A Linear Multi-Sector Equilibrium Model with Taxation

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Abstract

This paper presents a linear multi-sector equilibrium model with taxation, which includes two types of consumers (i.e. laborers and capitalists). In the model capitalists exploit laborers by levying taxes. Hence, the issues of the income distribution between laborers and capitalists can be studied in the form of tax issues. This model covers the two types of simple reproduction models as its special cases, that is, the simple reproduction model wherein agents exchange commodities by labor values (i.e. the surplus value model), and the simple reproduction model wherein agents exchange commodities by production prices (i.e. the profit-price model). Hence this is a more general framework for the analyzing of the simple reproduction. After the development of the model, the effects of levying the wage tax, value-added tax and the turnover tax are discussed. And the possible efficiency loss of levying the turnover tax is also illustrated.

Keywords: Tax, Linear Model, General Equilibrium, Marxian Economics

1 Introduction

Linear multi-sector models adopting Leontief-type production functions and utility functions have relative simple mathematical structures and quite abundant economic implications, which are convenient to be analyzed with the matrix theory and can be used to explain and study a variety of economic issues. Hence such models were widely applied and studied not only by early classical economists, e.g. Marx's reproduction model (Marx, 1956), but also by contemporary economists, e.g. Leontief (1936, 1941), Dorfman, Samuleson and Solow (1958), Sraffa (1960), Gale (1960) and in other voluminous works of the input-output analysis.

Marx (1956) and Sraffa (1960) investigated linear multi-sector simple reproduction models without joint production, and such type of models may also be referred to as zero-growth linear multi-sector models without joint production. They analyzed two types of simple reproduction models. The first type of the models assumes that capitalists yield equal profit from each unit of labor, and in this case the equilibrium price of each commodity equals the labor value it contains when the wage rate is set equal to unity. The second type of model assumes that capitalists yield equal profit from each unit of capital, and in this case the equilibrium price of each commodity equals its production price. Samuelson (1971) pointed out that the profit in the two cases can be regarded as distinct types of tax. The profit in the first case can be regarded as the value-added tax, and the profit in the second case can be regarded as the turnover tax. However, Samuelson didn't develop the idea into a mathematical model. And this idea appears to receive little further attention after Samuelson (1971).

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Almost at the same time as Samuelson (1971), Metcalfe and Steedman (1971) analyzed the effects of taxation in the Sraffian linear price-profit model. Their model contains two producer sectors. Later Semmler (1983) extended the work of Metcalfe and Steedman (1971), and discussed the effects of wage taxes, profits taxes and indirect taxes in a more general linear price-profit model. Then Gehrke and Lager (1995) applied the linear price-profit model to studying the effects of environmental taxes on relative prices and on the choice of technique. These researches on the linear taxation models and the effects of taxation are mainly based on the Sraffian framework, that is, the price-profit framework, and the surplus model is ignored at large. This paper will try to attach more attention to the surplus value model and develop the idea of Samuelson (1971) further by building a linear tax model, and the model will cover the two types of simple reproduction models as its special cases.

The following notations and terminologies will be used. $\mathbf{1}$ denotes the vector $(1,1,\ldots,1)^T$. A vector $\mathbf{x}$ is called positive (or nonnegative) and we write $\mathbf{x} \gg \mathbf{0}$ (or $\mathbf{x} \geq \mathbf{0}$) if all its components are positive (or nonnegative). $\mathbf{x}$ is called semipositive and we write $\mathbf{x} > \mathbf{0}$ if $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$. For vectors $\mathbf{x}$ and $\mathbf{y}$, we write $\mathbf{x} \gg \mathbf{y}$, $\mathbf{x} > \mathbf{y}$ and $\mathbf{x} \geq \mathbf{y}$ analogously. Such notations and terms are also used for matrices. $(\mathbf{x};\mathbf{y})$ denotes $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$.

2 The Model

Let's consider an n-sector, n-commodity economy. The n-th commodity is labor and is supplied by the laborer sector (i.e. the n-th sector). The other $n-1$ commodities are producible goods and supplied respectively by the $n-1$ producer sectors. All producers possess Leontief-type production functions. The laborer sector consists of homogeneous $\omega(>0)$ laborers, and each laborer supplies one unit of labor. A laborer has no income other than wage, which is paid before production. Suppose there is another type of consumer (i.e. capitalists) in the economy besides laborers, whose income comes from taxation (i.e. transfer payment). Because the number of capitalists is insignificant for the analysis in this paper, let's suppose there is only one capitalist for simplicity. Laborers and the capitalist expend all their income on consumption. Suppose laborers and the capitalist have constant demand structures, that is, they have Leontief-type utility functions. Hence the commodity bundle purchased by each laborer can be represented by $u^L\mathbf{d}^L$, wherein the exogenous semipositive vector $\mathbf{d}^L$ satisfying $\mathbf{1}^T\mathbf{d}^L = 1$ indicates the consumption structures of each laborer, and the nonnegative real number u^L represents the consumption level (i.e. the utility level) of each laborer. The commodity bundle purchased by the capitalist is $u^C\mathbf{d}^C$. Similarly, the exogenous semipositive vector $\mathbf{d}^C$ satisfying $\mathbf{1}^T\mathbf{d}^C = 1$ indicates the demand structure of the capitalist, and the nonnegative real number u^C represents the consumption level of the capitalist.

The technology coefficient of $n-1$ producer sectors and the consumption bundle of each laborer constitute an input coefficient matrix with a parameter u^L as follows:

$$\mathbf{A}(u^L) = \begin{bmatrix} \bar{\mathbf{A}} & u^L\mathbf{d}^L \\ \bar{\mathbf{l}} & 0 \end{bmatrix}$$

wherein $\bar{\mathbf{A}}$ is the $(n-1)\times(n-1)$ intermediate input coefficient matrix, and $\bar{\mathbf{l}}$ is the labor input coefficient vector. The consumption level $u^L(>0)$ of each laborer is a parameter of the input coefficient matrix. In this paper we make the following assumptions for the input coefficient matrix.

Assumption 1 The labor input coefficient vector is positive, i.e. $\bar{\mathbf{l}} \gg \mathbf{0}$.

Assumption 2 $u^L > 0$, and $\mathbf{A}(u^L)$ is an indecomposable nonnegative matrix.

Assumption 3 The spectrum radius of $\bar{\mathbf{A}}$ is smaller than 1.

Assumption 3 is the productivity assumption, which is widely-used in various linear economic models. By the properties of the M-matrix in the matrix theory (Horn, Johnson, 1990;

Bapat, Raghavan, 1997), the following lemma holds.

