Subsistence induced and complementarity induced irrelevance in preferences

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Abstract

In a two-good setting we axiomatize (a) preferences with subsistence consumption and (b) a generalized version of Leontief preferences. Our notion of subsistence allows for different levels of subsistence and captures the presence of poverty and prosperity. Our axioms are based on the irrelevance of one of the goods at certain consumption bundles. For subsistence, the irrelevance is induced by the subsistence requirement and for generalized Leontief, it is induced by complementarity. We capture this difference using the notion of unhappy sets.

JEL Classifications: D11, O12, O15
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1 Introduction

Subsistence is the minimum amount of basic necessities essential for a person's survival. Depending on the context, it can be expressed alternatively in terms of income (e.g., $1.25 per day) or in terms of nutrition such as a certain daily calorie requirement. It forms the basis of poverty measurements: "...absolute poverty lines are often based on estimates of the cost of basic food needs (i.e., the cost a nutritional basket considered minimal for the healthy survival of a typical family), to which a provision is added for non-food needs." (World Bank, 2013). As extreme poverty and hunger continue to pose a major global challenge, subsistence remains a useful concept for policymakers. For instance, effective policies to end hunger require knowledge of not only the number of hungry people, but also their *food deficit* or the depth of hunger, which is measured by "comparing the average amount of dietary energy that undernourished people get from the foods they eat with the minimum amount of dietary energy they need to maintain body weight and undertake light activity." (Food and Agricultural Organization of the United Nations, 2014).

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The Stone-Geary utility function is widely used to model subsistence.[1] However, under this utility function a consumer is compelled to consume above the subsistence level, thus assuming away the problem of poor people. Moreover the level of subsistence is assumed to be the same across individuals. This is also unrealistic as subsistence requirement does depend on factors such as age, gender, habitat and physical characteristics (see, e.g., Svedberg [11], Jensen and Miller [3]).

This paper seeks to provide a micro-foundation of subsistence consumption in a consumer theory framework. We axiomatize subsistence consumption in a setting where an individual makes consumption choices over two goods: a basic good which is a necessity such as food and a non-basic good which can represent a composite of other commodities.[2] In developing our theory, we appeal to two distinct aspects of a basic necessity. First, a minimum critical level of this good is required for the individual. This is the subsistence requirement. The other commodities can benefit the individual only if the consumption of the basic good exceeds the subsistence requirement. The second aspect is saturation. Once the individual has consumed sufficiently large amounts of the basic good, consuming more of it is of no benefit.[3]

Observe that subsistence and saturation generate *irrelevance* of one of the goods. The non-basic good is irrelevant when the subsistence requirement is not met, while the basic good becomes irrelevant when its saturation is reached. Incorporating these features, we define a preference with the subsistence requirement which we call the *subsistence induced irrelevance* (SII) preference. For such preferences there are three zones in the commodity space. Apart from the two zones where one of the goods is irrelevant, there is an intermediate region (where the consumption of the basic good has exceeded the subsistence level but not yet reached saturation) in which none of the goods is irrelevant. In this region the individual has a standard consumer preference where two goods can be imperfectly substitutable. SII preferences thus enrich consumer theory by allowing for the existence of poverty and prosperity in different regions of the commodity space. Furthermore, in contrast to the Stone-Geary utility function, these preferences allow for different levels of subsistence across individuals. In Theorem 1 we axiomatize SII preferences.

Irrelevance of a good in SII preferences is induced by subsistence and saturation. However, irrelevance can also be induced by complementarity between the two goods. If an individual prefers two spoons of sugar with every cup of tea and has one cup of tea, then sugar becomes irrelevant after two spoons. For such preferences (called Leontief preferences), complementarity between the goods implies that at any consumption bundle one of the two goods is irrelevant. Theorem 2 axiomatizes a generalized version of the Leontief preference (GL preferences).

Apart from the notion of irrelevance, the other key concept that is central for our

[1] Rebelo [6] and Steger [10]) use the Stone-Geary function to study the role of subsistence in economic growth. See Sharif [9] for a survey of measurement issues of subsistence. Subsistence consumption has also been associated with Giffen behavior, i.e., upward sloping demand curve (see, e.g., Jensen and Miller [2]).

[2] Jensen and Miller [3] also consider a two-good setting to study subsistence behavior. However, both goods in their model are basic goods (food items that contribute calories) and there is substitutability among them.

[3] We use a weaker notion of saturation in our model.

axiomatizations is an *unhappy set*. A set of consumption bundles is said to be an unhappy set if every bundle outside this set is preferred to all bundles inside the set. This captures the state of a poor person who has extreme urge to come out of poverty. To see how the notions of irrelevance and unhappy sets are connected in our axioms, call a set of consumption bundles irrelevant in a certain good if that good is irrelevant at all bundles of the set. For SII preferences, the zone where the subsistence requirement is not met is the largest unhappy set that is irrelevant in the non-basic good. But for GL preferences, if a set is irrelevant in any good, it can never be an unhappy set. Thus roughly speaking, SII and GL preferences are characterized by the presence or absence of unhappiness in irrelevance. It is a case of too little versus too much. Irrelevance of the non-basic good in SII preference stems from the fact that there is too little of the basic good. For GL preference, irrelevance of a good is driven by too much of that good in relation to the other good.

Conceptually unhappy sets are closely related to the notion of 'strong external instability' of von-Neumann and Morgenstern [12]. To see this define a set of consumption bundles to be a happy set if for any bundle outside this set we can find a bundle in this set which is preferred to it. Thus, a happy set is that set of bundles that are externally stable. So a set is not happy set if there exists a commodity bundle outside this set which is at least as good as all bundles in this set. Our definition of unhappy set strengthens this notion of not happy set since we require that each bundle in an unhappy set is dominated (not just weakly but strictly) by all bundles outside the set (and not just by one bundle outside this set).

To the best of our knowledge subsistence requirement has never been incorporated in the preference based approach of consumer behavior. One can find an axiomatization of the lexicographic preferences in Fishburn [1]. Milnor [5] axiomatizes the max, min and sum criteria. Maskin [4] provides a characterization of the sum and max-min criteria and also relaxes the continuity assumption to provide characterizations of lexicographic max-min and lexicographic max-max criteria. Segal and Sobel [8] characterize the min and max criteria and their combination with sum criteria. The main difference of our approach from the above mentioned literature is that our axioms are on the regions of irrelevance embedded in SII and GL preferences.

