Endogenous selection of aspiring and rational rules in coordination games

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Abstract

The paper studies an evolutionary model where players from a given population are randomly matched in pairs each period to play a coordination game. At each instant, a player can choose to adopt one of the two possible behavior rules, called the *rational rule* and the *aspiring rule*, and then take actions prescribed by the chosen rule. The choice between the two rules depends upon their relative performance in the immediate past. We show that there are two stable long run outcomes where either the rational rule becomes extinct and all players in the population achieve full efficiency, or that both the behavior rules co-exist and there is only a partial use of efficient strategies in the population. These findings support the use of the aspiration driven behavior in several existing studies and also help us take a comparative evolutionary look at the two rules in retrospect.

Keywords: Co-evolution, Aspirations, Best-response, Random matching, Coordination games.

JEL Classification: C72, D83.

1 Introduction

It is well accepted today that models of rational behavior do not fare well in many experiments. This is perhaps because in relatively complex environments, expected payoff maximization requires either too much knowledge about the environment or that the available information is hard to analyze. Moreover, it is at times extremely difficult to gather information required to form reasonable beliefs regarding the nature of the opponents with whom a player plays a game. If one is allowed to assume that in face of such complexities, rational behavior entails a cost of implementation, it is not surprising why many of us at times may prefer to use more simple behavior rules.

Various researchers have assumed alternative behavior rules, which are simple and where players do not necessarily use best responses when deciding about their strategies. One such approach that has gained much popularity is the so called reinforcement or stimulus learning behavior. This approach assumes that a player takes actions which are relatively more successful and discard those which are not. A celebrated behavior rule that describes such reinforcement methods is the one driven by payoff aspirations. According to this rule, a player has a payoff aspiration and takes an action. If the current action meets his payoff aspiration, he continues with it while if he is disappointed (because it fails to meet his current aspiration), he experiments with other available actions. There are many outstanding studies of evolution of play where players are assumed to be aspiration driven (for an excellent survey, see Bendor et al. [2001]; also see the section on literature review below). All of the existing papers assume that all players involved in the studied environments are aspiration driven. Given this, the present paper asks the following question: *are naive behavior rules, like the one based on aspirations, strong enough to survive evolutionary pressures from more efficient rules like that of expected payoff maximization if implementing such efficient rules is not too costly?*

To answer this question, we study a model where a 2×2 coordination game is played infinitely many times by players living in a given population who are randomly matched in pairs to take part in this game. There are two behavioral rules that are available: one is the myopic expected payoff maximizing rule (in short called the *rational rule*) and the other is the aspiration based reinforcement learning rule (in short called the *aspiring rule*). At any instant of time, each player is allowed to adopt one of the two rules and take actions that the adopted rule prescribes for that instant. Hence, those who adopt the rational rule venture into the society to gather information regarding the likelihood of the type of the opponent they would be matched with so that they can compute reasonable payoff expectations for each of the two available pure strategies and use that which yields a higher expected payoff. Gathering such information is assumed to be costly, no matter how small.[1] On the other hand, those who adopt the aspiring rule give up this search for costly information about their opponents and simply adopt a population wide fixed aspiration and take actions to check if the action taken meets this adopted aspiration. Thus at any instant of time, the population can be divided into two groups, those who are currently following the rational rule and those who are currently aspiring. We assume that the difference of the average performances of these two groups of players is publicly available. If at any instant, the average performance of one group exceeds that of the other, then there is some (but not necessarily full) flow of players from the low performing group into the more successful one. Time is continuous and this process is left to go on for ever. This environment gives rise to a two-dimensional dynamic system where we keep track of the fraction of aspiring players in the population and the fraction of aspiring players who play the efficient equilibrium strategy. It turns out that we are also able to report on what rational players do. We show that the limit behavior of our system depends crucially on whether the social aspiration level is high or low. In case of a high social aspiration, there are only two stable rest points of the system: the first where the rational rule gets extinct and all players follow the aspiring rule and play the efficient equilibrium strategy, and the second where both rules survive but all players who use the rational rule play the inefficient equilibrium strategy while those who use the aspiring rule have a positive fraction who play the efficient equilibrium strategy. Unlike in the case with high aspirations, we show that with a low social aspiration, there is no chance of a stable mixed population – the rational rule gets extinct independent of the intial conditions.

1.1 Related Literature

Models where players use reinforcement learning rules first appear in the mathematical psychology literature with studies by Estes [1954], Bush et al. [1954], Bush and Mosteller [1955] and Suppes and Atkinson [1960]. The computer science and engineering literature has also used such models representing various natures of automata learning as in Lakshmivarahan [1981], Narendra and Mars [1983], Narendra and Thathachar [1989] and Papavassilopoulos [1989].

This area in economics was pioneered by Simon [1955, 1957, 1959], Cross [1973] and Nelson and Winter [1982]. The focus since then has been on re-

[1] We would like to mention here that if the environment involves a game with a strictly dominant strategy, like the Prisoners' Dilemma, then playing a best response requires no information at all and hence the cost of implementing the rational rule should be zero. Then, by definition of rationality, the average payoff of rational players in the given population cannot fall below that of any other group using some other behavior rule, and hence the rational rule will always survive. In this sense our approach makes sense only in games without dominant strategies. As a starting point, we have analyzed the coordination game.

peated games (and in particular the repetition of the Prisoners' Dilemma) and most models typically show that players learn to coordinate on efficient outcomes, which may not (or may) be strategic equilibrium points [see for example, Bendor et al. [1994], Karandikar et al. [1998], Kim [1995] and Pazgal [1995]]. Limited work with aspiration driven rules has been done with a large population of players who are matched in pairs to play 2×2 games. Two most important works in this area are by Dixon [2000] and Palomino and Vega-Redondo [1999]. Dixon's framework assumes that a continuum of players are once and for all matched in pairs to form stable partnerships and play bilateral games (including the Prisoners' Dilemma or the Cournot market games). In his setup, individual players form aspirations from some statistic that signals the performance of players in other pairs (or markets). He shows that play in each pair converges to joint payoff maximization regardless of initial conditions. Palomino and Vega-Redondo consider a non-repeated setting where the Prisoners' Dilemma is played by randomly matched aspiring players with matchings taking place independently at each instant of time (and hence their approach is evolutionary rather than repeated and in this sense their work is closest to ours). In their setting, aspiration of each player depends on the payoff experiences of the entire population (and this dependence is symmetric across all players rendering a common aspiration for each of them) and that this social aspiration evolves over time. In our case as well, the aspiration level is modeled as a social attribute though it remains fixed over time. They show that in the long run a positive (but less than 1) fraction of the population are able to cooperate. In view of Palomino and Vega-Redondo, Dixon's work shows that stability of partnership is essential for full efficiency in the long run.

There is little work addressing situations where aspiration driven players play players of other types. An exception to this is a recent work by Roy [forthcoming] which studies repeated interaction between a myopic best respondent and an aspiring player with evolving aspiration. A class of 2×2 games is studied, which could either be common interest or coordination. He shows that if the speed of adjustment of aspirations is sufficiently fast, the two players are able to coordinate on the Pareto efficient outcome most of the time in the long run. However the work considers that these two players are fixed as are their behavior rules. In this sense, our work is an extension of Roy. And this brings us to another branch of literature which studies evolution of learning rules. Kirchkamp [1996] analyzes (through computer simulations) a model where players play a sequence of changing 2×2 games and learn rules rather than following some rule given to them exogenously. Among the various rules that a player can learn include cooperative, forgiving, tit-for-tat, grim, defective, etc. None of these are reinforcement learning rules but are rather repeated game strategies themselves. Moreover, learning is local in the sense that players adopt rules which are more successful within a given neighborhood of players with whom (and only with whom) each player interacts.[2] Binmore and Samuelson [1997] study endogenous learning rules in a global setting, as in ours. They view the evolutions of actions, guided by a particular learning rule, as a process that proceeds at a speed which is rapid compared to the evolution of the learning rule itself. In some sense though, in their model players use a learning rule forever and then evaluate what they obtain in this first round of infinite regress – called the long run, and then adopt a new rule and this process goes on for infinite rounds of these infinite regresses – called the ultra long run. Also, they identify a learning rule with the aspiration level which it incorporates. In our view, this is essentially a single learning rule, the aspiring rule, but with a very slow and experimental aspiration updating mechanism. Nevertheless, they show that this double infinity regress leads to the selection of the risk dominant equilibrium. This in itself is a very important result.

The rest of the paper is structured as follows. In the following section we describe in details the model. The results are stated and proved in section 3 with some proofs moved to an appendix to maintain a smooth exposition. In section 4 we report some simulated trajectories of our dynamic system to support some finer points of our analysis. The paper concludes in section 5.

2 The model

Consider a fixed set of individuals (or players) Ω with the cardinality of the continuum. At each time $t \in [0, \infty)$ individuals in Ω are randomly matched in pairs to play a coordination game presented in the figure 1, where $0 < \delta < \sigma$.

	H	L
H	σ, σ	0, 0
L	0, 0	δ, δ

Figure 1: Coordination game

At any instant of time t, the set Ω consists of two distinct sets, the set of *rational players* R, and the set of *aspiration driven players* A. We use a to denote the fraction (*with respect to* Ω) of aspiration driven players in the population and μ to denote the fraction (*with respect to* A) of aspiration driven players playing H.

Aspiration driven players use either the pure strategy H or the pure strategy L – the exact behavioral assumptions are stated later in this section. We assume that rational players are myopic and always play pure strategy

[2]There is a large literature on local learning models; see for example, Lindgren and Nordahl [1994], Nowak et al. [1994], Eshel et al. [1996] and Kirchkamp [1995]. However these works study the evolution of strategies rather than learning rules.

best responses. Let $p \in [0,1]$ be the probability belief held by a generic rational player for the event that in case his opponent is also a rational player then his opponent will play the pure strategy H. Hence, the probability beliefs held by a generic rational player that his opponent (who could either be a rational player or an aspiring player) will play pure strategies H and L, respectively, are $\Pr(H) = a\mu + (1-a)p$ and $\Pr(L) = 1 - \Pr(H)$. Given a, μ and p, the expected payoff of a generic rational player from playing H and L, respectively, are

$$\begin{aligned} E_r(H; a, \mu, p) &= \sigma \left[a\mu + (1-a)\,p\right], \text{ and} \\ E_r(L; a, \mu, p) &= \delta \left[1 - (a\mu + (1-a)\,p)\right]. \end{aligned}$$

Hence this generic rational player plays H with probability 1 whenever

$$\begin{aligned} \sigma \left[a\mu + (1-a)\,p\right] &\geq \delta \left[1 - (a\mu + (1-a)\,p)\right], \text{ that is} \\ a\mu + p\,(1-a) &\geq \frac{\delta}{\sigma + \delta}. \end{aligned}$$

Since the above inequality is true for any $p \in [0,1]$, we can conclude that if $a\mu \geq \frac{\delta}{\sigma+\delta}$, then the above inequality is satisfied. Similarly, it is easy to see that if $a\mu \leq a - \frac{\sigma}{\sigma+\delta}$, then the reverse of the above inequality is satisfied and this generic rational player plays L with probability 1. Hence, given the game, in the region of $(a-\mu)$ plane where $a\mu \geq \frac{\delta}{\sigma+\delta}$, a generic rational player always plays H while in the region of $(a-\mu)$ plane where $a\mu \leq a - \frac{\sigma}{\sigma+\delta}$, this generic rational player always plays L, independent of the belief probability p. Now let us concentrate on the remaining region of the $(a-\mu)$ plane with $a - \sigma/(\delta+\sigma) < a\mu < \delta/(\sigma+\delta)$ where beliefs about what "other" rational players play do matter in deciding upon a generic rational best response. Given a, μ and p, we impose symmetry across all rational strategies. Hence, p is to be interpreted as the probability with which all rational players play H. Given this, the expected payoff of a generic rational player is given by

$$E_r(p; a, \mu, p) = \sigma(a\mu + (1-a)p)p + \delta(a(1-\mu) + (1-a)(1-p))(1-p).$$

It is easy to check that for any value of p,

$$E_r(p; a, \mu, p) \leq \max\left\{\sigma(1 - a(1-\mu)), \delta(1 - a\mu)\right\}.$$

On the other hand, notice that

$$E_r(p; a, \mu, p) = \begin{cases} \sigma(1 - a(1-\mu)) & \text{if } p = 1, \\ \delta(1 - a\mu) & \text{if } p = 0. \end{cases}$$

We assume that in the region $a - \sigma/(\delta+\sigma) < a\mu < \delta/(\sigma+\delta)$, rational players always symmetrically choose that value of p which is efficient. By that we mean that all rational players play H with probability 1 if $\sigma(1-a(1-\mu)) \geq \delta(1-a\mu)$ and otherwise play L with probability 1. Notice that whenever the condition $a\mu \geq \frac{\delta}{\sigma+\delta}$ is satisfied, so is the condition $\sigma(1-a(1-\mu)) \geq \delta(1-a\mu)$. Similarly, whenever the condition $a\mu \leq a-\frac{\sigma}{\sigma+\delta}$ is satisfied, so is the condition $\sigma(1-a(1-\mu)) \leq \delta(1-a\mu)$.

Choosing between these rational best responses requires on part of any rational player to constantly update his information regarding the values of a and μ. We assume that acquiring this information is costly, no matter how small, and we denote this cost that each rational player has to incur as $\varrho \in (0, \delta)$.

During the whole process each player can change his behavior rule, that is either adopt the rational rule described above or adopt the aspiring one which we shall soon describe. This choice depends on the difference between average payoffs of the particular types of players and the exact procedure for this change is provided below. For the moment, notice that if $\varrho = 0$, the average payoff of rational players, by the very definition of rationality, can never be less than the average payoff of aspiration driven players.

