Dynamic Production Theory under No-Arbitrage Constraints

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I propose a dynamic production model under the joint constraints of technology, budget and no arbitrage. Comparative static and dynamic analysis indicates that this model is consistent with the behavior of firms in reality, and can explain a wide range of economic phenomena. Compared with classical production theory, this model confers some methodological advantages: (i) it turns out to be a natural generalization of classical production theory; (ii) it constitutes a marriage of production theory and finance; (iii) it constructs a bridge between microeconomics and macroeconomics; and (iv) it successfully reconciles some long-standing contradictions arising from classical theory. *JEL* Codes: D24, E23.

I. INTRODUCTION

The goal of classical production theory is to study the behavior of the profit-maximizing firms. It is shown that in the long run competition will drive the economic system to the equilibrium state in which the profit-maximizing firms are earning a *zero economic profit,* with no incentive to either enter or leave the industry.

However, the limitations of classical production theory cannot be neglected:

1. It lacks of empirical meaning in practice. To precisely calculate economic profit on an investment we have to know the corresponding opportunity cost, including labor and capital. But, to know the true opportunity cost necessitates precisely measuring capital.

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This theoretic difficulty will inevitably lead to the longstanding Capital Controversy: in what units capital is measured? (see Robinson 1971)

2. It is fundamentally microeconomic rather than macroeconomic in character. Money plays no significant role in classical production theory. This contradicts the fact that monetary policy has a profound impact on production decision in practice. In fact, the dichotomy of real and monetary economics has been extensively debated and criticized (see Modigliani 1963).

3. It is based on partial equilibrium rather than general equilibrium in theory. The classical production theory only analyzes a single industry, taken prices and wages as given. This will unconditionally lead to zero-profit equilibrium in the industry alone, regardless of the existence of risk-free assets in a whole economy with multiple interacting markets. In fact, the existence of risk-free assets may help to set a lower bound to the rate of return on investment and hence prevent the return on investment from being driven to zero. [1]

4. It is essentially static rather than dynamic in nature. [2] It tries to dodge the difficult problem of specifying the timing of inputs and related outputs by assuming stationary conditions. But we have no right to assume that there is no lag between expenditure and revenue. Further, risk may arise in the gap between investing money and receiving profits because unexpected events may occur which may alter the value of profit. To abstract from uncertainty means to postulate that no such events occur, so that the expected returns on investment never differs from the actual returns. The absence of risk and uncertainty shows itself particularly in the absence of asset preference (see Tobin 1958).

Indeed, the theory of optimal allocation of resources under uncertainty has had much less systematic attention (see Lucas and Prescott 1971). Under uncertainty, rational firms will hold portfolio which makes their wealth grow at the fast rate of *expected* return.[3] So,

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[1] It has been shown that the concept of a minimal rate of return on capital (a required rate of profit) plays a key role in the theory of growth. For details, see Tobin (1965).

[2] A close examination of the classical production theory will reveal that dynamic element have appeared, thanks to the device of " short- and long-run equilibrium", the oldest device of developing a dynamical theory with a static apparatus.

[3] The validity of this statement depends on the assumption that rational firms are risk neutral, so that the degree of uncertainty (measured by Variance) will not affect investment decisions. The behavior of competitive firm under price uncertainty and risk aversion has been studied by Sandmo (1971).

given any investment opportunity in certain industry, rational firms compare its expected rate of return with the risk-free interest rate and will choose to put their wealth in the asset with the higher yield. If the expected rate of return on investment exceeds the risk-free interest rate, then rational firms will enter the industry. Otherwise, if the expected rate of return on investment is lower than the risk-free interest rate, then rational firms will leave the industry to guarantee risk-free returns instead. In short, rational firm adjusts his investment budget so that its marginal rate of return is equal to the risk-free interest rate (see Tobin 1961). As a result, in the long run the economy will tend toward arbitrage equilibrium, rather than zero economic profit equilibrium.[4]

On the other hand, the development of finance has shown that no arbitrage is more primitive than competitive equilibrium (see Dybvig and Ross 2008). First, the absence of arbitrage does not require the economy to be in stable equilibrium, though a competitive equilibrium is invariably arbitrage-free. Second, the absence of arbitrage does not require all agents to be rational. Now that the absence of arbitrage turns out to be just a necessary condition for a competitive equilibrium (see Ang, Dong and Piazzesi 2007), there is a gap between the classical production theory and the production practice.

To fulfill this gap, we can go one step back. Since economic profit cannot be measured directly (in the sense of Capital Controversy), we can approximate it indirectly by consider the accounting profit instead. So, we can relax the equilibrium condition of zero economic profit to the more general one of risk-free rate of return on investment, or risk-free accounting profit for short. In other words, we can go back one further step to generalize the assumption of profit maximization to that of no arbitrage. This no-arbitrage approach means to develop a *general equilibrium* in multiple interacting markets. As a result, we will get arbitrage equilibrium instead of competitive equilibrium. This technical route can be shown in the following scheme in which the horizontal arrows ($\Rightarrow$) represent implication and the downward arrows ($\downarrow$) represent generalization.

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[4] The similarity between the absence of arbitrage and the zero economic profit condition for a firm has been noted by Dybvig and Ross (2008). The theoretical distinction between a zero profit condition and the absence of arbitrage is the distinction between commerce and simply trading under the price system, namely that commerce requires production. In practice, the distinction blurs.

$$\begin{array}{ccccc} \textit{profit Maximization} & \Rightarrow & \textit{Competitive Equilibrium} & \Rightarrow & \textit{Zero Economic profit} \\ \downarrow & & \downarrow & & \downarrow \\ \textit{No Arbitrage} & \Rightarrow & \textit{Arbitrage Equilibrium} & \Rightarrow & \textit{Risk - free Accounting profit} \end{array}.$$

Following this technical route a dynamic production model is built under joint constraints of technology, budget [5] and no arbitrage. In *ex ante* analysis, all of the three constraints are equilibrium conditions, but in *ex post* analysis, they turn out to be *accounting identities*. This is done essentially by assuming the existence of a fundamental time lag: the lag between the time when investment is taking place and the time when the resulting revenue is available. Dynamically, rational firm invests its total budget at the *beginning* of each period and gains a risk-free rate of return on investment at the *end* of each period, or equivalently, at the beginning of the succeeding period. The very bridge that links this time lag is the risk-free interest rate, which is the bridge between present and future (see Fisher 1930, Chapter 1).[6]

Comparative static analysis and dynamic analysis indicate that this model is consistent with the behavior of firms in reality, and can explain a wide range of economic phenomena. It will be seen that this no-arbitrage based production theory is more fundamental and is logically prior to the profit-maximizing production theory.

The rest of the paper proceeds as follows. In section **II**, the basic model of a closed economy under stationary state is constructed based on the joint constraints of technology, budget and no arbitrage. Section **III** focuses on the comparative static analysis of solutions for the model, with emphasis on micro-foundations for some empirical laws of macroeconomics. In section **IV** we extend the basic model to the case of stationary open

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[5] The traditional analytical distinction between firms and households is that firms are not supposed to be subject to budget constraints. But in practice, the existence and importance of a budget constraint becomes patently clear, and the traditional distinction is blurred and perhaps vanished (see Becker 1962). In reality, firms like consumers are subject to budget constraints, to which the Arrow-Debreu framework for general equilibrium theory (Arrow and Debreu 1954) has paid little attention. Much work in corporate finance has been devoted to the study of the firm's budget constraint (see Kornai 1979; Bolton and Dewatripont 1995; Kuga 1996).

[6] In Keynes' words this means that "*the importance of money essentially flows from its being a link between the present and the future*…. Money in its significant attributes is, above all, a subtle device for linking the present to the future; and we cannot even begin to discuss the effect of changing expectations on current activities except in monetary terms. We cannot get rid of money even by abolishing gold and silver and legal tender instruments. So long as there exists any durable asset, it is capable of possessing monetary attributes and, therefore, of giving rise to the characteristic problems of a monetary economy." (Keynes 1936, Chapter 21)

economies. In section **V** we extend the basic model to the general case in order to study dynamic economies. Section **VI** concludes this paper with some methodology remarks.

II. STATIONARY CLOSED ECONOMIES

To keep things as simple as possible, in this section we assume a stationary closed economy.

II.A. *The Basic Model*

The building blocks from which the no-arbitrage production model is constructed are three in number: 1. technology Constraints; 2. budget constraints; and 3. no-arbitrage constraints.

1. Technology Constraints.—The production function represents the possibilities afforded by an exogenous technology. A production function relates physical inputs to physical outputs, not involving prices. If Q represents physical output, and K and L represent capital and labor in physical units, then the production function is a function of two variables

(1) $$Q = AF(K, L),$$

where A stand for productivity. The production function fitted to empirical data actually reflects the accounting identity between values of inputs and outputs (see Simon 1979).

