Dynamic asset allocation for bank under stochastic interest rates.

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Abstract: This paper considers the optimal asset allocation strategy for bank with stochastic interest rates when there are three types of asset: Bank account, loans and securities. The asset allocation problem is to maximize the expected utility from terminal wealth of a bank's shareholders over a finite time horizon. As a consequence, we apply a dynamic programming principle to solve the Hamilton-Jacobi-Bellman (HJB) equation explicitly in the case of the CRRA utility function. A case study is given to illustrate our results and to analyze the effect of the parameters on the optimal asset allocation strategy.

Keywords: Bank asset allocation, Stochastic interest rates, Dynamic programming principle, HJB equation, CRRA utility.

1 Introduction

We study an optimal asset allocation problem with a stochastic interest rates which take into account specific features of bank. The goal is to present the numerical aspects of the resolution of the HJB equation and to focus on the results of the optimal asset allocation model taking a practicle viewpoint.

This is motivated by the need for banks to invest in assets with an acceptable level of risk and high return. For instance, if the return on a specific loan turns out to be very high at the end of a loan contract period, the bank might regret not having allocated a large enough portion of its capital to that loan type. A dynamic portfolio position is particularly important in bank risk management, since most banks select an initial loan portfolio at the beginning of a loan period but often do not actively manage their portfolio thereafter unless a possibility of default arises. Another motivation for discussing bank optimal asset allocation is that the failures spark risk management strategies and regulatory prescripts to mitigate this risk. One of these prescriptions is the Basel Accord on capital adequacy requirements[1], which mandates that all major international banks hold capital in proportion to their perceived risks. An internal model may be used by banks to make an assessment of their portfolio risk and to determine the capital requirement.

In our study, we propose to apply the model in a simplified framework in order to find an analytically tractable solution for the bank asset allocation problem. In particular, the representative bank dynamically allocates her wealth among the following assets: bank account, loans and securities. (i) The asset prices assumed to satisfy the geometric Brownian motion hypothesis which implies that asset prices are stationary and log-normally distributed. All expected asset returns are given as the interest rate plus a constant risk premium (ii) the interest rates are described by an Ornstein Uhlenbeck process and notably the case of Vasicek model; (iii) and the optimal asset allocation strategy are derived with power utility function. Then, a dynamic programming principle is used to derive the HJB equation. We find a closed form solution for the optimal asset allocation problem. Furthermore, we try to provide, through a case study on a Tunisian bank, a new insight of the model in terms of practical use.

The rest of the paper is organised as follows: In Section 2, we present the relevant literature. In Section 3, we introduce the asset allocation model for bank. In Section 4, we define and solve the optimization problem in the power utility case. In Section 5, we illustrate numerically our results and the last section we draw the conclusion.

2 Literature review

The bank asset allocation plays an increasingly important role in banks and other financial institutions and the attention paid to this topic has grown commensurately in recent years. An important problem in asset allocation is to characterize the optimal rebalancing pattern of assets through time. The method to deal with it has been the maximization of

[1]Basel III regulation establishes procedures for assessing credit, market, and operational risk (See, BCBS, 2011)

expected utility from terminal wealth. Mulaudzi et al. [20] consider an optimal allocation problem for bank funds in treasuries and loans in a risk and regret theoretic framework. A second asset allocation problem is considered in Fouche et al. [8], they illustrate that it is possible to use an analytic approach to optimize asset allocation strategies for banks. They formulate an optimal bank valuation problem via optimal choices of loan rate and demand which leads to maximal deposits, provisions for deposit withdrawals and bank profitability subject to cash flow, loan demand, financing, and balance sheet constraints. Also, several studies have investigated the asset allocation problems using stochastic control theory, developed by Merton [13, 14] in discrete and continuous-time setting (See, for instance, Sørensen, [21], Kim and Omberg, [11], Wachter, [22], Campbell and Vieceira, [3], and Munk, Sørensen and Vinther,[20]). This approach must solve the nonlinear partial differential HJB equation to find the closed form solution for the value function, which is typically hard to solve. In particular, Mukuddem-Petersen and Petersen [16] examines a problem related to the optimal risk management of banks in a stochastic dynamic setting. They minimize market and capital adequacy risk that involves the safety of the assets held and the stability of sources of capital, respectively. The solution is determined by means of the dynamic programming algorithm. Similar to this, [18] determine an optimal rate at which additional debt and equity should be raised and strategy for the allocation of bank equity. The dynamic programming algorithm for stochastic optimization is employed to verify the results. Among others, a general case of maximization problem with CRRA utility function is discussed in Mukuddem-Petersen et al. [17] that determine an analytical solution for the associated HJB equation in the case where the utility functions are either of power, logarithmic or exponential type. In this case, the control variates are the depository consumption, value of the depository financial institutions investment in loans, and provisions for loan losses.

Furthermore, some recent papers using martingale approach in analyzing the behavior of bank . In Gideon et al. [10], by considering a theoretical quantitative approach for bank liquidity provisioning, the authors used martingale approach to solve a nonlinear stochastic optimal liquidity risk management problem for subprime originators with deposit inflow rates and marketable securities allocation as controls. In this case, they provide an explicit expression for the aggregate liquidity risk when a locally risk minimizing strategy is utilized. Thus, this martingale method frequently appears in research into the optimal design and asset allocation of a pension fund or life insurance policy (Boulier et al. [2], Deelstra et al. [5], Battocchio and Menoncin [1]...). Indeed, the partial differential equation derived is much simpler to solve than the highly non linear HJB equation which comes from the usual dynamic programming method.

However, the bank asset allocation problem with stochastic interest rates have been only discussed in the work of Witbooi et al.[23]. The main novel feature of their research is the combination of the interest rates model of Cox-Ingersoll and Ross and the Cox-Huang methodology to a banking fund portfolio consisting of three assets such as, treasuries, securities and loans. They obtain an explicit solution for the optimal equity allocation strategy that will optimize the terminal utility of the banks shareholders under a power utility function.