As the values of a and μ evolve, rational players switch between strategies H and L, depending on which of them is currently a best reply (as defined above). We assume that this switching process is not instantaneous – the whole rational population does not switch at once, but that there is a fraction of rationals that do so while the remaining are "about to do so". This is motivated by the fact that current actions and rules have an inertia, no matter how small and to capture this notion of inertia, we incorporate a strictly increasing and continuously differentiable function $\xi(a, \mu)$ on the interval $[-\varepsilon, \varepsilon]$, representing the probability with which a rational player plays H, such that

$$\xi(a,\mu)\begin{cases} =0 & \text{if } E_r\left(1;a,\mu,1\right)-E_r\left(0;a,\mu,0\right) \leq -\varepsilon, \\ \in(0,1) & \text{if } -\varepsilon < E_r\left(1;a,\mu,1\right)-E_r\left(0;a,\mu,0\right) < \varepsilon, \\ =1 & \text{if } E_r\left(1;a,\nu,1\right)-E_r\left(0;a,\mu,0\right) \geq \varepsilon. \end{cases} \tag{1}$$

Here $\varepsilon > 0$ reflects the size of the area in which the rational population is in a switching phase.[3] In the remaining part of the paper we will write ξ instead of $\xi(a, \mu)$ for convenience.

As mentioned above, the change of players' behavior rules depends on the difference between the average payoffs of populations different types. So, given a, μ and ξ, the *average payoff* $\overline{\pi}_a$ *of the aspiration driven population*

[3] Another interpretation of $\xi(a, \mu)$ is that the rational players play almost best-responses in situations where the difference between the expected payoffs from H and L is positive but insignificant. As we do not want to deal in this paper with the behavior of the system when this rational switching between H and L occurs, we will assume that ε is some arbitrarily small (although fixed) number. This assumption, from a technical point of view, keeps our system smooth.

at any instant is given by

$$\overline{\pi}_a(\mu, a) = \sigma\mu(a\mu + (1-a)\xi) + \delta(1-\mu)(a(1-\mu) + (1-a)(1-\xi)), \quad (2)$$

while the *average payoff* $\overline{\pi}_r$ *of the rational population* at any instant is given by

$$\overline{\pi}_r(\mu, a) = \sigma\xi(a\mu + (1-a)\xi) + \delta(1-\xi)(a(1-\mu) + (1-a)(1-\xi)) - \varrho. \quad (3)$$

We assume that the difference $\overline{\pi}_a(\mu, a) - \overline{\pi}_r(\mu, a)$ is publicly observed. The probability rate at which a player changes his type is modelled by use of a strictly increasing (on $[0, +\infty)$) and continuously differentiable function $g(\psi)$ satisfying

$$g(\psi) \begin{cases} = 0 & \text{if } \psi \leq 0, \\ > 0 & \text{if } \psi > 0. \end{cases} \quad (4)$$

From the above assumptions it follows that $g'(\psi) = 0$ for $\psi \leq 0$. Given this function $g(\cdot)$, the change in time of the fraction of aspiration driven players is given by

$$\dot{a} = A(a, \mu) = \kappa_1 \left\{ \begin{array}{l} -g(-\psi(a,\mu))a \\ +g(\psi(a,\mu))(1-a) \end{array} \right\}, \quad (5)$$

where $\kappa_1 > 0$ is some arbitrary speed adjustment parameter and $\psi(a, \mu) = \overline{\pi}_a(\mu, a) - \overline{\pi}_r(\mu, a)$.

We are now in a position to describe the behavioral assumptions on the aspiring rule. A player who has decided to follow the aspiring rule essentially sets a payoff aspiration equal to a given (and fixed over time) aspiration level which is common to all players in the aspiring population. We denote this *social aspiration* level as α. With this payoff aspiration, a player who is currently following the aspiring rule takes an action, which in our game is either H or L, and receives an individual payoff of $\pi \in \{0, \delta, \sigma\}$. This social aspiration α and the realization of his current payoff π gives rise to an individual *dissatisfaction level* of $\chi = \alpha - \pi$. The probability rate at which an aspiration driven player with dissatisfaction level χ, who still wishes to follow the aspiring rule, changes his current strategy is given by a strictly increasing (on $[0, +\infty)$), continuously differentiable function $f(\chi)$ satisfying

$$f(\chi) \begin{cases} = 0 & \text{if } \chi \leq 0, \\ > 0 & \text{if } \chi > 0 \end{cases} \quad (6)$$

which captures the notion that if an aspiring player is satisfied with his current payoff (that is $\chi \leq 0$), and if he decides to remain an aspiring player in the next instant, then he sticks to his current action in the next instant; otherwise, if he still wants to remain an aspiring player, he experiments with the other available action with a positive probability.[4]

[4] We would like to note that our results are independent of the exact nature of the functions $f(\cdot)$ and $g(\cdot)$, as long as they satisfy our general assumptions.

Given the probability rate f, we can now describe the change in time of the fraction μ of aspiration driven players playing H. In doing so the following observations/assumptions are made:

- There are three factors that affect μ, (i) the switch of actions amongst current aspiration driven players who choose to remain aspiration driven, (ii) the mass of current aspiration driven players who chose to become rational and (iii) the mass of currently rational players who choose to become aspiring.[5]
- In case of (i), this is entirely determined by the function f.
- We assume that those aspiring players who leave the aspiring population are uniformly distributed over the two available current actions H and L.
- We assume that those rational players who join the aspiring population become aspiration driven and start out by playing H with probability μ.

In our settings it is reasonable to consider $0 < \alpha \leq \sigma$ which we assume henceforth. The above observations and assumptions lead us to the following expression for $\dot{\mu}$ given by

$$\dot{\mu} = M(a,\mu) = \kappa_2 \left\{ \begin{array}{l} -f(\alpha)\mu(a(1-\mu)+(1-a)(1-\xi)) \\ +f(\alpha)(1-\mu)(a\mu+(1-a)\xi) \\ +f(\alpha-\delta)(1-\mu)(a(1-\mu)+(1-a)(1-\xi)). \end{array} \right\}, \quad (7)$$

where $\kappa_2 > 0$ is some arbitrary speed adjustment.

Remark 1. *There is one important remark that must be made about our model. It can be used only if $a \neq 0$, since calling μ a fraction of aspiration driven players is not valid otherwise. However, as we shall see in the next section, the dynamic system described by the above differential equations (5) and (7) never reaches a state where $a = 0$, when started from any state where $a \neq 0$. Thus we can draw valid conclusions about the phenomenon we study using the above proposed model.*

In this paper we study the two-dimensional dynamic system in the $(a - \mu)$ plane characterized by the two equations of motion (5) and (7). Our goal is to find stable rest points of this system and identify their basins of attraction. This analysis is done in the following section.

[5] Given our function $g(\cdot)$, there can be no two-way flow in our model. Hence, at any instant of time, either no one changes his current behavior rule, or that either only some aspiring players become rational or some rational players become aspiring.

3 The analysis

The area under consideration is $\mathcal{Z} = [0,1] \times [0,1]$, with $(a,\mu) \in \mathcal{Z}$. The behavior of rational players gives rise to a partition of $\mathcal{Z}$ into two areas, one where rationals play H while the other where they play L. Let

$$s(a,\mu) = \sigma(1 - a(1-\mu)) - \delta(1 - a\mu),$$

and first define the regions

$$\begin{aligned} \mathcal{MH} &= \{(a,\mu) :\ (a,\mu) \in M \text{ and } s(a,\mu) > 0\}, \\ \mathcal{ML} &= \{(a,\mu) :\ (a,\mu) \in M \text{ and } s(a,\mu) < 0\} \end{aligned}$$

and their subsets respectively as

$$\begin{aligned} \mathcal{H} &= \{(a,\mu) :\ (a,\mu) \in Z \text{ and } a\mu > \delta/(\delta+\sigma)\} \text{ and} \\ \mathcal{L} &= \{(a,\mu) :\ (a,\mu) \in Z \text{ and } a\mu < a - \sigma/(\delta+\sigma)\}. \end{aligned}$$

Notice that $\mathcal{H} \subset \mathcal{MH}$, $\mathcal{L} \subset \mathcal{ML}$, $\mathcal{MH} \cap \mathcal{ML} = \emptyset$ and $\mathcal{MH} \cup \mathcal{ML} = \mathcal{Z}$. Morover while in $\mathcal{MH}$, all players using the rational rule play H. Similarly while in $\mathcal{ML}$, all players using the rational rule play L. All the areas discussed above are represented in figure 2.

Since in the neighborhood of the line $s(a,\mu) = 0$, rational players switch between strategies, so it will be called the *switch line*. The area $\mathcal{S}_\varepsilon = \{(a,\mu) \in \mathcal{Z} :\ -\varepsilon \le s(a,\mu) \le \varepsilon, \varepsilon > 0\}$ where, as exemplified in (1), rational switching takes place will be called the *switch area* of size ε. We will assume that ε is as small as possible and drop the subscript ε and denote this switching area as $\mathcal{S}$. Since ε is arbitrarily small, we will ignore analysis withing this area.

Let $\mathbf{v} : \mathcal{Z} \to \mathbb{R}^2$ be the vector field defined by (5) and (7). Since $\mathbf{v}$ is continously differentiable on $\mathbb{R}^2$ and $\mathcal{Z}$ is closed and bounded, so $\mathbf{v}$ is Lipschitz.[6] Notice that since $\mathbf{v}$ does not point outward on the boundary of $\mathcal{Z}$, so any trajectory starting from $\mathcal{Z}$ remains in $\mathcal{Z}$. Moreover, since $\mathbf{v}$ is Lipschitz, there exists a unique solution $\phi_{\mathbf{v}}(t, \mathbf{x_0})$ of the system of differential equations (5), (7) for any initial condition $\mathbf{x_0} \in \mathcal{Z}$. Notice also that any trajectory starting from initial condition $(0,0)$ remains on the line $a = 0$ until $\mu > (\sigma - \varrho)/\sigma$. Moreover $\mu > (\sigma - \varrho)/\sigma$ implies $\dot{a} > 0$ and $\mu = 0$ implies $\dot{\mu} > 0$. These observations, together with the uniqueness of solutions and the fact that our system is autonomous guarantee that every trajectory starting from initial conditions (a,μ) where $a \in (0,1]$ never reaches a state where $a = 0$. This justifies remark 1.

Our analysis will be divided into two cases – when the social aspiration level is high, that is $\delta < \alpha \le \sigma$, and when it is low, that is $0 < \alpha \le \delta$. In both cases we will analyse the areas $\mathcal{MH}$ and $\mathcal{ML}$ separately.

[6] This is true on the basis of standard lemmas (see for example [Hirsch and Smale, 1974, pp. 161,173]).

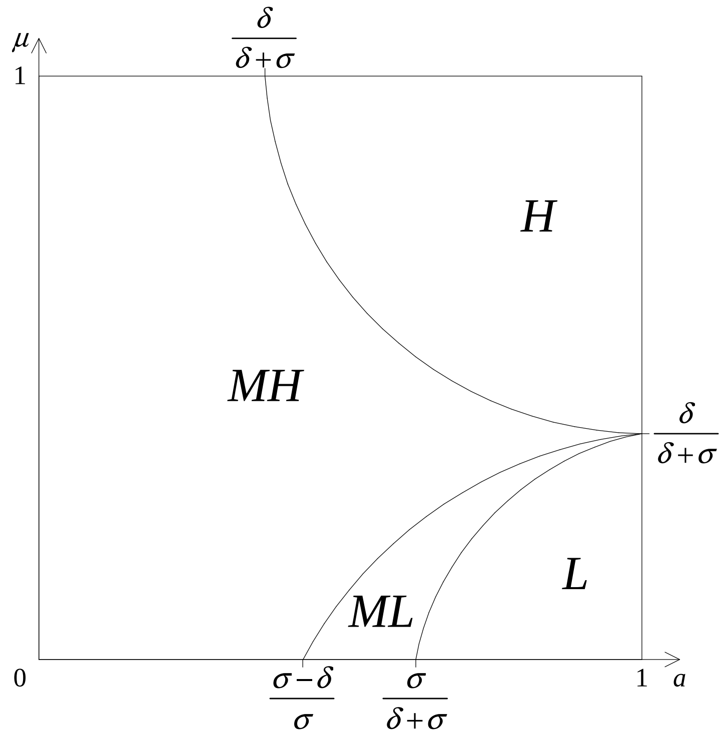

Figure 2: Areas of different rational best responses.

3.1 High aspiration level

Throughout this section we will assume that the aspiration level is high, that is $\delta < \alpha \leq \sigma$.

3.1.1 The analysis with an initial condition in $\mathcal{MH}$

In this section we will study the behavior of the system with initial conditions in the region $\mathcal{MH}$ where all rationals play H. Our main result for this case is as follows

Main Result 1. *Consider any coordination game such that* $0 < \delta < \sigma$ *and an aspiration level* $\alpha \in (\delta, \sigma]$. *Let* $\mathcal{X} = [0, 1] \times [\delta/(\delta + \sigma) + \varepsilon, 1] \supset \mathcal{H} - \mathcal{S}$, $\overline{\mathbf{x}} \in \mathcal{X}$ *and let* $\phi_{\mathbf{v}}(t, \overline{\mathbf{x}})$ *be a solution of the system of differential equations (5), (7). Then* $\lim_{t \to +\infty} \phi_{\mathbf{v}}(t, \overline{\mathbf{x}}) = (1, 1)$.

Remark 2. *Notice that the size of the area* $\mathcal{X}$ *(which is within the basin of attraction for the point* $(1, 1)$*) depends on the ratio* $\delta/(\delta + \sigma) < 1/2$ *(it depends also on* ε*, but we assume it to be arbitrarily small and small enough not to affect our results). So whenever the system enters (or starts) from points where* $\mu \geq 1/2$*, it converges for sure to the restpoint* $(1, 1)$.

Remark 3. *It is also important to note that the result is independent of the value of ϱ, provided that $\varrho \in (0, \delta)$. Thus no matter how small ϱ is, as long as it is positive, the system starting in $\mathcal{X}$ will converge to the restpoint $(1, 1)$.*

If the system starts in the area $\mathcal{MH} \subset \mathcal{Z}$, then the expressions for ψ and M have the following form:

$$\psi(a, \mu) = \psi^H(a, \mu) = a(1-\mu)^2(\delta + \sigma) - (1-\mu)\sigma + \varrho, \tag{8}$$

$$M(a, \mu) = M^H(a, \mu) = \kappa_2(f(\alpha - \delta)a(1-\mu)^2 + f(\alpha)(1-a)(1-\mu)). \tag{9}$$

The phase plane diagram for this situation is presented in figure 3 below.