In principle, production function may be distinguished for all sorts of commodities produced and for all sorts of production processes. In practice, production function can be estimated either for a single firm, or for an entire industry, or even for a nation as a whole.

2. Budget Constraints.—Since resources are scarce, each firm is, at any period of time, constrained by its total wealth. Formally, assume that the total budget in terms of money is M at the *beginning* of each period, or equivalently, at the end of the previous period.

Then the efficient allocation of labor L and capital K at the beginning of each period must satisfy the budget constraint imposed by total wealth

$$iK + WL = M \quad (2)$$

Here, W is the wage of labor, and i is the rental price of capital. Note that in competitive economy the equilibrium value of the rental price of capital (measured in terms of money) will equal the nominal interest rate.

The budget constraint is an *ex ante* behavioral regularity, which exerts an influence on the firm's decision. Given the total budget M , the budget identity amounts to the *budget line* of the firm.

3. No-Arbitrage Constraints.—In simple terms, an arbitrage opportunity is a money pump. The Fundamental Theorem of Finance derives the implications of the absence of such arbitrage opportunities (see Ross 2004). According to the Efficient-Market Hypothesis, real economy is arbitrage-free, given the information available at the time the investment is made (see Fama 1970).

The no-arbitrage constraint means there is no such things as free launch. Thus, in equilibrium the rate of return is necessarily equal to the risk-free interest rate, and is the same no matter in terms of what it is measured. To be precise, let the risk-free interest rate to be r, then the total wealth at the *end* of each period always equals $M(1+r)$ in terms of money. On the other hand, at the end of each period the firm's total wealth is divided into two parts: the physical output (Q) and the capital stock (K). In equilibrium the market value of the physical output and the capital stock must add up to the total wealth at the end of each period. Were this not so an arbitrage process would be set in motion.

But, to establish no-arbitrage constraint capital depreciation must be considered. Formally, let the depreciation rate of capital be δ , which is a physic attribution of the capital and is less than unity. Thus, at the end of each period, the capital stock equals $K(1-\delta)$.

Assuming that output is sold at the end of each period,[7] then the no-arbitrage constraint gives the following accounting identity

$$PQ + iK(1-\delta) = M(1+r)\,, \tag{3}$$

where P stands for the price level.

So far, we already have identified the three components of a complete economic model. In *ex ante* analysis, all of the three equations are equilibrium condition, but in *ex post* analysis, they turn out to be accounting identities. Since accounting identities must hold for any values of the variables we can interpret r as *actual* or *expected* rate of return on investment whenever necessary.

Now we put these three components together into a single framework that allows us to analyze them simultaneously. This means to develop a *general equilibrium* in all four markets: labor, capital, goods, and asset (include money market). Since there have three fundamental equations in general, to close the system three variables must be endogenously determined. For convenience, the three endogenously determined variables will be called *decision variables*. Other exogenously determined variables will be called *state variables* and taken as given.

II.B. *Analytic Solutions*

In this subsection we study the existence and calculation of analytical solution of the system of equation (1)-(3). In general, the solution of the system gives each of the three physical variables (K, L, Q) as *multivariable functions* of state variables. No maximum problem need be studied, and no derivatives need be taken.

To see the existence of solutions, just note that both the budget constraint and no-arbitrage constraint are given by linear equations which jointly determine a straight line. The intersections of this straight line with the surface described by the production function determine the solution of the input and output. Specially, if the production

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[7] We shall assume that in stationary economy the production period coincides with the maturity of risk-free interest rate. Time-to-build technology (see Kydland and Prescott 1982) will not be considered in this paper.

function is linear then the corresponding surface degenerated into a plane, and hence analytic solution can be found explicitly.

Now if the production function happens to be, or can be approximated by, a linear function

$$(4) \qquad Q = F(L,K) = aK + bL\,.$$

then analytic solution can be found by solving the following system of linear equations in three unknowns K, L, Q

$$(5) \qquad \begin{cases} aK & + & bL & -Q & = & 0 \\ iK & + & WL & & = & M \\ i(1-\delta)K & + & & PQ & = & M(1+r) \end{cases}\,.$$

The coefficient matrix of this system of linear equations is a square matrix

$$(6) \qquad \Lambda = \begin{pmatrix} \mathrm{a} & b & -1 \\ i & W & 0 \\ i(1-\delta) & 0 & P \end{pmatrix}.$$

The determinant of this coefficient matrix is $|\Lambda| = P(aW - bi) + Wi(1-\delta)$. According to Cramer's rule, if the coefficient determinant satisfies $|\Lambda| \neq 0$, then this system of linear equations has a *unique* solution[8] given by

$$(7) \qquad \begin{pmatrix} K \\ L \\ Q \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ M \\ M(1+r) \end{pmatrix} = M \cdot \Lambda^{-1} \begin{pmatrix} 0 \\ 1 \\ 1+r \end{pmatrix}.$$

where Λ^{-1} is the inverse of the coefficient matrix. Note that, under linear production function, if we take the technique parameters (a, b) and price variables (i, W, P) as fixed,

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[8] It is routine to check that when the coefficient determinant equals zero ($|\Lambda| = 0$) the system of linear equations has no solution since the rank of coefficient matrix is not equivalent to the rank of augmented matrix.

then the equilibrium values of real variables (K, L, Q) turn out to be determined essentially by monetary condition rather than by real factors. *Other things being the same*, a rise in total budget will cause real variables to increase.

In general, if the production function is non-linear, then the system can not be explicitly solved for decision variables, i.e., do not have analytic solutions. Further, numerical analysis has shown that there may have two positive solutions in the case of the Cobb–Douglas production function. These two solutions have different capital-labor ratio and different level of output, and hence stand for different type of firms. [9] As a result, heterogeneous firms can coexist in arbitrage equilibrium. So, in general an economic system at a particular macroscopic state may occupy a number of microscopic states. This result differs from the classical production theory, but agrees with the *thermodynamic equilibrium* of physical system. Indeed, according to Boltzmann's definition, the logarithm of the number of possible microstates of a system in thermodynamic equilibrium is proportional to its thermodynamic Entropy, which is a function of state, independent of the microscopic details of the system (see Feynman *et al*. 2013).

II.C. *Zero-Profit Equilibrium*

In this subsection we show that the zero-profit equilibrium turns out to be a special case of the arbitrage equilibrium.

Recall that in classical production theory the profit-maximization problem can break into two steps: First, find the minimum costs of producing any given level of output, and then choose the most profitable level of output. When a particular production is specified, solution of the profit-maximization problem yields the optimal decisions concerning the supply of output and the demand for labor and capital. However, this indirect approach cannot apply within the no-arbitrage framework since decision variables are simultaneously determined by the three constraints.

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[9] Further details are available upon request.

Firstly, let us consider the technology constraint, from which the *cost function* can be derived by solving the cost minimization problem

(8) $$C(Q) = \min_{AF(K,L)=Q} iK + WL .$$

Note that for any particular production function there is a particular cost function.

Secondly, the budget constraint forces the firm to make rational choices, behaving in the same way like a consumer. In equilibrium, the total cost must equal the total budget, or mathematically

(9) $$C(Q) = iK + WL = M .$$

Thus the budget constraint can also be regarded as *cost constraint*.

Finally, we consider the no-arbitrage constraint. By definition, the net profit equals the difference in total revenue and total cost, that is,

(10) $$\pi(Q) = PQ - C(Q) = PQ - M = Mr - iK(1-\delta) .$$

So, under the no-arbitrage constraint we can get the condition for zero-profit equilibrium

(11) $$\pi(Q) = 0 \Rightarrow Mr = iK(1-\delta) .$$

This condition for zero-profit equilibrium can be interpreted as follows: in zero-profit equilibrium, the total risk-free interests are equal to the depreciated value of the capital stock. Thus there is *no* positive cash flow in zero-profit equilibrium.

Solving for K and then for L we get

(12) $$K = \frac{Mr}{i(1-\delta)} , L = \frac{M(1-\delta - r)}{W(1-\delta)} .$$

From this it follows that, in zero-profit equilibrium, the choice of technique is independent of final demand (Q).[10] Therefore, in zero-profit equilibrium the economy will not substitute inputs (capital and labor) when final demand changes, since all

[10] The existence of effective arbitrage roughly equated the supply and demand. Thus, in no-arbitrage equilibrium the supply roughly equals final demand, and vice versa.

desirable substitutions have been made by the competitive market (see Samuelson 1951; Koopmans 1951; Arrow 1951; Mirrlees 1969; Stiglitz 1970).

Rate of Profit. [11]—The zero-profit equilibrium is just an idea state, corresponding to the "frictionless world" in Physics. In general, the net profit in arbitrage equilibrium does not tend to coincide with that in zero-profit equilibrium except by chance, since there is no mechanism that insures this coincidence. So it is natural to define the rate of profit by

$$\frac{\pi(Q)}{C(Q)} = \frac{PQ - C(Q)}{C(Q)} = \frac{P}{C(Q)/Q} - 1 . \tag{13}$$

Note that $C(Q)/Q$ is precisely the *average cost* in arbitrage equilibrium.

