existence. This reinforces the imperative to design policy rules that ensure existence of stationary REEs, even in the presence of the ZLB.

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A. Proofs

Proof of Condition 1. Consider a general rational expectations system of the form:

$$E_t\left[\mathbf{f}\left(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{z}_{t-1}, \boldsymbol{\varepsilon}_t\right)\right] = 0 \tag{31}$$

where $\mathbf{z}_t$ is a vector of endogenous variables, $\boldsymbol{\varepsilon}_t$ is a vector of exogenous variables, and $\mathbf{f}(\bullet)$ is a vector-valued function. The model of Section 2 can be cast in this form. Existence of stationary REEs requires that (31) holds in every state of nature. Writing out the conditional expectations in (31) as integrals:

$$\int \mathbf{f}\left(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{z}_{t-1}, \boldsymbol{\varepsilon}_t\right) g\left(\mathbf{z}_{t+1} | \mathbf{s}_t\right) d\mathbf{z}_{t+1} = 0 \tag{32}$$

where g is the density of $\mathbf{z}_{t+1}$ conditional on the state, $\mathbf{s}_t \equiv \left(\mathbf{z}'_{t-1}, \boldsymbol{\varepsilon}'_t\right)'$.

The unconditional expectation of (31) is:

$$E\left[\mathbf{f}\left(E_t\mathbf{z}_{t+1}, \mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{z}_{t-1}, \boldsymbol{\varepsilon}_t\right)\right] = \int \left[\int \mathbf{f}\left(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{z}_{t-1}, \boldsymbol{\varepsilon}_t\right) g\left(\mathbf{z}_{t+1} | \mathbf{s}_t\right) d\mathbf{z}_{t+1}\right] h\left(\mathbf{s}_t\right) d\mathbf{s}_t \tag{33}$$

where h is the unconditional density of the state vector. Note that by (32) the term in square brackets must be equal to zero in every state of nature, thus:

$$E\left[\mathbf{f}\left(\mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{z}_{t-1}, \boldsymbol{\varepsilon}_t\right)\right] = 0 \tag{34}$$

This proves that for any rational expectations system of the form (31), existence requires that the system hold in unconditional expectation.[5] ■

Proof of Lemma 1. Note that if the shocks are such that $i_t^d < 0$, then the endogenous variables will satisfy:

$$\begin{aligned} i_t &= 0 && (36) \\ x_t &= x^e + \frac{1}{\gamma}\left(\pi^e + \overline{r}_t\right) && (37) \\ \pi_t &= (1-\beta)\,\overline{\pi} + \beta\pi^e + \kappa x_t && (38) \end{aligned}$$

[5] In the case in which the desired nominal interest rate depends on expected future variables, it is straightforward to extend this result to a system of the form:

$$E_t\left[\mathbf{f}\left(E_t\mathbf{z}_{t+1}, \mathbf{z}_{t+1}, \mathbf{z}_t, \mathbf{z}_{t-1}, \boldsymbol{\varepsilon}_t\right)\right] = 0 \tag{35}$$

is and if the shocks are such that $i_t^d \geq 0$, the endogenous variables will satisfy:

$$
\begin{aligned}
i_t &= \overline{r} + \overline{\pi} + \eta_\pi \left(\pi_t - \overline{\pi}\right) + \eta_x x_t && (39)\\
x_t &= x^e - \frac{1}{\gamma}\left(i_t - \pi^e - \overline{r}_t\right) && (40)\\
\pi_t &= \left(1-\beta\right)\overline{\pi} + \beta\pi^e + \kappa x_t && (41)
\end{aligned}
$$

The solution for i_t in the case that $i_t^d < 0$ is trivially $i_t = 0$. In the other case, substituting (40) and (41) into (39) and using (14) yields:

$$
i_t = \overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right) + \alpha_1\left(\overline{r}_t - \overline{r}\right) \tag{42}
$$

where,

$$
\begin{aligned}
\alpha_0 &\equiv 1 + \left(\frac{\alpha_1\gamma}{\eta_\pi}\right)\left[\eta_\pi + \eta_x\left(\frac{1-\beta}{\kappa}\right) - 1\right]\\
\alpha_1 &\equiv \left(\frac{\eta_\pi}{\gamma + \eta_\pi\kappa + \eta_x}\right)\left(\kappa + \frac{\eta_x}{\eta_\pi}\right)
\end{aligned}
$$