The model applied in this paper is close to the models of Korn [12], explore an optimal portfolio problem with defaultable assets in the framework of Merton's firm value model (See Merton [15]). A drawback of this approach is the assumption of a deterministic interest rate. The main difference between Kraft[13] and our model is he consider mixed defaultable bond and stock portfolio problems and give an explicit solutions for the value function and the optimal strategies while this study includes the bank loans and securities into asset portfolio management.

3 Bank asset allocation model

In this section, we show that the bank's assets may be modelled as random variables that are driven by an associated standard and independent Brownian motions and can be bought and sold without incurring any transaction costs or restriction on short sales. The uncertainty is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where, $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the filtration generated by the Brownian motions $W \equiv \{W(t), t \geq 0\}$.

The dynamics of a security price, $S(t)$, are presented by the following stochastic differential equation (SDE):

$$\frac{dS(t)}{S(t)} = (r(t) + \lambda_S)dt + \sigma_S dW_S(t) \tag{3.1}$$

Where, σ_S, is the securities volatility, λ_S, is the risk premium. Under the Capital Asset Pricing Model (CAPM) it could be quantified by the relation $\lambda_S = \beta[E(R_m) - R_f]$ with, $E(R_m)$ is the expected return of the market, R_f is the risk-free rate of interest and β is the sensitivity of the expected excess asset returns to the expected excess market returns.

The instantaneous interest rate dynamics $r(t)$ are described by an Ornstein-Uhlenbeck process:

$$dr(t) = \theta(\mu - r(t))dt + \sigma_r dW_r(t) \tag{3.2}$$

Where, the parameters θ, μ and σ_r are strictly positive constants and correspond to the degree of mean-reversion, the long-run mean and the volatility of the interest rates.

The interest rate term structure has the same form as in Vasicek (1977). In particular, the price of a zero coupon bond with time to maturity $(T - t)$ is given by:

$$P(r, t, T) = e^{-a(T-t)-b(T-t)r} \tag{3.3}$$

Where;

$$a(\tau) = R(\infty)((T-t) - b(T-t)) + \frac{\sigma_r^2}{4\theta}(b(T-t))^2$$

$$b(T-t) = \frac{(1-e^{-\theta(T-t)})}{\theta}$$

and $R(\infty) = \theta + \frac{\sigma_r \lambda_r}{\theta} - \frac{1}{2}\frac{\sigma_r^2}{\theta^2}$ represents the yield to maturity of a zero-coupon bond and λ_r denotes the constant interest rate risk premium.

Any loan is essentially an interest rate contingent claim and by Itô lemma the dynamics of the loan price $L(t)$ are assumed as the following form:

$$\frac{dL(t)}{L(t)} = (r(t) + \lambda_L)dt + \sigma_L dW_r(t) \tag{3.4}$$

We suppose that the bank grants loans at the interest rate on loans or loan rate as a sum of instantaneous interest rates, the market price of risk and the default risk primium. Here, $\lambda_L = \lambda_r \sigma_L + \delta$. As in Merton [15] the default risk premium, δ, is the credit spread charged by the bank and is the function[2] of the probability of default, PD, and the loss given default of loans, LGD. We assume that the loans available for the customer has a constant duration, D, of the interest rate contingent claim. Hence, the volatility of the loans is also constant and given by $\sigma_L = \sigma_r D$.

Let $X(t)$ denote the value of the bank asset portfolio at time $t \in [0, T]$ and $\pi_L(t)$, $\pi_S(t)$ are the proportions invested in the loans and securities, respectively. Then, $(1 - \pi_L(t) - \pi_S(t))$ is the proportion invested in the bank account. Owing to the independence of the Brownian motions and the self-financing assumptions the value of the asset portfolio can be expressed as the following stochastic process:

$$\begin{aligned}\frac{dX(t)}{X(t)} &= (1-\pi_L(t)-\pi_S(t))\frac{dB(t)}{B(t)} + \pi_L(t)\frac{dL(t)}{L(t)} + \pi_S(t)\frac{dS(t)}{S(t)} \\ &= (r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S)dt + (\pi_L(t)\sigma_L dW_r(t) + \pi_S(t)\sigma_S dW_S(t))\end{aligned} \tag{3.5}$$

Where, $X(0) = X_0$ stands for an initial wealth.

4 Bank optimization problem

Shareholders of a bank expect a good return on their capital investment while minimizing their risk. In fact, bank management needs to strategically allocate the shareholders equity in order to maximize the terminal wealth of the shareholders. However, the changes in the banks asset value are reflected in changes in the shareholders equity this incites the bank to maximize asset portfolio return versus risk. In this regard, the shareholders utility function is assumed to belong to the class of constant risk aversion (CRRA) utility functions.

[2] Spread = PD*LGD

The optimization problem on a time interval [0,T] is defined by:

$$\max_{\pi(t)} \mathbb{E}[U(X_T^{t,r,x})] \tag{4.1}$$

Where,

$$U(X(T)) = \frac{X(T)^{1-\gamma}}{1-\gamma} \tag{4.2}$$

For a well posed optimization problem one needs additional assumptions about admissible controls set to the effect that the SDE (3.5) below admits a unique, strong and almost surely positive solution. The set of admissible controls is given by:

$$\mathcal{A} = \left\{ \pi = \pi(t)_{t\in[0,T]}, \mathcal{F}\text{- adapted}, \int_0^T (\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2 \mathrm{d}t < +\infty, \mathbb{P} - a.s. \right\} \tag{4.3}$$

In this case, the basic source of uncertainty are due to changes in the interest rate or the value of the asset portfolio. Moreover, the state variables in equation (3.5) can be identified as the interest rates, $r(t)$, and the value of the asset portfolio, $X(t)$, and the control variables is the optimal proportion $\pi(t)$.