In a sense this is a natural definition, but there exists a difficult problem: how to determine the output levels Q in this definition? Obviously, this output level cannot be derived from profit maximization. The reason is that profit maximization will result in a state of zero-profit output, regardless of the constraints of technology, budget and no arbitrage.

To avoid this deep-seated difficulty, we define the rate of profit on the basis of arbitrage equilibrium, rather than competition equilibrium. To be precise, denote the initial budget by M , then the equilibrium quantities Q is solved from the budget constraint, or equivalently, the cost constraint

$$C(Q) = M . \tag{14}$$

As an illustration, consider the Cobb–Douglas production function $Q = AK^{\alpha}L^{\beta}$. Then the cost function will be

$$C(Q) = [(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}}] i^{\frac{\alpha}{\alpha+\beta}} W^{\frac{\beta}{\alpha+\beta}} (\frac{Q}{A})^{\frac{1}{\alpha+\beta}} \equiv D(\frac{Q}{A})^{\frac{1}{\alpha+\beta}} . \tag{15}$$

[11] Historically, the term rate of profit was introduced by Marx (1894) in Volume III of *Capital* for the ratio of profit to total capital invested in a given cycle of reproduction. But here we adopt the conventional uses of the term "rate of profit", which is similar to the concept of the rate of return on investment.

Remember that D is a function of factor price and does not depend on the output. Solve the budget constraint equation $C(Q)=M$ to get $Q=A(\frac{M}{D})^{\alpha+\beta}$, and then we obtain the average cost

(16) $$\frac{C(Q)}{Q}=\frac{M}{A(\frac{M}{D})^{\alpha+\beta}}=\frac{D^{\alpha+\beta}}{A}M^{1-\alpha-\beta}.$$

Substituting into the formation of the rate of profit to obtain

(17) $$\frac{\pi(Q)}{C(Q)}=\frac{P}{C(Q)/Q}-1=\frac{AP}{D^{\alpha+\beta}}M^{\alpha+\beta-1}-1.$$

If $\alpha+\beta=1$, then we have $\pi(Q)/C(Q)=AP/D-1$, which is independent of M. This is a remarkable phenomenon: if the production function exhibits constant returns to scale, then the rate of profit is independent of the initial budget.[12]

In general, the higher the rate of profit in equilibrium, the more efficient is the economy. So the rate of profit can provide an actual measurement of the production efficiency. In open economies, those countries with higher rate of profit on identical goods will have Competitive Advantage.[13] For more details please skip directly to section **IV**.

II.D. *Modigliani–Miller Theorem*

In this subsection we show that the Modigliani-Miller Theorem can be viewed as a consequence of arbitrage equilibrium. Now that the Modigliani-Miller Theorem

[12] The empirical evidence approximately supports constant return to scale. In a majority of cases, the sum of the exponents of the labor and capital factors of the fitted Cobb–Douglas function is close to unity, and hence fitted Cobb–Douglas functions are very nearly homogeneous of the first degree. For details, see Simon (1979).

[13] This term is borrowed from Michael E. Porter (1985). The term competitive advantage seeks to address some of the criticisms of comparative advantage. It has been criticized that comparative advantage may lead countries to specialize in exporting primary goods and raw materials that trap countries in low-income economies due to terms of trade. The principle of competitive advantage attempts to correct for this issue.

represents one of the first formal uses of a *no arbitrage proof* in the context of the modern theory of finance (see Miller 1988), maybe this is not coincidental.[14]

The Modigliani-Miller Theorem is a cornerstone of modern corporate finance. At its heart, the theorem is an irrelevance proposition: a firm's financial decisions do not affect its market value. The assumptions of Modigliani-Miller theorem deal with various types of capital market frictions that are at the heart of effective arbitrage (Modigliani and Miller 1958).

As is well known, the market value of the firm is determined by its cash flows (with profits as major component). Under the condition of arbitrage equilibrium, the net profits satisfy

$$\pi(Q) = PQ - C(Q) = PQ - M = Mr - iK(1-\delta)\,. \tag{18}$$

Now consider the capital structure of the firm, i.e., the proportion of debt and equity used to finance the firm's operations. In general, the total budget at the beginning of each period may either be *accumulated* or *financed* during the past periods. In principle, the accumulated part of total budget can be taken as equity (E), and the financed part can be viewed as debt (D). When interpreting in this way, the total budget can be divided into two parts $M = E + D$. Substitute into net profit we get

$$\pi(Q) = Mr - iK(1-\delta) = (E+D)r - iK(1-\delta)\,. \tag{19}$$

From this it follows that a firm's capital structure does not affect its net profit if it can not affect the capital stock in equilibrium. In other words, firms with the same inputs and outputs will have the same net profit. This is essentially the Modigliani-Miller Proposition.

More fundamentally, if, under certain conditions, the financial policy indeed affects the net profits in equilibrium, then the Modigliani-Miller irrelevance proposition fails to hold in general. This is what Miller (1988) emphasizes when he says that "showing what *doesn't* matter can also show, by implication, what *does*." The systematic analysis of such

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14 Diamond (1967) has pointed out that the result of Modigliani-Miller theorem is a consequence of competitive equilibrium with price takers facing the same prices.

assumptions led to an expansion of the frontiers of economics and finance (see Modigliani 1988).

II.E. *The Equation of Exchange*

As pointed out by Fisher (1911), the equation of exchange merely expresses in form convenient for analysis the fact that the currency paid for goods is the equivalent of the value of the goods bought. In view of this, and in view of the fact that all of the three fundamental equations in our model are just accounting identities, our model can be a viewed as a natural generalization of the equation of exchange.

But, to enable our model to work at the macro level, all the variables in our model must be interpreted as the corresponding *aggregate* variables in macroeconomics. For example, the technology constraint must be reinterpreted as aggregate production function, and the total budget as the quantity of money demanded in arbitrage equilibrium[15], and so on. As long as we insist on practicing macro-economics we shall need aggregate relations.

When reinterpreted in this way, an equation similar to the equation of exchange can be derived from our model within the framework of endogenous growth (see Lucas 1988). The key point is that, in the theory of endogenous growth, the concept of capital has been broadened to include human capital. In such circumstance, the aggregate production function becomes into $Q = AF(K)$, where K embodies both physical capital and human capital, and A represents total factor productivity (see Prescott 1998). Correspondingly, the budget constraint degenerated into $iK = M$. Substitute M for iK in no-arbitrage equation, we get

$$M(1+r) = PQ + iK(1-\delta) = PQ + M(1-\delta) . \tag{20}$$

Or equivalently

$$PQ = M(1+r) - M(1-\delta) = M(r+\delta) . \tag{21}$$

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15 The "demand for money" has been taken to be the equilibrium quantity of money which people with some rules of behavior and given patterns of cash flow will hold. For details, see Akerlof (1973).

To see what does this equation mean, rearrange the right-hand side of it to get $M(r+\delta) = iKr + iK\delta$, which equals the interest cost (iKr) plus the depreciation cost ($iK\delta$). It follows that in arbitrage equilibrium the value of physical output can just cover the user cost of capital.

On the other hand, it is easy to see that this identity looks suspiciously like the *equation of exchange*

$$PQ = MV , \tag{22}$$

where V is the velocity of money. From the viewpoint of equation of exchange, the foregoing equation simply implies that in arbitrage equilibrium the velocity of money must satisfy $V = r + \delta$.

Further, it can be shown that $r + \delta$ turns out to be the *minimum* of the velocity of money. This is because that in arbitrage equilibrium no firms have motives to invest more than the user cost of capital. All money exceeding the user cost of capital will be deposited in asset market to get risk-free interests, and hence the velocity of this part of money equals zero within each period. “You can lead a horse to water, but you can’t make him drink.” Analogously, you can force money on the system, but you can’t make the money circulate against new goods and new jobs. On the other hand, if economic system allows for profitable arbitrage, then the demand for money tends to increase since more arbitrage trade will take place. Against a given quantity of money, this means that the velocity of money will tend to rise. Thus the velocity of money indeed reaches its minimum in arbitrage equilibrium.

The demonstration of this minimum of the velocity of money is of central importance. In a sense, it shows that arbitrage equilibrium behaves in much the same way as the *thermodynamics equilibrium*, rather than *mechanics equilibrium*. Indeed, the equation of exchange of Fisher looks suspiciously like the *equation of state* of idea gas, which shows the relationship between the pressure, volume, and temperature for a fixed amount of idea gas. In view of this, it seems that the velocity of money can be viewed as the temperature of the economic system. The limiting circumstance of minimum velocity of money seems like the *absolute zero of temperature*, which is impossible for any process to approach in a finite number of operations. Physically, the *third law of thermodynamics* states that the entropy of a closed system at absolute zero is exactly equal to zero (see Feynman *et al*. 2013).