Let $\overline{r}^*$ be the value of $\overline{r}_t$ such that $i_t = 0$:

$$
\overline{r}^* = \overline{r} - \frac{1}{\alpha_1}\left[\overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right)\right] \tag{43}
$$

Moreover, given the definition of $\overline{r}^*$, it is clear that $i_t^d < 0$ if and only if $\overline{r}_t < \overline{r}^*$, and $i_t^d \geq 0$ if and only if $\overline{r}_t \geq \overline{r}^*$. Thus, the equilibrium law of motion for i_t is:

$$
i_t = \begin{cases} 0 & \forall \overline{r}_t < \overline{r}^* \\ \overline{r} + \overline{\pi} + \alpha_0\left(\pi^e - \overline{\pi}\right) + \alpha_1\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t \geq \overline{r}^* \end{cases} \tag{44}
$$

Solving (36)-(38) and (39)-(41), and using (14), gives the equilibrium laws of motion for x_t and π_t.

$$
x_t = \begin{cases} \left(\frac{1-\beta}{\kappa}\right)\left(\pi^e - \overline{\pi}\right) + \frac{1}{\gamma}\left(\pi^e + \overline{r}\right) + \frac{1}{\gamma}\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t < \overline{r}^* \\ \left[\left(\frac{1-\beta}{\kappa}\right) - \frac{1}{\gamma}\left(\alpha_0 - 1\right)\right]\left(\pi^e - \overline{\pi}\right) + \frac{1}{\gamma}\left(1-\alpha_1\right)\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t \geq \overline{r}^* \end{cases} \tag{45}
$$

$$
\pi_t = \begin{cases} \left(1 + \frac{\kappa}{\gamma}\right)\pi^e + \frac{\kappa}{\gamma}\overline{r} + \frac{\kappa}{\gamma}\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t < \overline{r}^* \\ \pi^e + \frac{\kappa}{\gamma}\left(1-\alpha_0\right)\left(\pi^e - \overline{\pi}\right) + \frac{\kappa}{\gamma}\left(1-\alpha_1\right)\left(\overline{r}_t - \overline{r}\right) & \forall \overline{r}_t \geq \overline{r}^* \end{cases} \tag{46}
$$

So, the aggregate law of motion has the form stated in the lemma.

Furthermore, evaluating (16) at $\overline{r}_t = \overline{r}^*$ makes both linear pieces equal to zero. Similarly,evaluating (45) and (46) at $\overline{r}_t = \overline{r}^*$ makes both linear pieces equal at $x_t = x^*$ and $\pi_t = \pi^*$, where:

$$x^* \equiv \left[\left(\frac{1-\beta}{\kappa}\right) + \frac{1}{\gamma}\left(1 - \frac{\alpha_0}{\alpha_1}\right)\right](\pi^e - \overline{\pi}) + \frac{1}{\gamma}\left(1 - \frac{1}{\alpha_1}\right)(\overline{r} + \overline{\pi})$$

$$\pi^* \equiv \pi^e - \left(\frac{\kappa}{\gamma}\frac{\alpha_0}{\alpha_1}\right)(\pi^e - \overline{\pi}) + \left(\frac{\kappa}{\gamma}\right)(\overline{r} + \pi^e) - \left(\frac{\kappa}{\gamma}\frac{1}{\alpha_1}\right)(\overline{r} + \overline{\pi})$$

This proves the equality of the two linear pieces of the aggregate law of motion at $\overline{r}_t = \overline{r}^*$. ■

Proof of Lemma 2. The first derivative is:

$$\begin{aligned}\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial \pi^e} &= -\Phi'\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\left[\overline{r} + \overline{\pi} + \alpha_0(\pi^e - \overline{\pi}) + \alpha_1\sigma_{\overline{r}}\underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\left(\frac{1}{\sigma_{\overline{r}}}\right)\frac{\partial \overline{r}^*}{\partial \pi^e} \\ &\quad + \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_0 \\ &\quad + \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_1\sigma_{\overline{r}}\underline{\lambda}'\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\left(\frac{1}{\sigma_{\overline{r}}}\right)\frac{\partial \overline{r}^*}{\partial \pi^e}\end{aligned}$$