The paper is organized as follows. After providing the preliminaries in Section 2 we define SII preferences and discuss some implications of the notion of irrelevance. Axiomatization of SII preferences is presented in Section 3. In Section 4 we present axiomatization of GL preferences.

2 Preliminaries

Consider the problem of an agent in a two-good economy where the set of goods is $\{1, 2\}$. The agent has a consumption set $X = X_1 \times X_2$ where $X_i = \mathbb{R}_+$ for $i \in \{1, 2\}$, and $X = \mathbb{R}^2_+$. A consumption bundle is $x = (x_1, x_2) \in X$ where x_i stands for the amount of good i. Generic points in X will be denoted by x, y, z. If for $i \in \{1, 2\}$: (a) $x_i > y_i$, then we say $x > y$, (b) $x_i \geq y_i$, then $x \geq y$ and (c) $x_i = y_i$, then $x = y$.

The agent's preference on X is defined using the binary relation "at least as good as". We say $x \in X$ is at least as good as $y \in X$ and write it as $x \succsim y$. The preference relation $\succsim$ on X is rational (i.e., complete and transitive).[4] The strict preference is defined as $x \succ y \Leftrightarrow [x \succsim y]$ and *not*$[y \succsim x]$. The indifference relation is defined as $x \sim y \Leftrightarrow [x \succsim y]$ and $[y \succsim x]$. The preference relation $\succsim$ on X is *continuous* if $\{(x^n, y^n)\}$ is a sequence of pairs of elements in X such that $x^n \succsim y^n$ for all n and $\lim_{n\to\infty} x^n = x, \lim_{n\to\infty} y^n = y$, then $x \succsim y$. The preference relation $\succsim$ on X is *monotone* if for any $x, y \in X$ such that $x > y$, $x \succ y$.

Axiom 0 The preference relation $\succsim$ on $X = \mathbb{R}^2_+$ is rational, continuous and monotone.

Throughout the paper we shall assume that Axiom 0 holds, so it will not be stated separately in the ensuing statements.

2.1 SII preferences

Definition 1 We say good 2 is *irrelevant at a bundle* x if $x \sim (x_1, y_2)$ for all $y_2 > x_2$. Similarly good 1 is *irrelevant at a bundle* x if $x \sim (y_1, x_2)$ for all $y_1 > x_1$. A good is *relevant at a bundle* x if it is not irrelevant there.

Let $i, j \in \{1, 2\}$ and $i \neq j$. We say that *a bundle* y *involves* x_i if $y_i = x_i$. Thus, the set of all bundles involving x_i is $\{y \in X | y_i = x_i, y_j \in \mathbb{R}_+\}$.

Definition 2 *The preference relation* $\succsim$ *on* X *is a subsistence induced irrelevance preference* (*or an SII preference*) *with respect to good* 1 *if it satisfies the following properties.*

(I) *Subsistence:* $\exists\ \underline{Q} \in (0, \infty)$ *such that*

- (a) *Subsistence zone* $[0, \underline{Q}]$*: if* $x_1 \in [0, \underline{Q}]$, *then good* 2 *is irrelevant at all bundles involving* x_1;
- (b) *Non-subsistence zone* $(\underline{Q}, \infty)$*: if* $x_1 > \underline{Q}$, *then* $\exists\ y_1 \in (\underline{Q}, x_1)$ *such that good* 2 *is relevant at some bundle involving* y_1.

(II) *Weak saturation:* $\exists\ x_2 \in X_2$ *and* $\overline{Q} \in \mathbb{R}_+$ *such that good* 1 *is irrelevant at* x *if* $x_1 \geq \overline{Q}$ *and it is relevant at* x *if* $x_1 < \overline{Q}$.

Definition 2 has zones of subsistence and weak saturation in preferences. In this definition good 1 is the basic good which is a necessity and $\underline{Q}$ stands for the subsistence threshold. Good 2 is the non-basic good. For instance, if good 1 represents food, then $\underline{Q}$ stands for the amount of food that gives the minimum daily calorie requirements for the individual. The subsistence zone specifies that if the consumption of good 1 is below this critical level, then consumption of good 2 does not have any benefit (property I(a)). The non-basic good is beneficial to the consumer provided the consumption of the basic good exceeds the threshold level $\underline{Q}$ (property I(b)). Property (II) of the definition specifies another threshold $\overline{Q}$ of the basic good 1, beyond which it has no benefit to the consumer. This captures the saturation aspect of a standard basic good like food item.

[4]The preference relation $\succsim$ on X is *complete* if for any $x, y \in X$, either $x \succsim y$ or $y \succsim x$ or both. It is *transitive* if for any $x, y, z \in X$, $[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$.

The SII preference has two implications that are stated in Observation 1. First, there is a natural order between the threshold of subsistence and saturation (that is, $\underline{Q} \leq \overline{Q}$). Second, if the amount of the basic good exceeds the saturation level (that is, $x_1 > \overline{Q}$), then the non-basic good is necessarily beneficial. Formally, call a subset $S \subseteq X_1$ a *strong non-subsistence zone* if whenever $x_1 \in S$, then good 2 is relevant at some bundle involving x_1. We show that the interval $(\overline{Q}, \infty)$ is indeed a strong non-subsistence zone, that is if $x_1 > \overline{Q}$, then good 2 is relevant at some bundle involving x_1.

Observation 1 *For any SII preference $\underline{Q} \leq \overline{Q}$, with strict inequality if $x_2 > 0$ in* (II)*. Moreover, the weak saturation property implies that the interval $(\overline{Q}, \infty)$ is a strong non-subsistence zone.*

2.2 Irrelevance: some implications

We define two functions $f_1, f_2 : X \to \{0, 1\}$ that captures the notion of irrelevance.

$$f_1(x) \equiv \begin{cases} 0 \text{ if } x \sim (y_1, x_2) \text{ for all } y_1 \geq x_1, \\ 1 \text{ otherwise.} \end{cases}$$

$$f_2(x) \equiv \begin{cases} 0 \text{ if } x \sim (x_1, y_2) \text{ for all } y_2 \geq x_2, \\ 1 \text{ otherwise.} \end{cases}$$

The function $f_1(x)$ captures irrelevance of good 1 at bundle x. Similarly, the function $f_2(x)$ captures irrelevance of good 2 at bundle x. Observation 2 shows that if good i is irrelevant at $x = (x_1, x_2)$, then it continues to remain so for all bundles where quantity of good j is increased keeping x_i unchanged. This is immediate. Observation 2 also shows that the converse is true which is proved using continuity of the preference.