III. COMPARATIVE STATIC ANALYSIS

Only the simplest production functions admit solutions given by explicit formulas; however, some properties of solutions for a given production function may be determined without finding their exact form. It is the task of comparative statics to show the determination of the equilibrium values of decision variables in arbitrage equilibrium with state variables being specified (see Samuelson 1941).

In fact, it turns out that, under given conditions of technique the behavior of the economic system is governed by the subsystem of the budget constraint and no-arbitrage constraint

$$\text{(23)} \qquad \begin{cases} WL + iK = M \\ iK(1-\delta) + PQ = M(1+r) \end{cases}.$$

Comparative static analysis is primarily concerned with this subsystem. In this subsystem, under given conditions of technique, real variables and nominal variable are not independent quantities; they are connected by the system of two linear equations. Since both equations are accounting identities, any change in any one of the variables must show up somewhere, resulting in a corresponding change in at least one of the other variables. Another remarkable characterization of this subsystem is that there will always be positive *degrees of freedom*. [16]

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[16] In theory as well as in practice, the higher the degrees of freedom, the more complex will the system likely to be. In general, the number of degree of freedom of the system is decreased as we proceed from the general to the more particular cases. However, since the economic system has been proved to be an Evolving Complex System (see Anderson *et al.* 1988; Arthur *et al.* 1997; Blume and Durlauf 2006), those macroeconomic models with zero degree of freedom entailed a methodological error.

III.A. *Neutrality of Money*

Neutrality of money is the idea that prices respond proportionally to changes in the quantity of money (see Lucas 1996). If money is neutral then the economy exhibits the classical dichotomy: money affect only nominal variables, with no effect on real variables.

Fortunately, since real variables and nominal variables dually appeared in both of these two equations of constraints, it immediately eliminated the classical dichotomy: it is impossible to break down the system into the real sector and the monetary sector. Both real variables and nominal variables are determined in truly general equilibrium manner—by the system as a whole. This complex interaction of monetary and real forces completely freed us of the troublesome classical dichotomy.

Further, it will be shown that the neutrality of money imposed very strong constraint on the behavior of the economic system. To see this, taken the physical variables (L, K, Q) as fixed and then consider the system of linear equations with price variables W, i, P . Thus we have only two independent equations to determine three price variables: the system is not determinate. It is the existence of this positive degree of freedom that enables us to escape the classical conclusion that money is neutral.

In fact, it is easy to see that price vector (W, i, P) and the quantity of money (M) are so related that if we multiply price vector by a factor then the quantity of money will indeed be increased the same proportion in order to preserve arbitrage equilibrium. On the other hand, a change in the quantity of money leads the corresponding system of linear equations to having a solution given by a proportional change in the price vector. However, this is just a *specific solution* of the system.

To get the *general solution* of the system, solving W, P as functions of i

$$\begin{cases} W = -\dfrac{K}{L}i + \dfrac{M}{L} \\ P = -\dfrac{K(1-\delta)}{Q}i + \dfrac{M(1+r)}{Q} \end{cases} . \tag{24}$$

It follows that, if the quantity of money is *exogenously* determined, then the rental price of capital (i) is in indeterminate and serves as a *free variable*. This positive degree of freedom allows for the non-neutrality of money. [17]

To summarize, even taking the real variables (L, K, Q and r) as fixed, the neutrality of money is just a specific solution of the system and hence does not tend to hold in general. Neutrality of money is a situation that is the exception and not the rule.

One way illustrating the failure of neutrality is to consider the theory of *endogenous money*. To this end, assume that the quantity of money is endogenously determined, then we can eliminate M from the subsystem to get the following accounting identity[18]

(25) $$PQ = iK(r+\delta) + WL(1+r) .$$

This identity amounts to saying that the total revenue of output (PQ) equals the user cost of capital ($iK(r+\delta)$) pluses the cost of labor measured at the end of each period ($WL(1+r)$). From this *revenue-expenditure identity* it follows that the arbitrage equilibrium condition indeed imposed an essential constraint on the pattern of behavior of price variables. This pattern is much more complicated than the relation predicted by the classical monetary theory.

However, from the viewpoint of thermodynamics, this revenue-expenditure identity behaves in a way similar to the equation of state of ideal gas. It is well known that experimental gas laws, such as Boyle's law, Charles' law and Gay-Lussac's law, can be considered as special cases of the *equation of state* of idea gas, with one or more of the thermodynamic variables (temperature, pressure, and volume) held constant (see Feynman *et al.* 2013). Similarly, this revenue-expenditure identity also contains some

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[17] The economical interpretation of this positive degree of freedom is that the system can adjust to any value of the rental price of capital, and eventually approach a state of no-arbitrage equilibrium. On the other hand, the finding that the nominal interest rate is indeterminate in no-arbitrage equilibrium agrees with both empirical and theoretic evidences. For example, Friedman (1968) showed that the relation between the quantity of money and interest rate is much complex and monetary policy cannot peg interest rate. Similar result was also obtained by Tobin (1958), who showed that the direction of the relationship between the rate of interest and the demand for money is somewhat ambiguous. In fact, by now there is no general agreement on the theory of the relation between money and interest rate, as indicated by the Gibson paradox.

[18] If the concept of capital has been generalized to the broad sense to include human capital ($L=0$), then this revenue-expenditure identity degenerated into the equation of exchange discussed in section **II**.

empirical macroeconomic laws as special cases and hence can provide micro-foundations for them, such as the Gibson paradox, Okun's law, Phillips curve, and Keynesian doctrine.

III.B. *The Gibson Paradox*

The Gibson paradox is an empirical regularity that the general price level and nominal interest rate are positively correlated. It was regarded as a paradox because it seemed to contradict the prediction of classical monetary theory. According to the quantity theory of money, the price level would be expected to be rising if the quantity of money is increasing. In addition, by the theory of liquidity preference, the nominal interest rate should be falling when the quantity of money is increasing. If both these classical doctrines are true, the general price level and nominal interest rate should be *negatively* correlated. However, Gibson observed the empirical tendency for the general price level and nominal interest rate to apparently move together. This empirical evidence was believed to be a paradox because it seemed to constitute a disconfirmation of one of the important predictions of classical monetary theory.

To our knowledge, the Gibson paradox still remains an empirical phenomenon without a widely accepted theoretic explanation (see Keynes 1930; Fisher 1930; Friedman 1968; Sargent 1973; Fama 1975). However, it turns out that Gibson paradox is consistent with the pattern of behavior of price variables determined by arbitrage equilibrium. To see this, solve for the price level from the revenue-expenditure identity and take partial derivatives with respect to nominal interest rate to get

$$P=\frac{iK(r+\delta)+WL(1+r)}{Q}\Rightarrow\frac{\partial P}{\partial i}=\frac{K(r+\delta)}{Q}>0\,. \tag{26}$$

So, *other things being the same*,[19] general price level and nominal interest rate are positively correlated. This is essentially the positive correlation noted by Gibson in 1923. Within our framework, it becomes into a consequence of the arbitrage equilibrium. In

[19] In practice, if "other things" cannot hold constant, then we have to take total derivatives rather than partial derivatives. It is unnecessary and impossible to adopt a prior classification of variables into "endogenous" and "exogenous". How economic variables have in fact been related to each other can only be tested by practice.

view of this, the no-arbitrage framework thus has reconciled the long-standing contradiction between the quantity theory of money and the theory of liquidity preference.

Historically, Keynes (1930) first used the term Gibson paradox to emphasize the observations that nominal interest rates were highly correlated with the general price level but approximately uncorrelated with inflation as contradicting Irving Fisher's equation linking interest rates to expected inflation. Indeed, the classical monetary theory had expected a correlation between the nominal interest rates and the rate of change, rather than the general level, of prices. Yet, as indicated by the Gibson paradox, empirical data contradicted this view. On the other hand, Fama (1975) pointed out that the finding there are *no* relationships between interest rates and rates of inflation is in fact inconsistent with the Efficient-Market Hypothesis.

We shall show that this inconsistency can also be reconciled within the no-arbitrage framework. Indeed, a dynamical version of the revenue-expenditure identity turns out to be generally consistent with both the Fisher effect and the Gibson paradox within the framework of rational expectation. For details see section **V**.

III.C. *Okun's Law*

In macroeconomics, Okun's law is an empirically observed relationship relating unemployment to losses in potential GDP (Okun 1962). [20] Since the nominal GDP exactly equals PQ at the macro level, we can derive Okun's law from the revenue-expenditure identity as special case. Once again, it is not coincidental.