or,

$$\begin{aligned}\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial \pi^e} &= \phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\left[\frac{\overline{r} + \overline{\pi} + \alpha_0(\pi^e - \overline{\pi})}{\alpha_1\sigma_{\overline{r}}} + \underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_0 \\ &\quad + \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_0 - \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\underline{\lambda}'\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\alpha_0\end{aligned}$$

where I have used the fact that (17) implies $\partial\overline{r}^*/\partial\pi^e = -(\alpha_0/\alpha_1)$ and the fact that $\phi(\bullet)$ is the normal p.d.f. Using the definitions of $\underline{\lambda}(\bullet)$ and $\overline{r}^*$, I can write:

$$\begin{aligned}\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial \pi^e} &= \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right] \\ &\quad \times \left\{\underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\left[\underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right) - \frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right]\alpha_0 + \alpha_0 - \underline{\lambda}'\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\alpha_0\right\}\end{aligned}$$

Using the fact that the derivative of the standard normal p.d.f. is $\phi'(z) = -z\phi(z)$, it is straightforward to verify that $\underline{\lambda}'(z) = \underline{\lambda}(z)[\underline{\lambda}(z) - z]$. Then,

$$\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial \pi^e} = \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right]\left\{\underline{\lambda}'\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\alpha_0 + \alpha_0 - \underline{\lambda}'\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\alpha_0\right\}$$

or,

$$\frac{\partial M\left(\pi^{e};\sigma_{\overline{r}}\right)}{\partial\pi^{e}}=\left[1-\Phi\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_{0} \tag{47}$$

Thus, the first derivative is positive if, and only if, α_0 is positive. It is easy to verify that $\alpha_0 > 1$ as long as the Taylor principle, (7), is satisfied. This proves that $M\left(\pi^e;\sigma_\varepsilon\right)$ is increasing in π^e.

The second derivative is:

$$\frac{\partial^{2} M\left(\pi^{e};\sigma_{\overline{r}}\right)}{\partial\left(\pi^{e}\right)^{2}}=\phi\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\left(\frac{\alpha_{0}^{2}}{\alpha_{1}\sigma_{\overline{r}}}\right)>0 \tag{48}$$

This proves that $M\left(\pi^e;\sigma_{\overline{r}}\right)$ is convex in π^e. ■

Proof of Lemma 3. The result follows immediately from (47) and the definition of $\overline{r}^*$. ■

Proof of Lemma 4. Since $\partial M/\partial\pi^e$ is continuous, the existence of π^{e*} follows from the Lemma 3 and the Intermediate Value Theorem. To derive the expression for π^{e*}, set the derivative in (47) equal to unity and solve for π^{e*}:

$$\left[1-\Phi\left(-\frac{\overline{r}+\overline{\pi}+\alpha_{0}\left(\pi^{e*}-\overline{\pi}\right)}{\alpha_{1}\sigma_{\overline{r}}}\right)\right]\alpha_{0}\equiv 1$$

where I have used the definition of $\overline{r}^*$. Solving for π^{e*} gives the result. ■

Proof of Lemma 5. Using (19) to take the derivative of $M\left(\pi^e;\sigma_{\overline{r}}\right)$ with respect to $\sigma_{\overline{r}}$:

$$\begin{aligned}\frac{\partial M\left(\pi^{e};\sigma_{\overline{r}}\right)}{\partial\sigma_{\overline{r}}} = & -\Phi'\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\left(-\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}^{2}}\right)\left[\overline{r}+\overline{\pi}+\alpha_{0}\left(\pi^{e}-\overline{\pi}\right)+\alpha_{1}\sigma_{\overline{r}}\underline{\lambda}\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\\ & +\left[1-\Phi\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_{1}\underline{\lambda}\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\\ & +\left[1-\Phi\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_{1}\sigma_{\overline{r}}\underline{\lambda}'\left(\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}}\right)\left(-\frac{\overline{r}^{*}-\overline{r}}{\sigma_{\overline{r}}^{2}}\right)\end{aligned}$$

Simplifying and using the definition of $\phi(\bullet)$:

$$\begin{aligned}\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial\sigma_{\overline{r}}} &= \phi\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\left[\frac{\overline{r}+\overline{\pi}+\alpha_0(\pi^e-\overline{\pi})}{\sigma_{\overline{r}}}+\alpha_1\underline{\lambda}\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\\ &\quad+\left[1-\Phi\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_1\underline{\lambda}\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\\ &\quad-\left[1-\Phi\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\alpha_1\underline{\lambda}'\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\end{aligned}$$