Observation 2 (i) $f_2(x) = 0 \Leftrightarrow f_2(x_1, y_2) = 0$ *for all* $y_2 > x_2$ *and* (ii) $f_1(x) = 0 \Leftrightarrow f_1(y_1, x_2) = 0$ *for all* $y_1 > x_1$.

Let $i, j \in \{1, 2\}$ and $i \neq j$. Let $A_i \subseteq X_i$ be the set of all elements x_i for which there exists a bundle involving x_i at which good j is irrelevant, that is, $A_i := \{x_i \in X_i | f_j(x) = 0 \text{ for some } x_j \in X_j\}$. Let $B_i (\subseteq X)$ be the set of all bundles at which good j is irrelevant, that is, $B_i := \{x \in X | f_j(x) = 0\}$. We conclude from Observation 2 that for every $x_i \in A_i$, $\exists\ \alpha_i(x_i) \in X_j = \mathbb{R}_+$ such that

$$f_j(x) = \begin{cases} 0 \text{ if } x_j \geq \alpha_i(x_i), \\ 1 \text{ otherwise.} \end{cases} \tag{1}$$

It follows from (1) that $B_i = \{x \in X | x_i \in A_i, x_j \geq \alpha_i(x_i)\}$. For $x_i \in A_i$, let $B_i(x_i)$ be the set of all bundles involving x_i at which good j is irrelevant, that is, $B_i(x_i) := \{y \in X | y_i = x_i, y_j \geq \alpha_i(x_i)\}$. It is immediate that $B_i = \cup_{x_i \in A_i} B_i(x_i)$. For any $x_i \in X_i$, define the set of all bundles involving x_i as $M_i(x_i) := \{y \in X | y_i = x_i, y_j \in X_j\}$. Observe that for any $x_i \in A_i$, the set of bundles $B_i(x_i) \subseteq M_i(x_i)$. Moreover $B_i(x_i) = M_i(x_i)$ if and only if $\alpha_i(x_i) = 0$. The last equality implies that good j is irrelevant at *all* bundles involving x_i.

Consider any two arbitrary bundles at both of which good j is irrelevant. The first part of Observation 3 shows that the preference ordering of these two bundles is completely determined by amounts of good i. The second part shows that if for any $y_i < x_i$, good j is irrelevant at all bundles involving y_i, then good j is also irrelevant at all bundles involving x_i.

Observation 3

(i) *Let $x_i, y_i \in A_i$ and $y_i < x_i$. Then $x \succ y$ for any $x \in B_i(x_i)$ and $y \in B_i(y_i)$.*

(ii) *Let $x_i > 0$. If $B_i(y_i) = M_i(y_i)$ for all $y_i \in [0, x_i)$, then $x_i \in A_i$ and $B_i(x_i) = M_i(x_i)$.*

3 Axiomatization of SII preferences

We begin with some definitions which will be useful for our axiomatizations.

Definition 3 A set $K \subseteq X$ is an *unhappy set* (or a *U-set*) if for any $y \notin K$, $y \succ x$ for every $x \in K$.

Definition 4 B_i has (i) *property P_U* if it has a non-empty subset of positive area which is a U-set and (ii) *property $\overline{P}_U$* if it does not have property P_U.

Definition 5 For $i = 1, 2$, a set $S \subseteq B_i$ is a *maximal U-subset* of B_i if (a) S is a U-set and (b) $\nexists\, T \subseteq B_i$ such that T is a U-set and $S \subset T$.

We characterize SII preferences using unhappy set and property P_U. Axiom 1 requires that the set of bundles at which good 2 is irrelevant satisfies property P_U and there exists at least one bundle where good 1 is irrelevant. Theorem 1 shows that this requirement uniquely characterizes SII preferences.

Axiom 1 *B_1 has property P_U and B_2 is non-empty.*

Theorem 1 *The following statements are equivalent.*

(SII1) *The preference relation $\succsim$ on X satisfies Axiom 1.*

(SII2) *The preference relation $\succsim$ on X is a SII preference with respect to X_1.*

Proof of Theorem 1

Lemma 1 will be used to prove Theorem 1. Part (I) of Lemma 1 shows that if for some $x_1 > 0$, good 2 is irrelevant at all bundles involving any $y_1 \in [0, x_1]$ then B_1 satisfies property P_U. Part (II) shows that the converse is also true. Moreover, if B_1 satisfies P_U, then it has a unique maximal U-subset $\overline{S}$ which has the property that if $x = (x_1, x_2) \in \overline{S}$, then $(x_1, 0) \in \overline{S}$ and consequently good 2 is irrelevant at all bundles involving x_1. Part (III) shows that if B_1 satisfies property P_U, then B_2 cannot.

Given Axiom 1, an immediate consequence of Lemma 1(I) is that the set $T = \{y \in X \mid y_1 \in [0, x_1]\} \subseteq B_1$ is a U-set and the indifference curves in T are all parallel to the X_2 axis.

Lemma 1 (I) *If $x_1 > 0$, $[0, x_1] \subseteq A_1$ and $B_1(y_1) = M_1(y_1)$ for all $y_1 \in [0, x_1]$, then B_1 has property P_U.*

(II) *Suppose B_1 has property P_U.*

(i) *Let $S \subseteq B_1$ be a U-set. If $x \in S$, then $\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$ and $\cup_{y_1 \in [0,x_1]} B_1(y_1) = \cup_{y_1 \in [0,x_1]} M_1(y_1) \subseteq S$.*

(ii) *B_1 has a unique maximal U-subset $\overline{S}$, which has the following properties: Either* (a) *$\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ or* (b) *$\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ for some $\overline{x}_1 \in (0, \infty)$, or* (c) *$\overline{S} = \cup_{y_1 \in \mathbb{R}_+} M_1(y_1) = \mathbb{R}^2_+$.*

(iii) *Suppose* (a) *or* (b) *of* (ii) *holds. Then for every $x_1 > \overline{x}_1$, $\exists\ y_1 \in (\overline{x}_1, x_1)$ such that either $y_1 \notin A_1$, or $y_1 \in A_1$ and $\alpha_1(y_1) > 0$.*

(III) *If B_1 has property P_U, then B_2 must have property $\overline{P}_U$.*

Proof of Theorem 1: We first prove (SII1)⇒ (SII2).

Proof of subsistence property: Since B_1 has property P_U, by Lemma 1(II)(ii), B_1 has a unique maximal U-subset $\overline{S}$.