Denote the total labor force by $\overline{L}$, and let the unemployment rate in each period be u, which ranges between 0 and 1. Then in each period we have $L = \overline{L}(1-u)$. Now substituting for L in the revenue-expenditure identity and taking the partial derivative of PQ with respect to u we have

[20] The original work of Okun (1962) addressed the measurement of potential GNP, which coincides with the equilibrium value of nominal GDP in a closed economy. Okun's basic technique consisted of a leap from the unemployment rate to potential output, rather than a series of steps involving the underlying factors. Indeed, the latter technical route has been adopted in this paper and hence can provide micro-foundations for statistical estimates.

(27) $$PQ = iK(r+\delta) + W\bar{L}(1-u)(1+r) \Rightarrow \frac{\partial(PQ)}{\partial u} = -W\bar{L}(1+r) < 0\,.$$

From this it follows that, *other things the same*, an increase in the unemployment rate means a loss in equilibrium GDP. This is the essence of Okun's law.

Note that the stability and usefulness of Okun's law has been disputed. The reason is that Okun's law is approximate because factors other than employment may also affect potential output. In fact, any variable in the revenue-expenditure identity can influence the level of output.

III.D. *Phillips Curve*

Phillips curve doctrine implies that lower unemployment can be purchased at the cost of higher inflation. Hence there would be a trade-off between inflation and unemployment. Since its inception, the Phillips curve has been criticized for its lack of foundations in microeconomics and general equilibrium theory (see Lucas 1976). Also, a good deal of efforts went into developing theoretic foundations for this empirical observation.

In this subsection we shall establish the concrete function relationship between unemployment and the rate of change of wages, which turn out to be similar to the original Phillips curve fitted to 1861-1913 data (Phillips 1958).

Now denote the wages in the previous period by W_0 and the change rate of money wages in the current period by w, which ranges from -1 to $+1$ so that $W = W_0(1+w)$. Substitute for W and L in the revenue-expenditure identity and rearrange it to get

(28) $$(1-u)(1+w) = \frac{PQ - iK(r+\delta)}{W_0\bar{L}(1+r)}\,.$$

Taking logarithm, we get the function relationship between unemployment and the rate of change of wages

$$(29) \qquad \log(1-u)+\log(1+w)=\log\frac{PQ-iK(r+\delta)}{W_0\bar{L}(1+r)}.$$

It is easy to note that this equation looks suspiciously like the original Phillips curve fitted to 1861-1913 data, which reads (in our notations[21])

$$(30) \qquad 1.394\log(u*100)+\log(0.9+w)=0.984.$$

To continue, we shall state the conditions under which there exists a trade-off between wage increases and unemployment. Mathematically, this amounts to saying that the *total* derivative of wages with respect to unemployment must be negative. Using the fact that the increase of unemployment may affect nominal GDP (PQ) according to Okun's law and taking the total derivative of w with respect to u we get

$$(31) \qquad \frac{dw}{du}=\frac{\frac{d(PQ)}{du}(1-u)+[PQ-iK(r+\delta)]}{W_0\bar{L}(1+r)(1-u)^2}<0.$$

Thus the Phillips curve slopes downward only if

$$(32) \qquad \frac{d(PQ)}{du}<-\frac{PQ-iK(r+\delta)}{1-u}=-\frac{WL(1+r)}{1-u}.$$

In terms of elasticity we have

$$(33) \qquad \frac{u}{PQ}\frac{d(PQ)}{du}<-\frac{WL(1+r)}{PQ}\frac{u}{1-u}.$$

So, to guarantee that the Phillips curve slopes downward, *ceteris paribus*, the elasticity of GDP with respect to unemployment must be sufficiently large. This amounts to saying that an increase in unemployment will result in *dramatic* losses in GDP, which in turn will lead the wages to going down.

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[21] Note that in the original equation of the Phillips curve the level of unemployment (u) is ranged from 0 to 100, instead of from 0 to 1. It is for this reason that the original function $\log u$ had to be replaced by $\log(u*100)$.

On the other hand, if the Phillips curves slopes upward instead then there will be *stagflation*, a situation where high unemployment and high inflation steadily coexist. To be precise, stagflation will happen only if

$$\frac{u}{PQ}\frac{d(PQ)}{du} > -\frac{WL(1+r)}{PQ}\frac{u}{1-u} > -\frac{u}{1-u}. \tag{34}$$

This amounts to saying that the increase of unemployment can just result in moderate decrease in GDP. As a result, stagflation reared its ugly head.

In conclusion, there does not exist an *unconditional* trade-off between inflation and unemployment, whether short or long, temporary or permanent (see Friedman 1968). Further, the condition under which stagflation tend to occur is itself of interest, which perhaps justifies the no-arbitrage production theory as a useful framework.

III.E. *The Keynesian Doctrine*

According to modern Keynesian doctrine, the Keynesian revolution was a revolution in method. A key element in all Keynesian models is a "tradeoff" between inflation and output: the higher is the inflation rate, the higher is output (see Keynes 1936). That view is embodied most directly in the negatively sloped Phillips curve: the higher is the inflation rate, the lower is the rate of unemployment. As a result, the effects of aggregate demand policies tend to move inflation rates and output (relative to trend) in the same direction, or alternatively, unemployment and inflation in opposite direction.

However, it is found that in practice the typical inflation-output relation is the reverse, that prices and output tend to be related negatively, rather than positively (see Lucas 1973; Friedman and Schwartz 1982). This clearest conflict between empirical evidence and Keynesian doctrine was widely regarded as the failure of Keynesian revolution (see Lucas and Sargent 1978). In this subsection, we shall show that there does not exist an *unconditional* trade-off between inflation and output, thus reconcile this long-standing confliction.

To see this, denote the price level in the previous period by P_0 and the rate of change of price level in the current period by ρ, which ranges from -1 to $+1$ so that $P = P_0(1+\rho)$. Now solve for the output from the revenue-expenditure identity and substitute P to get

$$Q = \frac{iK(r+\delta) + WL(1+r)}{P_0(1+\rho)}. \tag{35}$$

Take the *total* derivative of output with respect to inflation

$$\frac{dQ}{d\rho} = \frac{\frac{dL}{d\rho}W(1+r)(1+\rho) - [iK(r+\delta) + WL(1+r)]}{P_0(1+\rho)^2} > 0. \tag{36}$$

So output and inflation move together only if

$$\frac{dL}{d\rho} > \frac{iK(r+\delta) + WL(1+r)}{W(1+r)(1+\rho)} = \frac{PQ}{W(1+r)(1+\rho)}. \tag{37}$$

This amounts to saying that the elasticity of the labor with respect to inflation must satisfy

$$\frac{\rho}{L}\frac{dL}{d\rho} > \frac{PQ}{WL(1+r)}\frac{\rho}{(1+\rho)} > \frac{\rho}{1+\rho}. \tag{38}$$

To see what this condition means, we shall express this condition in terms of the *partial* elasticity of output to inflation. First, take the *partial* derivative of output with respect to inflation to get

$$\frac{\partial Q}{\partial \rho} = -\frac{iK(r+\delta) + WL(1+r)}{P_0(1+\rho)^2} = -\frac{PQ}{P_0(1+\rho)^2} = -\frac{Q}{1+\rho}. \tag{39}$$

Second, rewrite the output-inflation tradeoff condition using the partial elasticity of output to inflation

$$\frac{\rho}{L}\frac{dL}{d\rho} >> \frac{\rho}{1+\rho} = -\frac{\rho}{Q}\frac{\partial Q}{\partial \rho}. \tag{40}$$

Thus, *other things the same*, there is a tradeoff between inflation and output only if the elasticity of labor to inflation is larger than the partial elasticity of output to inflation, or equivalently, only if the *direct* effect of inflation on the decrease in output can be cancelled out by the effect of inflation on the increase in labor demand.

IV. STATIONARY OPEN ECONOMIES

In this section we first use the basic model to calculate the Purchasing Power Parity exchange rate. Then we generalize the basic model to an open economy and use it to explain the Balassa–Samuelson effect.

IV.A. *Purchasing Power Parity*

The Purchasing Power Parity (PPP) doctrine has been used as a guide in establishing equilibrium exchange rate. Historically, Cassel (1918) first formulated the PPP hypothesis by arguing that "the rate of exchange between two countries is primarily determined by the quotient between the internal purchasing power against goods of the money of each country."

Contrary to the traditional consumption-based comparison of PPP, we shall calculate PPP on the basis of no-arbitrage framework for production theory. [22] It will be shown that our production based PPP calculation confers methodological advantages in that it constitute a marriage of the Keynes arbitrage version and the production-cost version. Within our framework, however, both these tradition versions of PPP become implications of our model, as opposed to assumptions (see Samuelson 1964).

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[22] Historically, Keynes had interpreted the PPP doctrine as the doctrine of spatial arbitrage for every goods (in the absence of transport costs) early in World War I. See Samuelson (1994).

