A Note on the Computation of the Pre-Kernel for Permutation Games

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To determine correctly a non-convex pre-kernel for TU games with more than 4 players can be a challenge full of possible pitfalls, even to the experienced researcher. Parts of the pre-kernel can be easily overlooked. In this note we discuss a method to present the full shape of the pre-kernel for a permutation game as discussed by Solymosi (2014). By using the property in which the pre-kernel is located in the least core for permutation games, the least core can be covered by a small collection of payoff equivalence classes as identified by Meinhardt (2013d) to finally establish the correct shape of the pre-kernel.

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1 INTRODUCTION

Although the axiomatic foundation of the pre-kernel made during the last decades a considerable progress, and even the computational aspects of singling out a pre-kernel point by an efficient and systematical method was satisfactorily solved by the recent book of Meinhardt (2013d), the determination of a non-convex pre-kernel is still a challenge full of possible pitfalls, even to the experienced researcher. But even when the connected parts of the pre-kernel get correctly specified, there still exists the possibility that a disconnected part has been overlooked. According to our knowledge the sole example of a disconnected pre-kernel have been studied by Kopelowitz (1967); Stearns (1968), we have nevertheless found some evidence that a disconnected pre-kernel occurs more frequently than expected. Not for TU games with a sufficient symmetrical structure as in many examples presented in the literature, but for games with loose symmetries among coalitions. The more asymmetries one has to consider the more challenging is the determination of the pre-kernel for games with more than 4 players, and some parts could get easily overlooked as in the case of the recent publication by Solymosi (2014). There, for a five person game from the class of permutation games a pre-kernel was specified as a non convex V-shaped set. However, it is in fact N-shaped as we shall see.

In this note we shall delineate a procedure of how we can correctly determine the pre-kernel for permutation games. One aspect of finding the whole pre-kernel is to rely on pre-kernel catchers. In the literature, several of them have been discussed by Maschler et al. (1979); Funaki (1986); Chang and Kan (1992); Chang and Driessen (1995); Chang and Lian (2002). But none of these pre-kernel catchers is useful for our purpose, since the proposed sets, in which the pre-kernel is located, are simply too large[1]. This does not mean that these sets are of no help at all, this is definitely not the case. These results are simply be too general, they specify for each TU game a set in which the pre-kernel is located. Selecting a pre-imputation from such a set may boost the computation of a pre-kernel element, since the possible number of iteration steps to successfully converge to the pre-kernel diminishes by such a choice. Unfortunately, this branch of research came to halt for more than a decade. Only recently, by the work of Solymosi (2014), this line of research has registered again a major contribution. There, it was shown that the pre-kernel for permutation games is located in the least core. This kind of pre-kernel catcher is exploitable for our purpose to unequivocally establish that the pre-kernel is completely determined. This set is sufficiently small to be covered by some payoff equivalence classes as identified by Meinhardt (2013d, Sec. 5.4). Establishing that each imputation inside of the least core belongs to one of these payoff sets that must be mapped by a linear transformation either to the pre-kernel or to another payoff set – which sends its imputations to the pre-kernel – shall allow us to specify the full shape of the pre-kernel. Thus, we shall ascertain a non-convex pre-kernel by a systematical method rather than by guesswork.

This note is organized as follows: Section 2 introduces some basic definitions. Whereas Section 3 provides the concepts of the indirect function and presents a dual pre-kernel characterization in terms of solution sets. Section 4 discusses a procedure to fully determine the pre-kernel for permutation games. We close this note by some final remarks in Section 5 and an Appendix 6 provides crucial game data to specify the shape of the pre-kernel.

[1]To get some idea how large such a set can be, we refer the interested reader to Meinhardt (2005) or `http://members.wolfram.com/jeffb/visualization/gametheory.shtml`. Calculating several pre-kernel catchers can be accomplished with our Mathematica package TuGames 2013a.

2 Preliminaries

A n-person cooperative game with side-payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1, 2, \ldots, n\}$ represents the player set and v is the characteristic function with $v : 2^N \rightarrow \mathbb{R}$, and the convention that $v(\emptyset) := 0$. Elements of N are denoted as players. A subset S of the player set N is called a coalition. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. However, the cardinality of the player set N is given by $n := |N|$, and that for a coalition S by $s := |S|$. We assume throughout that $v(N) > 0$ and $n \geq 2$ is valid. Formally, we identify a cooperative game by the vector $v := (v(S))_{S\subseteq N} \in \mathcal{G}^n = \mathbb{R}^{2^{|N|}}$, if no confusion can arise, whereas in case of ambiguity, we identify a game by $\langle N, v\rangle$.

A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a feasible vector $\mathbf{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\mathbf{x}$ with the corresponding measure on S, such that $x(S) := \sum_{k\in S} x_k$. The set of vectors $\mathbf{x} \in \mathbb{R}^n$ which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by

$$\mathcal{I}^0(v) := \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}, \tag{2.1}$$

where an element $\mathbf{x} \in \mathcal{I}^0(v)$ is called a pre-imputation.