Now we show that $\overline{S} \neq \mathbb{R}^2_+$. To see this, first note that since B_1 has property P_U, by Lemma 1(III), B_2 has property $\overline{P}_U$. Moreover, by Axiom 1, B_2 is non-empty and so is A_2. Let $x_2 \in A_2$, $y_1 > x_1 \geq \alpha_2(x_2)$ and $y_2 = x_2$. Then $x, y \in B_2(x_2)$, so that $x \sim y$. If $\overline{S} = \mathbb{R}^2_+$, then $x, y \in \overline{S} \subseteq B_1$. As $x \in M_1(x_1) = B_1(x_1)$, $y \in M_1(y_1) = B_1(y_1)$ and $y_1 > x_1$, by Obs. 3(i) we have $y \succ x$, a contradiction. So we must have $\overline{S} \neq \mathbb{R}^2_+$.

From the preceding paragraph and by Lemma 1(II)(ii) we conclude that either $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ or $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ for some $\overline{x}_1 \in (0, \infty)$. In either case, by Obs. 3(ii) we have $\alpha_1(y_1) = 0$ for all $y_1 \in [0, \overline{x}_1]$. Taking $\underline{Q} = \overline{x}_1$ proves part (a) of the *subsistence* property. Part (I)(b) of SII preference with respect to good 1 follows from Lemma 1(II)(iii).

Proof of weak saturation property: Since B_2 is non-empty, $\exists\ x_2 \in X_2$ and $\alpha_2(x_2) \geq 0$ such that good 1 is relevant at x if $x_1 < \alpha_2(x_2)$ and it is irrelevant at x if $x_1 \geq \alpha_2(x_2)$. Taking $\overline{Q} = \alpha_2(x_2)$ proves the *weak saturation* property. From continuity and monotonicity of preference it also follows that $\underline{Q} = \overline{x}_1 \leq \overline{Q} = \alpha_2(x_2)$ and the inequality is strict if $x_2 > 0$.

We now prove (SII2)⇒(SII1). We consider the SII preference with respect to good 1 and show that it satisfies Axiom 1. Observe from the subsistence property that $[0, \underline{Q}] \subseteq A_1$ and $B_1(x_1) = M_1(x_1)$ for all $x_1 \in [0, \underline{Q}]$. Then by Lemma 1(I), it follows that B_1 has property P_U. Next observe from the weak saturation property that $\{x | x_1 \geq \overline{Q}\} \subseteq B_2$ so that B_2 is non-empty. This proves that Axiom 1 holds. ■

3.1 Robustness of the axiom

Axiom 1 requires that the set B_2 (for which P_U does not hold) must be non-empty. This is useful not only to generate weak saturation, but it is also necessary for the existence of a non-subsistence zone. Without it, a non-subsistence zone might not exist. Without a reference to a situation of non-subsistence, the notion of subsistence may not be meaningful. To see this, consider the following variation of Axiom 1.

Axiom 1A B_1 has property P_U.

Corollary 1 *The following statements are equivalent.*

(S1) *The preference relation* $\succsim$ *on* X *satisfies Axiom* 1A*.*

(S2) *For the preference relation* $\succsim$ *on* X*, either property (I) of Definition 2 holds, or good* 2 *is irrelevant at any bundle* x*.*

Proof: We first prove (S1)⇒ (S2). Since B_1 has property P_U (Axiom 1A), by Lemma 1(II)(ii), B_1 has a unique maximal U-subset $\overline{S}$. If either (a) or (b) of Lemma 1(II)(ii) holds, then property (I) of Definition 2 holds. So suppose (c) of Lemma 1(II)(ii) holds, i.e., $\overline{S} = \mathbb{R}^2_+$. Then $A_1 = \mathbb{R}_+$ and $\alpha_1(x_1) = 0$ for all $x_1 \in \mathbb{R}_+$, implying that good 2 is irrelevant at any bundle x.

To prove (S1)⇒ (S2), if property (I) of Definition 2 holds, then from the proof of Theorem 1 it follows that B_1 has property P_U. Otherwise, $B_1 = \mathbb{R}^2_+$, which is itself an U-set of positive area. So property P_U holds for B_1. ■

Recall that in property (I) of Definition 2, the subsistence zone is $[0, \underline{Q}]$ for $0 < \underline{Q} < \infty$, which results in a non-subsistence zone $(\underline{Q}, \infty)$. The preference in $\overline{\text{S2}}$ of Corollary 1 includes the case where $\underline{Q} = \infty$, in which case there is no non-subsistence zone with respect to good 1, rendering the other good 2 to be irrelevant at any bundle.

4 Generalized Leontief preferences

Definition 6 The preference relation $\succsim$ on X is a *generalized Leontief preference* (or a GL preference) if there exists an onto (surjective)[5] and increasing function $F : X_1 \to X_2$ with $F(0) = 0$ such that for any $x_1 \in X_1$:

(i) at any bundle $(x_1, F(x_1))$, both goods X_1 and X_2 are irrelevant,

(ii) good 1 is relevant at any bundle $(y_1, F(x_1))$ for $y_1 < x_1$, and

(iii) good 2 is relevant at any bundle (x_1, y_2) for $y_2 < F(x_1)$.

Observe that since F is onto and increasing, it is also one-to-one and continuous. The domain of the inverse function of F is X_2. For the standard Leontief preference $F(x_1)$ is a linear function.

Axiom 2 $B_1 \cup B_2 = X$.

Definition 7 B_i has (i) *property* P_G if $A_i = X_i = \mathbb{R}_+$.

[5]A function $F : X_1 \to X_2$ is an *onto* or a *surjective* function if for any $x_2 \in X_2$, $\exists\ x_1 \in X_1$ such that $F(x_1) = x_2$.

Observe that if B_i has property P_G, then for all $x_i \in X_i$, the function $\alpha_i(x_i)$ is well defined, i.e., $\exists$ a function $\alpha_i(x_i)$ such that $B_i(x_i) = \{y \in X | y_i = x_i, y_j \geq \alpha_i(x_i)\}$.

Axiom 3 *For $i = 1, 2$, the set B_i has both properties $\overline{P}_U$ and P_G.*

Theorem 2 *The following statements are equivalent.*

(GL1) *The preference relation $\succsim$ on X satisfies Axiom 2 and Axiom 3.*

(GL2) *The preference relation $\succsim$ on X is a generalized Leontief preference.*

Proof of Theorem 2

To prove Theorem 2 we will use the following lemmas. Given Axiom 2, Lemma 2 shows that if a good is irrelevant (relevant) at a bundle and its amount is decreased (increased), then it continues to be irrelevant (relevant) at the new bundle.