Lemma 1 Given a positive real number s and a nonnegative square matrix $\bar{\mathbf{A}}$, the following statements are equivalent:

(i) The spectrum radius of $\bar{\mathbf{A}}$ is smaller than s;

(ii) There exists $\mathbf{x} > \mathbf{0}$ such that $\bar{\mathbf{A}}\mathbf{x} \ll s\mathbf{x}$ holds;

(iii) There exists $\mathbf{x} > \mathbf{0}$ such that $\mathbf{x}\bar{\mathbf{A}} \ll s\mathbf{x}$ holds;

(iv) There exists $\mathbf{x} \geq \mathbf{0}$ such that $\bar{\mathbf{A}}\mathbf{x} \ll s\mathbf{x}$ holds;

(v) There exists $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{x}\bar{\mathbf{A}} \ll s\mathbf{x}$ holds;

(vi) $s\mathbf{I} - \bar{\mathbf{A}}$ is nonsingular;

(vii) $\left(s\mathbf{I} - \bar{\mathbf{A}}\right)^{-1}$ exists and is a semipositive matrix;

(viii) $\left(s\mathbf{I} - \bar{\mathbf{A}}\right)^{-1}$ exists and is equal to $\left(\mathbf{I} + \sum_{k=1}^{\infty}\left(\bar{\mathbf{A}}/s\right)^k\right)/s$.

Let the wage rate (i.e. the price of a unit of labor) be 1. Thus the income of each laborer equals 1. Let the n-dimensional semipositive vector $\mathbf{p} = (\bar{\mathbf{p}}, 1)$ represent the prices of all commodities. The $(n-1)$-dimensional semipositive row vector $\bar{\mathbf{p}}$ represents the prices of $n-1$ products, and is referred to as the product price vector. When the consumption level u^L of each laborer, the input coefficient matrix $\mathbf{A}(u^L)$ and the price vector $\mathbf{p}$ are given, the vector $\mathbf{pA}(u^L)$ is the unit cost vector excluding tax. The i-th $(i = 1, 2, \cdots, n-1)$ component of $\mathbf{pA}(u^L)$ indicates the tax-exclusive cost of sector i for producing a unit of product, and the n-th component indicates the tax-exclusive living cost (i.e. the consumption expenditure) of a laborer.

Here let's suppose the capitalist obtains profit or surplus value by levying tax on production sectors and/or laborers, and all tax are expended by capitalist on his consumption. Suppose the tax system can be represented by a linear tax function $\tau\mathbf{pT}$. Here $\tau \geq 0$ is the exogenous tax rate. $\mathbf{T}$ is an exogenous $n \times n$ semipositive matrix, which indicates the methods of levying tax and is referred to as the tax structure matrix. In a more general model, the tax function can be a non-linear function $\tau\mathbf{t}(\mathbf{p})$, or adopts an even more complex form (e.g. $\tau\mathbf{t}(\mathbf{p}, u^L)$). However, in this paper we restrict our focus to the above-mentioned linear tax function.

Hence, the tax-inclusive unit cost vector of each agent is $\mathbf{pA}(u^L) + \tau\mathbf{pT}$. In a zero growth equilibrium, the revenue of each producer obtained by selling one unit of commodity equals the tax-inclusive unit cost, and the consumption expenditure plus tax of each laborer is equal to his wage. Thus we have the following price equilibrium equation (i.e. the cost-revenue equation):

$$\mathbf{pA}(u^L) + \tau\mathbf{pT} = \mathbf{p} \tag{1}$$

Now let's turn to the quantity aspect of the equilibrium.

Let $\mathbf{y} := (\bar{\mathbf{y}}; \omega)$ stand for the n-dimensional semipositive supply vector, which is also referred to as the output vector. The $(n-1)$-dimensional semipositive row vector $\bar{\mathbf{y}}$ denotes the supply amounts (i.e. the output amounts) of $n-1$ types of products, and $y_n = \omega$ denotes the exogenous supply amount of labor.

When the consumption level u^L of each laborer, the input coefficient matrix $\mathbf{A}(u^L)$ and the output vector $\mathbf{y}$ are given, the vector $\mathbf{A}(u^L)\mathbf{y}$ is the demand vector (and the input vector as well). The i-th component of $\mathbf{A}(u^L)\mathbf{y}$ indicates the input amount of commodity i required by the production processes of producers and the consumption processes of laborers under the given conditions. In the tax-inclusive case, a part of the output is consumed by capitalist, and the remainder is used for production and laborers' consumption. In a zero-growth economic equilibrium, the sum of the input vector $\mathbf{A}(u^L)\mathbf{y}$ and the consumption bundle $u^C\mathbf{d}^C$ of the capitalist equals the output vector. That is, in equilibrium the demand for each commodity is equal to its supply. Hence we have the following quantity equilibrium equation (i.e. the demand-supply equation):

$$\mathbf{A}(u^L)\mathbf{y} + u^C\mathbf{d}^C = \mathbf{y} \tag{2}$$

Formula (1) and (2) constitute a tax-inclusive linear multi-sector zero-growth equilibrium model, wherein u^L, u^C, $\mathbf{p}$ and $\mathbf{y}$ are unknown variables. By Formula (1) and (2), we can solve the equilibrium price vector, the equilibrium output vector and the equilibrium consumption levels of laborers and the capitalist.

By right multiplying both sides of Formula (1) with $\mathbf{y}$, and left multiplying both sides of Formula (2) with $\mathbf{p}$, we see that $\tau\mathbf{pTy} = \mathbf{p}u^C\mathbf{d}^C$ holds. That is, the total tax amount in equilibrium is equal to the consumption expenditure of the capitalist.

3 Equilibrium without Taxation

3.1 Equilibrium Equations without Taxation

When the tax rate equals 0, all net outputs are consumed by laborers and there is no exploitation. In such a case the equilibrium equations (1) and (2) become as follows:

$$\mathbf{pA}(u^L) = \mathbf{p} \tag{3}$$

$$\mathbf{A}(u^L)\mathbf{y} = \mathbf{y} \tag{4}$$

The price equation (3) can be written as

$$\bar{\mathbf{p}}(\mathbf{I} - \bar{\mathbf{A}}) = \bar{\mathbf{l}} \tag{5a}$$

$$u^L = 1/(\bar{\mathbf{p}}\mathbf{d}^L) \tag{5b}$$

The quantity equation (4) can be written as

$$(\mathbf{I} - \bar{\mathbf{A}})\bar{\mathbf{y}} = \omega u^L \mathbf{d}^L \tag{6a}$$

$$\bar{\mathbf{l}}\bar{\mathbf{y}} = \omega \tag{6b}$$

By Formulas (3)-(6), the existence and uniqueness of equilibrium can be analyzed conveniently by utilizing the properties of the nonnegative square matrix and the Perron-Frobenius theorem in the matrix theory.