Using the definitions of $\underline{\lambda}(\bullet)$ and $\overline{r}^*$, and rearranging:

$$\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial\sigma_{\overline{r}}}=\left[1-\Phi\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\left\{\begin{array}{c}\alpha_1\left[\underline{\lambda}\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)-\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\underline{\lambda}\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\\ +\alpha_1\underline{\lambda}\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)-\alpha_1\underline{\lambda}'\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\end{array}\right\}$$

Using $\underline{\lambda}'(z)=[\underline{\lambda}(z)-z]\,\underline{\lambda}(z)$ gives the result:

$$\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial\sigma_{\overline{r}}}=\alpha_1\left[1-\Phi\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\underline{\lambda}\left(\frac{\overline{r}^*-\overline{r}}{\sigma_{\overline{r}}}\right)$$

To evaluate this derivative at $\pi^e=\pi^{e*}$, use (17) to eliminate $\overline{r}^*$ and then use the expression for π^{e*}. ■

Proof of Proposition 2. Let g be the gap between the interest rate implied by the Fisher condition and that implied by the monetary policy rule:

$$g(\pi^e;\sigma_{\overline{r}})\equiv M(\pi^e;\sigma_{\overline{r}})-F(\pi^e)$$

Then at $\pi^e=\pi^{e*}$ we have:

$$\begin{aligned}g(\pi^{e*}(\sigma_{\overline{r}});\sigma_{\overline{r}}) &\equiv M(\pi^{e*}(\sigma_{\overline{r}});\sigma_{\overline{r}})-F(\pi^{e*}(\sigma_{\overline{r}}))\\ &= i^{e*}(\sigma_{\overline{r}})-\overline{r}-\pi^{e*}(\sigma_{\overline{r}})\end{aligned}$$

where I have made explicit the dependence of i^{e*} and π^{e*} on σ_{ε}. Given Lemmas 2 and 3, and the definitions of i^{e*} and π^{e*}, I can prove (i) by proving that there exists a $\sigma^*_{\overline{r}}$ such that:

$$g(\pi^{e*}(\sigma^*_{\overline{r}});\sigma^*_{\overline{r}})\equiv 0$$

or,

$$\overline{r}+\pi^{e*}=\left[1-\Phi\left(\frac{\overline{r}^*(\pi^{e*})-\overline{r}}{\sigma_{\overline{r}}}\right)\right]\left[\overline{r}+\overline{\pi}+\alpha_0(\pi^{e*}-\overline{\pi})+\alpha_1\sigma_{\overline{r}}\underline{\lambda}\left(\frac{\overline{r}^*(\pi^{e*})-\overline{r}}{\sigma_{\overline{r}}}\right)\right]$$

Substituting $\pi^{e*}(\sigma^*_\varepsilon)$ from Lemma 4 and using the definition of $\overline{r}^*$:

$$\begin{aligned} \overline{r}+\overline{\pi}-\left(\frac{\overline{r}+\overline{\pi}}{\alpha_0}\right)-\left(\frac{\alpha_1}{\alpha_0}\right)\sigma_{\overline{r}}\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right) &= -\left(\frac{\alpha_1}{\alpha_0}\right)\sigma_{\overline{r}}\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right) \\ &\quad +\left(\frac{\alpha_1}{\alpha_0}\right)\sigma_{\overline{r}}\underline{\lambda}\left(\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right)\right) \end{aligned}$$

Rearranging and using the definition of $\underline{\lambda}(\bullet)$:

$$\left(1-\frac{1}{\alpha_0}\right)(\overline{r}+\overline{\pi})=\alpha_1\sigma_{\overline{r}}\phi\left(\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right)\right)$$

Solving for σ^*_ε:

$$\sigma^*_{\overline{r}}=\frac{\left(1-\frac{1}{\alpha_0}\right)(\overline{r}+\overline{\pi})}{\alpha_1\phi\left(\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right)\right)}$$

Thus, as long as the Taylor principle (7) is respected[6], there exists a unique $\sigma^*_{\overline{r}}>0$ which makes $M(\pi^e;\sigma^*_{\overline{r}})$ tangent to $F(\pi^e)$ at $\pi^e=\pi^{e*}$. This proves (i). The fact that M and F are tangent at this point, implies that there is only one solution to $M(\pi^{e*};\sigma^*_{\overline{r}})=F(\pi^{e*})$. This proves (ii).