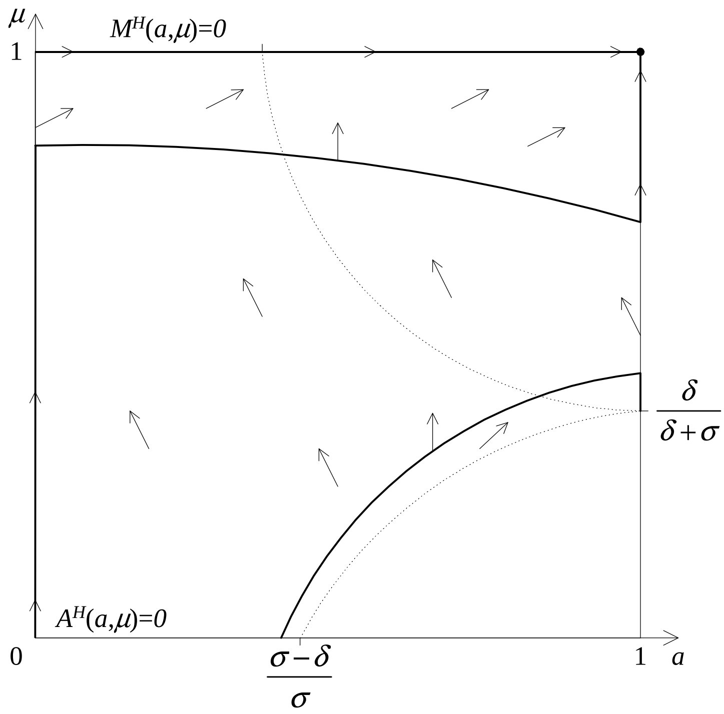

Figure 3: Phase plane diagram when all rationals play H.

The following lemma shows that if all rational players play H, there is only one rest point in $\mathcal{Z}$ which is asymptotically stable[7] by proving that the rest point is a *generic saddle node* (see [Hubbard and West, 1991, part II, p. 281] or theorem 2 in the appendix for the characterisation of generic saddle nodes).

[7] A rest point is asymptotically stable if every trajectory that starts in the neighborhood of the restpoint converges to it as $t \to +\infty$. Thus an asymptotically stable rest point may be viewed as a state of the system that is considered robust, with regard to small enough perturbations.

Lemma 1. *Suppose the system begins in $\mathcal{MH}$. Then the point* $(1,1)$, *where the entire population consists of only aspiring players playing* H, *is the only rest point and it is asymptotically stable.*

Proof. It is easily seen that for any $a \in [0,1]$, if $\mu \in [0,1)$ then $M^H(a,\mu) > 0$ and if $\mu = 1$ then $M^H(a,\mu) = 0$. On the other hand $\psi(a,1) = \varrho > 0$, so $A(a,1) > 0$ for all $a \in [1,0)$ and $A(1,1) = 0$. Thus $(1,1)$ is the only rest point.

To study the stability of this rest point we will linearize of our system at this point. The partial derivatives of A and M^H at $(1,1)$ are as follows:

$$A_a(1,1) = -\kappa_1 g(\varrho), \quad A_\mu(1,1) = 0,$$
$$M_a^H(1,1) = 0, \quad M_\mu^H(1,1) = 0.$$

Thus the Jacobian of the vector field $\mathbf{v}$ at $(1,1)$ is

$$D\mathbf{v}(1,1) = \begin{bmatrix} -\kappa_1 g(\varrho) & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

The eigenvaules of it are $\lambda_1 = -\kappa_1 g(\varrho) < 0$, $\lambda_2 = 0$ and the corresponding eigenvectors are $\mathbf{v_1} = [1,0]^T$, $\mathbf{v_2} = [0,1]^T$. Since one of eigenvalues is zero, the other is nonzero and eigendirection of the zero eigenvalue is parallel to the μ-axis, we have to check the sign of the coefficient of μ^2 in the Taylor expansion of our differential equation, that is, the sign of $M_{\mu\mu}^H = 2f(\alpha-\delta)a$. Since it is positive, $(1,1)$ is a *generic saddle node.* This means in particular, that since $\lambda_1 < 0$, there exists unique trajectory within $\mathcal{Z}$ that tend to $(1,1)$ tangentially to $\mathbf{v_1}$, which is the separatrix of the generic saddle node. This trajectory goes along the line $\mu = 1$ (notice that $M^H(a,1) = 0$ and $A(a,1) > 0$ for $a \in [0,1)$). All other trajectories tend to $(1,1)$ tangentially to the line $a = 1$ (forming a pony tail). Since our system is restricted to area $\mathcal{Z}$, the trajectory that emanates form $(1,1)$ stay outside $\mathcal{Z}$, and thus $(1,1)$ is an asymptotically stable rest point. □ □

To complete the proof of Main Result 1, what remains to be shown is that all trajectories starting from within the area $\mathcal{H} - \mathcal{S}$ end up in the rest point $(1,1)$. To show this we will consider a broader set $\mathcal{X}$ containing $\mathcal{H}-\mathcal{S}$, and show that if the system attains a state $\overline{\mathbf{x}} = (\overline{a},\overline{\mu}) \in \mathcal{X}$ such that $\overline{\mu} \geq \delta/(\delta+\sigma)+\varepsilon$, then the system converges for sure to the rest point $(1,1)$.

Main Result 1. Since if all rational players play H we have $M(a,\mu) > 0$ for $a \in [0,1]$ and $\mu = \delta/(\delta+\sigma)+\varepsilon$, so for any $\overline{\mathbf{x}} \in \mathcal{X}$, $\phi_{\mathbf{v}}(t,\overline{\mathbf{x}})$ will stay within $\mathcal{X}$. Moreover, as $A(a,1) > 0$ for $a \in [0,1)$ and $(1,1)$ is an asymptotically stable rest point, so $\lim_{t\to+\infty} \phi_{\mathbf{v}}(t,\overline{\mathbf{x}}) = (1,1)$. This completes the proof of Main result 1. □ □

The above result confirms that in the region $\mathcal{MH}$, there is only one asymptotically stable rest point $(1,1)$ with a fairly large basin of attraction that contains the set $\mathcal{H} - \mathcal{S}$ and any intial condition (a,μ) with $\mu \geq 1/2$.

3.1.2 The analysis with an initial condition in $\mathcal{ML}$

If our system starts from any point in the region $\mathcal{ML}$ where all rational players play L, the analysis shows that for any coordination game we consider here and for any function f, there is an aspiration level $\alpha \in (\delta, \sigma]$ and cost $\varrho \in (0, \delta)$ such that there exists an asymptotically stable restpoint of the system in which a mixed population lives forever. However, the existing aspiring population at this rest point is unstable in the sense that there is a constant (and of equal size) switch of aspiring players between the two pure strategies. This is stated formally below.

Main Result 2. *For any coordination game with $0 < \delta < \sigma$ and any function f, there exists $\overline{\alpha} \in (\delta, \sigma]$ such that for any an aspiration level $\alpha \in (\delta, \overline{\alpha})$ there is an implementation cost $\varrho \in (0, \delta)$ such that there is a rest point $\mathbf{x_s} \in \mathcal{L}$ (or $\mathbf{x_s} \in \mathcal{ML}$) of the system of differential equations (5), (7) which is asymptotically stable.*

If the system starts in the area $\mathcal{ML} \subset \mathcal{Z}$, then the expressions for ψ and M have the following form:

$$\psi(a,\mu) = \psi^L(a,\mu) = a\mu^2(\delta+\sigma) - \mu\delta + \varrho, \tag{11}$$
$$M(a,\mu) = M^L(a,\mu) = \kappa_2(f(\alpha-\delta)(1-\mu)(1-a\mu) - f(\alpha)\mu(1-a)). \tag{12}$$

The phase plane diagram for this situation is presented in figure 4

Since in the case of all rationals playing L we are only interested in the behavior of the system within the area $\mathcal{ML}$, and $(a,\mu) \in \mathcal{ML}$ implies $\mu < \delta/(\delta+\sigma) < 1/2$ and $a > (\sigma-\delta)/\sigma > 0$, so the only restpoints we are interested in are the crossing points (intersections) of the curves (11) and (12). The existence of those restpoints within the areas $\mathcal{L}$ and $\mathcal{ML}$, for given payoffs and function f, depends on the cost ϱ and an aspiration level α. We will not give precise conditions that guarantee existence of crossing points within areas $\mathcal{L}$ or $\mathcal{ML}$ here.[8] Instead, the following lemma will be sufficient for our purposes (see appendix for the proof of the lemma).

Lemma 2. *For any given environment with $0 < \varrho < \delta < \alpha \leq \sigma$ and f, the system of equations*

$$\begin{cases} \psi^L(a,\mu) = 0, \\ M^L(a,\mu) = 0 \end{cases} \tag{13}$$

either has zero, or one or two solutions. In case where it has only one solution, this solution will be denoted by $\mathbf{x_u} = (a_u, \mu_u)$. In case where it has two solutions, they will be denoted by $\mathbf{x_u} = (a_u, \mu_u)$ and $\mathbf{x_s} = (a_s, \mu_s)$), with $a_s < a_u$ and $\mu_s < \mu_u$.

[8] To calculate these conditions one has to solve a few quadratic equations and inequalities.

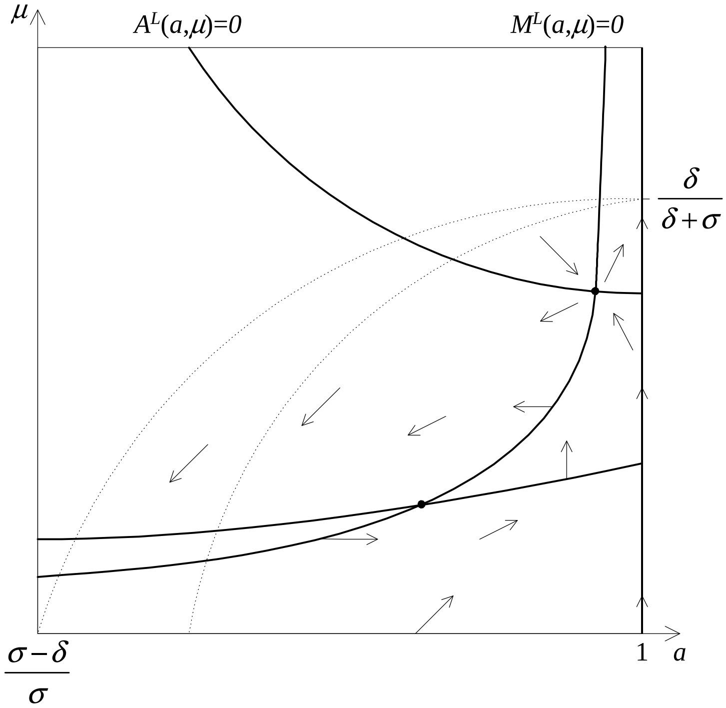

Figure 4: Phase plane diagram when all rationals play L.

Moreover, for any $0 < \delta < \sigma$ *and* f *there is an aspiration level* $\overline{\alpha} \in (\delta, \sigma)$ *such that for any* $\alpha \in (\delta, \overline{\alpha})$ *there exist* $\overline{\varrho} \in (0, \delta)$ *and* $\varepsilon \in (0, \overline{\varrho})$ *such that for any* $\varrho \in (\overline{\varrho} - \varepsilon, \overline{\varrho})$ *the system of equations (13) has two distinct solutions, both lying within the area* $\mathcal{L}$[9]*.*

From the above lemma we know that there are at most two restpoints within the area $\mathcal{ML}$. It turns out that the rest point $\mathbf{x_u}$ is *unstable*, as stated by the following proposition (see appendix for the proof of the proposition).

Proposition 1. *The rest point* $\mathbf{x_u}$ *is unstable.*

The other restpoint that may exist within the area $\mathcal{ML}$ turns out to be *asymptotically stable*. To show this we will use the method adopted from Palomino and Vega-Redondo [1999], which is based on the following result in the theory of ordinary differential equations (see for example [Arnold, 1973, p. 198]).

Theorem 1 (Liouville's Theorem)**.** *Let* $\dot{x}(t) = \mathbf{H}(x(t))$ *be a dynamical system defined on a certain open subset* $U \subseteq \mathbb{R}^n$*, where* $\mathbf{H}$ *is a differentiable*

[9] This of course means that these solutions lie within the area $\mathcal{ML}$. One can also show that that it is possible to have a solution that lie within $\mathcal{ML}$, but not within $\mathcal{L}$. It will not be shown here, because it is not needed.

vector field. If $S \subseteq U$ has a volume $V \equiv \int_S dx$, then the volume $V(t)$ of the set $S(t) = \{y = x(t) : x(0) \in S\}$ satisfies:

$$\dot{V}(t) = \int_{S(t)} \text{div}\, \mathbf{H}(x) dx,$$

where the divergence of the vector field $\mathbf{H}$ *is defined as the trace of the Jacobian of* $\mathbf{H}$ *given by*

$$\text{div}\, \mathbf{H}(x) \equiv \sum_{i=1}^{n} \frac{\partial H_i(x)}{\partial x_i}.$$

We now state and prove our next result.

Proposition 2. *The rest point* $\mathbf{x_s}$ *is an asymptotically stable rest point.*

Proof. To show that $\mathbf{x_s}$ is an asymptotically stable rest point, we will construct a set $S \subseteq \mathcal{ML}$ such that (i) $\mathbf{x_s} \in S$ (and it is the only restpoint in S), (ii) the vector field $\mathbf{v}$ points inwards on the boundary of S and (iii) div : $\mathbf{v}(\mathbf{x}) < 0$ for all $\mathbf{x} \in S$. Under (i) – (iii), we know that for any t and $S(t)$ defined as in theorem 1 we will have $S(t) \subseteq S$, and thus div : $\mathbf{v}(\mathbf{x}) < 0$ for all $\mathbf{x} \in S(t)$. By theorem 1 this would imply that the volume of the set $S(t)$ is decreasing. Furthermore, by the Poincar-Bendixson theorem (see for example Hirsch and Smale [1974] or Hubbard and West [1991]) we know that limit sets of solutions of two dimensional differential equations either include a rest point or are closed orbits. Since the vector field $\mathbf{v}$ points inwards on the boundary of S, so any solution starting within S, remains there. Suppose its limit set is a closed orbit. Then the region enclosed by this orbit is invariant, and so is its volume. This contradicts the fact that the volume of $S(t)$ is decreasing[10]. Thus limit set of any solution starting from within our constructed S must contain $\mathbf{x_s}$, and so $\mathbf{x_s}$ must be an asymptotically stable rest point. So to prove the proposition, what remains to be shown is the existence of S. In what follows we will show how to construct a set S satisfying the three properties (i) – (iii) postulated above.