By Lemma 1 and Assumption 1 (i.e. $\bar{\mathbf{l}}$ is a positive vector), there exists nonnegative vector $\bar{\mathbf{p}}$ satisfying the price equation (5) if and only if the spectrum radius of the nonnegative matrix $\bar{\mathbf{A}}$ is smaller than 1. Let $\boldsymbol{\lambda} := \bar{\mathbf{l}}(\mathbf{I} - \bar{\mathbf{A}})^{-1}$, which is the labor value vector (Fujimori, 1982). By Lemma 1, $\boldsymbol{\lambda}$ is a positive vector. Furthermore, by Lemma 1 and Formula (6) we see that the equilibrium product price vector, denoted by $\bar{\mathbf{p}}^*$, is equal to $\boldsymbol{\lambda}$, and there is no other nonnegative vector $\bar{\mathbf{p}}$ satisfying the price equation (5). Hence we know that there exists the unique positive vector $\bar{\mathbf{p}}^* = \boldsymbol{\lambda}$ and the unique $u^{L*} = 1/(\boldsymbol{\lambda}\mathbf{d}^L) > 0$ satisfying the price equilibrium equation (5).

Recall that $\mathbf{A}(u^L)$ is assumed to be indecomposable when $u^L > 0$ holds. By Formula (3) and the Perron-Frobenius theorem the P-F (i.e. Perron-Frobenius) eigenvalue of $\mathbf{A}(u^{L*})$ equals 1, and $\mathbf{p}^* = (\bar{\mathbf{p}}^*, 1)$ is a left P-F eigenvector of $\mathbf{A}(u^{L*})$.

Furthermore, by the Perron-Frobenius theorem there exists a unique positive vector $\mathbf{y}^*$ satisfying Formula (4), i.e. $\mathbf{y}^* = (\bar{\mathbf{y}}^*; \omega) = (\omega(\mathbf{I} - \bar{\mathbf{A}})^{-1} u^{L*}\mathbf{d}^L; \omega)$, and there is no other nonnegative output vector satisfying Formula (4). Since $\bar{\mathbf{y}}^*$ is positive, $(\mathbf{I} - \bar{\mathbf{A}})^{-1}\mathbf{d}^L$ is also positive. The above discussion can be summarized as the following proposition.

Proposition 1 Under Assumption 1-3, the positive real number $u^{L*} = 1/(\boldsymbol{\lambda}\mathbf{d}^L)$, the positive vectors $\mathbf{p}^* = (\boldsymbol{\lambda}, 1)$ and $\mathbf{y}^* = (\omega(\mathbf{I} - \bar{\mathbf{A}})^{-1} u^{L*}\mathbf{d}^L; \omega)$ satisfy the price equilibrium equation (3) and the quantity equilibrium equation (4). Moreover, there is no other nonnegative real number u^L, semipositive vector $\mathbf{p} = (p_1, \cdots, p_{n-1}, 1)$ and semipositive vector $\mathbf{y} = (y_1, \cdots, y_{n-1}, \omega)^T$ satisfying Formula (3) and Formula (4).

3.2 Optimality of the Equilibrium without Taxation

In the equilibrium without taxation, all net outputs in the economy are consumed by laborers and there is no exploitation. This is the most favorable case for laborers. In other words, there is no other simple reproduction method to raise the consumption level of laborers. Or more formally, the following proposition holds.

Proposition 2 There are no nonnegative vector $\mathbf{y}' = (\bar{\mathbf{y}}', \omega)$ and real number $u^{L'}$ satisfying $\mathbf{A}(u^{L'})\mathbf{y}' = \mathbf{y}'$ and $u^{L'} > u^{L*} = 1/(\boldsymbol{\lambda}\mathbf{d}^L)$.

Proof Proof by contradiction. Suppose there exist a nonnegative vector $\mathbf{y}' = (\bar{\mathbf{y}}', \omega)$ and a real number $u^{L'}$ satisfy $\mathbf{A}(u^{L'})\mathbf{y}' = \mathbf{y}'$ and $u^{L'} > u^{L*} = 1/(\boldsymbol{\lambda}\mathbf{d}^L)$, then by the Perron-Frobenius theorem $\mathbf{y}'$ is positive. Hence $\mathbf{A}(u^{L*})\mathbf{y}' < \mathbf{y}'$ holds. Let $\mathbf{p}^*$ be a left P-F eigenvector of $\mathbf{A}(u^{L*})$, then by Proposition 1 $\mathbf{p}^*$ is positive. Hence we have $\mathbf{p}^*\mathbf{A}(u^{L*})\mathbf{y}' < \mathbf{p}^*\mathbf{y}'$, that is, $\mathbf{p}^*\mathbf{y}' < \mathbf{p}^*\mathbf{y}'$. There is a contradiction. Thus the proposition holds. ■

The above proposition can be regarded as a special case of the first welfare theorem.

3.3 Comparative Statics of Equilibrium without Taxation

When the exogenous values of $\bar{\mathbf{A}}$ and $\bar{\mathbf{l}}$ in the input coefficient matrix $\mathbf{A}(u^{L*})$ changes, the equilibrium consumption level u^{L*} of laborers will change accordingly. By Assumption 2, $\mathbf{A}(u^{L*})$ is an indecomposable nonnegative square matrix, and the P-F eigenvalue of an indecomposable nonnegative square matrix has the following properties (Horn, Johnson, 1990).

Lemma 2 Let $\mathbf{M}$ be an indecomposable nonnegative square matrix, and $\rho(\mathbf{M})$ denotes its P-F eigenvalue. When $\rho(\mathbf{M})$ is regarded as a function of all elements of $\mathbf{M}$, $\rho(\mathbf{M})$ is a continuous, strictly increasing function (i.e. $\mathbf{M} < \mathbf{M}'$ implies $\rho(\mathbf{M}) < \rho(\mathbf{M}')$). Moreover, when an element of $\mathbf{M}$ approaches infinity and the other elements keep constant, $\rho(\mathbf{M})$ will approach infinity.

Since the P-F eigenvalue of the input coefficient matrix $\mathbf{A}(u^{L*})$ is always 1, by Lemma 2 u^{L*} is a strictly decreasing functions of elements of $\bar{\mathbf{A}}$ and $\bar{\mathbf{l}}$. Moreover, when a component of $\bar{\mathbf{l}}$ approaches infinity, u^{L*} will approach 0. That is to say, if the labor amount required by a sector to produce a unit of product approaches infinity, the equilibrium consumption level of laborers will approach 0.

When the greatest element of $\bar{\mathbf{l}}$ approaches 0, the greatest component of $\boldsymbol{\lambda}\mathbf{d}^L$ will also approach 0, and $u^{L*} = 1/(\boldsymbol{\lambda}\mathbf{d}^L)$ will approach infinity. That is to say, when the labor amount required to produce a unit of product approaches 0, the equilibrium consumption level of laborers will approach infinity.