Given a vector $\mathbf{x} \in \mathcal{I}^0(v)$, we define the **excess** of coalition S with respect to the pre-imputation $\mathbf{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S, \mathbf{x}) := v(S) - x(S). \tag{2.2}$$

A non-negative (non-positive) excess of S at $\mathbf{x}$ in the game $\langle N, v\rangle$ represents a gain (loss) to the members of the coalition S unless the members of S do not accept the payoff distribution $\mathbf{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game $v \in \mathcal{G}^n$. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any pre-imputation $\mathbf{x} \in \mathcal{I}^0(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S\in\mathcal{G}_{ij}} e^v(S, \mathbf{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{2.3}$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\mathbf{x} \in \mathcal{I}^0(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathcal{P}r\mathcal{K}(v) := \left\{\mathbf{x} \in \mathcal{I}^0(v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\right\}. \tag{2.4}$$

The least core formalized by Maschler et al. (1979) is the $\epsilon_0(v)$-core of the game v, where $\epsilon_0(v)$ is the critical number at which the ϵ-core still exists, that is, for $\epsilon < \epsilon_0(v)$ the ϵ-core is empty. This critical number is specified by

$$\epsilon_0(v) := \min_{\vec{x}\in\mathcal{I}^0(v)} \max_{S\neq\emptyset,N} e(\vec{x}, S). \tag{2.5}$$

3 A Dual Pre-Kernel Representation

Theorem 3.1 (Martinez-Legaz (1996))**.** *The indirect function $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ of any n-person TU game is a non-increasing polyhedral convex function such that*

(i) $\partial\pi(\mathbf{x}) \cap \{-1, 0\}^n \neq \emptyset \qquad \forall \mathbf{x} \in \mathbb{R}^n$,

(ii) $\{-1,0\}^n \subset \bigcup_{\mathbf{x}\in\mathbb{R}^n} \partial\pi(\mathbf{x})$, *and*

(iii) $\min_{\mathbf{x}\in\mathbb{R}^n} \; \pi(\mathbf{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \rightarrow \mathbb{R}$ *satisfies* (i)*-*(iii) *then there exists an unique* n*-person TU game* $\langle N, v\rangle$ *having* π *as its indirect function, its characteristic function is given by*

$$v(S) = \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \pi(\mathbf{x}) + \sum_{k\in S} x_k \right\} \qquad \forall\, S \subset N. \tag{3.1}$$

According to the above result, the associated **indirect function** $\pi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is given by:

$$\pi(\mathbf{x}) = \max_{S\subseteq N} \left\{ v(S) - \sum_{k\in S} x_k \right\} \qquad \forall \mathbf{x} \in \mathbb{R}^n, \tag{3.2}$$

whereas $\partial\pi$ is the subdifferential of the function π. Hence, $\partial\pi(\mathbf{x})$ is the set of all subgradients of π at $\mathbf{x}$, which is a closed convex set. A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). Here, we present this representation in its most general form, although we restrict ourselves to the trivial coalition structure $\mathcal{B} = \{N\}$.

The pre-imputation that comprises the possibility of compensation between a pair of players $i, j \in N, i \neq j$, is denoted as $\mathbf{x}^{\,i,j,\delta} = (x_k^{\,i,j,\delta})_{k\in N} \in \mathcal{I}^0(v)$, with $\delta \geq 0$, which is given by

$$\mathbf{x}^{\,i,j,\delta}_{N\setminus\{i,j\}} = \mathbf{x}_{N\setminus\{\,i,j\}}, \; x_i^{\,i,j,\delta} = x_i - \delta \quad \text{and} \quad x_j^{\,i,j,\delta} = x_j + \delta$$

Proposition 3.1 (Meseguer-Artola (1997))**.** *For a TU game with indirect function* π*, a pre-imputation* $\mathbf{x} \in \mathcal{I}^0(v)$ *is in the pre-kernel of* $\langle N, v\rangle$ *for the coalition structure* $\mathcal{B} = \{B_1, \ldots, B_l\}$*,* $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v, \mathcal{B})$*, if, and only if, for every* $k \in \{1, 2, \ldots, l\}$*, every* $i, j \in B_k,\; i < j$*, and some* $\delta \geq \delta_1(v, \mathbf{x})$*, one receives*

$$\pi(\mathbf{x}^{\,i,j,\delta}) = \pi(\mathbf{x}^{\,j,i,\delta}).$$

Meseguer-Artola (1997) was the first who recognized that based on the result of Proposition 3.1 a pre-kernel element can be derived as a solution of an over-determined system of non-linear equations. Every over-determined system can be equivalently expressed as a minimization problem. The set of global minima coalesces with the pre-kernel set. For the trivial coalition structure $\mathcal{B} = \{N\}$ the over-determined system of non-linear equations is given by