First we present a fact and an observation characterizing the functions ψ^L and M^L in the neighborhood of (a_s, μ_s) (see appendix for the proof of the fact).

Fact 1. *In the neighbourhood of (a_s, μ_s), the partial derivatives of ψ^L and M^L have the following properties:*

1. $\psi^L_a(a, \mu) > 0$ *and* $\psi^L_\mu(a, \mu) < 0$,
2. $M^L_a(a, \mu) > 0$ *and* $M^L_\mu(a, \mu) < 0$,

[10]This argument comes from Corchn and Mas-Colell [1996], and we used it following Palomino and Vega-Redondo [1999].

3. $-\psi^s_a/\psi^s_\mu < -M^s_a/M^s_\mu$*, where* ψ^s_a*,* ψ^s_μ*,* M^s_a *and* M^s_μ *denote partial derivatives of* ψ^L *and* M^L *at* (a_s, μ_s)*.*

Observation 1. *Since in the neighbourhood of* (a_s, μ_s) *we have* $\psi^L_\mu(a,\mu) < 0$ *and* $M^L_\mu(a,\mu) < 0$*, it follows that equations* $\psi^L(a,\mu) = 0$ *and* $M^L(a,\mu) = 0$ *implicitly define the functions* $\mu = h_\psi(a)$*,* $\mu = h_M(a)$*. Moreover, as* $g'_\psi = -\psi^L_a/\psi^L_\mu > 0$ *and* $g'_M = -M^L_a/M^L_\mu > 0$*, both those functions are increasing. We also have* $g'_\psi < g'_M$*, as* $-\psi^L_a/\psi^L_\mu < -M^L_a/M^L_\mu$*. Also, the inequalities* $\psi^L(a,\mu) < 0$ *and* $M^L(a,\mu) < 0$ *are equivalent to* $\mu > h_\psi(a)$*,* $\mu > h_M(a)$ *(which follows from signs of partial derivatives) (see also figure 5).*

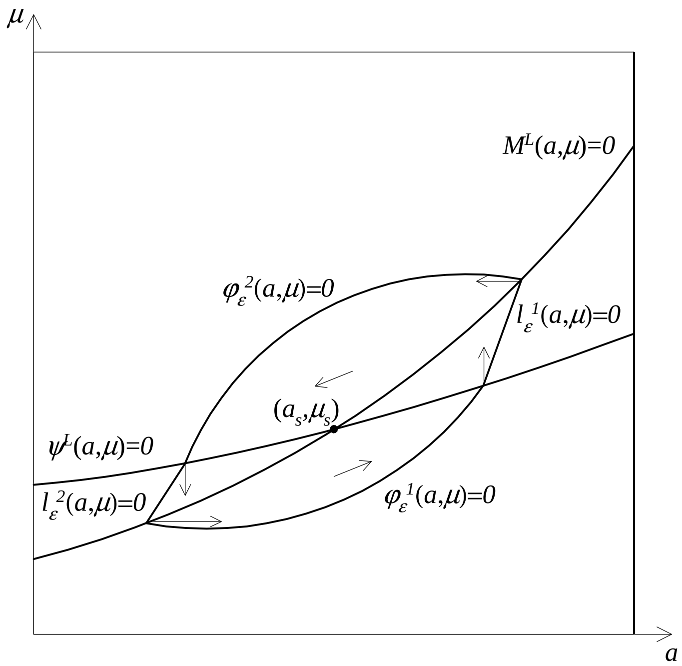

Figure 5: Construction of the set S.

The following three facts are important for further steps in the construction of S (see appendix for proofs of these facts). To state these facts, first let

$$K_1 = -\frac{M^s_\mu}{M^s_a - M^s_\mu}, \quad K_2 = \frac{M^a_\mu}{M^s_a - M^s_\mu}, \quad K = 1 + \max\{1, K_1, K_2\}. \tag{14}$$

Notice that $K_1, K_2, K > 0$ since $M^s_a > 0$ and $M^s_\mu < 0$. Consider the following curves:

$$\begin{aligned}
\varphi^1_\varepsilon(a,\mu) &= M^L(a,\mu)^2 - 2\kappa_1\kappa_2 f(\alpha)\eta(\varepsilon)(a - a_s + \mu - \mu_s + \varepsilon) = 0,\\
\varphi^2_\varepsilon(a,\mu) &= M^L(a,\mu)^2 + 2\kappa_1\kappa_2 f(\alpha)\eta(\varepsilon)(a - a_s + \mu - \mu_s - \varepsilon) = 0,
\end{aligned}$$

where $\varepsilon > 0$ and

$$\begin{aligned} \eta(\varepsilon) &= \max_{(a,\mu)\in C(K\varepsilon)} g(|\psi(a,\mu)|) \text{ where} \\ C(x) &= [a_s - x, a_s + x] \times [\mu_s - x, \mu_s + x]. \end{aligned}$$

Fact 2. *If ε is close to 0, then $\eta(\varepsilon) = o(\varepsilon) > 0$.*

Fact 3. *There exists $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon})$ there is an intersection point $(a_M^1(\varepsilon), \mu_M^1(\varepsilon))$ of the curves $\varphi_\varepsilon^1(a,\mu) = 0$ and $M^L(a,\mu) = 0$ and an intersection point $(a_M^2(\varepsilon), \mu_L^2(\varepsilon))$ of the curves $\varphi_\varepsilon^2(a,\mu) = 0$ and $M^L(a,\mu) = 0$. Moreover, these intersection points have the following properties:*

1. *$\psi^L(a_M^1(\varepsilon), \mu_M^1(\varepsilon)) > 0$ and $-K\varepsilon < a_M^1(\varepsilon) - a_s < 0$, $-K\varepsilon < \mu_M^1(\varepsilon) - \mu_s < 0$, and*

2. *$\psi^L(a_M^2(\varepsilon), \mu_M^2(\varepsilon)) < 0$ and $0 < a_M^1(\varepsilon) - a_s < K\varepsilon$, $0 < \mu_M^1(\varepsilon) - \mu_s < K\varepsilon$.*

Fact 4. *There exists $\overline{\varepsilon} > 0$ satisfying fact 3 and such that for all $\varepsilon \in (0, \overline{\varepsilon})$*

1. *there is an intersection point $(a_\psi^1(\varepsilon), \mu_\psi^1(\varepsilon))$ of the curves $\varphi_\varepsilon^1(a,\mu) = 0$ and $\psi^L(a,\mu) = 0$ such that $M(a_\psi^1(\varepsilon), \mu_\psi^1(\varepsilon)) > 0$, $a_s < a_\psi^1(\varepsilon) < a_M^1(\varepsilon)$ and $\mu_s < \mu_\psi^1(\varepsilon) < \mu_M^1(\varepsilon)$;*

2. *if $M^L(a,\mu) > 0$, $\psi^L(a,\mu) > 0$ and $\varphi_\varepsilon^1(a,\mu) = 0$, then $a < a_s + K\varepsilon$ and $\mu > \mu_s - K\varepsilon$;*

3. *there is an intersection point $(a_\psi^2(\varepsilon), \mu_\psi^2(\varepsilon))$ of the curves $\varphi_\varepsilon^2(a,\mu) = 0$ and $\psi^L(a,\mu) = 0$, such that $M(a_\psi^2(\varepsilon), \mu_\psi^2(\varepsilon)) < 0$, $a_M^2(\varepsilon) < a_\psi^2(\varepsilon) < a_s$ and $\mu_M^2(\varepsilon) < \mu_\psi^2(\varepsilon) < \mu_s$;*

4. *if $M^L(a,\mu) < 0$, $\psi^L(a,\mu) < 0$ and $\varphi_\varepsilon^2(a,\mu) = 0$, then $a > a_s - K\varepsilon$ and $\mu < \mu_s + K\varepsilon$.*

Now consider the curves

$$\begin{aligned} \ell_\varepsilon^1(a,\mu) &= (\mu_M^2(\varepsilon) - \mu_\psi^1(\varepsilon))(a - a_\psi^1(\varepsilon)) - (a_M^2(\varepsilon) - a_\psi^1(\varepsilon))(\mu - \mu_\psi^1(\varepsilon)) = 0, \\ \ell_\varepsilon^2(a,\mu) &= (\mu_M^1(\varepsilon) - \mu_\psi^2(\varepsilon))(a - a_\psi^2(\varepsilon)) - (a_M^1(\varepsilon) - a_\psi^2(\varepsilon))(\mu - \mu_\psi^2(\varepsilon)) = 0, \end{aligned}$$

which are lines going through points $(a_\psi^1(\varepsilon), \mu_\psi^1(\varepsilon))$, $(a_M^2(\varepsilon), \mu_M^2(\varepsilon))$ and $(a_\psi^2(\varepsilon), \mu_\psi^2(\varepsilon))$, $(a_M^1(\varepsilon), \mu_M^1(\varepsilon))$ respectively. Define the sets

$$\begin{aligned} S_\varepsilon^1 &= \{(a,\mu) : \varphi_\varepsilon^1(a,\mu) \le 0 \text{ and } A(a,\mu) \le 0 \text{ and } M^L(a,\mu) \le 0\}, \\ S_\varepsilon^2 &= \{(a,\mu) : \varphi_\varepsilon^2(a,\mu) \le 0 \text{ and } A(a,\mu) \ge 0 \text{ and } M^L(a,\mu) \ge 0\}, \\ S_\varepsilon^3 &= \{(a,\mu) : \ell_\varepsilon^1(a,\mu) \le 0 \text{ and } A(a,\mu) \ge 0 \text{ and } M^L(a,\mu) \le 0\}, \\ S_\varepsilon^4 &= \{(a,\mu) : \ell_\varepsilon^2(a,\mu) \le 0 \text{ and } A(a,\mu) \le 0 \text{ and } M^L(a,\mu) \ge 0\}. \end{aligned}$$

Let $S_\varepsilon = S^1_\varepsilon \cup S^2_\varepsilon \cup S^3_\varepsilon \cup S^4_\varepsilon$. Facts 3 – 4 guarantee that there exists $\overline{\varepsilon}$ such that for any $\varepsilon \in (0, \overline{\varepsilon})$ the set S_ε is nonempty, $(a_s, \mu_s) \in S_\varepsilon$ and $S_\varepsilon \subseteq C(\overline{K}\varepsilon)$ (refer to observation 1 and figure 5). The following lemma helps us to show that if $\varepsilon < \overline{\varepsilon}$, then all solutions that enter S_ε must stay there (see appendix for the proof of the lemma).

Lemma 3. *Let $\overline{\varepsilon}$ be such that facts 3 – 4 hold. Then for any $\varepsilon \in (0, \overline{\varepsilon})$ the vector field $\mathbf{v}$ points inwards on the boundary of the set S_ε.*

Notice that, since $A_a(a_s, \mu_s) = 0$ (as $g(0) = 0$) and $M_\mu(a, \mu) < 0$ around (a_s, μ_s) and at (a_s, μ_s), so there is $\varepsilon^* > 0$ such that $A_a(a, \mu) + M_\mu(a, \mu) < 0$ for all $(a, \mu) \in C(\varepsilon^*)$. Thus the set $S = S_\varepsilon$, where $\varepsilon \in (0, \varepsilon^*)$ and is such that facts 3 – 4 are satisfied, satisfies all the postulates at the beginning of the proof of the lemma. This completes the proof that (a_s, μ_s) is an asymptotically stable restpoint. □ □

Now can prove Main Result 2.

Main Result 2. This is an immediate consequence of lemma 2 and proposition 2. Additionally, by proposition 1 we know that there can be at most one stable restpoint in the area $\mathcal{L}$. This completes the proof of Main Result 2. □ □

We end this section with the following remark.

Remark 4 (The aspiring population in an eternal flux)**.** *Consider the asymptotically stable rest point in the region $\mathcal{ML}$ which we have discovered above. Since it is a rest point, $\dot{a} = 0$. In our environment, this means that there is no player in the population who switches behavior rules at this rest point. Moreover, at this point we have that $0 < \mu < 1$ and therefore some aspiring players play H while others play L. Also, since this rest point is in the region $\mathcal{ML}$, we know that all players using the rational rule play L. Hence, given that the level of the social aspiration of the aspiring population is between δ and σ, it must be that some aspiring players are satisfied with their current strategy, while some are not. Those who are not must be switching their strategies. In spite of that, since $\dot{\mu} = 0$, the size of the aspiring players who move from L to H must be the same as the size who move from H to L at each instant of time. It seems therefore that this stable restpoint puts the aspiring population in a rather confused individual state or an eternal flux (as observed in Palomino and Vega-Redondo, though in the Prisoners' Dilemma games) – although they do not give up the aspiring rule, they are unable to settle on a single pure strategy. Computer simulations (see section 4) reveal that such a confused state has a fairly large basin of attraction (which includes the set S we have constructed above) within the region $\mathcal{ML}$.*

3.2 Low aspiration level

Throughout this section we will assume that the aspiration level is low, that is $0 < \alpha \le \delta$. In this case an expression for $\dot{\mu}$ simplifies to the following:

$$\dot{\mu} = M(a, \mu) = \kappa_2 f(\alpha)(1 - a)(\xi - \mu). \tag{15}$$

As the following analysis will show, with low social aspirations, the relatively costly rational rule (no matter how small its associated cost may be) has no chance of survival in the long run.

3.2.1 The analysis with an initial condition in $\mathcal{MH}$

Similar to the high aspiration case we will first study the behavior of the system with initial conditions in the region $\mathcal{MH}$ where all rationals play H. Unlike in the case of high aspiration, there is no asymptotically stable rest point when the aspiration level is low. This is because $M(a, \mu) = 0$ for $a = 1$, so all points lying within the set $\{(1, \mu) : \delta/(\delta + \sigma) < \mu \le 1\}$ are rest points. The following analysis and simulations show the existence of asymptotically stable sets (in forms of intervals contained in the set $\{(1, \mu) : 0 \le \mu \le 1\}$).