Lemma 2 *Suppose $\succsim$ satisfies Axiom 2.*

(I) *Let $i, j \in \{1, 2\}$ and $i \neq j$. For any $x_i \in X_i$, $f_i(x)$ is non-decreasing in x_j.*

(II) *If $x_i \in A_i$, then $y_i \in A_i$ and $\alpha_i(y_i) \leq \alpha_i(x_i)$ for all $y_i \in [0, x_i)$.*

Since P_G holds for any B_i (by Axiom 3), $\alpha_i(.)$ is defined for any $x_i \in X_i$. Lemma 3 derives properties of this function and as a consequence we get the $F(.)$ function specified in the definition of GL preference.

Lemma 3 *Suppose the preference relation $\succsim$ on X satisfies Axiom 2 and Axiom 3. The following hold for $i, j \in \{1, 2\}, i \neq j$.*

(I) *$\alpha_i(x_i) > 0$ for any $x_i > 0$.*

(II) *$\alpha_i(0) = 0$.*

(III) *$\alpha_j(\alpha_i(x_i)) = x_i$.*

(IV) *$\alpha_i(x_i)$ is increasing for all $x_i \geq 0$.*

(V) *$\alpha_i(x_i)$ is an onto function from X_i to X_j, i.e., for every $x_j \in X_j$, $\exists$ $x_i \in X_i$ such that $\alpha_i(x_i) = x_j$.*

Proof of Theorem 2: $(L1) \Rightarrow (L2)$ By Axiom 3, for $i = 1, 2$, B_i has property P_G. Hence $A_i = X_i$ and $\alpha_i(x_i)$ is well defined for all $x_i \in X_i$. Note from Lemma 3 that $\alpha_1(.) : X_1 \to X_2$ is an increasing and onto function with $\alpha_1(0) = 0$ (the same property holds for $\alpha_2(.) : X_2 \to X_1$ and $\alpha_2(.)$ is the inverse function of $\alpha_1(.)$). Taking $F(x_1) = \alpha_1(x_1)$, by Lemma 3(III) it follows that (i)-(iii) of Definition 6 hold.

$(L2) \Rightarrow (L1)$ Suppose the preference is generalized Leontief. Then for $i = 1, 2$, $A_i = X_i = \mathbb{R}_+$, so property P_G holds. For any $x_1 \in X_1$, we have $\alpha_1(x_1) = F(x_1)$ and for any $x_2 \in A_2$, we have $\alpha_2(x_2) = F^{-1}(x_2)$, and $F(0) = 0$. Hence $B_1(x_1) = \{(x_1, x_2) | x_2 \geq$

$F(x)\}$ and $B_2(x_2) = \{(x_1, x_2)|x_1 \geq F^{-1}(x_2)\}$. So we have $B_i = \cup_{x_i \in \mathbb{R}_+} B_i(x_i)$ for $i = 1, 2$, and $B_1 \cup B_2 = X$, so Axiom 2 holds.

It remains to show that B_i has property $\overline{P}_U$ for $i = 1, 2$. If for some $i = 1, 2$, $\exists$ $S \subseteq B_i$ such that S is a U-set of positive area, then $\exists$ $x \in S$ such that $x_i > 0$. By Lemma 1(II)(i), this will imply that $\alpha_i(x_i) = 0$ for all $y_i \in [0, x_i]$, a contradiction since $\alpha_i(y_i) > 0$ for all $y_i > 0$. This shows that B_i has property $\overline{P}_U$ for $i = 1, 2$. Since B_i also has property P_G for $i = 1, 2$, we conclude that Axiom 3 holds. ■

4.1 Robustness of axioms

Axiom 2 and Axiom 3 have three requirements: (i) $B_1 \cup B_2 = X$, (ii) B_1, B_2 both have property P_G and (iii) B_1, B_2 both have property $\overline{P}_U$. In each of the following examples, only one of requirements (i)-(iii) is violated, and we see that we do not get the generalized Leontief preference. These examples show that without all of these three conditions, we are not guaranteed to get a Leontief preference.

Example 1 Consider the preference represented by utility function u where $K > 0$.

$$u(x_1, x_2) = \begin{cases} \min\{x_1/(K - x_1), x_2\} & \text{if } x_1 < K, \\ x_2 & \text{if } x_1 \geq K \end{cases}$$

Some indifference curves of this preference are drawn in Figure 1. For this example, $B_1 \cup B_2 = X$, so Axiom 2 holds. But Axiom 3 does not hold. This is because $A_1 = [0, K)$ and $A_2 = \mathbb{R}_+$, so B_1 does not have property P_G (although both B_1, B_2 have property $\overline{P}_U$). For this example, $\alpha_1(.) : [0, K) \to \mathbb{R}_+$ is defined as $\alpha_1(x_1) = x_1/(K - x_1)$ and $\alpha_2(.) : \mathbb{R}_+ \to \mathbb{R}_+$ as $\alpha_2(x_2) = Kx_2/(1+x_2)$. Note that α_1 is not an onto function, so we do not get a generalized Leontief preference. We get "locally Leontief" (for $x_1 < K$) and saturation at $x_1 = K$.

Example 2 Consider the preference represented by utility function u where $K > 0$.

$$u(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 \leq K, \\ \min\{x_1 - K, x_2\} & \text{if } x_1 > K \end{cases}$$

Some indifference curves of this preference are drawn in Figure 2. For this example, $B_1 \cup B_2 = X$, so Axiom 2 holds. Moreover, $A_i = \mathbb{R}_+$ for $i = 1, 2$, so both B_1, B_2 have property P_G. However, Axiom 3 does not hold since B_1 does not have property $\overline{P}_U$. The set $\{(x_1, x_2)|x_1 \in [0, K], x_2 \in \mathbb{R}_+\} \subset B_1$ is a U-set of positive area. For this example, $\alpha_i(.) : \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha_1(x_1) = \max\{x_1 - K, 0\}$, $\alpha_2(x_2) = K + x_2$ for $i = 1, 2$. We do not get a generalized Leontief preference. We get "locally Leontief" (for $x_1 > K$) and subsistence for $x_1 \leq K$.