Now assume the price level of the same product in foreign country to be P^*, in terms of the *foreign* currency.[23] Then the net profit of identical goods abroad can be denoted by $\pi^*(Q^*) = P^*Q^* - C^*(Q^*)$. Thus the rate of profit in foreign country is

$$(41) \qquad \frac{\pi^*(Q^*)}{C^*(Q^*)} = \frac{P^*Q^* - C^*(Q^*)}{C^*(Q^*)} = \frac{P^*}{C^*(Q^*)/Q^*} - 1.$$

Thanks to the no-arbitrage constraint, in equilibrium the rate of profit in different country will be the same. Thus we can get the PPP identity in terms of the rate of profit

$$(42) \qquad \frac{\pi(Q)}{C(Q)} = \frac{PQ - C(Q)}{C(Q)} = \frac{P^*Q^* - C^*(Q^*)}{C^*(Q^*)} = \frac{\pi^*(Q^*)}{C^*(Q^*)}.$$

By virtue of this identity, the ratio of price level can be solved as a function of average cost

$$(43) \qquad \frac{P}{P^*} = \frac{C(Q)/Q}{C^*(Q^*)/Q^*}.$$

Thus in equilibrium the ratio of general price levels must equal the ratio of the average costs between these two countries. Denote the equilibrium ratio of the average cost by

$$(44) \qquad \varepsilon = \frac{C(Q)/Q}{C^*(Q^*)/Q^*},$$

then we can get the familiar expression of PPP

$$(45) \qquad P = \varepsilon P^*.$$

From this it follows that, *other things equal*, the PPP exchange rate (ε) between two countries is determined by the relative average cost of identical goods in these two countries. If the average cost of domestic goods is relatively *low* in arbitrage equilibrium, the purchasing power of domestic currency will be relatively *high*. To emphasize this

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[23] Throughout the asterisk (*) is used to denote the corresponding variables of foreign country.

note that different countries, at the same period of time, have different production functions even apart from differences in natural resource endowment.

Recall that both of the output levels Q and Q^* in the PPP identity are assumed to be solved from the budget constraint (or cost constraint) respectively. To this end, denote the *nominal* exchange rate by e, which means that one unit of foreign currency equals e units of domestic currency. In arbitrage equilibrium, unit currency must have the same returns in different countries. Further, assume that the initial budget is $M^* = 1$ measured in foreign currency, then the equilibrium quantities Q and Q^* must satisfy

(46) $$C^*(Q^*) = 1,\ C(Q) = e\,.$$

As an illustration, reconsider the Cobb–Douglas technology, which give rise to production functions $Q = AK^{\alpha}L^{\beta}$ and $Q^* = A^*K^{*\alpha^*}L^{*\beta^*}$ at home and abroad, where A and A^* stand for productivity respectively. Then the relative average cost determines the PPP exchange rate[24]

(47) $$\varepsilon = \frac{C(Q)/Q}{C^*(Q^*)/Q^*} = \frac{A^*}{A}\frac{D^{\alpha+\beta}}{D^{*\alpha^*+\beta^*}}e^{1-\alpha-\beta} \equiv h(e)\,.$$

If $\alpha + \beta \neq 1$, then ε will be a function of e, denoted by $\varepsilon = h(e)$. In this case the *fixed point* of this function can determine the *equilibrium* exchange rate $\varepsilon = h(\varepsilon)$, which can be calculated by iteration algorithm.[25]

Suppose, at one extreme, that technologies in both countries are perfect complements. It means that the production function of domestic country is $Q = A\min\{K, L\}$, where A stands for productivity. Then the cost function will be $C(Q) = (i + W)Q/A$ and hence

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[24] In the sense of David Ricardo, the ratio A/A^* can be interpreted as the degree of comparative advantage of domestic country. The higher the comparative advantage at home, the higher will be the purchasing power of domestic currency. See Samuelson (1964).

[25] It is easy to see that $\varepsilon = e$ implies $P = \varepsilon P^* = eP^*$. In this case the law of one price holds and hence the fixed point indeed defines an equilibrium exchange rate.

the average cost equals $C(Q)/Q=(i+W)/A$, which equals unit factor costs. By symmetry we get the PPP exchange rate

(48) $$\varepsilon=\frac{C(Q)/Q}{C^*(Q^*)/Q^*}=\frac{A^*}{A}\frac{i+W}{i^*+W^*}.$$

This brings us back to the production-cost version. If technologies in both countries are perfect complements, the equilibrium exchange rate must be equal to the ratio of unit cost of production.

However, in practice no one would expect \$1 to buy the same level of real goods in different countries. Given the opportunity to freely invest \$1 in different countries, *other things equal*, the country with higher rate of profit will have competitive advantage (see Porter 1985). Thus to get competitive advantage in trade means that

(49) $$\frac{\pi(Q)}{C(Q)}>\frac{\pi^*(Q^*)}{C^*(Q^*)}\Rightarrow\frac{C(Q)/Q}{C^*(Q^*)/Q^*}<\frac{P}{P^*}.$$

As a result, competitive advantage can be achieved only if the relative average cost in arbitrage equilibrium is lower than the relative price of domestic goods. So competitive advantage in international trade can only be maintained by endlessly reducing the cost of production. It is the irreducible differential in costs that leads to importing rather than producing at home.

In conclusion, the competitive advantage that stems from the difference in the rate of profits has much to do with the pattern of international trade.

IV.B. *Balassa-Samuelson Effect*

The Balassa–Samuelson effect is the observation that the currencies of developed countries would generally appear to be greatly overvalued. This phenomenon has been studied at various levels of abstraction (see Balassa 1964; Samuelson 1964, 1994). Here we give a comparative-statics explanation on the basis of arbitrage equilibrium.

To this end, we need to extend our basic model to an open economy at first. In a sense this is an analogy of the extension of the IS-LM Model (Hicks 1937) to the Mundell–Fleming model (Mundell 1963; Fleming 1962).

Assume the fraction of exported products to be x. Then the no-arbitrage identity is essentially the same as in stationary economy

$$(50) \qquad PQ(1-x) + eP^{*}Qx + iK(1-\delta) = M(1+r)\,.$$

The domestic market can be in equilibrium only if this accounting identity is satisfied.[26]

For our present purposes solve for the exchange rate from the no-arbitrage identity

$$(51) \qquad e = \frac{M(1+r) - iK(1-\delta) - PQ(1-x)}{P^{*}Qx} = \frac{M(1+r) - iK(1-\delta)}{P^{*}Qx} - \frac{P(1-x)}{P^{*}x}\,.$$

Note that richer country usually tends to have higher capital accumulation. So the Balassa-Samuelson effect can be explained by means of capital stock. To see this, taking the partial derivative of e with respect to K to get

$$(52) \qquad \frac{\partial e}{\partial K} = \frac{-i(1-\delta)}{P^{*}Qx} < 0\,.$$

In terms of the elasticity of nominal exchange rate to capital stock we have

$$(53) \qquad \frac{K}{e}\frac{\partial e}{\partial K} = \frac{-iK(1-\delta)}{eP^{*}Qx} < 0\,.$$

Thus, *other things the same*, higher capital stock will yield lower nominal exchange rate. Or equivalently, the increase of capital accumulation will stimulate domestic currency to appreciate. As a consequence, the domestic currency of the developed country with higher capital stock would generally appear to be overvalued in arbitrage equilibrium.

To sum up, it is the no-arbitrage constraint that is crucial, rather than the law of one price. Therein lies the essence and rationale of the Balassa-Samuelson effect.

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[26] The other two constraints, namely the technology constraint and budget constraint, are remaining unchanged and are still at work.

V. DYNAMIC ECONOMIES

So far the economy has been assumed to be stationary. Actually our framework turns out to be very convenient to study dynamic economies. We may now pass over to the dynamic analysis of economic system by building a multi-period model. For simplicity, we discuss the closed economy only. The general principles carry over to open economies.

V.A. *The General Model*

Similar as in the stationary case, the axiomatic basis of the dynamic model consists of three fundamental constraints of technology, budget and no arbitrage. We proceed directly to the fundamental equations, which give rise to a system of difference equations

$$(54) \qquad \begin{cases} Q_t = A_t F(K_t, L_t) \\ i_t K_t + W_t L_t = M_t \\ P_t Q_t + i_t K_t (1-\delta) = M_t (1+r_t) \end{cases} .$$

This system of difference equations includes the stationary system of section **II** as special case.[27]

From the point of view of recursive macroeconomic theory (see Ljungqvist and Sargent 2012), the three fundamental accounting identities constitute a *decision function* which maps the state variables into decision variables of the economy. Since accounting identity must hold for any values of the variables, its structure is time-invariant. Thus, under given conditions of technique the decision function is also time-invariant.[28]

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[27] Variables with subscript represent the corresponding values at period t. As in stationary economy, variables without subscript denote the corresponding steady state variables in what follows.