We are going to solve this problem via stochastic control.

Theorem 4.1 *: Suppose that $J \in C^{1,2}$ is a solution of the HJB equation*

$$J(t,r,x) = \max_{\pi(t)\in\mathcal{A}} \mathbb{E}[U(X_T^{t,r,x})] \tag{4.4}$$

and the optimal investment strategy $(\pi_L^*(t), \pi_S^*(t))$, if it exists

$$(\pi_L^*(t), \pi_S^*(t)) = \arg\max_{\pi(t)\in\mathcal{A}} J(t,r,X)$$

Proof. We provide a mere outline of the proof.

Let $\pi(t) \in \mathcal{A}$ and $X(t)$ the corresponding asset portfolio value process, hence, the Hamilton Jacobi Bellman (HJB) equation associated with the optimization problem is:

$$\begin{aligned} J_t + J_r(\theta(\mu - r(t))) + \frac{1}{2}J_{rr}\sigma_r^2 + \max_{\pi(t)\in\mathcal{A}}[XJ_X(r(t) + \pi_L(t)\lambda_L + \pi_S(t)\lambda_S) \\ + \frac{1}{2}J_{XX}X^2(\pi_L(t)\sigma_L)^2 + (\pi_S(t)\sigma_S)^2 + J_{Xr}X[(\pi_L(t)\sigma_L + \pi_S(t)\sigma_S)\sigma_r] = 0 \end{aligned}$$

Here, J_t, J_r, J_X, J_{rr}, J_{XX}, and J_{Xr} denote the first and second order partial derivatives with respect to t, r and X in the normal way.

By applying the first order conditions we get:

$$\begin{aligned} \pi_S^*(t) &= -\frac{\lambda_S}{\sigma_S^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_r}{\sigma_S}\frac{J_{Xr}}{XJ_{XX}} \\ \pi_L^*(t) &= -\frac{\lambda_L}{\sigma_L^2}\frac{J_X}{XJ_{XX}} - \frac{\sigma_r}{\sigma_L}\frac{J_{Xr}}{XJ_{XX}} \end{aligned}$$

The standard approach to solve this kind of PDE is to try for separability condition. In Merton ([13, 14]), the condition of separability in wealth by product represents a common assumption in the attempt to solve explicitly optimal portfolio problems. Specifically, in order to obtain smooth analytic solution to the maximization problem, we choose a power utility function. The value function J can be rewritten as:

$$J(t, r, X) = X^\gamma(T) f(t, r) \tag{4.5}$$

Substituting the partial derivatives of the value function (4.5) and the optimal proportions $\pi(t)^*$ into HJB equation, leads to a second-order PDE for f of the form:

$$\begin{aligned}(\gamma-1) f f_t + (\gamma-1) f f_r(\theta(\mu - r(t))) + \frac{1}{2}(\gamma-1) f f_{rr}\sigma_r^2 + \\ \gamma(\gamma-1) f^2 r - \frac{1}{2}\gamma f^2\left[\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right] - \gamma f f_r\left[\frac{\lambda_L}{\sigma_L} + \frac{\lambda_S}{\sigma_S}\right]\sigma_r - \gamma f_r^2\sigma_r^2 = 0\end{aligned}$$

With the terminal condition $f(T, r) = 1$ for all r.

Therefore, by conjencture, a soltion of J have the following form: $f(t, r) = g(t)\exp\left(A(t)r\right)$[3] that satisfies: $g(T) = 1, A(T) = 0$.

Simplifications yields:

$$\begin{aligned}&(\gamma-1).g' + \underbrace{(\gamma-1)(A'(t) + \gamma - A(t)\theta)}_{=:\alpha(t)}.rg + \\ &\underbrace{\left\{(\gamma-1)\theta\mu A(t) - \frac{1}{2}(\gamma+1)A^2(t)\sigma_r^2 - \gamma A(t)\left[\frac{\lambda_L}{\sigma_L} + \frac{\lambda_S}{\sigma_S}\right]\sigma_r - \frac{1}{2}\gamma\left[\frac{\lambda_L^2}{\sigma_L^2} + \frac{\lambda_S^2}{\sigma_S^2}\right]\right\}}_{=:h(t)}.g = 0\end{aligned}$$

The conjencture for f is only meaningful, if A can be calculated so the factor $\alpha(t)$ becomes zero. As a result we have to solve the inhomogeneous ODE for A which has the following form:

$$A'(t) = \theta A(t) - \gamma$$

With $A(t) = 0$ leading to: $A(t) = \frac{\gamma}{\theta}[1 - exp(\theta(T - t))]$.

Choosing A as calculated we again get a first order homogenuous ODE for g:

$$(\gamma-1)g' + h(t)g = 0$$

with $g(T) = 1$. Hence,

$$g(t) = exp\left[\frac{1}{\gamma-1}H(t) - H(T)\right]$$

[3]is the solution of PDE with g and A are regular function

with $H(t)$ is the primitive of $h(t)$,(See appendix A).

Therefore,

$$J(t,r,X) \quad = \quad X^{\gamma} exp\left[\left(\frac{1}{\gamma-1}H(t)-H(T)\right)+\frac{\gamma}{\theta}(1-exp(\theta(T-t)))r\right]$$

Then, the optimal solution of the asset allocation problem:

$$\begin{aligned}
\pi_S^*(t) &= \frac{1}{1-\gamma}\frac{\lambda_S}{\sigma_S^2} - \frac{\gamma\sigma_r k(t)}{1-\gamma} \\
\pi_L^*(t) &= \underbrace{\frac{1}{1-\gamma}\frac{\lambda_L}{\sigma_L^2}}_{\text{Merton result}} - \underbrace{\frac{\gamma\sigma_r k(t)}{1-\gamma}}_{\text{Correction term}}
\end{aligned} \tag{4.6}$$

With, $k(t) = \frac{1-e^{-\theta(T-t)}}{\theta}$.