Main Result 3. *Consider any coordination game such that $0 < \delta < \sigma$ and an aspiration level $\alpha \in (0, \delta]$. Let $\mathcal{X} = [0, 1] \times [\delta/(\delta + \sigma) + \varepsilon, 1] \supset \mathcal{H} - \mathcal{S}$, $\overline{\mathbf{x}} \in \mathcal{X}$ and let $\phi_{\mathbf{v}}(t, \overline{\mathbf{x}})$ be a solution of the system of differential equations (5), (7). Then $\lim_{t\to+\infty} \phi_{\mathbf{v}}(t, \overline{\mathbf{x}}) = (1, \mu^*)$, where $\mu^* \in [\delta/(\delta + \sigma) + \varepsilon, 1]$.*

Moreover if $\varrho \in (0, \sigma^2/(4(\delta + \sigma)))$, then for $\overline{\mathbf{x}} = (\overline{a}, \overline{\mu}) \in \mathcal{X}$ such that $\overline{a} < 1$ and $\overline{\mu} > \mu_1$ we have $\mu^* \ge \mu_2$, where

$$\mu_1 = \frac{\delta}{\delta + \sigma} + \frac{\sigma - \sqrt{\sigma^2 - 4\varrho(\delta + \sigma)}}{2(\delta + \sigma)}, \quad \mu_2 = \frac{\delta}{\delta + \sigma} + \frac{\sigma + \sqrt{\sigma^2 - 4\varrho(\delta + \sigma)}}{2(\delta + \sigma)}.$$

Proof. Obviously we have $M^H(a, \mu) = 0$ iff $a = 1$ or $\mu = 1$. Moreover $M^H(a, \mu) \ge 0$ and $\psi^H(a, 1) = \varrho$ for $a \in [0, 1)$, so $A^H(a, 1) > 0$ for $\mu \in [0, 1)$. It also holds that $A^H(1, \mu) = 0$ for $a = 1$ (cf. proof of lemma 1).

This shows that all points $(1, \mu^*)$ such that $\mu^* > \delta/(\delta + \sigma)$ are the only rest points within the area $\mathcal{MH}$. Since there are no cycles in the area $\mathcal{X}$ (as $M^H(a, \mu) \ge 0$ for all $(a, \mu) \in \mathcal{X}$ and whenever $M^H(a, \mu) = 0$ it holds that $A^H(a, \mu) = 0$) and all trajectories starting within $\mathcal{X}$ will remain there, so by Poincar-Bendixson theorem we have $\lim_{t\to+\infty} \phi_{\mathbf{v}}(t, \overline{\mathbf{x}}) = (1, \mu^*)$.

For the second part of the theorem, let us consider the crossing points of lines $\psi^H(a, \mu) = 0$ and $M^H(a, \mu) = 0$. To find them it is enough to solve the quadratic equation

$$\psi^H(1, \mu) = \mu^2(\delta + \sigma) - \mu(2\delta + \sigma) + \delta + \varrho = 0.$$

The discriminant of the equation is $\varDelta = \sigma^2 - 4\varrho(\delta + \sigma)$. It is clear that $\varDelta \ge 0$ for $\varrho \le \sigma^2/(4(\delta + \sigma))$ and that $\varDelta \le \sigma^2$. If $\varrho < \sigma^2/(4(\delta + \sigma))$ the equation has two solutions

$$\mu_1 = \frac{\delta}{\delta+\sigma} + \frac{\sigma - \sqrt{\sigma^2 - 4\varrho(\delta+\sigma)}}{2(\delta+\sigma)}, \quad \mu_2 = \frac{\delta}{\delta+\sigma} + \frac{\sigma + \sqrt{\sigma^2 - 4\varrho(\delta+\sigma)}}{2(\delta+\sigma)}.$$

Moreover $\psi^H(1,\mu) < 0$ iff $\mu \in (\mu_1, \mu_2)$ so the set of points satisfying the inequality $\psi^H(a,\mu) < 0$ contains the set of points $(1,\mu)$, where $\mu \in (\mu_1, \mu_2)$. Thus no trajectory starting from points $(a_0, \mu_0) \in \mathcal{X}$, such that $a_0 < 1$ will reach a rest point in this set. So any trajectory starting from any point $(a_0, \mu_0) \in \mathcal{Y} = [0,1] \times [\mu_1, 1] \subset \mathcal{X}$, such that $a < 1$ and $\mu > \mu_1$ will remain in $\mathcal{Y}$ (as the vector field points inwards on its boundary, everywhere apart from points where it vanishes) and will reach the rest point $(1, \mu^*)$ such that $\mu^* > \mu_2$. □ □

Remark 5. *Observe that $\mu_2 \to 1$ and $\mu_1 \to \delta/(\delta+\sigma)$ as $\varrho \to 0$. Thus the smaller ϱ we take, the bigger the area $\mathcal{Y}$ from where all trajectories approach restpoints of the form $(1, \mu^*)$. In particular, if ϱ is small enough, the population starting from the state where less than a half of aspiration driven players play H will reach the state where most of them play H and there are no rationals.*

3.2.2 The analysis with an initial condition in $\mathcal{ML}$

In case of initial conditions within $\mathcal{ML}$, the fraction of aspiration driven players playing H decreases continually over time and we are able to show that if the trajectory stays within $\mathcal{ML}$, it will reach the rest point $(1, \mu)$, where $\mu \in [0, \delta/(\delta+\sigma) + \varepsilon]$.

Main Result 4. *Consider any coordination game such that $0 < \delta < \sigma$ and an aspiration level $\alpha \in (0, \delta]$. Any trajectory that starts in the are $\mathcal{ML}$ and remains there reaches the rest point $(1, \mu^*)$, where $\mu^* < \delta/(\delta+\sigma)$.*

Proof. Obviously we have $M^L(a,\mu) = 0$ iff $a = 1$ or $\mu = 0$. Moreover $M^L(a,\mu) \leq 0$ and $\psi^L(a, 0) = \varrho$ for $a \in [0,1)$, so $A^L(a,0) > 0$ for $a \in [0,1)$. It also holds that $A^L(1,\mu) = 0$.

This shows that all points $(1, \mu^*)$ such that $\mu^* < \delta/(\delta+\sigma)$ are the only rest points within the area $\mathcal{ML}$. Since there are no cycles in the area $\mathcal{ML}$ (as $M^L(a,\mu) \leq 0$ for all $(a,\mu) \in \mathcal{ML}$ and whenever $M^L(a,\mu) = 0$ it holds that $A^L(a,\mu) = 0$), so for any trajectory starting within $\mathcal{ML}$ that remains there, by the Poincar-Bendixson theorem, we have $\lim_{t\to+\infty} \phi_{\mathbf{v}}(t, \overline{\mathbf{x}}) = (1, \mu^*)$. □
□

Remark 6. *The result above is a conditional statement, and we cannot say more without studying the behavior in the switching region, where we cannot exclude cyclic behavior. However simulations show that there are scenarios in which most of the trajectories starting within $\mathcal{ML}$ and remaining there* *converge to the rest point* $(1, \mu^*)$. *Moreover, the set of trajectories that remain in the* $\mathcal{ML}$ *area is non empty, as any trajectory starting from the point* $(a, 0)$ *that is within* $\mathcal{ML}$ *remains there and reaches* $(0, 0)$ *point (as* $M^L(a, 0) = 0$*).*

4 Simulations

In this section we will demonstrate how the system under consideration behaves in four different scenarios – two with high aspiration levels and two with low aspiration levels.

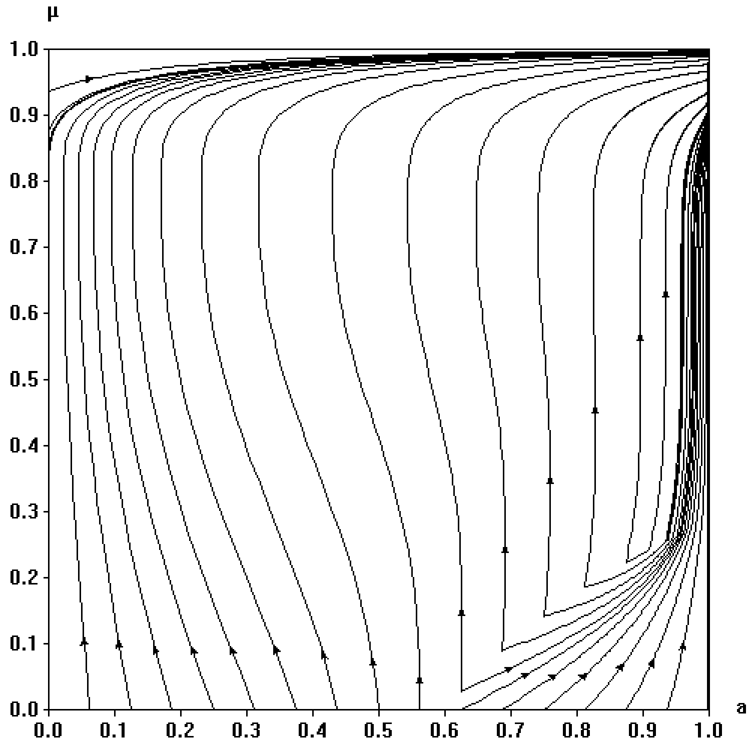

Figure 6: $\alpha = 2.7$, $\varrho = 1.0$.

Our analytical results show that any qualitatively different behavior of the system depends neither on the payoffs of the game nor on the probability functions affecting willingness of individuals to change the rule or the action, but on the aspiration level α and the cost of rationality ϱ. Thus in scenarios considered in this section we fix the game to the one with $\delta = 2$ and $\sigma = 5$ and we also fix the speed adjustment parameters to $\kappa_1 = \kappa_2 = 1$. We also consider the probability functions of the form $f(x) = dx^2$, $g(x) = b(x^2 - cx^3)$. Parameters d, b and c are chosen to satisfy the conditions $f(\sigma) = 1$, $g(\sigma - \varrho) = 1$ and $g'(x) > 0$ for $x \in (0, \sigma - \varrho]$ (notice that σ is the maximal value for α and $\sigma - \varrho$ is the maximal absolute difference between average payoffs of rational and aspiration driven populations).[11]

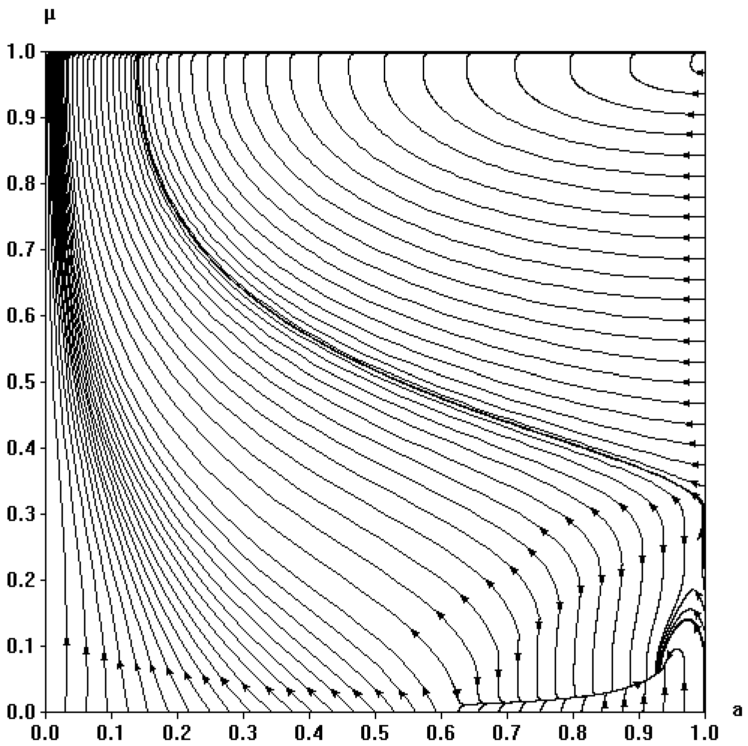

Figure 7: $\alpha = 2.15$, $\varrho = 0.1$.

Since the value of d depends only on the game, it will be fixed to $d = 0.04$. The value of c depends on the cost of rationality and will be calculated separately for each scenario using the same formula $c = 2.0/(3.1(\sigma - \varrho))$. The value of b depends on the value of c and will be calculated using the formula $b = 1.0/((\sigma - \varrho)^2 - c(\sigma - \varrho)^3)$.

In the first scenario we demonstrate the behavior of the system when the aspiration level and the cost of rationality are high enough to prevent the existence of rest points in the $\mathcal{ML}$ area. As we can see in the figure 6 the fraction of aspiration driven players playing H increases constantly until the system reaches the only rest point $(1, 1)$.

In the second scenario we demonstrate the behavior of the system when the aspiration level is above δ, but close to it and the cost of rationality is small enough to enforce the existence of rest points in the $\mathcal{ML}$ area. We can see in figure 7 that in this scenario most of the area $\mathcal{ML}$ is the basin of

[11]The function f satisfies all the requirements given in the description of the model. In case of g this requirement is not satisfied (since the proposed function is not strictly increasing on $[0, +\infty)$). However, it is enough in practice that the function be strictly increasing on $[0, \sigma - \varrho]$ and we guarantee that.

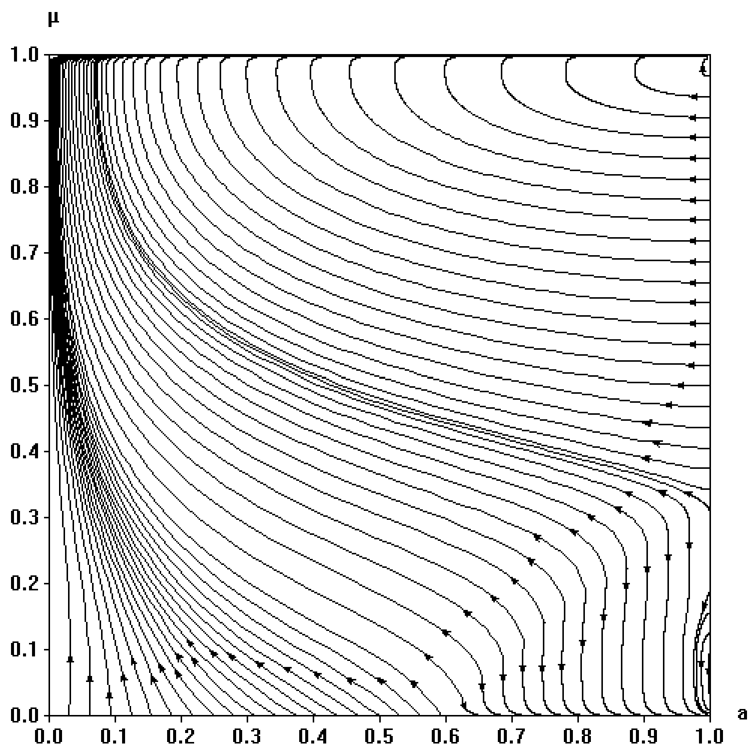

Figure 8: $\alpha = 1.8$, $\varrho = 0.12$.

attraction of the stable rest point $\mathbf{x_s}$. So most of the initial conditions lying within the $\mathcal{ML}$ region remain there and lead to a stable population where rational and aspiration driven players coexist.