$$\begin{cases} f_{ij}(\mathbf{x}) = 0 & \forall i, j \in N, i < j \\ f_0(\mathbf{x}) = 0 \end{cases} \tag{3.3}$$

where, for some $\delta \geq \delta_1(\mathbf{x}, v)$,

$$f_{ij}(\mathbf{x}) := \pi(\mathbf{x}^{\,i,j,\delta}) - \pi(\mathbf{x}^{\,j,i,\delta}) \qquad \forall i, j \in N, i < j, \tag{3.3-a}$$

and

$$f_0(\mathbf{x}) := \sum_{k\in N} x_k - v(N). \tag{3.3-b}$$

$$h(\mathbf{x}) := \sum_{\substack{i,j\in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 \geq 0 \qquad \forall\, \mathbf{x} \in \mathbb{R}^n. \tag{3.4}$$

For further details see Meinhardt (2013d, Chap. 5 & 6). Then one can establish the subsequent result

Corollary 3.1 (Meinhardt (2013d)). *For a TU game $\langle N, v\rangle$ with indirect function π, it holds that*

$$h(\mathbf{x}) = \sum_{\substack{i,j\in N\\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 = \min_{\mathbf{y}\in \mathcal{I}^0(v)} h(\mathbf{y}) = 0, \tag{3.5}$$

if, and only if, $\mathbf{x} \in \mathcal{P}r\mathcal{K}(v)$.

To identify a partition of the domain of function h into payoff equivalence classes we first define the set of **most effective** or **significant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathcal{C}_{ij}(\mathbf{x}) := \left\{ S \in \mathcal{G}_{ij} \;\middle|\; s_{ij}(\mathbf{x}, v) = e^v(S, \mathbf{x}) \right\}. \tag{3.6}$$

When we gather for all pair of player $i, j \in N, i \neq j$ all these coalitions that support the claim of a specific player over some other players, we have to consider the concept of the collection of most effective or significant coalitions w.r.t. $\mathbf{x}$, which we define as in Maschler et al. (1979, p. 315) by

$$\mathcal{C}(\mathbf{x}) := \bigcup_{\substack{i,j\in N\\ i\neq j}} \mathcal{C}_{ij}(\mathbf{x}). \tag{3.7}$$

Notice that the set $\mathcal{C}_{ij}(\mathbf{x})$ for all $i, j \in N, i \neq j$ does not have cardinality one, which is required to identify a partition on the domain of function h. Now let us choose for each pair $i, j \in N, i \neq j$ a descending ordering of the set most effective coalitions in accordance with their size, and within of such a collection of most effective coalitions having smallest size the lexicographical minimum is single out, then we obtain the required uniqueness to partition the domain of h. This set is denoted by $\mathcal{S}_{ij}(\mathbf{x})$ for all pairs $i, j \in N, i \neq j$, and gathering all these collections we are able to specify the set of lexicographically smallest most effective coalitions w.r.t. $\mathbf{x}$ through

$$\mathcal{S}(\mathbf{x}) := \left\{ \mathcal{S}_{ij}(\mathbf{x}) \;\middle|\; i, j \in N, i \neq j \right\}. \tag{3.8}$$

This set will be denoted in short as the set of **lexicographically smallest coalitions**. Given the correspondence $\mathcal{S}$ on the payoff space we say that two payoff vectors $\mathbf{x}$ and $\mathbf{y}$ are equivalent w.r.t. the binary relation $\sim$ iff $\mathcal{S}(\mathbf{x}) = \mathcal{S}(\mathbf{y})$. This binary relation induces a partition on the payoff space. Having identified payoff equivalence classes, we can select an arbitrary payoff vector to get a unique quadratic and convex function. To see this, select payoff vector $\mathbf{x}$ from payoff equivalence class $[\vec{\gamma}]$, then we get the set $\mathcal{S}(\mathbf{x})$ from which a rectangular matrix $\mathbf{E}$ can be constructed through $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n,\ \forall i, j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. Notice that in this respect the characteristic vector for $\mathbf{x} \in \mathbb{R}^n$ is given by $x_k = 1$ if $k \in S$ and $x_k = 0$ whenever $k \notin S$. Let $q = \binom{n}{2} + 1$ combining these q-column vectors, we can construct matrix $\mathbf{E}$ as a $(n \times q)$-matrix in $\mathbb{R}^{n\times q}$, which is given by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{n-1,n}, \mathbf{E}_0] \in \mathbb{R}^{n\times q}. \tag{3.9}$$

A matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ can now be expressed as $\mathbf{Q} = 2 \cdot \mathbf{E}\,\mathbf{E}^\top$, a column vector $\mathbf{a}$ as $2 \cdot \mathbf{E}\,\vec{\alpha} \in \mathbb{R}^n$. Moreover, defining $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}\ \forall i, j \in N, i < j$ and $\alpha_0 := v(N)$. Finally, the scalar α is given by $\|\vec{\alpha}\|^2$, whereas $\mathbf{E} \in \mathbb{R}^{n\times q}, \mathbf{E}^\top \in \mathbb{R}^{q\times n}$ and $\vec{\alpha} \in \mathbb{R}^q$. For the details to construct the above set, matrix and vector we refer the reader to Meinhardt (2013d, Chap. 5 & 6).