The optimal proportions of power utility function are continuous function of time and directly related to the interest rate. The first term of (4.6) coincides with the classical optimal one in Merton ([13, 14]) when the coefficients are deterministic. The second term can be interpreted as a correction term which positive and monotonously decreasing to zero up to the terminal date T. Moreover, we discuss three optimal strategies for asset allocation that follow from theorem (4.1), viz, borrowing, short selling and overcapitalization. In the optimal solution (4.6) we observe that the bank borrows money in order to invest in securities in the case where the proportion of the securities is below loans. Short selling is what happens when the bank believes that the proportion of a securities decreasing. This security is then sold at a price with the purpose of buying it back some time in the future for less than its current value. In the case where overcapitalization occurs, the optimal strategy is to issue more loans and securities than the assets value.

5 Case study

This section considers the particular case with a bank portfolios loans and securities as a specific illustration of the general bank asset allocation results.

The parameters' estimation related to the risk and return of each asset is a more difficult task since the confidential nature and the little data sample. The estimation is based on maximum likelihood method[4] using historical data collected from over the counter market and the Tunisian stock exchange for the period 2004-2012. The interest rate parameters are adopted from the estimation of the Vasicek term structure[5] using the 52 weeks Treasury bond yields. Most bank loans are divided into corporate and consumer categories, respectively. However, we focus on corporate loans only because is no information

[4]See Appendix B and C for more detail

[5]See Chakroun and Abid, [4]

for consumer loans. The loans have constant duration similar to a 10 year zero-coupon bond. Using the historical bank financial statement data (Balance sheet and Cash flow statement) and by calibrating the Merton's [15] model we get the parameter of the loans volatility and the default risk premium[6]. Concerning the securities are represented by a portfolio of three SICAV [7] (SICAV Prosperity, SICAV Opportunity and SICAV Tresor) available on the Financial Market Council (CMF). The investment horizon is set to 10 years and the degree of risk aversion is $\gamma = 0.5$. The estimation results are displayed in Table 1.

[Insert Table 1 about here]

Fig.1 highlights how the evolution of the optimal asset allocation strategy is actually affected by the realization of the stochastic variables characterizing the economy. The optimal asset allocation strategy, shows that the optimal proportion invested in the bank account (represented by the downward sloping curve) decreases from 65.44% to 55%. On the other hand, the optimal proportion invested in the securities and loans increase with respect to time. In particular, the loans proportion raise from an initial value of about 19.78% to just above 25%, while the proportion invested in the securities increases from an initial value close to 14.78% to about 20%.

[Insert Figure 1 about here]

However, the bank account play a residual role in the optimal portfolio composition. At the beginning of the investment period, the need of an conservative strategy for creating a higher wealth level and a lower risk leads to a high proportion of bank account in the optimal portfolio, while the investment in the loans and securities asset is very low. Consistently, as the time approaches to the maturity T, a shift of wealth from the investment in bank account to the risky assets. The riskiness of strategy increases both the investment in securities and loans increases and the proportion of wealth invested in the bank account decreases. Then, the bank manager maintain diversified portfolio until maturity with a high percentage of wealth is allocated to the loans. These results is very intuitive

[6]

$$\text{Default risk premium} = R_i(\tau) - r = -\frac{1}{\tau}\ln\left\{\phi(h_2) + \frac{1}{d}\phi(-h_1)\right\}$$

Where;

$$\begin{aligned} d &= \frac{De^{-R(\tau)}}{L} \\ h_1 &= [\ln(d) - (r + \frac{1}{2}\sigma_L^2)\tau]/\sigma_L\sqrt{\tau} \\ h_2 &= h_1 - \sigma_L\sqrt{\tau} \end{aligned}$$

We denote by: $R(\tau)$ is the yield to maturity on the loans, $\tau = T - t$ is length of time until maturity, σ_L is the volatiltiy of the loans, D is the deposit reimbursed at time T and ϕ is the cumulative Normal distribution function.

[7] Investment company with variable capital held by the banque

and reasonable since it indicates that the bank optimal strategy is to borrow money to invest in securities.

To test how sensitive the optimal strategy is to changes in the different underlying variables, we have performed a sensitivity analysis, keeping the parameters in Table 1 and changing each time the value of one parameter.

Fig.2 shows for a given value of gamma ($\gamma = 0.5$), how the proportions are modified as time passes. With horizons arranging from 5 years to 15 years the proportion in bank account increase and remain positive. However, the allocation to loans and securities decrease as the investment horizon increases. As a result, it seems that a long horizon bank manager behaves more conservatively.

[Insert Figure 2 about here]

Fig.3 presents the effect of varying the degree of risk aversion. The optimal asset allocation strategy is quite sensitive to the risk aversion. For given time horizon the proportion invested in securities and loans increases with risk aversion. For shorter horizon and higher risk aversion, the proportion in bank account remain positive. However, The allocation to the asset are constant across time for a risk aversion lower than 0.8 and the investment behavior seems to be stabilized until maturity.

[Insert Figure 3 about here]

Fig.4 shows the effect of changes in the degree of mean-reversion parameter on the proportions of the optimal asset allocation strategy. Obviously, increasing the mean reverssion of the interest rate has a same impact on the allocation of the loans and securities. At time proche to maturity, the asset allocation is relatively insensitive to the interest rate mean reversion parameter. As a consequence, a strong correlation has been established between assets and interest rate. In order to monitor the fluctuations in the interest rate, in practice, the securities may partially be used to hedge real interest rate uncertainty.

[Insert Figure 4 about here]

Increasing interest rate volatility causes the bank manager to shift money from securities and loans into bank account. Therefore, this result can be explained by the fact that the loans and securities becomes more risky for the same risk premium.