In the last two scenarios we demonstrate the behavior of the system when the aspiration level is low. In both cases we fix it to the same value and show how the behavior of the system depends on the cost of rationality. In figure 8 we can see that if ϱ is small then almost all trajectories starting within $\mathcal{MH}$ end up in the state where rationals die out and the fraction of aspiration driven players playing H is close to 1. Moreover almost all trajectories starting within $\mathcal{ML}$ end up in the state where rationals die out and the fraction of aspiration driven players playing H is close to 0.

Figure 9 shows the behavior of the system with a higher cost of rationality. One can observe that there are trajectories starting within $\mathcal{MH}$ which end up in the state where all rationals die out and the fraction of aspiration driven players playing H is relatively small. We can also see that most of the trajectories starting within $\mathcal{ML}$ remain there, but reach states of various levels of fractions of players playing H. A careful look at the line $a = 1$ reveals the existence of stable and unstable sections. For example, for μ very low, very high and around 1/2, there are stable sections while in the remaining ranges of μ, the sections are unstable.

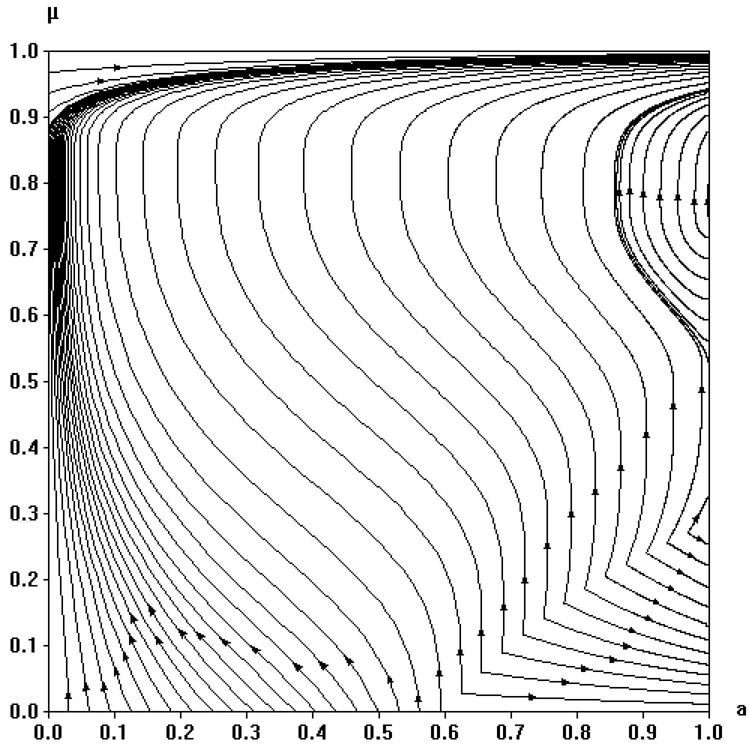

Figure 9: $\alpha = 1.8$, $\varrho = 0.8$.

5 Concluding Remarks

In this paper we study an evolutionary model of equilibrium and behavior-rule selection. We have kept the analysis as simple and tractable as possible by restricting attention to the simplest possible class of games where playing best response requires a rational player to gather information, the cost of which we assumed to be positive no matter how small. Players could avoid this small implementation cost of rationality by adopting an alternative and simple behavior rule, which we call the aspiring rule, which incorporates a fixed and common aspiration across all players who adopt this rule. We show that there are two stable long run outcomes where either the relatively costly rational rule becomes extinct and (if aspiration level is high enough) all players in the population achieve full efficiency, or that both the behavior rules co-exist and there is only a partial use of efficient strategies in the population. These findings rationalize the use of the aspiration driven behavior in several existing studies in the literature and also helps us take a comparative evolutionary look at the two rules in retrospect.

Our next goal is to allow for the social aspiration level to evolve as in Palomino and Vega-Redondo, which is a hard problem in our case since it involves dynamic systems in dimensions higher than 2. We reserve this issue for future research.

Acknowledgement We thank Sandro Brusco, Huw Dixon, Rajeeva Karandikar and Jorgen Weibull for various comments and suggestions during several stages of this research. The usual disclaimer applies.

A Appendix

Theorem 2. *Suppose that an autonomous differential equation* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ *has a rest point* $\mathbf{x_0} = (x_0, y_0)$*, its linearization has eigenvalues* 0*,* λ*, the* x*-axis is the eigendirection for the eigenvalue* 0*, the* y*-axis is the eigendirection for the eigenvalue* λ *and that*

first nondegeneracy condition 0 *is a simple eigenvalue of the linearization, the other eigenvalue* λ *is nonzero,*

second nondegeneracy condition *the coefficient* $p_{2,0}$ *of* x^2 *in the Taylor expansion of* x' *is nonzero.*

Assume further that $\lambda < 0$ *and* $p_{2,0} > 0$*. Then there exist unique trajectories which tend to* $\mathbf{x_0}$ *tangentially to the line of eigenvectors with eigenvalue* λ *from both sides. Moreover,*

(i). these trajectories, together with $\mathbf{x_0}$*, form a smooth curve called the separatrix of the saddle node;*

(ii). all trajectories to the left of this separatrix tend to 0 *tangentially to the line of eigenvectors with eigenvalue* 0*, forming a pony tail; and*

(iii). there is a unique exceptional trajectory to the right of this separatrix which emanates from $\mathbf{x_0}$*, also tangentially to the to the line of eigenvectors with eigenvalue* 0*.*

Lemma 2. It will be convenient to make the following substitutions:

$$d := f(\alpha)/(f(\alpha-\delta)+f(\alpha)), \quad \tau := \sigma/\delta, \quad \upsilon := \varrho/\delta. \tag{16}$$

Then the system of equations (13) takes the following form:

$$\begin{cases} a\mu^2(1+\tau) - \mu + \upsilon = 0, \\ (1-\mu)(1-a\mu) - d(1-d)\mu(1-a) = 0 \end{cases} \tag{17}$$

where $\tau > 1$, $0 < \upsilon < 1$ and $1/2 < d < 1$.

Notice that the line $\mu = 0$ is an asymptote of both curves defined by the above equations, and so the following analysis will be conducted for $\mu \neq 0$. By solving for a from these two equations we get

$$\begin{cases} a = \dfrac{\mu - \upsilon}{\mu^2(\tau+1)} \\ a = \dfrac{\mu - (1-d)}{\mu(2d-1+\mu(1-d))} \end{cases} \tag{18}$$

Comparing the right side (for $\mu \neq 0$) we get the following quadratic equation:

$$\mu^2(d+\tau) + \mu(d(\tau - \upsilon - 1) - \tau + \upsilon) + \upsilon(2d-1) = 0. \tag{19}$$

Thus the system of equations (13) has zero, one or two solutions.

The discriminant of equation (19) is

$$\varDelta = \upsilon^2(1-d)^2 - 2\upsilon(d^2(\tau+1) + (2d-1)(\tau+d)) + (\tau(1-d)+d)^2. \tag{20}$$

Solving the inequality $\varDelta \geq 0$ for υ we get

$$\upsilon_1 = \frac{d^2(\tau+1) + (2d-1)(\tau+d) - \sqrt{\varGamma}}{(1-d)^2},$$

$$\upsilon_2 = \frac{d^2(\tau+1) + (2d-1)(\tau+d) + \sqrt{\varGamma}}{(1-d)^2},$$

where $\varGamma = 4d^2(d+\tau)(\tau+1)(2d-1) > 0$ for any $\tau > 1$ and $1/2 < d < 1$. Notice also that $\sqrt{(\varGamma)} < d^2(\tau+1) + (2d-1)(\tau+d)$, as $4(1-d)^2((\tau(1-d)+d)^2) > 0$ and $d^2(\tau+1) + (2d-1)(\tau+d) > 0$. Moreover $d^2(\tau+1) + (2d-1)(\tau+d) > (1-d)^2(\tau+1)$, since $d^2 > (1-d)^2$. Hence we can conclude that $\varDelta \geq 0$ for $\upsilon \leq \upsilon_1$ and $\upsilon \geq \upsilon_2$, where $0 \leq \upsilon_1 < \upsilon_2$ and $\upsilon_2 > \tau+1$. Because we restrict ourselves to $0 < \upsilon < 1$, the condition reduces to $\upsilon \leq \upsilon_1$.

To show that it is always possible to have one or two solutions within the area $\mathcal{L}$ we will first show that there is always some $d \in (1/2, 1)$ such that $\upsilon_1 < 1$. This inequality (under our assumptions) reduces to

$$(d-1)^2(d^2(\tau - 2\sqrt{\tau+1})(\tau+2\sqrt{\tau+1}) - 2d\tau(\tau+1) + (\tau+1)^2) < 0$$

and further to

$$(d-1)^2\left(d - \frac{\tau+1}{\tau+2\sqrt{\tau+1}}\right)\left(d - \frac{\tau+1}{\tau-2\sqrt{\tau+1}}\right) < 0, \text{ if } \tau \neq 2\sqrt{\tau+1}$$

and

$$(d-1)^2(\tau+1)(2d\tau - \tau - 1) > 0, \text{ if } \tau = 2\sqrt{\tau+1}.$$

Let

$$d_1^* = \frac{\tau+1}{\tau+2\sqrt{\tau+1}} = \frac{1}{2} + \frac{(\sqrt{\tau+1}-1))^2}{2(\tau+2\sqrt{\tau+1})} < 1,$$

$$d_2^* = \frac{\tau+1}{\tau-2\sqrt{\tau+1}}.$$

It is easy to see that if $\tau - 2\sqrt{(\tau + 1)} \neq 0$ then either $d_2^* < 0$ or $d_2^* > 1$, depending on sign of $\tau - 2\sqrt{(\tau + 1)}$. Thus for $d \in (0, 1)$, $\upsilon_1 < 1$ if $d > d_1^*$. So by making d big enough one can always make sure that it is possible to choose $\upsilon = \upsilon^* \in (0, 1)$ such that system of equations (13) has exactly one solution.

After the substitution

$$A := d + \tau, \quad B := d(\upsilon + \tau), \quad C := \upsilon(2d - 1), \tag{21}$$

the equation (19) takes the form

$$\mu^2 A + \mu(B - A - C) + C = 0, \tag{22}$$

where $A > 3/2$, $B > 1/2$ and $0 < C < 1$. The discriminant of the above equation is $\Delta = (A - (B - C))^2 - 4AC$ and the solutions are

$$\mu_s = \mu^* - \frac{\sqrt{\Delta}}{2A}, \quad \mu_u = \mu^* + \frac{\sqrt{\Delta}}{2A}, \tag{23}$$

where

$$\mu^* = \frac{A - (B - C)}{2A}.$$

Since $A > B$ (as $A - B = d(1 - \upsilon) + \tau(1 - d)$), so $A - (B - C) > 0$. Also $B > C$ (as $B - C = d(\tau - \upsilon) + \upsilon$). Moreover $\sqrt{(\Delta)} < A - (B - C)$. Thus $0 < \mu_s \leq \mu^* \leq \mu_u < 1 - (B - C)/2A < 1$.

Let $q : \mathbb{R}_+ \to \mathbb{R}$ be a function such that

$$q(\mu) = \frac{\mu - (1 - d)}{\mu(2d - 1 + \mu(1 - d))}.$$

It can be easily checked that for $d \in (1/2, 1)$, q is strictly increasing, so for $a_s = q(\mu_s)$, $a_u = q(\mu_u)$ and $\mu_s < \mu_u$ we have $a_s < a_u$.

To see if it is possible that $\{\mathbf{x_u}, \mathbf{x_s}\} \in \mathcal{L}$ we will first look for conditions on μ for which $(q(\mu), \mu) \in \mathcal{L}$, that is such that

$$q(\mu)\mu < q(\mu) - \sigma/(\delta + \sigma). \tag{24}$$

The inequality (24) simplifies to

$$\mu^2((2 - d)\tau + 1) - \mu((1 - d)(3\tau + 1) + 1) + (1 - d)(\tau + 1) < 0, \tag{25}$$

with the discriminant

$$\Gamma = d^2(5\tau^2 + 2\tau + 1) - 2d\tau(3\tau + 1) + \tau^2.$$

So for the inequality (24) to have two solutions it must be true that $d < d_1^L$ or $d > d_2^L$ where

$$\begin{aligned} d_1^L &= \frac{\tau(3\tau + 1 - 2\sqrt{\tau(\tau + 1)})}{5\tau^2 + 2\tau + 1}, \\ d_2^L &= \frac{\tau(3\tau + 1 + 2\sqrt{\tau(\tau + 1)})}{5\tau^2 + 2\tau + 1} = 1 - \frac{(\tau + \sqrt{\tau(\tau + 1)})^2 + 1}{5\tau^2 + 2\tau + 1}. \end{aligned}$$

It is easy to see that $d_1^L < 1/2$ and $d_2^L < 1$, so for the values of d we consider, the condition for $\Gamma > 0$ reduces to $d \in (\max(d_2^*, d_2^L), 1)$.