Example 3 Consider the preference represented by the utility function

$$u(x_1, x_2) = \begin{cases} x_2 & \text{if } x_2 \leq x_1/2, \\ (x_1 + x_2)/3 & \text{if } x_1/2 < x_2 < 2x_1, \\ x_1 & \text{if } x_2 \geq 2x_1 \end{cases}$$

Some indifference curves of this preference are drawn in Figure 3. For this example, B_i have properties $P_G, \overline{P}_U$ for $i = 1, 2$, so Axiom 3 holds. However, $B_1 \cup B_2 \neq X$, so Axiom 2 does not hold. For this example, $\alpha_i(.) : \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha_i(x_i) = 2x_i$ for $i = 1, 2$. Here also we do not get a generalized Leontief preference.

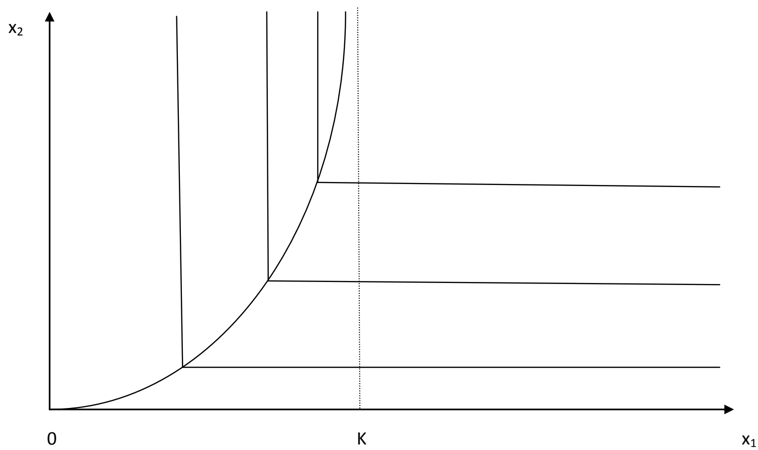

Figure 1

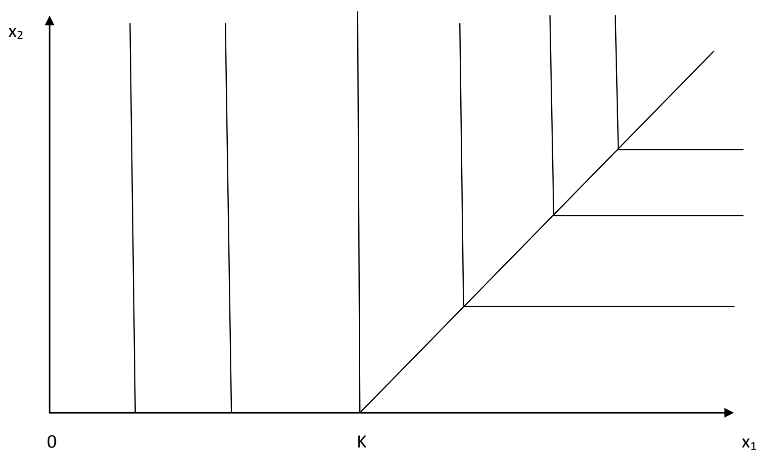

Figure 2

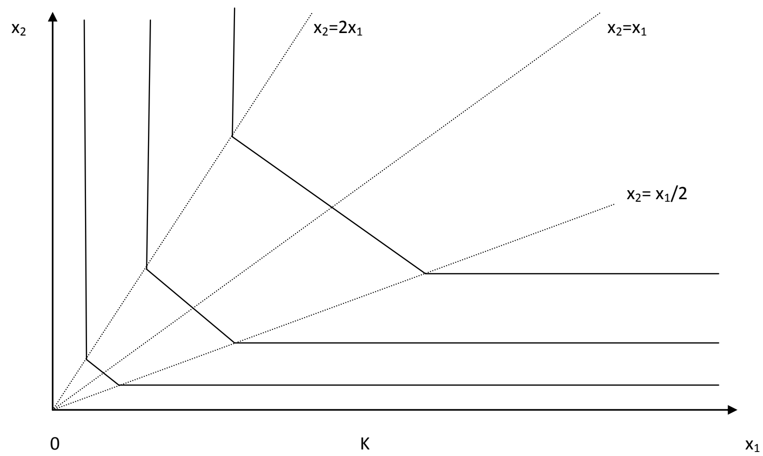

Figure 3

14

Appendix

Proof of Observation 1: For the first part suppose, on the contrary, $\underline{Q} > \overline{Q}$. Then $(\overline{Q}, x_2) \sim (\underline{Q}, x_2)$ (by (II)) and $(\underline{Q}, x_2) \sim (\underline{Q}, y_2)$ for any $y_2 > x_2$ (by (I)(a)), implying $(\overline{Q}, x_2) \sim (\underline{Q}, y_2)$ for any $y_2 > x_2$ which violates monotonicity. So we must have $\underline{Q} \leq \overline{Q}$. Let $x_2 > 0$. If $\overline{Q} = \underline{Q} = Q$, then $(Q, 0) \sim (Q, x_2)$ (by (I)(a)) and $(Q, x_2) \sim x$ for all $x_1 > Q$ (by (II)), implying $(Q, 0) \sim x$ for all $x_1 > Q$ which violates monotonicity. So we must have $\underline{Q} < \overline{Q}$ if $x_2 > 0$.

For the second part note that by (II) $\exists\, x_2$ and $\overline{Q}$ such that at x, good 1 is irrelevant if $x_2 \geq \overline{Q}$ and relevant if $x_1 < \overline{Q}$. Hence $x_2 \in A_2$ and $\alpha_2(x_2) = \overline{Q}$. We show that $(\overline{Q}, \infty)$ is the strong non-subsistence zone with respect to good X_1. For this we have to show that for any $x_1 > \overline{Q}$, either (a) $x_1 \notin A_1$ or (b) $x_1 \in A_1$ and $\alpha_1(x_1) > 0$.

Suppose, on the contrary, $\exists\, x_1 > \overline{Q}$ such that neither (a) nor (b) holds, i.e., $x_1 \in A_1$ and $\alpha_1(x_1) = 0$. Then for any $y_2 > x_2$ we have $x \sim (x_1, y_2)$. But since $x_1 > \overline{Q}$, we have $x, (\overline{Q}, x_2) \in B_2(x_2)$, hence $x \sim (\overline{Q}, x_2)$. By transitivity, $(x_1, y_2) \sim (\overline{Q}, x_2)$, which violates monotonicity, a contradiction. ∎

Proof of Observation 2: We prove (i), proof of (ii) is similar. Let $f_2(x) = 0$. Then $x \sim (x_1, y_2)$ for $y_2 > x_2$. Hence $(x_1, y_2) \sim (x_1, z_2)$ for any $z_2 > y_2 > x_2$, implying that $f_2(x_1, y_2) = 0$.