[Insert Figure 5 about here]

Varying the long run mean of interest rate has no effect. This is linked to the assumption of CRRA utility, which yields proportions that are mainly a function of the risk premium and independent of the interest rate level.

6 Conclusion

This paper addresses the problem of optimal asset allocation for a bank. The bank shareholders has a power utility function and can invest in the bank account, loans and securities and tries to maximize her utility from terminal wealth in a complete market setting where the Vasicek term structure model applies. The solution approach is based on dynamic programming principle. Indeed, a verification theorem claim that the related HJB equation has a closed form solution under the separation condition. The estimation of parmaters is based on the maximum likelihood method. With this parameterization a case study confirms the practical potential of the results and shows that this model can adequately account for the essential aspects of the bank. The sensitivity analysis highlights the importance of dynamic considerations in optimal asset allocation depends on the stochastic characteristics of the investment opportunity set.

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References

[1] Battocchio, P. and Menocin, F. (2004). Optimal Pension Management in a Stochastic Framework. *Insurance: Mathematics and Economics*. 34, 79-95.

[2] Boulier, J.F. and Huang, S.J. and Taillard, G. (2001). Optimal Management under Stochastic Interest Rates: the Case of a Protected Defined Contribution Pension Fund. *Insurance: Mathematics and Economics*. 28,173-189

[3] Campbell, J.Y. and Viceira, L.M. (2002). Strategic Asset Allocation: Portfolio Choice for Long-Term Investors. *Oxford University Press*

[4] Chakroun, F. and Abid, F. (2013). A methodology to estimate the interest rates yield curve in illiquid market: the Tunisian case. *The Macrotheme Review*. 2, 18-37

[5] Deelstra, G. and Grasselli, M. and Koehl, P.F. (2003). Optimal investment strategies in the presence of a minimum guarantee. *Insurance: Mathematics and Economics*. 33, 189-207

[6] Duan, J.C. (1994). Maximum likelihood estimation using price data of the derivative contract. *Mathematical Finance*. 4,155-167

[7] Duan, J.C.,Gauthier, G. and Simonato, J.G. (2004). On the equivalence of the KMV and maximum likelihood methods for structural credit risk models. *Working Paper, University Toronto*

[8] Fouche, C.H. and Mukuddem- Petersen, J. and Petersen, M.A. and Senosi, M.C. (2008). Bank valuation and its connections with the subprime mortgage crisis and basel II capital accord. *Discrete Dynamics in Nature and Society*. 33, 189-207

[9] Gideon, F. and Mukuddem-Petersen, J. and Petersen, M.A. (2007). Minimizing banking risk in a lvy process setting. *Journal of Applied Mathematics*.

[10] Gideon, F. and Mukuddem-Petersen, J. and Petersen, M.A and Bernadine De Waal (2012). Bank liquidity and the global financial crisis. *Journal of Applied Mathematics*.

[11] Kim, T. S. and Omberg, E. (1996). Dynamic Nonmyopic Portfolio Behavior*Review of Financial Studies*, 9(1), 141-161

[12] Korn, R. (2008). Optimal portfolios: new variations of an old theme. *Computational Management Science*. 5, 289-304

[13] Merton, R.C. (1969). Lifetime portfolio selection under uncertainty: The continuous case. *The Review of Economics and Statistics*. 51, 247-257

[14] Merton, R.C. (1971). Optimal consumption and portfolio rules in a continuous time model. *The Journal of Economic Theory*. 3, 373-413

[15] Merton, R.C. (1974). On the Pricing of Corporate Debt: the Risk Structure of Interest Rates. *Journal of Finance*. 29,449-470

[16] Mukuddem-Petersen, J. and Petersen, M.A. (2006). Bank management via stochastic optimal control. *Automatica*. 42, 1395-1406

[17] Mukuddem-Petersen, J. and Petersen, M.A. and Schoeman, I.M. and Tau, B.A (2007). An application of stochastic optimization theory to institutional finance. *Applied Mathematical Sciences*. 1359-1385

[18] Mukuddem-Petersen, J. and Petersen, M.A. (2008). Optimizing asset and capital adequacy management in banking. *Journal of Optimization Theory and Applications*. 137-160

[19] Munk, C.,Sørensen, C., Vinther,T. N. (2004). Dynamic Asset Allocation under Mean-Reverting Returns, Stochastic Interest Rates and Inflation Uncertainty Are Popular Recommendations Consistent with Rational Behavior? *International Review of Economics and Finance*, 13, 141-166.

[20] Mulaudzi, M.P. and Petersen, M.A. and Schoeman, I.M. (2001). Optimal allocation between bank loans and treasuries with regret. *Optimization letters*. 6, 59-69

[21] Sørensen, C. (1999). Dynamic asset allocation and fixed income management. *Journal of Financial and Quantitative Analysis*, 34(4), 513-531.

[22] Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: an exact solution for complete markets. *Journal of Financial and Quantitative Analysis*. 37(1), 63-91

[23] Witbooi, Peter.J. and Garth, J. van Schalkwyk and Grant, E.Muller (2011). An optimal investment strategy in bank management. *Mathematical methods in the applied sciences*. 6, 59-69.