The inequality (24) is satisfied if $\mu \in (\mu_1^{\mathcal{L}}, \mu_2^L)$, where

$$\mu_1^L = \frac{(1-d)(3\tau+1)+1-\sqrt{\Gamma}}{2(1+\tau(2-d))}, \qquad \mu_2^L = \frac{(1-d)(3\tau+1)+1+\sqrt{\Gamma}}{2(1+\tau(2-d))}$$

Since $(\mu_1^L, \mu_2^L) \to [0, 1/(\tau+1)]$ and $\mu^* \to 1/(2(\tau+1))$ when $d \to 1$, so there is $\overline{d} \in (\max(d_2^*, d_2^L), 1)$ and $\overline{\upsilon}$ such that $\mu^* \in (\mu_1^L, \mu_2^L)$. Moreover there is $\varepsilon > 0$ such that for all $\upsilon \in (\overline{\upsilon} - \varepsilon, \overline{\upsilon})$, $\mu_s < \mu_u$ and $(\mu_s, \mu_u) \subseteq (\mu_1^L, \mu_2^L)$. Thus there is $\overline{\alpha} \in (\delta, \sigma)$ (for having d big enough one can take α close to δ) such that for any $\alpha \in (\delta, \overline{\alpha})$ there is $\overline{\varrho}$ and $\varepsilon \in (0, \overline{\varrho})$ such that for any $\varrho \in (\overline{\varrho} - \varepsilon, \overline{\varrho})$ the system of equations (13) has two distinct solutions $\mathbf{x_s}$, $\mathbf{x_u}$ such that $\{\mathbf{x_s}, \mathbf{x_u}\} \subseteq \mathcal{L}$. □ □

Proposition 1. The linearization of the system of differential equations (5), (7) at $\mathbf{x_u}$ has one zero eigenvalue, so we cannot simply use it to reason about properties of this restpoint. However we can study a linearization of the modified system of differential equations

$$\begin{aligned} \dot{a} &= \psi^L(a, \mu), \\ \dot{\mu} &= M^L(a, \mu), \end{aligned} \tag{26}$$

to reason about stability of the rest point (notice that one can view the system of linear differential equations (5), (7) as a perturbed system (26) where the perturbation preserves the signs of $\dot{a}$ and $\dot{\mu}$). We first prove the following fact.

Fact 5. *The linearization of the system of differential equations (26) at* $\mathbf{x_u}$ *has two nonzero eigenvalues of opposite signs, that is,* $\mathbf{x_u}$ *is a saddle node of the system.*

Proof. Partial derivatives of ψ^L and M^L at (a_u, μ_u) are as follows:

$$\begin{aligned} \psi_a^u &= \psi_a^L(a, \mu)|_{(a_u, \mu_u)} = [\mu^2(\delta + \sigma)]|_{(a_u, \mu_u)}, \\ \psi_\mu^u &= \psi_\mu^L(a, \mu)|_{(a_u, \mu_u)} = [2a\mu(\delta + \sigma) - \delta]|_{(a_u, \mu_u)}, \\ M_a^u &= M_a^L(a, \mu)|_{(a_u, \mu_u)} = [\kappa_2 \mu(f(\alpha) - f(\alpha - \delta)(1 - \mu))]|_{(a_u, \mu_u)}, \\ M_\mu^u &= M_\mu^L(a, \mu)|_{(a_u, \mu_u)} \\ &= [-\kappa_2(f(\alpha - \delta)(a(1 - \mu) + 1 - a\mu) + f(\alpha)(1 - a))]|_{(a_u, \mu_u)}. \end{aligned} \tag{27}$$

The characteristic polynomial of the Jacobian of the vector field $\mathbf{v}$ at (a_u, μ_u) is

$$\lambda^2 - (\psi_a^u + M_\mu^u)\lambda - (M_a^u \psi_\mu^u - \psi_a^u M_\mu^u), \tag{28}$$

with the discriminant $\Delta = (\psi_a^u - M_\mu^u)^2 + 4M_a^u\psi_\mu^u$. It is obvious that $\psi_a^u > 0$, $M_a^u > 0$ and $M_\mu^u < 0$. We will show that $\Delta > 0$, by showing that $\psi_\mu^u \geq 0$. Consider the equation

$$a\mu^2(\delta + \sigma) - \mu\delta + \varrho = 0,$$

satisfied by (a_u, μ_u). Solving it with respect to μ we get that

$$\mu_u \geq \frac{\delta}{2a_u(\delta + \sigma)}$$

(notice that by lemma 2, if there are two solutions to this equation, μ_u is the bigger one). This immediately leads to conclusion that $\psi_\mu^u \geq 0$.

Since $M_a^u\psi_\mu^u - \psi_a^u M_\mu^u > 0$, so $\sqrt{(\Delta)} > |\psi_a^u + M_\mu^u|$. Thus the characteristic polynomial has two real and distinct solutions $\lambda_1 < 0 < \lambda_2$. □ □

The rest of the proof of proposition 1 below is based on the proof of theorem 8.3.2 from [Hubbard and West, 1991, part II, pp 155–159], stating that if a rest point is a saddle node of a linearization then it is a saddle node.

To analyze the system we will first move the point (a_u, μ_u) to the origin and change the basis to eigenbasis of the linearization matrix of (26). After this transformation, the system of equations (26) becomes

$$\begin{aligned} \dot{x} &= \lambda_1 x + P(x, y), \\ \dot{y} &= \lambda_2 y + Q(x, y), \end{aligned} \tag{29}$$

where λ_1 and λ_2 are eigenvalues of the linearization of (26), P and Q are polynomials starting with at least quadratic terms from the expansion of ψ^L and M^L into a Taylor polynomial, and a and μ of the original system are changed to x and y, for clarity.

Similarly, the system of differential equations (5), (7) is transformed to

$$\begin{aligned} \dot{x} &= \widetilde{A}(x, y), \\ \dot{y} &= \lambda_2 y + Q(x, y), \end{aligned} \tag{30}$$

where

$$\widetilde{A}(x, y) = \kappa_1((1 - c(x, y))g(\lambda_1 x + P(x, y)) - c(x, y)g(\lambda_1 x + P(x, y))),$$

where $c(x, y) = \alpha x + \beta y + a_u$ is a after an affine transformation. Obviously in the neighborhood of $(0, 0)$, $\widetilde{A}(x, y)$ and $\lambda_1 x + P(x, y)$ have exactly the same sign (as it is in the case of A and ψ).

Following the proof from Hubbard and West [1991] we turn the system of differential equations (29) into first order equations, first for y as a function of x, and then for x as a function of y :

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \varphi_1(x,y) = \frac{\lambda_2 y + Q(x,y)}{\lambda_1 x + P(x,y)}, \tag{31}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \varphi_2(x,y) = \frac{\lambda_1 x + P(x,y)}{\lambda_2 y + Q(x,y)}. \tag{32}$$

We do the same with the system of differential equations (30) :

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \widetilde{\varphi}_1(x,y) = \frac{\lambda_2 y + Q(x,y)}{\widetilde{A}(x,y)}, \tag{33}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \widetilde{\varphi}_2(x,y) = \frac{\widetilde{A}(x,y)}{\lambda_2 y + Q(x,y)}. \tag{34}$$

The equations (31) and (33) will be considered only within areas where $\lambda_1 x + P(x,y) \neq 0$ (and also $\widetilde{A}(x,y) \neq 0$). The equations (32) and (34) will be considered only within areas where $\lambda_2 y + Q(x,y) \neq 0$. Under these assumptions all those differential equations satisfy Lipschitz condition, since the right hand sides are differentiable.

From Hubbard and West [1991] we observe that trajectories of solutions to the system (29) follow the graphs of solutions to the first order differential equations (31) and (32) (with time going backwards or forward, depending on the sign of λ_1 and λ_2, cf. Hubbard and West [1991]).

Similarly trajectories of solutions to the system (30) follow the graph of solutions to the first order differential equations (33) and (34) as in the neighborhood of $(0,0)$, $\lambda_1 x$ has the dominating effect on $\widetilde{A}(x,y)$ and g is strictly increasing.

In the analysis of the behaviour of solutions of equations (33) and (34) we will need the following notions and theorems from [Hubbard and West, 1991, part II, pp 511–515] (in what follows, $\mathbf{I}$ denotes an open or closed interval, possibly unbounded from either side).

Definition 1 (Lower fence). *For the differential equation $x' = f(t,x)$, we call a continuous and continuously differentiable function $\alpha(x)$ a* lower fence *if $\alpha'(t) \leq f(t,\alpha(t))$ for all $t \in \mathbf{I}$.*

Definition 2 (Upper fence). *For the differential equation $x' = f(t,x)$, we call a continuous and continuously differentiable function $\beta(t)$ an upper fence if $f(t,\beta(t)) \leq \beta'(t)$ for all $t \in \mathbf{I}$.*

Definition 3 (Antifunnel). [12] *If, for the differential equation $x' = f(t,x)$, over some t-interval $\mathbf{I}$, $\alpha(t)$ is a* lower fence *and $\beta(t)$ in an* upper fence *and*

[12] There is an analogical notion to this defined for x "going backward", called *backward antifunnel.*

if $\alpha(t) > \beta(t)$, then the set of points (t,x) for $t \in \mathbf{I}$ with $\alpha(t) \geq x \geq \beta(t)$ is called an antifunnel.

Theorem 3 (Antifunnel Theorem; Existence)**.** [13] *Let $\alpha(t)$ and $\beta(t)$, $\alpha(t) \leq \beta(t)$, be two fences defined for $t \in [a,b)$, where b might be infinite, that bound an* antifunnel *for the differential equation $x' = f(t,x)$. Furthermore, let $f(t,x)$ satisfy Lipschitz condition in the antifunnel. Then there exists a solution $x = u(t)$ that remains in the antifunnel for all $t \in [a,b)$ where $u(t)$ is defined.*

Suppose that $\lambda_1 > 0$. We will consider the system of differential equations (33) for $x < 0$ and we will show that there exist at least one solution $u(x)$ of the system (33) that approaches point $(0,0)$ as x increases, and moreover, that it is tangent to the x-axis.

Consider the two areas defined as follows

$$\begin{aligned} L_1(\varepsilon,\gamma) &= \{(x,y) : -\varepsilon < x < 0 \text{ and } -\gamma x^2 < y < \gamma x^2\} \\ L_2(\varepsilon) &= \{(x,y) : -\varepsilon < x < 0 \text{ and } x < y < -x\}. \end{aligned}$$

We will show that there exist $\varepsilon > 0$ and $\gamma > 0$ such that $L_1(\varepsilon,\gamma)$ and $L_2(\varepsilon)$ are both backward antifunnels for the equation (33).

The equation $\lambda_1 x + P(x,y) = 0$ implicitly defines a curve tangent to the y-axis (which, in the neighborhood of $(0,0)$, is exactly the same as the curve defined by $\widetilde{A} = 0$). Thus there exist $\varepsilon > 0$ and $\gamma > 0$ such that $\widetilde{A}(x,y) \neq 0$ and $\lambda_1 x + P(x,y) \neq 0$ within areas $L_1(\varepsilon,\gamma)$ and $L_2(\varepsilon)$.

First we will show that $y(x) = \gamma x^2$ is a lower fence. Notice that

$$\begin{aligned} \varphi_1(x,y) &= \frac{\lambda_2 y + Q(x,y)}{\lambda_1 x + P(x,y)} = \frac{\lambda_2 y + Q(x,y)}{\lambda_1 x}\left(\frac{1}{1 - \frac{P(x,y)}{(-\lambda_1)x}}\right) \\ &= \left(\frac{\lambda_2 y}{\lambda_1 x} + \frac{Q(x,y)}{\lambda_1 x}\right)\left(1 + \frac{P(x,y)}{(-\lambda_1)x} + \left(\frac{P(x,y)}{(-\lambda_1)x}\right)^2 + \cdots\right). \end{aligned}$$

Take $y(x) = \gamma x^2$. If $Q(x,y)$ does not have a term in x^2, then for sufficiently small ε the first term in the first parentheses dominates the second. Otherwise both terms are linear in x. In that case, since the coefficient of x^2 in $Q(x,y)$ is independent of γ, we can take γ sufficiently large such that for ε sufficiently small the first term will dominate the second. Then the sign of $\varphi_1(x,\gamma x^2)$ depends on the sign of $\lambda_1/(\lambda_2 x)$, so $\varphi_1(x,\gamma x^2) > 0$ and thus $\widetilde{\varphi}_1(x,\gamma x^2) > 0$.

Since $(\gamma x^2)' = 2\gamma x < 0$ for $x < 0$, so for sufficiently small ε, $-\varepsilon < x < 0$, and for γ sufficiently large, γx^2 is a lower fence for $\varphi_1(x,y)$. Similarly it can be shown that for $-\varepsilon < x < 0$, ε sufficiently small, and for γ sufficiently

[13] An analogical theorem holds for backward antifunnels.

large $\widetilde{\varphi}_1(x, -\gamma x^2) < 0$, so, since $(-\gamma x^2)' = -2\gamma x > 0$ for $x < 0$, $-\gamma x^2$ is an upper fence for $\varphi_1(x, y)$. Thus $L_1(\varepsilon, \gamma)$ is an antifunnel for $\varphi_1(x, y)$.

The proof that there is an ε such that $L_2(\varepsilon)$ is an antifunnel goes along the same lines and is easier. This is because the first term in the first parentheses is a constant and the second term is at most linear in x, and so it is dominated by the first term.

Thus, by theorem 3, for $x < 0$ there is a solution to the linear differential equation (33) such that it remains in the antifunnels L_1 and L_2 and approaches $(0, 0)$. Moreover it does so tangentially to the x-axis, as it remains in the antifunnel L_1. Since $x(t)$ decreases as t increases (cf. (29), (30)) so this means that there are solutions to (29) leaving $(0, 0)$ tangentially to the direction of eigenvector of λ_1.

One can show similar result for $x > 0$ analogically, using backward antifunnels being analogues for L_1 and L_2. Moreover, using (34) one can show that there are trajectories approaching $(0, 0)$ tangentially to eigenvector of eigenvalue λ_2, defining antifunnels that are regions complementary to those for equation (33) (cf. Hubbard and West [1991]).

If $\lambda_1 < 0$, one can show that there are trajectories leaving $(0, 0)$ tangentially to eigenvector of eigenvalue $\lambda_2 < 0$, and its proof is analogical to that above, again relying on the fact that $\widetilde{A}(x, y)$ has, in the neighborhood of $(0, 0)$, exactly the same sign as $\lambda_1 x + P(x, y)$. Thus we have shown that $\mathbf{x_u}$ is an unstable rest point. This ends the proof of proposition 1. □ □

Fact 1 of Proposition 2. To show that $\psi^L_\mu(a, \mu) < 0$, consider the equation

$$a\mu^2(\delta + \sigma) - \mu\delta + \varrho = 0,$$

with (a_s, μ_s) being its solution. Solving it with respect to μ we get that

$$\mu_s < \frac{\delta}{2a_s(\delta + \sigma)},$$

(notice that here we assume that there are two solutions to this equality). Thus in the neighborhood of (a_s, μ_s) $\psi^L_\mu(a, \mu) < 0$.