From vector $\vec{\gamma}$ the set (3.8) is constructed and then matrix $\mathbf{Q}$, column vector $\mathbf{a}$, and scalar α are induced from which a quadratic and convex function can be specified through

$$h_{\gamma}(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}\,\mathbf{x}\,\rangle + \langle\, \mathbf{x}, \mathbf{a}\,\rangle + \alpha \qquad \mathbf{x} \in \mathbb{R}^n. \tag{3.10}$$

In view of Proposition 6.2.2 Meinhardt (2013d) function h as defined by (3.4) is composed of a finite family of quadratic and convex functions of type (3.10). For the the details, we again refer the interested reader to Meinhardt (2013d, Chap. 5 & 6). In accordance with Theorem 7.3.1 by Meinhardt (2013d, p. 137) a dual representation of the pre-kernel is obtained as a finite union of convex and restricted solution sets $M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})$ of a quadratic and convex function of type h_{γ_k}, that is,

$$\mathcal{P}r\mathcal{K}(v) = \bigcup_{k \in \mathcal{J}'} M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]}). \tag{3.11}$$

where $\mathcal{J}'$ is a finite index set such that $\mathcal{J}' := \{k \in \mathcal{J} \,|\, g(\vec{\gamma}_k) = 0\}$. In addition, $g(\vec{\gamma}_k)$ is the minimum value of a minimization problem under constraints of function h_{γ_k} over the closed convex payoff set $\overline{[\vec{\gamma}_k]}$. For the index set it is claimed that this minimum value is equal to zero on the closed payoff set $\overline{[\vec{\gamma}_k]}$. The solution sets $M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})$ are convex. Taking the finite union of convex sets may give us a non-convex set. Hence, the pre-kernel set is generically a non-convex set for games with more than 4 players. By the characterization of (3.11) we observe that it can be even disconnected. An example of a disconnected pre-kernel were discussed by Kopelowitz (1967); Stearns (1968). This example was recently reconsidered in Meinhardt (2013d, Sec. 8.5). According to our information, this is the sole example of a disconnected pre-kernel investigated in the literature. However, we have found some further evidence that a disconnected pre-kernel occurs more frequently as this example may suggest. Thus, it might be a rare event having a null measure, but it can be materialized even though the conditions under which such an event can be observed are still unclear. Caused by the fact that the pre-kernel is still not fully understood.

For the class of convex games and three person games we have $|\,\mathcal{J}'\,| = 1$, which implies that the per-kernel must be a singleton. In this respect, Meinhardt (2014) has established that whenever a default game has a unique pre-kernel satisfying the non-empty interior condition for a payoff set, then on a restricted subset of the game space constituted by the default game and a set of related games this point is the sole pre-kernel element. The pre-kernel correspondence is single-valued and constant on this subset.

4 Determining a Non-Convex Pre-Kernel

In this section we delineate a procedure of how one can conclude that the whole pre-kernel of a permutation game has been determined. As we have outlined above, even the experienced researcher has quickly overlooked a part of the pre-kernel. But even tough the connected parts of the pre-kernel set has been found, it might still exist the possibility that an element or a line segment that is disconnected from the main part of the pre-kernel has not been detected. To avoid such an accident, the resort on a good pre-kernel catcher is indispensable. For permutation games an extremely useful pre-kernel catcher has been discovered by Solymosi (2014). The knowledge that the pre-kernel is in the least core in connection with some characterization results from Meinhardt (2013d) allow us to unequivocally identify the pre-kernel for the class of permutation games.

To this end we consider a mapping that sends a point $\vec{\gamma}$ to a point $\vec{\gamma}_{\circ} \in M(h_{\gamma})$ through

$$\Gamma(\vec{\gamma}) := -\Big(\mathbf{Q}^{\dagger}\,\mathbf{a}\Big)(\vec{\gamma}) = -\Big(\mathbf{Q}^{\dagger}_{\vec{\gamma}}\,\mathbf{a}_{\vec{\gamma}}\Big) = \vec{\gamma}_{\circ} \in M(h_{\gamma}) \qquad \forall \vec{\gamma} \in \mathbb{R}^n, \tag{4.1}$$

where $\mathbf{Q}_\gamma$ and $\mathbf{a}_\gamma$ are the matrix and the column vector induced by vector $\vec{\gamma}$, respectively. Notice that matrix $\mathbf{Q}_{\vec{\gamma}}^{\dagger}$ is the pseudo-inverse of matrix $\mathbf{Q}_\gamma$. In addition, the set $M(h_\gamma)$ is the solution set of function h_γ. Under a regime of orthogonal projection this mapping induces a cycle free method to evaluate a pre-kernel point for any class of TU games. We restate here Algorithm 8.1.1 of Meinhardt (2013d) in a more compactly written form by