4 Equilibrium with Taxation

4.1 Analysis of the Price Equilibrium Equation

The price equilibrium equation (1) can be written as

$$\mathbf{p}\left(\mathbf{A}(u^L) + \tau\mathbf{T}\right) = \mathbf{p} \tag{7}$$

By Assumption 2, if $u^L > 0$ then $\mathbf{A}(u^L)$ is indecomposable. Thus Formula (7) implies the P-F eigenvalue (i.e. the spectrum radius) of $\mathbf{A}(u^L) + \tau\mathbf{T}$ equals 1.

By Lemma 2, when a semipositive tax structure matrix $\mathbf{T}$ is given, if the tax rate τ is high enough, then it's possible that the spectrum radius of $\mathbf{A}(u^L) + \tau\mathbf{T}$ is greater than 1 for any positive u^L. In such a case there is no semipositive vector $\mathbf{p}$ satisfying Formula (7), i.e. there is no equilibrium. In other words, a high tax rate may lead to the nonexistence of equilibrium.

Note that there exists equilibrium when the tax rate equals 0. Therefore when the tax-exclusive equilibrium consumption level u^{L*} of laborers decreases, say, to $u^{L*}/2$, by Lemma 2 the P-F eigenvalue of $\mathbf{A}(u^{L*}/2)$ is smaller than 1. Hence we can increase the tax rate τ such that the P-F eigenvalue of $\mathbf{A}(u^{L*}/2)+\tau\mathbf{T}$ equals 1. That is to say there exists a positive tax rate such that there exists equilibrium and the equilibrium consumption level of laborers equals $u^{*L}/2$. A tax rate under which there exists equilibrium is called a feasible tax rate. Hence we see that there must be some positive feasible tax rates. And the following proposition shows that all feasible tax rates constitute an interval.

Proposition 3 If a tax rate $\bar{\tau}(>0)$ is feasible, then each $\tau \in (0,\bar{\tau})$ is also a feasible tax rate.

Proof Let $u^{L*\prime}$ and $u^{L*\prime\prime}$ denote the equilibrium consumption levels of laborers under tax rates $\bar{\tau}$ and 0 respectively. By Formula (7) we see that $\rho\left(\mathbf{A}(u^{L*\prime})+\bar{\tau}\mathbf{T}\right)=1$ and $\rho\left(\mathbf{A}(u^{L*\prime\prime})\right)=1$ hold. Hence $\rho\left(\mathbf{A}(u^{L*\prime})+\tau\mathbf{T}\right)<1$ and $\rho\left(\mathbf{A}(u^{L*\prime\prime})+\tau\mathbf{T}\right)>1$ hold for each $\tau\in(0,\bar{\tau})$. By Lemma 2, $\rho(\mathbf{A}(u^L)+\tau\mathbf{T})$ is a continuous function of u^L, hence there exists $u^{L*}(u^{L*\prime}<u^{L*}<u^{L*\prime\prime})$ such that $\rho\left(\mathbf{A}(u^{L*})+\tau\mathbf{T}\right)=1$ holds. ■

If we regard the laborer equilibrium consumption level u^{L*} as a function of tax rate, then by Lemma 2 $u^{L*}(\tau)$ is a strictly decreasing function defined on a convex set.

When we partition the matrix $\mathbf{T}$ as $\mathbf{T}\equiv\begin{bmatrix}\bar{\mathbf{T}} & \bar{\bar{\mathbf{t}}}\\ \bar{\mathbf{t}} & \end{bmatrix}$, the price equilibrium equation (1) can be written as

$$\bar{\mathbf{p}}\left(\mathbf{I}-(\bar{\mathbf{A}}+\tau\bar{\mathbf{T}})\right)=\bar{\mathbf{l}}+\tau\bar{\mathbf{t}} \tag{8a}$$

$$u^L=\frac{1-\tau\mathbf{p}\bar{\bar{\mathbf{t}}}}{\bar{\mathbf{p}}\mathbf{d}^L} \tag{8b}$$

Because (8a) implies $\bar{\mathbf{p}}(\bar{\mathbf{A}}+\tau\bar{\mathbf{T}})\ll\bar{\mathbf{p}}$, by Lemma 1 we see that (8a) has a solution if and only if the spectrum radius of the matrix $\bar{\mathbf{A}}+\tau\bar{\mathbf{T}}$ is smaller than 1, and the solution is $(\bar{\mathbf{l}}+\tau\bar{\mathbf{t}})\left(\mathbf{I}-(\bar{\mathbf{A}}+\tau\bar{\mathbf{T}})\right)^{-1}$. And if the solution can make the equilibrium consumption level of laborers be positive, then it's an equilibrium product price vector.

By Lemma 1 we know that the solution is a positive vector, and each component of that is an increasing function of the tax rate τ. That is, when the wage rate equals 1, the production price of each product is no less than its labor value, and we have $(\bar{\mathbf{l}}+\tau\bar{\mathbf{t}})\left(\mathbf{I}-(\bar{\mathbf{A}}+\tau\bar{\mathbf{T}})\right)^{-1}\geq\boldsymbol{\lambda}$ holds. Hence by Formula (8) it's clear that the equilibrium product price vector $\bar{\mathbf{p}}^*$ and the equilibrium consumption level u^{L*} of laborers satisfy $u^{L*}\boldsymbol{\lambda}\mathbf{d}^L\leq u^{L*}\bar{\mathbf{p}}^*\mathbf{d}^L\leq 1$.

4.2 Analysis of the Quantitative Equilibrium Equation

The quantitative equilibrium equation (2) can be written as

$$\omega u^L\mathbf{d}^L+u^C\mathbf{d}^C=(\mathbf{I}-\bar{\mathbf{A}})\bar{\mathbf{y}} \tag{9a}$$

$$\bar{\mathbf{l}}\bar{\mathbf{y}}=\omega \tag{9b}$$

Hence we have

$$\bar{\mathbf{y}}=(\mathbf{I}-\bar{\mathbf{A}})^{-1}(\omega u^L\mathbf{d}^L+u^C\mathbf{d}^C) \tag{10}$$

By Formula (9) and (10) we have

$$\boldsymbol{\lambda}(\omega u^L\mathbf{d}^L+u^C\mathbf{d}^C)=\omega \tag{11}$$

and

$$u^C=\frac{\omega-\omega u^L\boldsymbol{\lambda}\mathbf{d}^L}{\boldsymbol{\lambda}\mathbf{d}^C} \tag{12}$$

Formula (11) and Formula (12) indicate that the labor value amount of the total consumption bundle is ω, and there is a tradeoff between u^L and u^C. After finding the equilibrium consumption level u^{L*} of laborers by Formula (8), the equilibrium consumption level u^{C*} of the capitalist can be solved by Formula (12). Then by Formula (10) we can solve the equilibrium output vector. Because $(\mathbf{I}-\bar{\mathbf{A}})^{-1}\mathbf{d}^L$ is positive, by Formula (10) it's clear that the equilibrium output vector is also positive.