Conversely, let $f_2(x_1, y_2) = 0$ for all $y_2 > x_2$. Then $(x_1, y_2) \sim (x_1, z_2)$ for all $z_2 > y_2 > x_2$. Let $x^n = (x_1, x_2 + 1/n)$ and $y^n = (x_1, y_2 + 1/n)$ for $n = 1, 2, \ldots$. Then $x^n \sim y^n$, and hence $x^n \succsim y^n$ for $n = 1, 2, \ldots$. Since $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = (x_1, y_2)$, by continuity we have $x \succsim (x_1, y_2)$. Since $y_2 > x_2$, by Observation 1 we have $(x_1, y_2) \succsim x$. We then conclude that $x \sim (x_1, y_2)$ for any $y_2 > x_2$, proving that $f_2(x) = 0$. ∎

Proof of Observation 3: (i) Let $y \in B_1(y_1)$. Consider any $z_2 > \max\{y_2, \alpha_1(x_1)\}$. Then $(x_1, z_2) \in B_1(x_1)$. Since $x_1 > y_1$ and $z_2 > y_2$, by monotonicity $(x_1, z_2) \succ y$. Since $(x_1, z_2) \sim x$ for any $x \in B_i(x_i)$ the result follows from transitivity.

(ii) Consider any $x_2 > 0$ and two sequences $x^n = (x_1 - 1/n, x_2)$, $y^n = (x_1 - 1/n, 0)$ for $n > 1/x_1$. Since $y_i \in A_i$ and $\alpha_i(y_i) = 0$ for $y_i \in [0, x_i)$, we have $x^n, y^n \in M_1(x_1 - 1/n) = B_1(x_1 - 1/n)$. Hence $x^n \sim y^n$ and in particular, $y^n \succsim x^n$. Since $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = (x_1, 0)$, by continuity we have $(x_1, 0) \succsim x$. Since $x_2 > 0$, by Obs. 1 we have $x \succsim (x_1, 0)$, implying that $x \sim (x_1, 0)$ for any $x_2 > 0$. This proves the result. ∎

Proof of Lemma 1: (I) Let $y_1 \in [0, x_1]$. As $\alpha_1(y_1) = 0$, we have $B_1(y_1) = M_1(y_1)$. Let $T := \cup_{y_1 \in [0, x_1)} B_1(y_1) = \cup_{y_1 \in [0, x_1)} M_1(y_1) \subseteq B_1$.

To prove that T is a U-set, first we show that $x \succ y$ for any $y \in T$. Observe that $x \in M_1(x_1) = B_1(x_1)$. Let $y \in T$. Then $y \in M_1(y_1) = B_1(y_1)$ for some $y_1 < x_1$. By Obs. 3(i), we conclude that $x \succ y$.

To complete the proof we show that $z \succ y$ for any z such that $z_1 > x_1$. Continuity and monotonicity of preference imply that $z \succsim (x_1, 0)$ for any such z. From the preceding paragraph, we have $(x_1, 0) \succ y$ for any $y \in T$. By transitivity, $z \succ y$ for any $y \in T$. This proves that T is an U-set. As $x_1 > 0$, the area of T is positive. So B_1 has property P_U.

(II) If B_1 has property P_U, then $\exists\, S \subseteq B_1$ such that S is a U-set with a positive area. Since S has positive area, $\exists\, x \in S$ where $x_1 > 0$.

(i) Consider any $x \in S$. Since $y \sim x$ for all $y \in B_1(x_1)$ and S is a U-set, we must have $B_1(x_1) \subseteq S$.

Next observe that if $\alpha_1(x_1) > 0$ for some $x \in S$, we can find y such that $y_1 = x_1$ and $y_2 \in [0, \alpha_1(x_1))$. Then $y \notin B_1$, so we have $y \notin S$. But $x \succsim y$ (by continuity and monotonicity of $\succsim$), which contradicts that S is a U-set. Hence for any $x \in S$, we must have $\alpha_1(x_1) = 0$, implying that $B_1(x_1) = M_1(x_1) \subseteq S$.

Now we show that if $x \in S$, then $y \in S$ for any y such that $y_1 < x_1$. To see this, consider z such that $z_1 = x_1$ and $z_2 > y_2$. Since $B_1(x_1) = M_1(x_1) \subseteq S$, we have $z \in S$. By monotonicity, $z \succ y$. As S is a U-set, we must have $y \in S$.

From the preceding paragraphs we conclude that if $x \in S$, then $\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$ and $\cup_{y_1 \in [0,x_1]} B_1(y_1) = \cup_{y_1 \in [0,x_1]} M_1(y_1) \subseteq S$. This proves (i).

(ii) First observe that if S, T are two subsets of B_1 that are both U-sets, then either $S \subseteq T$ or $T \subseteq S$. If neither holds, then $\exists\ x \in S, y \in T$ such that $x \notin S, y \notin T$. If $x_1 = y_1$, then $y \in M_1(x_1) \subseteq S$, a contradiction. So $x_1 \neq y_1$. W.l.o.g., let $y_1 < x_1$. But then from the last paragraph, we have $y \in M_1(y_1) \subseteq S$, again a contradiction.

Therefore, if B_1 has property P_U, then it has a unique maximal U-subset $\overline{S}$ and this set has positive area. From part (i) we conclude that either $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ or $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ for some $0 < \overline{x}_1 < \infty$, or $\overline{S} = \cup_{y_1 \in \mathbb{R}_+} M_1(y_1) = \mathbb{R}^2_+$.

(iii) If (a) or (b) of (ii) holds, then $y_1 \in A_1$ and $\alpha_1(y_1) = 0$ for all $y_1 \in [0, \overline{x}_1]$ (for (b), the result for $y_1 = \overline{x}_1$ follows from Obs. 3(ii)). Suppose, on the contrary $\exists$ $x_1 > \overline{x}_1$ where the assertion (iii) does not hold. Then for every $y_1 \in (\overline{x}_1, x_1)$, we have $y_1 \in A_1$ and $\alpha_1(y_1) = 0$, so that $B_1(y_1) = M_1(y_1)$. Let $\widetilde{S}^* := \cup_{y_1 \in [0,x_1)} M_1(y_1)$. Then $\overline{S} \subset \widetilde{S}^* \subseteq B_1$. By part (I), $\widetilde{S}^*$ is a U-set, which contradicts (II)(ii).