Algorithm 4.1: Procedure to seek for a Pre-Kernel Element

Data: Arbitrary TU Game $\langle N, v \rangle$, and a payoff vector $\vec{\gamma}_0 \in \mathbb{R}^n$.
Result: A payoff vector s.t. $\vec{\gamma}_{k+1} \in \mathcal{PrK}(v)$.
begin
    $k \longleftarrow 0, \quad \mathcal{S}(\vec{\gamma}_{-1}) \longleftarrow \emptyset$
    Select an arbitrary starting point $\vec{\gamma}_0$
    if $\vec{\gamma}_0 \notin \mathcal{PrK}(v)$ **then** Continue
    else Stop
    Determine $\mathcal{S}(\vec{\gamma}_0)$
    if $\mathcal{S}(\vec{\gamma}_0) \neq \mathcal{S}(\vec{\gamma}_{-1})$ **then** Continue
    else Stop
    repeat
        if $\mathcal{S}(\vec{\gamma}_k) \neq \emptyset$ **then** Continue
        else Stop
        Compute $\mathbf{E}_k$ and $\vec{\alpha}_k$ from $\mathcal{S}(\vec{\gamma}_k)$ and v
        Determine $\mathbf{Q}_k$ and $\mathbf{a}_k$ from $\mathbf{E}_k$ and $\vec{\alpha}_k$
        Calculate by Formula (4.1) $\mathbf{x}$
        $k \longleftarrow k+1$
        $\vec{\gamma}_{k+1} \longleftarrow \mathbf{x}$
        Determine $\mathcal{S}(\vec{\gamma}_{k+1})$
    until $\mathcal{S}(\vec{\gamma}_{k+1}) = \mathcal{S}(\vec{\gamma}_k)$
end

Meinhardt (2013d, Theorem 8.1.2) establishes that this iterative procedure converges towards a pre-kernel point. In view of Meinhardt (2013d, Theorem 9.1.2) we even know that at most $\binom{n}{2} - 1$-iteration steps are sufficient to successfully terminate the search process. However, we have some empirical evidence that generically at most $n+1$-iteration steps are needed to determine an element from the pre-kernel set (cf. Meinhardt (2013d, Appendix A)).[2]

Consider the player set $N = \{1, \ldots, n\}$. Furthermore, let us denote the set of all permutations of the player set N by $\mathfrak{S}_N$, and notice that $a_{i,\sigma(i)}$ is the valuation of situation $\sigma(i)$ for player i under the permutation $\sigma \in \mathfrak{S}_N$. Player i is pushed in the position of player $\sigma(i)$ by permutation σ. Now for each coalition $S \subseteq N$ one may consider all restricted permutations σ that fix the position of all players in the complement of S, that is, we have $\sigma(i) = i$ for all $i \in N \backslash S$, and acts freely on the set of coalition S by $\sigma(i)$ for all $i \in S$. This set of restricted permutation is denoted by $\mathfrak{S}_S$. Then a TU game $\langle N, v \rangle$ is called a **permutation game** whenever there exists a matrix $\mathbf{A} := [a_{ij}]_{(i,j) \in N \times N}$ such that

$$v(S) := \max_{\sigma \in \mathfrak{S}_S} \sum_{i \in S} a_{i,\sigma(i)} \qquad \forall\, S \subseteq N, \tag{4.2}$$

[2] Algorithm 4.1 is implemented in our MATLAB toolbox MatTuGames 2013b. The documentation of the toolbox is given by Meinhardt (2013c) and ships with the toolbox.

is given. In the next step consider the cyclic matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 7 & 0 & 6 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 6 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 5 & 3 & 2 & 4 & 0 \end{bmatrix},$$

as introduced in the example by Solymosi (2014), then the associated permutation game is identified by Table 4.1.[3]

Table 4.1: Permutation Game[a,b]

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	1	$\{1,4\}$	6	$\{4,5\}$	6	$\{2,3,4\}$	9	{1,3,4,5}	14
$\{2\}$	0	$\{1,5\}$	5	$\{1,2,3\}$	11	$\{2,3,5\}$	9	{2,3,4,5}	14
$\{3\}$	0	$\{2,3\}$	8	$\{1,2,4\}$	8	$\{2,4,5\}$	6	N	23
$\{4\}$	0	$\{2,4\}$	0	$\{1,2,5\}$	13	$\{3,4,5\}$	11		
$\{5\}$	0	$\{2,5\}$	4	$\{1,3,4\}$	9	$\{1,2,3,4\}$	16		
$\{1,2\}$	8	$\{3,4\}$	8	$\{1,3,5\}$	5	$\{1,2,3,5\}$	14		
$\{1,3\}$	2	$\{3,5\}$	2	$\{1,4,5\}$	13	$\{1,2,4,5\}$	14		