Since the equilibrium consumption level u^{L*} of laborers is a strictly decreasing function of the tax rate τ, the equilibrium consumption level u^{C*} of the capitalist is a strictly increasing function of the tax rate τ. But the numerical example in Section 8 will show that this point may not hold if a producer sector has more than one technology (i.e. it has a non-Leontief-type production function).

5 Equilibrium Prices under Three Types of Taxes

The equilibrium prices may change with the tax system and the tax rate. Here we will focus mainly on three types of taxes as follows:

(i) Income tax levied on laborers, which is referred to as payroll tax in Metcalfe and Steedman (1971) and will be referred to as the wage tax hereafter in this paper;

(ii) Value-added tax levied on producer sectors;

(iii) Turnover tax levied on production sectors.

5.1 The Wage Tax

When $\tau(0 \leq \tau < 1)$ dollar tax is levied on each dollar of wage of laborers, the price equilibrium equation can be written as

$$\mathbf{pA}(u^L) + \tau\mathbf{p}\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix} = \mathbf{p} \tag{13}$$

wherein the tax structure matrix is $\mathbf{T} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}$. By Formula (13) and (8) we find the equilibrium price vector is $\mathbf{p}^* = (\boldsymbol{\lambda}, 1)$ and the equilibrium consumption level of laborers is $u^{L*} = (1-\tau)/(\boldsymbol{\lambda}\mathbf{d}^L)$.

Labor is a primary production factor whose supply amount is assumed to be exogenous. Levying tax directly on the income of a primary production factor is equivalent to a redistribution of ownership of the primary production factor. By the first welfare theorem, a redistribution of ownership (i.e. a transfer payment) won't lead to a loss of economic efficiency (Mas-Colell, Whinston, Green, 1995). Hence the equilibrium with a wage tax is also Pareto optimal. And an equilibrium price vector in this case is referred to as an optimal price vector. In this model an optimal price vector can be written as $\xi(\boldsymbol{\lambda}, 1), \xi > 0$.

Since laborers spend all income on consumption, levying tax on their income has the same effect with levying tax on their consumption expenditure (i.e. levying consumption tax). When levying $\tau'(\geq 0)$ dollar of tax on one dollar of expenditure, the price equilibrium equation is

$$\mathbf{pA}(u^L) + \tau'\mathbf{p}\begin{bmatrix}\mathbf{0} & u^L\mathbf{d}^L\\ \mathbf{0} & 0\end{bmatrix} = \mathbf{p}$$

And we have $\mathbf{p}^* = (\boldsymbol{\lambda}, 1)$ and $u^{L*} = \frac{1}{(1+\tau')\boldsymbol{\lambda}\mathbf{d}^L}$. Hence it's clear that when $\tau' = \frac{\tau}{1-\tau}$ holds, levying wage tax is equivalent to levying consumption tax.

5.2 The Value-added Tax

In the model of this paper, levying value-added tax means levying tax on the labor used in the production since there is no other primary production factor. In the simple reproduction model with a uniform rate of surplus value of Marx (1956), the surplus value can be regarded as the tax levied on labor. Producer sectors pay the same amount of tax for inputting a unit of labor, that is, a unit of labor used in different sectors will contribute the same amount of surplus value. Let $\mathbf{l} := (\bar{\mathbf{l}}, 0)$. Since the wage rate is assumed to be 1, now the price equilibrium equation is

$$\mathbf{pA}(u^L) + \tau\mathbf{l} = \mathbf{p} \tag{14}$$

Now the tax structure matrix is $\mathbf{T} = \begin{bmatrix}\mathbf{0}\\ \mathbf{l}\end{bmatrix}$. Formula (14) can also be written as

$$\mathbf{p}\begin{bmatrix}\bar{\mathbf{A}} & u^L\mathbf{d}^L\\ (1+\tau)\bar{\mathbf{l}} & 0\end{bmatrix} = \mathbf{p} \tag{15}$$

The equilibrium price vector and the equilibrium consumption level of laborers are computed to be

$$\mathbf{p}^* = ((1+\tau)\boldsymbol{\lambda}, 1) \tag{16a}$$

$$u^{L*} = \frac{1}{(1+\tau)\boldsymbol{\lambda}\mathbf{d}^L} \tag{16b}$$

That is, the current equilibrium consumption level of laborers is $(1+\tau)^{-1}$ times of the equilibrium consumption level of laborers in the zero-tax case.

From Formula (15) we see that in this model the tax-inclusive cost for sectors to use a unit of labor is $1+\tau$. Hence in equilibrium a producer sector faces the tax-inclusive price vector $(1+\tau)(\boldsymbol{\lambda}, 1)$ when calculating the cost, and the tax-inclusive price vector for a sector takes the distortionary effects of a tax into account.

Recall the optimal price vectors are $\xi(\boldsymbol{\lambda}, 1), \xi > 0$, here the tax-inclusive price vector is still an optimal price vector.

Since in this case the relative prices won't change with the tax rate, it's clear that even though a producer sector has more than one technology, in an equilibrium the producer sector won't change its technology in use when the value-added tax rate changes. That is to say, in that case there won't be any loss of economic efficiency. Since both levying value-added tax and levying wage tax won't lead to any loss of economic efficiency, those two tax systems will have the same economic effect when proper tax rates are chosen. In other words, the two types of tax are equivalent in our model. Hence hereafter we will only investigate the value-added tax and omit the wage tax.