Note that the definitions of π^{e*} and $g(\pi^e;\sigma_{\overline{r}})$ imply that $\pi^{e*}=\arg\min g(\pi^e;\sigma_{\overline{r}})$. As a consequence, $g(\pi^{e*};\sigma_{\overline{r}})>0$ implies that $M(\pi^e;\sigma_{\overline{r}})>F(\pi^e)$ for all π^e. Also, $g(\pi^{e*};\sigma_{\overline{r}})<0$ implies that $M(\pi^{e*};\sigma_{\overline{r}})<F(\pi^{e*})$. In this case, Lemmas 1 and 2 imply that M cuts F from above at some $\pi^{eL}<\pi^{e*}$ and M cuts F from below at some $\pi^{eH}>\pi^{e*}$, so there are two steady-state rational expectations equilibria. Thus, to prove (iii) and (iv) it suffices to prove that the following derivative is positive:

$$\begin{aligned} \left.\frac{\partial g(\pi^e;\sigma_{\overline{r}})}{\partial\sigma_{\overline{r}}}\right|_{\pi^e=\pi^{e*}} &= \left.\frac{\partial M(\pi^e;\sigma_{\overline{r}})}{\partial\sigma_{\overline{r}}}\right|_{\pi^e=\pi^{e*}}-\frac{\partial F(\pi^e)}{\partial\sigma_{\overline{r}}} \\ &= \alpha_1\phi\left(\Phi^{-1}\left(1-\frac{1}{\alpha_0}\right)\right)>0 \end{aligned} \tag{49}$$

The second line uses Lemma 5 and the fact that $\partial F(\pi^e)/\partial\sigma_{\overline{r}}=0$. The derivative in (49) is positive because $\alpha_1>0$ and $\alpha_0>1$ as long as (7) holds. This proves (iii) and (iv). ∎

[6] This is because $\alpha_0>1$ if and only if (7) holds.

Proof of Proposition 4. Let $\widehat{\pi}^e \in \left\{\pi^{eL}, \pi^{eH}\right\}$. In equilibrium,

$$M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right) \equiv F\left(\widehat{\pi}^e\right)$$

Totally differentiating this gives,

$$\frac{\partial M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right)}{\partial \widehat{\pi}^e} \partial \widehat{\pi}^e + \frac{\partial M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}} \partial \sigma_{\overline{r}} = \frac{\partial F\left(\widehat{\pi}^e\right)}{\partial \widehat{\pi}^e} \partial \widehat{\pi}^e$$

Using the definition of F,

$$\frac{\partial M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right)}{\partial \widehat{\pi}^e} \partial \widehat{\pi}^e + \frac{\partial M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}} \partial \sigma_{\overline{r}} = \partial \widehat{\pi}^e$$

Thus,

$$\left.\frac{\partial \widehat{\pi}^e}{\partial \sigma_{\overline{r}}}\right|_{M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right) = F\left(\widehat{\pi}^e\right)} = \left[1 - \frac{\partial M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right)}{\partial \widehat{\pi}^e}\right]^{-1} \frac{\partial M\left(\widehat{\pi}^e; \sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}} \tag{50}$$

Recall that from Lemmas 2 and 5 we know,

$$\frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \pi^e} = \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right] \alpha_0 > 0$$

$$\frac{\partial M\left(\pi^e; \sigma_{\overline{r}}\right)}{\partial \sigma_{\overline{r}}} = \alpha_1 \left[1 - \Phi\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right)\right] \underline{\lambda}\left(\frac{\overline{r}^* - \overline{r}}{\sigma_{\overline{r}}}\right) > 0$$

Since $\partial M / \partial \sigma_{\overline{r}}$ is always positive, the sign of (50) depends on the sign of the term in square brackets. Since M is convex and $\partial M / \partial \pi^e = 1$ when $\pi^e = \pi^{e*}$, the sign of the term in square brackets in (50) is,

$$sgn\left(1 - \frac{\partial M}{\partial \pi^e}\right) = \begin{cases} -1 & \pi^e > \pi^{e*} \\ 0 & \pi^e = \pi^{e*} \\ 1 & \pi^e < \pi^{e*} \end{cases} \tag{51}$$

The result in the proposition follows from (50), (51) and the fact that $\pi^{eH} > \pi^{e*}$ and $\pi^{eL} < \pi^{e*}$. ■