(III) Suppose on the contrary both B_1, B_2 have property P_U. Then by part (II), for $i = 1, 2$, $\exists\ x_i > 0$ such that $x_i \in A_i$ and $\alpha_i(x_i) = 0$. Then $(x_1, 0) \sim x$ (since $\alpha_1(x_1) = 0$) and $(0, x_2) \sim x \sim (y_1, x_2)$ for any $y_1 > x_1$ (since $\alpha_2(x_2) = 0$). This implies $(x_1, 0) \sim (y_1, x_2)$. But since $y_1 > x_1$ and $x_2 > 0$, by monotonicity we must have $(y_1, x_2) \succ (x_1, 0)$, a contradiction. This proves (III). ∎

Proof of Lemma 2: W.l.o.g. take $i = 1$ and $j = 2$.

(I) We have to show that $f_2(y_1, x_2) \leq f_2(x)$ for all $y_1 < x_1$ and $f_2(y_1, x_2) \geq f_2(x)$ for all $y_1 > x_1$. Since $f_2(.)$ equals 0 or 1, it is sufficient to show: (a) if $f_2(x) = 0$, then $f_2(y_1, x_2) = 0$ for all $y_1 < x_1$ and (b) if $f_2(x) = 1$, then $f_2(y_1, x_2) = 1$ for all $y_1 > x_1$. If (a) does not hold, then $\exists$ x and $y_1 < x_1$ such that $f_2(x) = 0$ and $f_2(y_1, x_2) = 1$, i.e., $(y_1, x_2) \notin B_1$. By Axiom 2, we must have $(y_1, x_2) \in B_2$, so that $\alpha_2(x_2) \leq y_1 < x_1$. Hence $(y_1, x_2), x \in B_2(x_2)$, implying $(y_1, x_2) \sim x$. Since $f_2(x) = 0$, we have $x \sim (x_1, z_2)$ for any $z_2 > x_2$. By transitivity, $(y_1, x_2) \sim (x_1, z_2)$ which violates monotonicity, so (a) must hold. If (b) does not hold, then $\exists$ z and $\widetilde{z}_1 > z_1$ such that $f_2(z) = 1$ and $f_2(\widetilde{z}_1, z_2) = 0$. Taking $x_1 = \widetilde{z}_1$, $x_2 = z_2$ and $y_1 = z_1$ contradicts (a). Hence (b) must hold.

(II) If $x_1 \in A_1$, then $\exists\ \alpha_1(x_1) = x_2$ such that $f_2(x_1, y_2) = 0\ \forall\ y_2 \geq x_2$. By Lemma 2 (I), for any $y_1 \in [0, x_1)$, we have $f_2(y_1, x_2) = 0\ \forall\ y_2 \geq x_2$. By definition of $\alpha_1(.)$, we have $\alpha_1(y_1) \leq x_2 = \alpha_1(x_1)$ for all $y_1 \in [0, x_1)$. ∎

Proof of Lemma 3: W.l.o.g., take $i = 1, j = 2$.

(I) Suppose on the contrary $\alpha_1(x_1) = 0$ for some $x_1 > 0$. Then by Lemma 2(II),

$\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$. Then by Lemma 1(I), B_1 has property P_U, contradicting Axiom 3.
(II) Suppose on the contrary $\alpha_1(0) = x_2 > 0$. Let $y_2 \in (0, x_2)$. Then $(0, y_2) \notin B_1$ (since $y_2 < \alpha_1(0)$) and $(0, y_2) \notin B_2$ (since $0 < \alpha_2(y_2)$, part (I)), i.e., $y_2 \notin B_1 \cup B_2$, which contradicts Axiom 2.
(III) By (II), the result clearly hold for $x_1 = 0$, so let $x_1 > 0$. Then $\alpha_1(x_1) > 0$ (by (I)). Let $x_2 \in [0, \alpha_1(x_1))$. Then $x \notin B_1$, so by Axiom 2 we must have $x \in B_2$, implying that $\alpha_2(x_2) \leq x_1$ for all $x_2 \in [0, \alpha_1(x_1))$. By continuity,[6] we have $\alpha_2(\alpha_1(x_1)) \leq x_1$.

Denote $\alpha_1(x_1) = y_2$ and $\alpha_2(y_2) = y_1$. If $y_1 < x_1$, then $y, (x_1, y_2) \in B_2(y_2)$, so that $y \sim (x_1, y_2)$. Let $z_2 > y_2 = \alpha_1(x_1)$. Then $(x_1, z_2), (x_1, y_2) \in B_1(x_1)$, implying $(x_1, z_2) \sim (x_1, y_2)$. By transitivity, $y \sim (x_1, z_2)$, a contradiction (since $x_1 > y_1$ and $z_2 > x_2$). Hence we must have $y_1 \geq x_1$, i.e., $\alpha_2(\alpha_1(x_1)) \geq x_1$. From the conclusion of the previous paragraph, we conclude that $\alpha_2(\alpha_1(x_1)) = x_1$.
(IV) Since $\alpha_1(0) = 0$ and $\alpha_1(x_1) > 0$ for any $x_1 > 0$, $\alpha_1(x_1)$ is increasing at $x_1 = 0$. By Lemma 2(II), $\alpha_1(x_1)$ is non-decreasing. If it is not increasing for all $x_1 > 0$, $\exists$ $x_1 > y_1 > 0$ such that $\alpha_1(x_1) = \alpha_1(y_1) = x_2 > 0$. By part (III), we then have $\alpha_2(x_2) = \alpha_2(\alpha_1(x_1)) = x_1$ and $\alpha_2(x_2) = \alpha_2(\alpha_1(y_1)) = y_1 < x_1$, a contradiction.
(V) By (II), the result holds for $x_2 = 0$. Suppose $\exists$ $x_2 > 0$ such that $\alpha_1(x_1) \neq x_2$ $\forall$ $x_1 \in X_1$. Since $\alpha_1(.)$ is continuous and $\alpha_1(0) = 0$, we must have $\alpha_1(x_1) < x_2$ for all $x_1 \in X_1$. By Axiom 3, B_2 has property P_G. Hence $x_2 \in A_2$ and $\alpha_2(x_2)$ is well defined. Taking $x_1 = \alpha_2(x_2)$ above, we have $\alpha_1(\alpha_2(x_2)) < x_2$, which contradicts (III). ■

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