[a] Pre-Nucleolus: $\{5, 5, 4, 5, 4\}$
[a] Shapley Value: $\{157/30, 93/20, 269/60, 93/20, 239/60\}$

Solymosi (2014) provided a non-convex pre-kernel solution given by the following two line segments $\mathcal{P}r\mathcal{K}(v) = conv\,\{\{7, 2, 7, 2, 5\}, \{6, 4, 5, 4, 4\}\} \cup conv\,\{\{6, 4, 5, 4, 4\}, \{4, 6, 3, 6, 4\}\}$. Thus, he calculated a V-shaped pre-kernel. According to our analysis, which will be discussed below, we come up with the conclusion that the line segment $conv\,\{\{4, 6, 3, 6, 4\}, \{3, 8, 1, 8, 3\}\}$ got overlooked, and that therefore the pre-kernel is literally N-shaped given as the union of the three line segments specified through:

$$\begin{aligned} S1 &= conv\,\{\{7, 2, 7, 2, 5\}, \{6, 4, 5, 4, 4\}\}, \\ S2 &= conv\,\{\{6, 4, 5, 4, 4\}, \{4, 6, 3, 6, 4\}\}, \\ S3 &= conv\,\{\{4, 6, 3, 6, 4\}, \{3, 8, 1, 8, 3\}\}. \end{aligned} \tag{4.3}$$

First, let us observe that the pre-kernel solution given by Solymosi (2014) cannot be the whole pre-kernel. To this end, we consider the end points of this set, namely the points $\mathbf{x}_1 := \{7, 2, 7, 2, 5\}$ as well as $\mathbf{x}_3 := \{4, 6, 3, 6, 4\}$. By proposition 6.6.1 Meinhardt (2013d) both points induce a convex and quadratic function of type (3.10), and therefore two square matrices of type $\mathbf{Q}$. The end point $\mathbf{x}_1$ induces a matrix $\mathbf{Q}_1$ with full rank indicating by Lemma 6.2.2 Meinhardt (2013d) that from this point no further line segment can be spread out, whereas for the end point $\mathbf{x}_3$ the induced matrix $\mathbf{Q}_3$ has only rank four with the orientation $\{120, -240, 240, -240, 120\}/449$ having the same direction as the line segment $S1$. Hence, the function value of h at $\mathbf{x}_3$ is zero, i.e. $h(\mathbf{x}_3) = 0$, and by Lemma 6.2.2 Meinhardt (2013d) the relation $h = h_{\mathbf{x}_3}$ must hold on payoff set $[\mathbf{x}_3]$. This implies that on the line segment $S3$ spread out by $\mathbf{x}_3$ we must have $h(\mathbf{x}) = 0$ for all $\mathbf{x} \in S3$. For the end point $\mathbf{x}_4 := \{3, 8, 1, 8, 3\}$ the induced matrix $\mathbf{Q}_4$ has full rank, and by the same reasoning as for point $\mathbf{x}_1$ we conclude that from this point no further line segment can be spread out. Since the line segments $S1$ and $S3$ have the same direction we can literally speak from a N-shaped pre-kernel.

[3]We have written a small MATLAB program to compute from an arbitrary assignment matrix the associated permutation game. This program can be made available upon request.

The intended reader may want check out by herself/himself that the set $S1 \cup S2 \cup S3$ is located inside the least core, that is, the pre-kernel catcher.

By the above argumentation we have found a connected part of the pre-kernel. But is this part also the whole pre-kernel of the game? Does exist an additional part that is disconnected from the identified part? To be unequivocal certain that the above solution (4.3) is the whole pre-kernel, we need to rely on the one hand on the pre-kernel catcher of permutation games, which is the least core in accordance with Solymosi (2014, Theorem 1) and on the other hand on Meinhardt (2013d, Proposition 5.4.1).

We start with the least core which is the convex hull of the 4 extreme points given by

$$\mathcal{LC}(v) = conv\,\{\{7,2,7,2,5\},\{7,4,5,4,3\},\{3,6,3,6,5\},\{3,8,1,8,3\}\}.$$

In view of Proposition 5.4.1 Meinhardt (2013d) the binary relation $\sim$ forms a partition on the domain of function h by payoff sets. Using this result, we can identify 10 payoff sets of type $[\vec{\gamma}]$ that cover the least core, that is, $\mathcal{LC}(v) \subseteq \bigcup_{i=1}^{10} [\vec{\gamma}_i]$. The specification of these sets have been relegated to the Appendix. By the discussion lead in the Appendix, we observe that all imputations within the payoff sets $[\vec{\gamma}_k]$ indexed by $k = 1,2,4,6,7,9,10$ are mapped by (4.1) to one of the pre-kernel segments $S1, S2$ or $S3$. However, the vectors of the sets $[\vec{\gamma}_k]$ indexed by $k = 3,5,8$ are mapped to the sets indexed by $k = 10,6,6$. This implies that within the least core all imputations are sent by Algorithm 4.1 after at most two iteration steps to a pre-kernel element located at one of the line segments $S1, S2$ or $S3$. From this observation we conclude that there cannot exist a disconnected pre-kernel part within the pre-kernel catcher. Therefore, we have completely determined by (4.3) the pre-kernel as a non-convex and N-shaped set.