[28] Our model differs from the recursive methods of Ljungqvist and Sargent (2012) in one major respect. In their recursive models the decision function (a time-invariant policy function) was solved from a functional equation known as Bellman equation, taken the transition function as given. Unfortunately, a fatal logical flaw may arise from their approach to dynamic economics due to the existence of competitive arbitrage. To see why, consider an economy in which we can take prices as state variables and quantities as decision variables. Now suppose that the transition function has predicted that asset price will increase in the future, then rational-expectation based arbitrage will drive asset price to rising dramatically rather than obeying the given transition law. This self-denying pattern means that the economy may not have a time-invariant transition function as common knowledge. Indeed, such dynamic-programming-based econometric policy evaluation procedures have been criticized by Lucas (1976) and Kydland and Prescott (1977).

To describe the dynamic evolution of the economic system the law of motion for state variables must be explicitly characterized by a *transition function*. Mathematically, the state transition function of a dynamic economy can be described by a difference equation

$$y_{t+1} = \varphi(y_0, y_1, \cdots, y_t; x_0, x_1, \cdots, x_t) \,. \tag{55}$$

where y_t's are vectors of state variables, x_t's are vectors of decision variables. The state transition function φ determines how the economy transit from current state y_t to future state y_{t+1}, which in conjunction with the decision function will determine the vector of decision variables x_{t+1}. This iteration process dynamically gives the entire time path of the economy. So, if the state transition function is specified, then the analysis of the dynamical evolution of the economy is a straightforward matter.

Specially, under the assumption of the Markov state (see Ljungqvist and Sargent 2012), the motion of the economy is determined by a first-order difference equation

$$y_{t+1} = \varphi(y_t, x_t) \,. \tag{56}$$

Ignoring random shocks, this difference equation is essentially the theoretical framework adopted by Lucas (1976) to criticize econometrical policy evaluation. In view of this, the Lucas critique just emphasizes the importance of state transition function. Indeed, it has been shown that even simple and deterministic first-order difference equations can exhibit an extraordinarily rich spectrum of dynamical behavior (see May 1976).

Since a change in policy necessarily affects the state of the economy in highly complex fashion, we should specify in advance the state transition function that governs the behavior of the dynamic economy if we want to predict the effect of a policy. Without knowledge as to which and how state transits as policy changes, we cannot assess alternative policies. The only scientific quantitative policy evaluations are to compare the consequences of alternative state transition function and to select that one with good operating characteristics. Thus, within the no-arbitrage framework of dynamic production theory, the choice of policies is equivalent to the choice of the state transition function of the system,[29] and macroeconomic policy evaluation is reduced into *scenario analysis* on the basis of rational expectation.

V.B. *Zero-profit Equilibrium as Limiting Case*

In this subsection, we consider the limiting case when the periods become sufficiently large, with emphasis on how the behavior of the economy is governed by the state transition equation. In order to simplify the task, our analysis proceeds under the assumption of constant risk-free interest rate. That is, $r_t \equiv r$ for any time period $t = 0,1,2,\cdots$.

To simplify the complication, we assume that all of the net profit at the end of each period will be automatically reinvested at the beginning of the next period, and firms cannot borrow money during the production process. Thus, the state transition equation for total budget was given by

$$M_{t+1} = \pi(Q_t)\,. \tag{57}$$

The cash flows in this system amount to the series of net profits $(\pi(Q_1), \pi(Q_2), \cdots, \pi(Q_t), \cdots)$. At any period t, the net profit satisfies

$$\pi(Q_t) = M_t r - i_t K_t (1-\delta) < M_t r\,. \tag{58}$$

Inductively, we get the following sequence of inequalities

$$\pi(Q_t) < M_t r < M_{t-1} r^2 \cdots < M_0 r^{t+1}\,. \tag{59}$$

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[29] This framework differs from that of Kydland and Prescott (1977). Formally, Kydland and Prescott supposed that the economy can be described by a vector of state variables, a vector of policy variables, a vector of decision variables, and a vector of random shocks. The movement over time of these variables is given by state transition equation in which the vector of policy variables explicitly appeared. However, as criticized by Lucas (1976), the change in policy induces change in the state transition function, which in turn induces a change in the policy rule, and so on. This iterative process indicates that policy-invariant state transition equations are inconsistent with the maximization postulate in dynamic settings. In fact, we have no reason to believe that state transition function is invariant under changes in policy and no reliable way to break it down into well-understood components. Thus it is necessary to view the policy as the state transition function itself, rather than preordained parameters in it. It is the state transition function that must be estimated, not just some of its parameters.

Here, M_0 is the initial budget at the begging of period 0.

Empirically, the risk-free interest rate can be assumed to satisfy $r < 1$, so $r^t \to 0$ as $t \to \infty$. Therefore, by the property of limit, we get

$$\lim_{t \to \infty} \pi(Q_t) = 0. \tag{60}$$

In conclusion, the traditional zero-profit equilibrium can be regarded as a limiting case of the arbitrage equilibrium. So, under given conditions of technique and so on, if firms cannot borrow money during the production process, the economic system must move ultimately to a zero-profit stationary state; which is the essential thesis of classical production theory.

In practice, however, there does not need infinite process of reinvesting. Typically, there may be an upper limit to the capital deepening. So when the accumulation of capital has been accomplished, new investment just need to cover the user cost of capital. In this case, there will be positive profits ever after. On the other hand, in such circumstance the state transition equation for budget must be modified accordingly. So the behavior of the economy is indeed governed by the state transition equation.

V.C. *Neoclassical Growth*

The usual neoclassical conditions of economic growth can be interpreted as state transition equations in a natural way within no-arbitrage framework. We proceed in the spirit of the Solow model (Solow 1956).

As a result of exogenous population growth the labor force increase at a constant rate n. Thus we get the state transition equation for labor

$$L_{t+1} = L_t(1+n). \tag{61}$$

Assume that part of the instant output is consumed and the rest is saved as capital accumulation.[30] Let the fraction of output saved be a constant s , then the following basic identity gives the state transition equation of capital

(62) $$K_{t+1} = K_t(1-\delta) + sQ_t \, .$$

These two state transition equations, in conjunction with the fundamental constraints, can trace out, step by step, the growth path of the economy.

Now denote the ratio of capital to labor by $R = K_t / L_t$. We need to determine its equilibrium value. First, we assume that all price variables take *constant* values. Second, we assume that the quantity of money M_t is endogenously determined and eliminate it by using the budget constraint and no-arbitrage constraint

(63) $$PQ_t = WL_t(1+r) + iK_t(r+\delta) \, .$$

This turns out to be the dynamical analogy of the revenue-expenditure identity for stationary economy. Now dividing L_t we get

(64) $$P\frac{Q_t}{L_t} = W(1+r) + i(r+\delta)R \, .$$

Solving for $Q_t = (K_{t+1} - K_t(1-\delta))/s$ and dividing L_t we get

(65) $$\frac{Q_t}{L_t} = \frac{1}{s}[\frac{K_{t+1}}{L_t} - \frac{K_t(1-\delta)}{L_t}] = \frac{R(1+n) - R(1-\delta)}{s} = \frac{R(n+\delta)}{s} \, .$$

Substituting it in equation (60) yields

(66) $$\frac{P(n+\delta)}{s}R = W(1+r) + i(r+\delta)R \, .$$

30 It follows that the quantity of capital is measured in physical units of output. This assumption is unnecessary in other parts of this paper.

This *linear* equation determines the equilibrium value of the capital-labor ratio, which in turn determines the capital accumulation path of the economy. This equation can be interpreted in the same way as that of the Solow model. Its right-hand side equals the *cost* of unit labor and R units of capital at the end of each period. As for its left-hand side, it is precisely the *revenue* of unit labor employed with R units of capital. As a result, the equilibrium capital-labor ratio is determined in such a way that the revenue equals the cost. [31]

V.D. *Synthesis of Schools of Macroeconomics*

Generally, different state transition function will result in different dynamic behavior of the economic system. In practice, more precise prediction of the dynamic behavior of the economic system can only be achieved if or when it becomes possible to write down the actual state transition function. Without detailed knowledge of the state transition function only very crude statement can be made.

In this subsection, it will be shown that some of the critical assumptions of different schools of macroeconomics, such as nominal rigidity and rational expectations, can serve as state transition equations. The difference between macroeconomics schools can be attributed to the differences in their state transition equations. Thus the no-arbitrage framework for production theory provides a unifying framework to synthesize different schools of macroeconomics.

1. Nominal Rigidity.—In practice, the adjustment of economic variable may not be instantaneous. Many economic processes, such as wage bargaining and Price adjustment, include time-delay phenomena in their inner dynamics. This time-delay characterization of wages and prices make nominal rigidity ideally suited to serve as the state transition equation of the economy.