Appendix A

We have:

$$\underbrace{\left\{(\gamma-1)\theta\mu A(t)-\frac{1}{2}(\gamma+1)A^2(t)\sigma_r^2-\gamma A(t)\left[\frac{\lambda_L}{\sigma_L}+\frac{\lambda_S}{\sigma_S}\right]\sigma_r-\frac{1}{2}\gamma\left[\frac{\lambda_L^2}{\sigma_L^2}+\frac{\lambda_S^2}{\sigma_S^2}\right]\right\}}_{=:h(t)}=0$$

Then; $H(t)$ is the primitive of $h(t)$ by replacing A(t) we get:

$$H(t)=\frac{\gamma}{\theta}(t-\frac{e^{\theta\tau}}{\theta})\left\{\theta\mu(\gamma-1)-\frac{1}{2}(\gamma+1)\sigma_r^2\frac{\gamma}{\theta}(t-\frac{2e^{\theta\tau}}{\theta}+\frac{e^{2\theta\tau}}{2\theta})-\gamma\left[\frac{\lambda_L}{\sigma_L}+\frac{\lambda_S}{\sigma_S}\right]\sigma_r\right\}-\frac{1}{2}\gamma\left[\frac{\lambda_L^2}{\sigma_L^2}+\frac{\lambda_S^2}{\sigma_S^2}\right]$$

Appendix B

The estimation of the Vasicek parameters which maximise the likelihood function. Given N observations of 52-week Treasury bond yields $\{r_{t_i}, i=1,...,N\}$. The likelihood function is as follows:

$$L(\psi)=\prod_{i=1}^{N-1}p(r_{t_{i+1}}|r_{t_i};\psi;\Delta) \tag{6.1}$$

with Δt time step, $\psi\equiv(\theta,\mu,\sigma)$ a parameter vector to be estimated and $p(r_i|\psi)$ defined as the transition function of the Vasicek and CIR process respectively. Then, the log-likelihood function is,

$$\ln L(\psi)=\sum_{i=1}^{N-1}\ln p(r_{t_{i+1}}|r_{t_i};\psi;\Delta t) \tag{6.2}$$

Therefore, the maximum likelihood estimator $\hat{\psi}$ of parameter vector ψ is:

$$\psi\equiv(\hat{\theta},\hat{\mu},\hat{\sigma})=\arg\max_{\psi}\ln L(\psi) \tag{6.3}$$

Moreover, the application of the maximum likelihood requires the specification of the transition function of each process. Hence, the conditional density function for Vasicek model is given by:

$$p(r_{t+\Delta t}|r_t;\psi,\Delta t) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{\left(r_{t_{i+1}} - r_{t_i}e^{-\theta\Delta t} - \mu(1-e^{-\theta\Delta t})\right)^2}{2\sigma^2}\right] \tag{6.4}$$

With, $\sigma^2 = \hat{\sigma}^2 \frac{1-e^{-2\theta\Delta t}}{2\theta}$. The corresponding log-likelihood function is:

$$\ln L(\psi) = -\frac{N}{2}\ln(2\pi) - N\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}\left[r_{t_{i+1}} - r_{t_i}e^{-\theta\Delta t} - \mu\left(1-e^{-\theta\Delta t}\right)\right]^2 \tag{6.5}$$

Appendix C

Let X, Y and Z be three random variables define the securities SICAV Opportunity, SICAV Prosperity and SICAV Tresor, respectively. In order to determine the maximum likelihood (ML) function, we only consider random variables with multivariate normal transition density,

$$f(x;\mu,\Sigma) = (2\pi)^{-n/2}\frac{1}{(\det\Sigma)^{1/2}}e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

Where, $\mu \in \mathbb{R}^N$ is the mean vector and $\Sigma \in \mathbb{R}^{N\times N}$ is symmetric and positive definite covariance matrix. Where,

$$\Sigma^{-1} = \begin{bmatrix} K_{1,1} & \cdots & K_{1,n} \\ \vdots & \ddots & \vdots \\ K_{n,1} & \cdots & K_{n,n} \end{bmatrix}$$

The maximum likelihood estimation (MLE) of the parameters (μ,Σ) of a multivariate distribution $X \sim N(\mu,\Sigma)$ can be solved efficiently as a convex optimization problem.

The log-likelihood function of the observations is:

$$L(x;\mu,\Sigma) = \ln\prod_i f(x_i) = -\frac{N}{2}\ln\det(\Sigma) - \frac{1}{2}\Sigma_i(x_i-\mu)^T\Sigma^{-1}(x_i-\mu)$$

Define the sample estimates $\bar{\mu} = \frac{1}{N}\sum_{i=1}^{N}x_i$ and $\bar{\Sigma} = \frac{1}{N}\Sigma_{i=1}^{N}(x_i-\bar{\mu})(x_i-\bar{\mu})^T$. Then the likelihood function can written as

$$L(x;\mu,\Sigma) = \frac{N}{2}(-\ln(\det(\Sigma) - \mathrm{Tr}(\Sigma^{-1}\bar{\Sigma}) - (\mu-\bar{\mu})^T\Sigma^{-1}(\mu-\bar{\mu}))$$

Thus, the maximum likelihood problem is as follows:

$$\max \ln\det(K) - \mathrm{Tr}(K\bar{\Sigma})$$

In our implementation we used the Tunisia Stock Market index (TUNINDEX)to estimate the securities risk premium.

Appendix D

The estimation of Merton's [21] parameters are proposed by Duan et al. ([6, 7]) based on the transformed-data MLE method.

The loan value process follows the geometric Brownian motion, we can derive its discrete-time form with time step $\tau_i - \tau_{i-1} = h$ as:

$$\ln L_{\tau_{i+1}} = \ln L_{\tau_i} + (\mu - \frac{\sigma_L^2}{2})h + \sigma\sqrt{h\epsilon_{i+1}}$$

Where, $\epsilon_i, i = 1, N$ are i.i.d standard normal random variables. We denote the log-likelihood function of observed data set under a specific model as $L(\theta; \text{data})$ where θ is the set of unknown parameters under the model. The MLE is to find the value of θ at which the data set has the highests likelihood of occurrence under Merton [17] model and assuming that one could directly observe the firm's loan values $\{L_0, L_k, ..., L_{nk}\}$ the log-likelihood function could be written as:

$$L^L(\mu, \sigma; L_0, L_k, ..., L_{nk}) = -\frac{n}{2}\ln(2\pi\frac{\sigma^2}{2}h) - \frac{1}{2}\sum_{k=1}^{n}\frac{(R_k - (\mu - \sigma^2)h)^2}{\sigma^2 h} - \sum_{k=1}^{n}\ln L_{kh}$$

Where, $R_k = \ln\left(\frac{L_{kh}}{L_{(k-1)h}}\right)$.