The rest of the claims in parts 1 and 2 follow directly from the expressions for partial derivatives (cf. (27)).

For part 3 we first transform the inequality to the following form

$$\psi^s_a M^s_\mu - \psi^s_\mu M^s_a > 0,$$

and further (for $\mu_s > 0$) to

$$-a_s\mu_s(\tau + 1)(2d - 1) - \mu_s(d + \tau) + (2d - 1) > 0,$$

(where d and τ are as in (16). Since a_s and μ_s satisfy (18), so this inequality is equivalent to

$$-\frac{(\mu_s - \upsilon)(2d - 1)}{\mu_s} - \mu_s(d + \tau) + (2d - 1) > 0,$$

which reduces to

$$\frac{C}{\mu_s} - \mu_s A > 0,$$

(where A and C are as in (21). Using the expression for (23) and facts that $\mu_s > 0$ and $A > 0$, the inequality can be transformed into

$$4AC - (A - (B - C) - \sqrt{(A - (B - C))^2 - 4AC})^2 > 0.$$

Since the left side of the inequality is equal to $2(A - (B - C))\sqrt{((A - (B - C))^2 - 4AC)}$, which is positive, as $\mu_s > 0$, so the inequality is satisfied. This completes the proof. $\square$ $\square$

Fact 2 of Proposition 2. Since in the neighborhood of (a_s, μ_s) we have $\psi^L_a(a, \mu) > 0$ and $\psi^L_\mu(a, \mu) < 0$ and g is strictly increasing for positive arguments, if ε is near 0 then we have

$$\max_{(a,\mu) \in C(K\varepsilon)} g(|\psi(a, \mu)|) = g(\max\{-\psi(a_s - K\varepsilon, \mu_s + K\varepsilon), \psi(a_s - K\varepsilon, \mu_s - K\varepsilon)\}),$$

One can easily check that

$$\max\{-\psi(a_s - K\varepsilon, \mu_s + K\varepsilon), \psi(a_s - K\varepsilon, \mu_s\mu_s - K\varepsilon)\} = L\varepsilon + o(\varepsilon),$$

where $L > 0$ is some constant. On the other hand expanding g in Taylor polynomials around 0 we get

$$g(x) = g(0) + g'(0)x + o(x) = o(x).$$

Thus if ε is close to 0,

$$g(\max\{-\psi(0, \mu_s + \varepsilon), \psi(1, \mu_s - \varepsilon)\}) = o(\varepsilon) > 0$$

since $g(x) > 0$ for $x > 0$ and so $\eta(\varepsilon) = o(\varepsilon) > 0$. $\square$ $\square$

Fact 3 of Proposition 2. From $M^L(a, \mu) = 0$ and $\varphi^1_\varepsilon(a, \mu) = 0$ we have that $a^1_M(\varepsilon) - a_s + \mu^1_M(\varepsilon) - \mu_s = -\varepsilon$. Expanding $M^L(a, \mu)$ in a Taylor polynomial around (a_s, μ_s) we get

$$M^L(a, \mu) = M^s_a(a - a_s) + M^s_\mu(\mu - \mu_s) + o(a - a_s) + o(\mu - \mu_s).$$

Since $a^1_M(\varepsilon) - a_s, \mu^1_M(\varepsilon) - \mu_s \to 0$ as $\varepsilon \to 0$, so using an expansion of M^L in a Taylor polynomial around (a_s, μ_s), for ε close to 0 we get

$$\begin{aligned}
&M^s_a(a^1_M(\varepsilon) - a_s) + M^s_\mu(\mu^1_M(\varepsilon) - \mu_s) + o(a^1_M(\varepsilon) - a_s) + o(\mu^1_M(\varepsilon) - \mu_s) = \\
&(M^s_a - M^s_\mu)(a^1_M(\varepsilon) - a_s) + M^s_\mu(a^1_M(\varepsilon) - a_s + \mu^1_M(\varepsilon) - \mu_s) \\
&+ o(a^1_M(\varepsilon) - a_s + \mu^1_M(\varepsilon) - \mu_s)) = \\
&(M^s_a - M^s_\mu)(a^1_M(\varepsilon) - a_s) - M^s_\mu\varepsilon + o(\varepsilon) = 0.
\end{aligned}$$

From this (and by a similar method for $\mu^1_M(\varepsilon) - \mu_s$) we get

$$\begin{aligned} a^1_M(\varepsilon) - a_s &= \frac{M^s_\mu}{M^s_a - M^s_\mu}\varepsilon + o(\varepsilon) > -K\varepsilon, \\ \mu^1_M(\varepsilon) - \mu_s &= -\frac{M^a_\mu}{M^s_a - M^s_\mu}\varepsilon + o(\varepsilon) > -K\varepsilon, \end{aligned}$$

for ε small enough (cf. (14)).

Similarly for $\varphi^2_\varepsilon(a, \mu) = 0$ we obtain

$$\begin{aligned} a^2_M(\varepsilon) - a_s &= -\frac{M^s_\mu}{M^s_a - M^s_\mu}\varepsilon + o(\varepsilon) < K\varepsilon, \\ \mu^2_M(\varepsilon) - \mu_s &= \frac{M^a_\mu}{M^s_a - M^s_\mu}\varepsilon + o(\varepsilon) < K\varepsilon. \end{aligned}$$

for ε small enough.

By observation 1 we have $\psi^L(a^1_M(\varepsilon), \mu^1_M(\varepsilon)) > 0$ and $\psi^L(a^2_M(\varepsilon), \mu^2_M(\varepsilon)) < 0$ (see also figure 5). □ □

Fact 4 of Proposition 2. Throughout the entire proof we will assume that any considered ε is small enough to satisfy fact 3. Let $M > 0$ and $a < a_s + K\varepsilon$, $\mu < \mu_s + K\varepsilon$. Since $\varphi^1_\varepsilon(a, \mu) \leq 0$, we have that

$$M^L(a, \mu) < \sqrt{2\kappa_1\kappa_2 f(\alpha)(2K+1)\eta(\varepsilon)\varepsilon}. \tag{35}$$

Let $d(\varepsilon) = \sqrt{(2\kappa_1\kappa_2 f(\alpha)(2K+1)\eta(\varepsilon)\varepsilon)}$. Since $\eta(\varepsilon) = o(\varepsilon)$, we have $d(\varepsilon) = o(\varepsilon)$. Therefore,

$$\lim_{\varepsilon\to 0} \frac{\sqrt{2\kappa_1\kappa_2 f(\alpha)(2K+1)\eta(\varepsilon)\varepsilon}}{\varepsilon} = \lim_{\varepsilon\to 0} \sqrt{\frac{2\kappa_1\kappa_2 f(\alpha)(2K+1)\eta(\varepsilon)}{\varepsilon}} = 0.$$

Consider the curve $M^L(a, \mu) = d(\varepsilon)$. Notice that for ε close to 0 there is an intersection point $(a_0(\varepsilon), \mu_0(\varepsilon))$ of that curve and the curve $\psi^L(a, \mu) = 0$ such that $(a_0(\varepsilon), \mu_0(\varepsilon)) \to (a_s, \mu_s)$ when $\varepsilon \to 0$ (this follows from observation 1). Using an expansion of M^L in a Taylor polynomial around (a_s, μ_s), for ε close to 0 and from the fact that

$$M^s_a(a_0(\varepsilon) - a_s) + M^s_\mu(\mu_0(\varepsilon) - \mu_s) + o(a_0(\varepsilon) - a_s) + o(\mu_0(\varepsilon) - \mu_s) = d(\varepsilon),$$

we get

$$\frac{M^s_a}{M^s_\mu}(a_0(\varepsilon) - a_s) + \mu_0(\varepsilon) - \mu_s = \frac{d(\varepsilon)}{M^s_\mu} + o(d(\varepsilon)). \tag{36}$$

Moreover, using the expansion of $\psi^L(a,\mu)$ in a Taylor polynomial around (a_s,μ_s) we get

$$
\begin{aligned}
&\psi_a^s(a_0(\varepsilon)-a_s)+\psi_\mu^s(\mu_0(\varepsilon)-\mu_s)+o(a_0(\varepsilon)-a_s)+o(\mu_0(\varepsilon)-\mu_s)=\\
&\left(\psi_a^s-\frac{\psi_\mu^s M_a^s}{M_\mu^s}\right)(a_0(\varepsilon)-a_s)+\psi_\mu^s\left(\frac{M_a^s}{M_\mu^s}(a_0(\varepsilon)-a_s)+\mu_0(\varepsilon)-\mu_s\right)\\
&+o\left(\frac{M_a^s}{M_\mu^s}(a_0(\varepsilon)-a_s)+\mu_0(\varepsilon)-\mu_s\right)=\\
&\left(\psi_a^s-\frac{\psi_\mu^s M_a^s}{M_\mu^s}\right)(a_0(\varepsilon)-a_s)+\frac{\psi_\mu^s}{M_\mu^s}d(\varepsilon)+o(d(\varepsilon))=0.
\end{aligned}
$$

Thus (and by similar method for $\mu_0(\varepsilon)$) we have

$$
\begin{aligned}
a_0(\varepsilon)-a_s &= -\frac{\psi_\mu^s}{\psi_a^s M_\mu^s-\psi_\mu^s M_a^s}d(\varepsilon)+o(d(\varepsilon))<K\varepsilon,\\
\mu_0(\varepsilon)-\mu_s &= \frac{\psi_a^s}{\psi_a^s M_\mu^s-\psi_\mu^s M_a^s}d(\varepsilon)+o(d(\varepsilon))<K\varepsilon,
\end{aligned}
$$

for ε close to 0, as $d(\varepsilon)=o(\varepsilon)$ (notice that by fact 1 $\psi_a^s M_\mu^s-\psi_\mu^s M_a^s>0$).

Suppose that ε is such that $a_0(\varepsilon)-a_s<K\varepsilon$ and $\mu_0-a_s(\varepsilon)<K\varepsilon$. Consider the function $c(a)=\varphi_\varepsilon^1(a,h_\psi(a))$. Since $c(a_s)<0$, $c(a_0(\varepsilon))>0$ and c is continous, there must be $a_\psi^1(\varepsilon)<a_0(\varepsilon)$ such that $c(a_\psi^1(\varepsilon))=0$. From this it follows that $\mu_\psi^1(\varepsilon)=h_\psi(a_\psi^1(\varepsilon))<\mu_0(\varepsilon)$. Thus for ε sufficiently small the intersection point $(a_\psi^1(\varepsilon),\mu_\psi^1(\varepsilon))$ exists and $a_\psi^1(\varepsilon)-a_s<K\varepsilon$ and $\mu_\psi^1(\varepsilon)-\mu_s<K\varepsilon$.

For point 2, suppose that $M(a,\mu)>0$ and $\psi^L(a,\mu)>0$. From the analysis above we know that, for $(a,\mu)\in C(K\varepsilon)$, $\varphi_\varepsilon^1(a,\mu)=0$ implies $a<a_0(\varepsilon)$ (notice that equation $M(a,\mu)=d(\varepsilon)$ implicitly defines the function $\mu=h_{M2}(a)$, which is strictly increasing). Moreover, from $\varphi_\varepsilon^1=0$ we get $\mu-\mu_0\geq-(a_0(\varepsilon)-a_s)-\varepsilon\geq-K\varepsilon$.

Points 3 and 4 can be shown similarly. □ □

Lemma 3 of Proposition 2. To check that the vector field $\mathbf{v}$ points inwards we will compare the direction of the vector field and the slope of the tangent to the boundary along all four parts of the boundary. That is we have to check whther

$$F_a(a,\mu)A^0(a,\mu)+F_\mu(a,\mu)M^L(a,\mu)<0 \tag{37}$$

is true along the entire boundary $F(a,\mu)=0$.

If $M^L(a,\mu)\geq 0$ and $A^0(a,\mu)\geq 0$, the corresponding part of the boundary is the curve $\varphi_\varepsilon^1(a,\mu)=0$ and we have

$$
\begin{aligned}
2A^0M^LM_a^L-\varepsilon^2A+2[M^L]^2M_\mu^L-2M^L\kappa_1\kappa_2f(\alpha)\eta(\overline{K},\varepsilon) &= \\
2M^L(A^0M_a^L-\kappa_1\kappa_2f(\alpha)\eta(\varepsilon))-\varepsilon^2A+2[M^L]^2M_\mu^L &< 0,
\end{aligned}
$$

since

$$
\begin{aligned}
M^L_a(a,\mu) &= \kappa_2\mu(f(\alpha) - f(\alpha-\delta)(1-\mu)) < \kappa_2 f(\alpha),\\
M^L_\mu(a,\mu) &= -\kappa_2(f(\alpha-\delta)(a(1-\mu)+1-a\mu) + f(\alpha)(1-a)) < 0,\\
A(a,\mu) &= \kappa_1(1-a)g(\psi^L(a,\mu)) < \eta(\varepsilon),
\end{aligned}
$$

for all $\varepsilon \in (0,\overline{\varepsilon})$.

Similarly it can be shown that condition (37) holds for $M^L(a,\mu) \leq 0$ and $A^0(a,\mu) \leq 0$, when the corresponding part of the boundary is the curve $\varphi^2_\varepsilon(a,\mu) = 0$.

If $M^L(a,\mu) \geq 0$ and $A^0(a,\mu) \leq 0$ the correspoding part of the boundary is the line $\ell^1_\varepsilon(a,\mu) = 0$ and we have

$$
(\mu^2_M(\varepsilon) - \mu^1_\psi(\varepsilon))A - (a^2_M(\varepsilon) - a^1_\psi(\varepsilon))M < 0,
$$

since, by fact 4, $\mu^2_M(\varepsilon) > \mu^1_\psi(\varepsilon)$ and $a^2_M(\varepsilon) > a^1_\psi(\varepsilon)$.

Similarly it can be shown that condition (37) holds for $M^L(a,\mu) \leq 0$ and $A(a,\mu) \geq 0$, when the corresponding part of the boundary is the curve $\ell^2_\varepsilon(a,\mu) = 0$. □ □

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