5.3 The Turnover Tax

In this case, $\tau(\geq 0)$ dollar tax is levied on each dollar of expenditure of each producer sector, That is, equal tax will be levied on each dollar of capital. Now the tax structure matrix is $\mathbf{T} = \begin{bmatrix}\bar{\mathbf{A}} & \mathbf{0}\\ \bar{\mathbf{l}} & 0\end{bmatrix}$. The price equilibrium equation can be written as

$$(1+\tau)\left(\bar{\mathbf{p}}\bar{\mathbf{A}} + \bar{\mathbf{l}}\right) = \bar{\mathbf{p}} \tag{17a}$$

$$u^L = 1/(\bar{\mathbf{p}}\mathbf{d}^L) \tag{17b}$$

And (17a) is the Sraffian equilibrium price equation (Sraffa, 1960), which also can be written as

$$\bar{\mathbf{p}}\left((1+\tau)^{-1}\mathbf{I} - \bar{\mathbf{A}}\right) = \bar{\mathbf{l}} \tag{18}$$

Because Formula (18) implies $\bar{\mathbf{p}}\bar{\mathbf{A}} \ll (1+\tau)^{-1}\bar{\mathbf{p}}$, by Lemma 1 we know Formula (18) has a nonnegative solution if and only if the spectrum radius of $\bar{\mathbf{A}}$ (i.e. $\rho(\bar{\mathbf{A}})$ is smaller than $(1+\tau)^{-1}$. That is, the feasible tax rate interval is $\left(0, 1/\rho(\bar{\mathbf{A}}) - 1\right)$. Under a feasible tax rate, the equilibrium price vector is

$$\mathbf{p}^* = \left(\bar{\mathbf{l}}\left((1+\tau)^{-1}\mathbf{I} - \bar{\mathbf{A}}\right)^{-1}, 1\right)$$

And $\bar{\mathbf{l}}\left((1+\tau)^{-1}\mathbf{I}-\bar{\mathbf{A}}\right)^{-1}$ is the production price vector.

Moreover, the turnover tax can also be levied in another way, that is, to levy $\tau'(0 \leq \tau' < 1)$ dollar tax on each dollar of sales revenue of each producer sector. In an equilibrium it's equivalent to levy $\tau' p_i$ dollar of tax on each unit of product of each producer sector. That is, the tax structure matrix is $\mathbf{T}' = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$. The price equilibrium equation can be written as

$$\frac{1}{1-\tau'}(\bar{\mathbf{p}}\bar{\mathbf{A}}+\bar{\mathbf{l}}) = \bar{\mathbf{p}} \tag{19a}$$

$$u^L = 1/(\bar{\mathbf{p}}\mathbf{d}^L) \tag{19b}$$

Comparing Formula (19) with (17), we see that when $\tau' = \frac{\tau}{1+\tau}$ holds the two methods of levying tax have the same effect.

Moreover, by Formula (17) we see that each producer sector faces the tax-inclusive price vector $(1+\tau)(\bar{\mathbf{p}}, 1)$, which is

$$(1+\tau)\mathbf{p}^* = (1+\tau)\left(\bar{\mathbf{l}}\left((1+\tau)^{-1}\mathbf{I}-\bar{\mathbf{A}}\right)^{-1}, 1\right)$$

in equilibrium. That price vector may not be an optimal price vector. Therefore, if a producer sector has more than one technology, the choice of the producer sector may differ from that one in a Pareto optimal case. That is, in such a case levying turnover tax may result in a loss of economic efficiency. This point will be illustrated by a numerical example in Section 8.

6 Marx's Numerical Example

Marx (1956) supposed that each unit of labor contributes equal profit in his simple reproduction model in Chapter 20, Volume 2 of Capital. And in Volume 3 Marx supposed each dollar of capital contributes equal profit, which is the same assumption as of Sraffa (1960).

Here let's investigate the numerical example of Marx in Chapter 20, Volume 2 of Capital with the new model. There are three kinds of commodities in that example, i.e.:

(i) Means of production (i.e. capital goods), which is represented by iron hereafter;

(ii) Articles of consumption (i.e. consumption goods), which is represented by wheat hereafter;

(iii) Labor.

Laborers and capitalists consume only wheat. The number of laborers is assumed to be 3000, i.e. the supply amount of labor is 3000 units. The input coefficient matrix is

$$\mathbf{A}(\mathbf{u}^L) = \begin{bmatrix} 2/3 & 2/3 & 0 \\ 0 & 0 & u^L \\ 1/3 & 1/3 & 0 \end{bmatrix}$$

wherein $\mathbf{d}^L = (0,1,0)^T$. And the labor value vector is computed to be $\boldsymbol{\lambda} = (1,1)$.

6.1 Equilibrium without Taxation

When the tax rate equals zero (i.e. $\tau = 0$), the equilibrium price vector and supply vector are computed to be $\mathbf{p}^* = (1,1,1)$ and $\mathbf{y}^* = (6000, 3000, 3000)^T$ respectively. And the equilibrium consumption level of laborers and the capitalist are $u^{L*} = 1$ and $u^{C*} = 0$ respectively. The equilibrium production and consumption processes are shown in Table 1.

Table 1. Equilibrium Input-Output Processes without Taxation

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	4000	2000	0	0	6000
Wheat Input	0	0	3000	0	3000
Labor Input	2000	1000	0	0	3000
Tax	0	0	0	0	0
Output	6000	1000	3000	0	

6.2 Equilibrium with the Value-added Tax

In the example of Marx, the tax rate of the value-added tax (i.e. the surplus value rate) is $\tau = 1$. By Formula (16) the equilibrium price vector is computed to be $\mathbf{p}^* = (2,2,1)^T$. The equilibrium supply vector is $\mathbf{y}^* = (6000, 3000, 3000)^T$. And the equilibrium consumption level of laborers and the capitalist are $u^{L*} = 0.5$ and $u^{C*} = 1500$ respectively. The equilibrium production and consumption processes are shown in Table 2 and Table 3.

Table 2. Equilibrium Input-Output Processes with Taxation

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	4000	2000	0	0	6000
Wheat Input	0	0	1500	1500	3000
Labor Input	2000	1000	0	0	3000
Tax	2000	1000	0	0	3000
Output	6000	3000	3000	0	

Table 3. Price-measured Equilibrium Input-Output Processes with Taxation

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	8000	4000	0	0	12000
Wheat Input	0	0	3000	3000	6000
Labor Input	2000	1000	0	0	3000
Tax	2000	1000	0	0	3000
Output	12000	6000	3000	0	

6.3 Equilibrium with the Turnover Tax

First let's compute the tax rate of the turnover tax which will lead to the same result of distribution as the previous subsection. By Formula (19) we find $p_2^* = 2$. Further, the tax rate of the turnover tax (i.e. the profit rate) is computed to be $\tau = 1/5$, and the equilibrium price of iron is $p_1^* = 2$.

Hence the equilibrium price vector is $\mathbf{p}^* = (2,2,1)$. The equilibrium supply vector and the consumption levels keep unchanged. It's clear that in the example of Marx the equilibrium price vectors under the value-added tax and the turnover tax are the same. And the equilibrium input-output processes are still as shown in Table 2 and Table 3.

7 A Numerical Example with Distinct Organic Compositions

Producer sectors in the numerical example of Marx have the uniform organic composition, and consequently the equilibrium price vectors under the value-added tax and the turnover tax are the same. In this section, we modify the input coefficient matrix in the numerical example of Marx as

$$\mathbf{A}(\mathbf{u}^L) = \begin{bmatrix} 1/5 & 2/3 & 0 \\ 0 & 0 & u^L \\ 4/5 & 1/3 & 0 \end{bmatrix}$$

so that the two producer sectors have distinct organic compositions.