Recognizing that Merton's [15] model implicitly provides a one to one smooth relationship between the equity $E(t)$ and loan values, one can invoke the standard transformations to derive the log-likelihood function solely based on the bank observed equity data. If we denote the density of the loan value as $f(L)$, the density associated with the equity will be given by $f(L)|\frac{\delta g(L;\sigma)}{\delta L}|$. Applying this knowledge yields the following log-likelihood function on the bank observed equity data:

$$L^E(\mu_L, \sigma_L; E_0, E_h, ..., E_{nh}) = L^L\left(\mu_L, \sigma_L; \hat{L}_0(\sigma_L), \hat{L}_h(\sigma_L), ..., \hat{L}_{nh}(\sigma_L)\right) - \sum_{k=1}^{n}\ln(\Phi(\hat{d}_{kh}(\sigma_L))) \tag{6.6}$$

Where, $\hat{L}_{kh}(\sigma_L) = g^{-1}(E_{kh}; \sigma_L)$ and $\hat{d}_{kh} = \frac{\ln(\hat{L}_{kh}(\sigma_L/F) + (r + \frac{\sigma_L^2}{2}(T-kh)}{\sigma_L\sqrt{T-kh}}$. One can easily find the maximum-likelihood estimates by numerically maximizing the function 6.6 based on the function fmincon in MATLAB. Then, the estimation procedure is as follows:

Step 1: Estimate the Merton[15] model using the MLE function in equation 6.6

Step 2: Compute λ_L which represent the point estimates for credit spread.

Table 1: Parameter values

Definition	Symbol	value	Standard Error
Interest rate premium	λ_r	0.0002	5.26×10^{-5}
Mean rate	μ	0.044	$6,96 \times 10^{-5}$
Volatility	σ_r	0.0003	4.47×10^{-5}
Mean reversion	θ	0.0037	2.53×10^{-4}
Risk premium	λ_S	0.007	2.27×10^{-4}
Securities volatility	σ_S	0.046	1.5×10^{-4}
Default risk premium	δ	0.023	3×10^{-4}
Investment horizon	T	10	

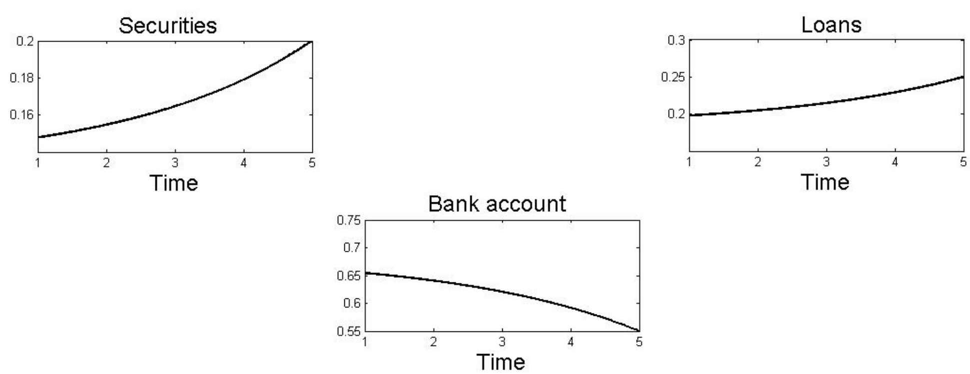

Figure 1: Optimal proportions invested in bank account, loans and securities.

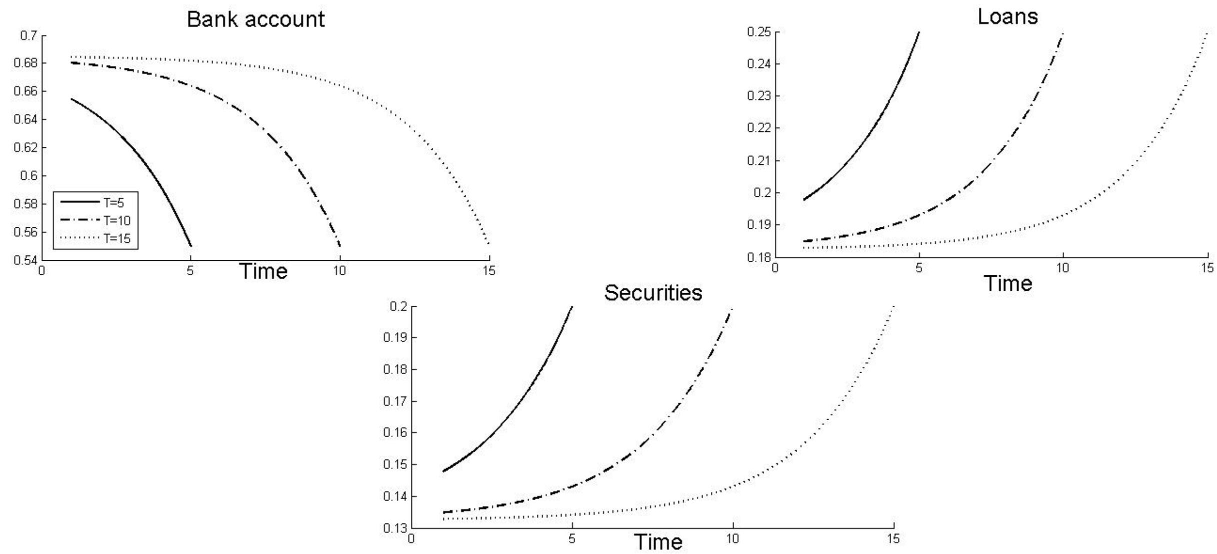

Figure 2: Effect of time.