5 Concluding Remarks

We illustrated a method to establish the correct shape of the pre-kernel using the least core which is the pre-kernel catcher of permutation games, and payoff equivalence classes that cover it. This approach demonstrates the usefulness of a tractable pre-kernel catcher to determine the full shape of a non-convex pre-kernel. Thus, more research effort is needed to find pre-kernel catchers that would allow us to specify the correct shape of a non-convex pre-kernel for every TU game with such a pre-kernel.

6 Appendix

In this Appendix we characterize the payoff equivalence classes that cover the least core and which help us to identify the full pre-kernel. Notice that for a better reading the pairs are ordered lexicographically. However, for computational purposes the order inside of the pairs have been reversed, this allows us to get immediately the correct sign of the vector components. Thus, in order to get the correct sign of vector components that sum up to an efficient payoff vector one has to reverse the order inside of each pair and the grand coalition must be added to the sets of lexicographically smallest coalitions (cf. Meinhardt (2013d, Sec. 6.9)).

In accordance with Proposition 5.4.1 Meinhardt (2013d) each vector $\mathbf{x}$ in $[\vec{\gamma}_1]$ induces the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_1(\mathbf{x}) =& \{\{\{1,4,5\},\{2,3\}\},\{\{1,2\},\{2,3\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{4,5\}\},\\ & \{\{1,2\},\{3,4\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{4,5\}\},\{\{2,3\},\{4,5\}\},\\ & \{\{2,3\},\{4,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

From the collection of lexicographically smallest coalitions matrix $\mathbf{Q}_1$ and column vector $\mathbf{a}_1$ can be derived. Then the Mapping (4.1) specifies a vector $\vec{\gamma}_1^*$ in the solution set $M(h_{\gamma_1})$. By Algorithm 4.1 a pre-kernel point will be mapped to its proper payoff set and non pre-kernel elements to a solution set contained into the complement set $[\vec{\gamma}]^c$ of $[\vec{\gamma}]$. The solution vector $\{7, 2, 7, 2, 5\}$ is located in its proper payoff set $[\vec{\gamma}_1]$, recall that it is the first end point of the pre-kernel set located in the line segment $S1$.

For each $\mathbf{x} \in [\vec{\gamma}_2]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_2(\mathbf{x}) = \{&\{\{1,4,5\},\{2,3\}\},\{\{1,2\},\{2,3\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{4,5\}\},\\ &\{\{1,2\},\{3,4\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\ &\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

Again, the Mapping (4.1) specifies a vector $\vec{\gamma}_2^*$ in the solution set $M(h_{\gamma_2})$. Once again the solution vector is $\{7, 2, 7, 2, 5\}$, which is located in the payoff set $[\vec{\gamma}_1]$ and in the line segment $S1$ of the pre-kernel solution.

For each $\mathbf{x} \in [\vec{\gamma}_3]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_3(\mathbf{x}) = \{&\{\{1,4,5\},\{2,3\}\},\{\{1,2\},\{2,3\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{3,4,5\}\},\\ &\{\{1,2\},\{3,4\}\},\{\{1,2\},\{3,4\}\},\{\{1,2\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\ &\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_3^*$ of the Mapping (4.1) is given by $\{563/114, 521/114, 503/114, 545/114, 245/57\}$ located in the payoff set $[\vec{\gamma}_{10}]$. This vector is not a pre-kernel element.

For each $\mathbf{x} \in [\vec{\gamma}_4]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_4(\mathbf{x}) = \{&\{\{1,4,5\},\{2,3\}\},\{\{1,2,5\},\{2,3\}\},\{\{1,2,5\},\{3,4\}\},\{\{1,2\},\{4,5\}\},\\ &\{\{1,2,5\},\{3,4\}\},\{\{2,3\},\{3,4\}\},\{\{2,3\},\{4,5\}\},\{\{2,3\},\{4,5\}\},\\ &\{\{2,3\},\{4,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_4^*$ of the Mapping (4.1) is again the pre-kernel element $\{7, 2, 7, 2, 5\}$ which is located in the payoff set $[\vec{\gamma}_1]$.

For each $\mathbf{x} \in [\vec{\gamma}_5]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_5(\mathbf{x}) = \{&\{\{1,4,5\},\{2,3\}\},\{\{1,2,5\},\{2,3\}\},\{\{1,2,5\},\{3,4\}\},\{\{1,2\},\{4,5\}\},\\ &\{\{1,2,5\},\{3,4\}\},\{\{2,3\},\{3,4\}\},\{\{2,3\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\ &\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_5^*$ of the Mapping (4.1) is given by $\{40/7, 32/7, 31/7, 32/7, 26/7\}$ located in the payoff set $[\vec{\gamma}_6]$. This vector is not a pre-kernel element.