And we still assume that the number of laborers (i.e. the amount of the labor supply) is 3000.

7.1 Equilibrium without Taxation

When the tax rate equals zero (i.e. $\tau = 0$), the labor value vector is computed to be $\boldsymbol{\lambda} = (1,\ 1)$. And the equilibrium price vector and the equilibrium supply vector are $\mathbf{p}^* = (1,1,1)$ and $\mathbf{y}^* = (2500, 3000, 3000)^T$ respectively. The equilibrium consumption levels of laborers and the capitalist are $u^{L*} = 1$ and $u^{C*} = 0$ respectively. The equilibrium production and consumption processes are shown in Table 4.

Table 4. Equilibrium Input-Output Processes without Taxation

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	500	2000	0	0	2500
Wheat Input	0	0	3000	0	3000
Labor Input	2000	1000	0	0	3000
Tax	0	0	0	0	0
Output	2500	3000	3000	0	

7.2 Equilibrium with the Value-added Tax

Let the tax rate (i.e. the surplus value rate) be $\tau = 1$. By Formula (18) we compute the equilibrium price vector to be $\mathbf{p}^* = (2,\ 2,\ 1)$. The equilibrium supply vector is $\mathbf{y}^* = (2500,\ 3000,\ 3000)^T$. The equilibrium consumption levels of laborers and the capitalist are $u^{L*} = 0.5$ and $u^{C*} = 1500$ respectively. The equilibrium production and consumption processes are shown in Table 5 and Table 6.

Table 5. Equilibrium Input-Output Processes with the Value-added Tax

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	500	2000	0	0	2500
Wheat Input	0	0	1500	1500	3000
Labor Input	2000	1000	0	0	3000
Tax	2000	1000	0	0	3000
Output	2500	3000	3000	0	

Table 6. Price-measured Equilibrium Input-Output Processes with the Value-added Tax

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	1000	4000	0	0	5000
Wheat Input	0	0	3000	3000	6000
Labor Input	2000	1000	0	0	3000
Tax	2000	1000	0	0	3000
Output	5000	6000	3000	0	

7.3 Equilibrium with the Turnover Tax

First let's compute the tax rate of the turnover tax which will produce the same effect of distribution as the previous subsection. By Formula (19) we know the equilibrium price of wheat is $p_2^* = 2$. Furthermore, the rate of the turnover tax (i.e. the profit rate) is solved to be $\tau = 3/7$ and the equilibrium price of iron is $p_1^* = 1.6$. That is, the equilibrium price vector is $\mathbf{p}^* = (1.6, 2, 1)$. The equilibrium supply vector and the consumption levels (i.e. utility levels) are unchanged. The price-measured equilibrium production and consumption processes are shown in Table 7.

Table 7. Price-measured Equilibrium Input-Output Processes with the Turnover Tax

	Iron Producer	Wheat Producer	Laborer	Capitalist	Total Input
Iron Input	800	3200	0	0	4000
Wheat Input	0	0	3000	3000	6000
Labor Input	2000	1000	0	0	3000
Tax	1200	1800	0	0	3000
Output	4000	6000	3000	0	

8 An Example of Efficiency Losses under the Turnover Tax

In the model discussed above both the production function of each producer and the utility function of each consumer are Leontief-type, which means the demand structure of both producers and consumers won't change with prices. Thus in this case, when the consumption level of laborers is given, the consumption level of the capitalist is unique and has nothing to do with the tax system. Hence it's clear that the tax system will affect prices but won't lead to any efficiency loss. However, if we extend the model to allow that a producer sector possesses more than one technology, levying turnover tax may result in efficiency loss. Next we will illustrate this point by a simple example.

Suppose there are two types of commodities in the economy, i.e. wheat and labor. And the pretax wage rate is assumed to be 1. There are a wheat producer, a laborer and a capitalist in the economy. The wheat producer has a Leontief-type production function, and the laborer and capitalist consume only wheat. The input coefficient matrix is $\mathbf{A}(u^L) = \begin{bmatrix} \alpha & u^L \\ 1 & 0 \end{bmatrix}$ wherein α is assumed to satisfy $0 < \alpha < \frac{2}{3}$.

8.1 Equilibrium with the Value-added Tax

When the value-added tax is levied, the tax-inclusive unit cost of the wheat producer is $\alpha p_1 + 1 + \tau$. Because $\alpha p_1 + 1 + \tau = p_1$ holds in equilibrium, the equilibrium price of wheat (i.e. the equilibrium unit cost of wheat) is $p_1^* = \frac{1+\tau}{1-\alpha}$. The equilibrium consumption level of the laborer is $\frac{1}{p_1^*} = \frac{1-\alpha}{1+\tau}$, and the equilibrium consumption level of the capitalist is $\frac{\tau(1-\alpha)}{1+\tau}$. The equilibrium production and consumption processes are shown in Table 8.

Table 8. Equilibrium Input-Output Processes with the Value-added Tax

	Wheat Producer	Laborer	Capitalist	Total Input
Wheat Input	α	$\frac{1-\alpha}{1+\tau}$	$\frac{\tau(1-\alpha)}{1+\tau}$	1
Labor Input	1	0	0	1
Tax	τ	0	0	τ
Output	1	1	0	

When the tax rate τ approaches infinity, the consumption levels of the laborer and the capitalist approach 0 and $1-\alpha$ respectively. The labor values in the wheat consumed by the laborer and the capitalist are $\frac{1}{1+\tau}$ and $\frac{\tau}{1+\tau}$ respectively.

If the wheat producer has another technology $(0.5\alpha, 2)$, that is, it can produce a unit of wheat by inputting 0.5α unit of wheat and 2 units of labor, then under the current tax-inclusive equilibrium prices the unit cost of this technology is $0.5\alpha\frac{1+\tau}{1-\alpha} + 2(1+\tau) = \frac{(2-1.5\alpha)(1+\tau)}{1-\alpha}$. Recall the assumption $0 < \alpha < \frac{2}{3}$, we have $\frac{(2-1.5\alpha)(1+\tau)}{1-\alpha} > \frac{1+\tau}{1-\alpha}$. Hence this technology won't be utilized.

8.2 Equilibrium with the Turnover Tax

When the turnover tax is levied, the tax-inclusive unit cost of the wheat producer is $(1+\tau)(\alpha p_1 + 1)$. The equilibrium price of wheat is $p_1^* = \frac{1+\tau}{1-\alpha-\alpha\tau}$. Obviously a feasible tax rate τ satisfies $0 \leq \tau < \frac{1}{\alpha} - 1$.