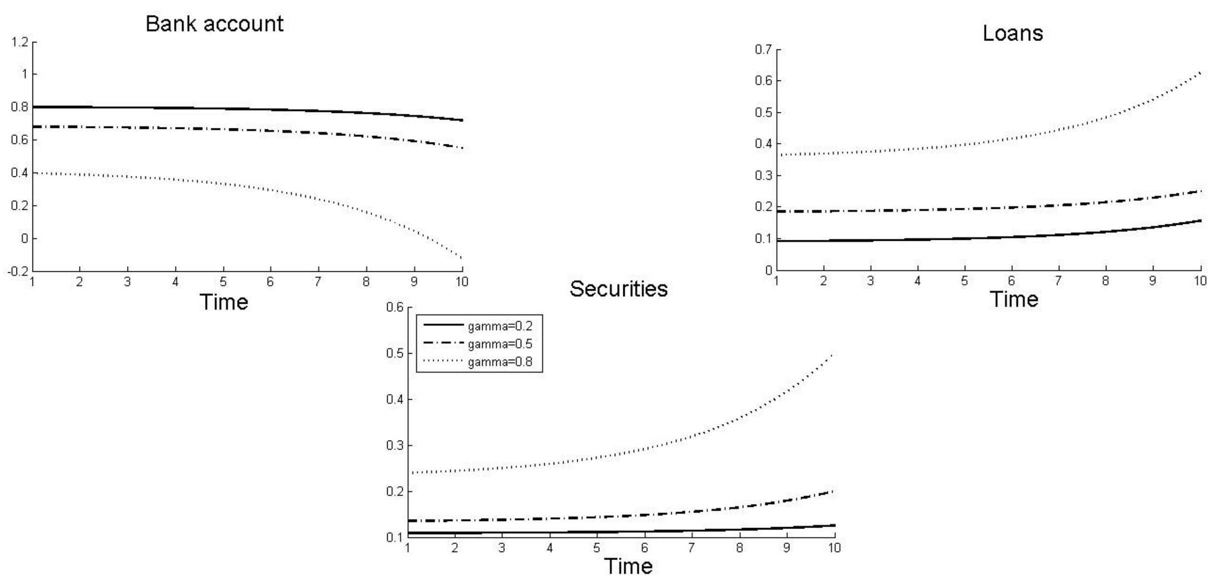

Figure 3: Effect of the degree of risk aversion.

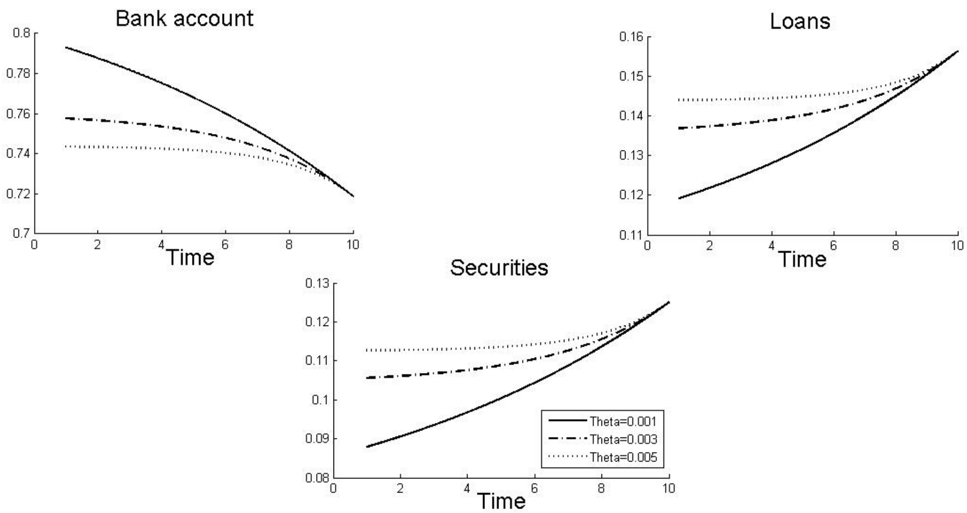

Figure 4: Effect of the degree of mean-reversion parameter.

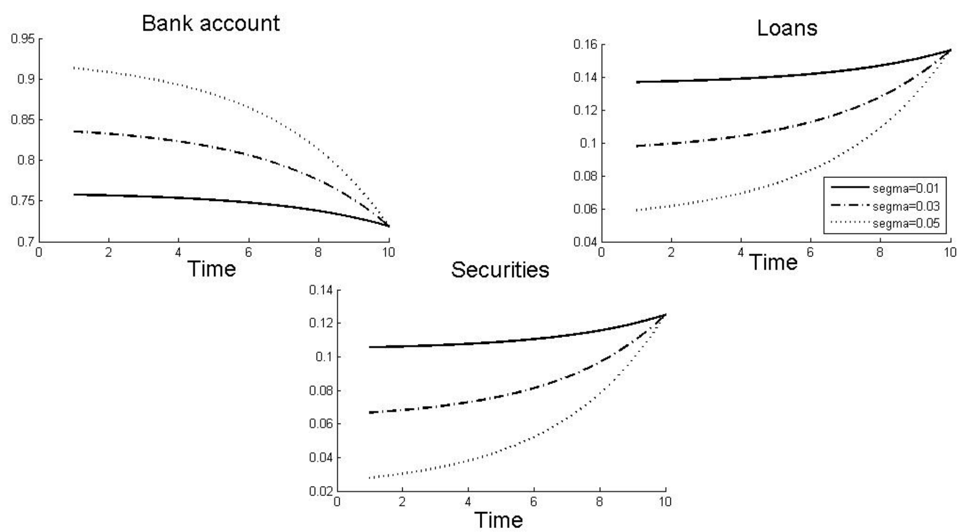

Figure 5: Effect of the interest rate volatility.