For each $\mathbf{x} \in [\vec{\gamma}_6]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_6(\mathbf{x}) = \{&\{\{1,4,5\},\{2,3\}\},\{\{1,2,5\},\{2,3\}\},\{\{1,2,5\},\{3,4\}\},\{\{1,2\},\{3,4,5\}\},\\ &\{\{1,2,5\},\{3,4\}\},\{\{2,3\},\{3,4\}\},\{\{2,3\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\ &\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_6^*$ of the Mapping (4.1) is the pre-kernel element $\{21/4, 19/4, 17/4, 19/4, 4\}$ located in the payoff set $[\vec{\gamma}_7]$ and in the pre-kernel segment $S2$.

For each $\mathbf{x} \in [\vec{\gamma}_7]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_7(\mathbf{x}) =&\{\{\{1,4,5\},\{2,3\}\},\{\{1,2,5\},\{2,3\}\},\{\{1,2,3\}\},\{3,4\},\{\{1,2,3\},\{3,4,5\}\},\\&\{\{1,2,5\},\{3,4\}\},\{\{2,3\},\{3,4\}\},\{\{2,3\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\&\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_7^*$ of the Mapping (4.1) is the pre-kernel element $\{3,8,1,8,3\}$ located in its proper payoff set $[\vec{\gamma}_7]$ and in the pre-kernel segment $S3$. Recall that this vector is the second end point of the pre-kernel solution.

For each $\mathbf{x} \in [\vec{\gamma}_8]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_8(\mathbf{x}) =&\{\{\{1,4,5\},\{2,3\}\},\{\{1,2,5\},\{2,3\}\},\{\{1,2,5\},\{3,4\}\},\{\{1,2,3\},\{3,4,5\}\},\\&\{\{1,2,5\},\{3,4\}\},\{\{2,3\},\{3,4\}\},\{\{2,3\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\&\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_8^*$ of the Mapping (4.1) is given by $\{33/7, 32/7, 31/7, 32/7, 33/7\}$ located in the payoff set $[\vec{\gamma}_6]$. This vector is not a pre-kernel element.

For each $\mathbf{x} \in [\vec{\gamma}_9]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_9(\mathbf{x}) =&\{\{\{1,5\},\{2,3\}\},\{\{1,5\},\{3\}\},\{\{1,5\},\{3,4\}\},\{\{1,2,3\},\{3,4,5\}\},\\&\{\{1,2,5\},\{3\}\},\{\{2,3\},\{3,4\}\},\{\{2,3\},\{1,5\}\},\{\{3\},\{1,4,5\}\},\\&\{\{3\},\{1,5\}\},\{\{3,4\},\{1,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_9^*$ of the Mapping (4.1) is again the pre-kernel element $\{3,8,1,8,3\}$ located in the payoff set $[\vec{\gamma}_7]$ and in the pre-kernel segment $S3$.

For each $\mathbf{x} \in [\vec{\gamma}_{10}]$ we can deduce the following set of lexicographically smallest coalitions

$$\begin{aligned}\mathcal{S}_{10}(\mathbf{x}) =&\{\{\{1,4,5\},\{2,3\}\},\{\{1,2,5\},\{2,3\}\},\{\{1,2,5\},\{3,4\}\},\{\{1,2\},\{3,4,5\}\},\\&\{\{1,2,5\},\{3,4\}\},\{\{1,2,5\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\{\{2,3\},\{1,4,5\}\},\\&\{\{2,3\},\{1,2,5\}\},\{\{3,4\},\{1,2,5\}\}\}.\end{aligned}$$

The solution vector $\vec{\gamma}_{10}^*$ of the Mapping (4.1) is the pre-kernel element $\{21/4, 19/4, 17/4, 19/4, 4\}$ located in the payoff set $[\vec{\gamma}_7]$ and in the pre-kernel segment $S2$. But this point is also located in the closure $\overline{[\vec{\gamma}_{10}]}$. Hence, it is a relative boundary point.

Finally, notice that in view of Section 6.6 Meinhardt (2013d) each payoff equivalence class $[\vec{\gamma}]$ induces a set of ordered bases $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ that spans the vector space $\mathcal{E}$. The positive general linear group $\mathrm{GL}^+(m)$ is acting on the collection of all ordered bases of the vector space $\mathcal{E}$, whereas a attached basis matrix is denoted as $E^\top \in \mathbb{R}^{q \times m}$. Thus, two payoff sets are considered as equivalent if there exists a transition matrix $X \in \mathrm{GL}^+(m)$ such that $E_1^\top = E^\top X$ holds true, with $3 \leq m \leq n$. The group action imposes a basis change. In this respect, we have $[\vec{\gamma}_5] \sim [\vec{\gamma}_6] \sim [\vec{\gamma}_8] \sim [\vec{\gamma}_{10}]$, therefore all these payoff sets can be considered as equivalent.

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