The consumption level of the laborer is $\frac{1}{p_1^*} = \frac{1-\alpha-\alpha\tau}{1+\tau}$. The consumption level of the capitalist is $\frac{\tau(\alpha p_1^*+1)}{p_1^*}$, i.e. $\frac{\tau}{1+\tau}$. The equilibrium production and consumption processes are shown in Table 9.

Table 9. Equilibrium Input-Output Processes with the Turnover Tax

	Wheat Producer	Laborer	Capitalist	Total Input
Wheat Input	α	$\frac{1-\alpha-\alpha\tau}{1+\tau}$	$\frac{\tau}{1+\tau}$	1
Labor Input	1	0	0	1
Tax	$\frac{\tau}{1-\alpha-\alpha\tau}$	0	0	$\frac{\tau}{1-\alpha-\alpha\tau}$
Output	1	1	0	

When the tax rate τ approaches $\frac{1}{\alpha} - 1$, the consumption levels of the laborer and the capitalist approach 0 and $1-\alpha$ respectively. In the equilibrium with turnover tax, the tax-inclusive price vector for the wheat producer is $\left(\frac{(1+\tau)^2}{1-\alpha-\alpha\tau}, 1+\tau\right)$, which is not one of those optimal price vectors $\left(\frac{\xi}{1-\alpha}, \xi\right)$, $\xi > 0$ for any $\tau > 0$.

Therefore, when the wheat producer has more than one technology, in equilibrium it may use a technology which won't be used in a Pareto optimal state. Hence a loss of the economy efficiency may follow. The following numerical example may illustrate this point.

Suppose the wheat firm has another technology $(0.5\alpha, 2)$. The unit cost under the current equilibrium prices is $(1+\tau)\left(\frac{0.5\alpha(1+\tau)}{1-\alpha-\alpha\tau} + 2\right)$. We can find that when $\frac{2}{3\alpha} - 1 < \tau < \frac{1}{\alpha} - 1$ holds that value is smaller than $\frac{1+\tau}{1-\alpha-\alpha\tau}$, that is, this technology costs less and will be utilized.

The unit cost of the firm using the second technology is $(1+\tau)(0.5\alpha p_1 + 2)$. Obviously the corresponding equilibrium price of wheat is $\frac{4(1+\tau)}{2-\alpha-\alpha\tau}$. Now the tax rate can be as high as $\frac{2}{\alpha} - 1$. When $\frac{2}{3\alpha} - 1 < \tau < \frac{2}{\alpha} - 1$ holds the unit cost of using technology$(0.5\alpha, 2)$ under that price is $\frac{4(1+\tau)}{2-\alpha-\alpha\tau}$, which is smaller than the unit cost of using technology $(\alpha, 1)$ (i.e. $(1+\tau)\left(\frac{4(1+\tau)}{2-\alpha-\alpha\tau}\alpha + 1\right) = \frac{(2+3\alpha(1+\tau))(1+\tau)}{2-\alpha-\alpha\tau}$). Now the consumption level of the laborer is $\frac{2-\alpha-\alpha\tau}{4(1+\tau)}$. The equilibrium production and consumption processes are shown in Table 10.

Table 10. Equilibrium Input-Output Processes with the Turnover Tax

	Wheat Producer	Laborer	Capitalist	Total Input
Wheat Input	0.25α	$\frac{2-\alpha-\alpha\tau}{4(1+\tau)}$	$\frac{\tau}{2(1+\tau)}$	0.5
Labor Input	1	0	0	1
Tax	$\frac{2\tau}{2-\alpha-\alpha\tau}$	0	0	$\frac{2\tau}{2-\alpha-\alpha\tau}$
Output	0.5	1	0	

When the tax rate τ approaches $\frac{2}{\alpha} - 1$, the consumption level of the laborer approaches 0, and the consumption level of the capitalist approaches $0.5 - 0.25\alpha$.

In this case the total amount of wheat consumed by the laborer and the capitalist is $0.5 - 0.25\alpha$, which contains $\frac{0.5-0.25\alpha}{1-\alpha}$ labor value. Because the total amount of labor utilized in the production is 1, it's clear that the loss of the labor value as a result of levying tax is $1 - \frac{0.5-0.25\alpha}{1-\alpha}$, i.e. $\frac{0.5-0.75\alpha}{1-\alpha}$, which indicates the amount of efficiency loss in this economy as a result of levying the turnover tax.

When the turnover tax is levied, there may exist an optimal tax rate (i.e. an optimal profit rate) for the capitalist. In this example, the wheat producer will use the first technology only if the tax rate satisfies $\tau \leq \frac{2}{3\alpha} - 1$. And in such a case when the tax rate equals $\frac{2}{3\alpha} - 1$ the consumption level of the capitalist reaches the greatest value $1-1.5\alpha$. When the tax rate satisfies $\frac{2}{3\alpha}-1<\tau<\frac{2}{\alpha}-1$, the second technology will be used, and in such a case when the tax rate approaches $\frac{2}{\alpha}-1$ the consumption level of the capitalist approaches the greatest value $0.5-0.25\alpha$. It's easy to know if $0<\alpha<0.4$, then we have $1-1.5\alpha>0.5-0.25\alpha$. That is to say, if $0<\alpha<0.4$ holds the optimal tax rate for the capitalist is $\frac{2}{3\alpha}-1$.

9 Concluding Remarks

In this paper we build a linear multi-sector equilibrium model with taxation, which contains only one kind of primary production factor (i.e. homogeneous labor). The tax in this model may stand for profit and surplus value. In the reality the capitalists yield profit and surplus value mainly by the market mechanism. However, from the viewpoint of theoretical research the profit and surplus value obtained by the market mechanism can always be obtained by levying tax instead. That is to say, the tax mechanism is a more general method of distribution than the market mechanism. Therefore, an issue of distribution can be transformed into an issue of tax to be analyzed.

In the model of this paper, when no tax is levied, the equilibrium price vector of products (with labor as numeraire) equals the labor value vector. The equilibrium with a wage tax and the equilibrium with the value-added tax are Pareto optimal. That is, levying wage tax or value-added tax is merely a redistribution process and won't lead to any loss of economy efficiency. Even if the model of this paper is extended to allow that a producer sector has more than one technology, this point will still hold. And when capitalists yield equal profit (i.e. surplus value) from each unit of labor input of producer sectors, it's equivalent to levy value-added tax on producer sectors.

However, in reality capitalists usually yield equal profit from each unit of capital input of producer sectors, which is equivalent to levy turnover tax on producer sectors. In such a case the exogenous profit rate (i.e. the turnover tax rate) can be interpreted as the natural interest rate. When a producer sector with more than one technologies is levied turnover tax, the equilibrium may not be Pareto optimal. That is to say, if there exists a positive natural interest rate in the economy and in equilibrium the profit rates of all producer sectors equals that natural interest rate, the equilibrium may not be Pareto optimal.

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