[31] A remarkable characterization of this equilibrium condition is that it is based on price variables. This enables the existence of Nominal Rigidity, which has been shunted aside in the Solow model.

Rigid Wages: The assumption of rigid wages played a critical role in explaining the consistency of economic equilibrium with the presence of involuntary unemployment, which is usually considered as the most important achievements of the Keynesian theory.

To be precise, suppose that there exist a time delay (τ) in the adjustment of wages, then the wages in each period t equals the wages τ periods ago. Mathematically, we have

$$W_t = W_{t-\tau} . \tag{67}$$

For example, if the wage in period $t-\tau$ is determined by the marginal product of labor, then we have $W_t = P_{t-\tau} \frac{\partial}{\partial L} AF(K_{t-\tau}, L_{t-\tau})$. It is easy to see that when the time-delay approaches zero ($\tau = 0$), we obtain the traditional marginal-productivity equation $W_t = P_t \frac{\partial}{\partial L} AF(K_t, L_t)$. [32]

Rigid Prices: As is well known, the assumption that prices are rigid is crucial to Keynesian doctrine. Sticky prices are an important part of macroeconomic theory since they may be used to explain why markets might not reach equilibrium in the short run or even possibly in the long-run.

Suppose that there exist a time delay (τ) in the adjustment of price level, and hence the price level at period t equals the price levels τ periods ago

$$P_t = P_{t-\tau} . \tag{68}$$

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[32] It is worth mentioning that this approach of describing the wage rigidity differs from the device adopted by Modigliani (1963), which relies on the notion of a "potential" supply function. This approach also differs from that of Akerlof (2007), where nominal rigidity was explained by assumption that employees have a *norm* for what wages and prices *should* be.

The classical assumption of price flexibility can also be formalized by setting the time delay equal to zero ($\tau = 0$). Indeed, the state transition equation of Keynesian economy differs from that of neoclassical economy, *just like ellipses differ from circles*. [33]

Note that to simplify notation we use the same symbol τ to represent the time delay for both wages and prices. However, in practice, different variable may have different time delay. In the extreme case, the economic system may simultaneously contain *random* time delays for both nominal and real variables, and hence would collapse into *chaos* (see May 1976).

2. Rational Expectation.—Modeling expectations is crucial in all models which study how firms make choices under uncertainty. Concrete analytical results must rest on concrete assumption about expectations. And it is well known that the macroeconomic predictions of the model may differ depending on the assumptions made about expectations.

To make dynamic economic models complete, various expectation formulas have been used. In response to perceived flaws in theories based on adaptive expectations, Muth (1961) advanced the hypothesis of Rational Expectations. As will be evident, the character of rational expectation formation makes it well-suited for the state transition equation of the economy.

For example, under rational expectations, the actual price vector at period t always equals the *mathematical expectation* of the price vector of the succeeding period. In this case, the corresponding budget constraint and no-arbitrage constraint becomes into

$$(69) \qquad i_t K_t + \mathrm{E}[W_{t+1}]L_t = M_t\,, \;\; i_t K_t(1-\delta) + \mathrm{E}[P_{t+1}]Q_t = M_t(1+r_t)\,,$$

---

[33] Due to the similarity between the flexibility in economic variables and symmetry in physical laws, it is worthwhile to quote the words of Feynman *et al.* (2013, Chapter 52) concerning Broken Symmetries:"We have, in our minds, a tendency to accept symmetry as some kind of perfection. In fact it is like the old idea of the Greeks that circles were perfect, and it was rather horrible to believe that the planetary orbits were not circles, but only nearly circles. The difference between being a circle and being nearly a circle is not a small difference, it is a fundamental change so far as the mind is concerned. There is a sign of perfection and symmetry in a circle that is not there the moment the circle is slightly off—that is the end of it—it is no longer symmetrical. Then the question is why it is only nearly a circle—that is a much more difficult question. ... Now the question is whether we have a similar problem here. The problem from the point of view of the circles is if they were perfect circles there would be nothing to explain, that is clearly simple. But since they are only nearly circles, there is a lot to explain, and the result turned out to be a big dynamical problem, and now our problem is to explain why they are nearly symmetrical…"

where E[·] is the mathematical expectation operator, conditioned on given information set. The crucial issue is what assumption to make concerning the information set. What kind of information is used and how it is put together to frame an estimate of future states is important to understand because the character of dynamic process is sensitive to the way expectations are influenced by the actual course of events.

As an illustration, assume that the price level in period t is governed by the anticipated rate of inflation in period $t+1$. Mathematically, let $P_{t+1} = P_t(1+\rho_{t+1})$ so that $E[P_{t+1}] = P_t(1+E[\rho_{t+1}])$, where $E[\rho_{t+1}]$ stands for the anticipated rate of inflation. As in the discussion of Gibson paradox, we can eliminate the quantity of money, replace $E[P_{t+1}]$ by $P_t(1+E[\rho_{t+1}])$, and solve for the nominal interest rate to get

(70) $$i_t = \frac{Q_t P_t (1+E[\rho_{t+1}]) - \mathrm{E}[W_{t+1}] L_t (1+r_t)}{K_t (r_t + \delta)} .$$

It follows that the one-period nominal interest rate (i_t) depends on, among other things, the price level (P_t) *and* the anticipated rate of inflation ($E[\rho_{t+1}]$). As a result, our no-arbitrage production theory under rational expectations is consistent with both the Fisher effect and the Gibson paradox. In view of this, the Fisher effect and the Gibson paradox each touched one part, but only one part, of *the* elephant. Though each was partly in the right, and both were in the wrong! [34]

VI. CONCLUSIONS

In this article I propose a dynamic production model under the joint constraints of technology, budget and no arbitrage. Comparative static and dynamic analysis indicates that this model is consistent with the behavior of firms in reality, and can explain a wide range of economic phenomena. Compared with classical production theory, this dynamic production model confers some methodological advantages:

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[34] This last sentence is borrowed from the famous poem "The Blind Men and the Elephant" by John Godfrey Saxe (1816–1887).

First, this no-arbitrage based production theory can be viewed as a natural generalization of classical production theory based on profit maximization. For example, it is shown that the zero-profit equilibrium induced by the profit-maximization turns out to be a special and at the same time a limiting case of the arbitrage equilibrium. At the macro level our no-arbitrage equilibrium equation can be a viewed as a natural generalization of the Equation of Exchange (Fisher 1911).

Second, this no-arbitrage framework for production emphasizes the general equilibrium of the economic system as a whole and constitutes a marriage of production theory and finance. For example, it is shown that the Modigliani-Miller Theorem (Modigliani and Miller 1958) can be derived as a consequence of arbitrage equilibrium. In essence this reflects the fundamental economic significance of the arbitrage equilibrium.

Third, this no-arbitrage based production theory constructs a bridge between microeconomics and macroeconomics, and at the macro level it can provide a unified framework to explain some empirical laws in macroeconomics. For instance, Okun's law can be derived from our model as special case. Also, we can establish the concrete function relationship between unemployment and the rate of change of wages, which turn out to be similar to the original Phillips curve fitted to 1861-1913 data (Phillips 1958). In open economies, the purchasing power parity exchange rate between two countries is shown to be the ratio of *average costs* in arbitrage equilibrium. Further, comparative-statics explanation of Balassa-Samuelson effect can also be given on the basis of arbitrage equilibrium (see Balassa 1964; Samuelson 1964, 1994).

Fourth, this no-arbitrage framework for production theory can successfully reconcile some long-standing contradictions arising from the classical theory. For example, the complex interaction of monetary and real forces in our model completely freed us of the troublesome classical dichotomy. Further, a dynamical version of our no-arbitrage production model turns out to be generally consistent with both the Fisher effect and the Gibson paradox (see Keynes 1930; Fisher 1930; Friedman 1968; Sargent 1973; Fama 1975) within the framework of rational expectation. Conditions for stagflation to occur are derived too. It is shown that there does not exist an *unconditional* trade-off between inflation and output (see Lucas 1973; Friedman and Schwartz 1982). This reconciles the long-standing confliction between Keynesian doctrine (see Keynes 1936) and the empirical evidence, which was widely regarded as the failure of Keynesian revolution (see Lucas and Sargent 1978).

Fifth, from the viewpoint of recursive macroeconomic theory (see Ljungqvist and Sargent 2012), the fundamental constraints of technology, budget and no arbitrage constitute a *decision function* which maps the state variables into decision variables of the economy. The dynamic evolution of the economic system can be characterized by a *state transition function*. Generally, different state transition function will result in different dynamic behavior of the economic system. It is shown that different behaviors of different kinds of economies can be explained by the differences in their state transition functions. Thus the no-arbitrage framework for production theory provides a unifying framework to synthesize different schools of macroeconomics.

In conclusion, these methodological advantages justify the no-arbitrage based production theory as a useful framework.

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