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Abstract

We study what topological assumptions should be added to the acyclicity of individual best response improvements in order to ensure the existence of a (pure strategy) Nash equilibrium, as well as the possibility to reach a Nash equilibrium in the limit of a best response improvement path.

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1 Introduction

Cournot tâtonnement is the oldest and one of the most natural dynamic scenarios of individual myopic adaptation in strategic games. It has been studied in various contexts and from various viewpoints, see, e.g., Topkis (1979), Bernheim (1984), Moulin (1984), Vives (1990), Milgrom and Roberts (1990), Kandori and Rob (1995), and Milchtaich (1996).

The introduction of the concept of a potential game by Monderer and Shapley (1996) stimulated studies of similarities and dissimilarities between better and best response dynamics. Since Monderer and Shapley paid most attention to the cardinal concept of an exact potential, they defined every kind of a potential as a *real-valued* function. When Voorneveld (2000) introduced a "best-response potential," he followed their lead. For a finite game, the restriction to numeric potentials is innocuous; in the general case, it is not so. And yet, nobody has demonstrated so far that the possibility of a numeric representation has anything to do with improvement dynamics.

Voorneveld's definition is over-exacting in another respect too, viz. it followed Monderer and Shapley's concept of an ordinal potential rather than a generalized ordinal one. Meanwhile, Monderer and Shapley (1996, Lemma 2.5) showed that it is the latter kind of a potential that is most relevant to the convergence of adaptive dynamics. Kukushkin (2004, Section 6) defined a "Cournot potential" as a partial order on the set of strategy profiles with respect to which every best response improvement by a single player pushes the current strategy profile upwards. In a finite game, the existence of such an order is equivalent to the "finite best response property" (Milchtaich, 1996). Naturally, Voorneveld's potential always defines a Cournot potential; the converse is generally wrong, even in a finite game (Monderer and Shapley, 1996, the example on p. 129).

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When attention is turned to infinite games, the acyclicity of (either better or best response) improvements does not imply even the existence of a Nash equilibrium, to say nothing of the convergence of adaptive dynamics. Nonetheless, the main theorem of Kukushkin (2011) showed that in a game with compact strategy sets and "continuous enough" preferences, the acyclicity of *all* individual improvements ensures the existence of a (pure strategy) Nash equilibrium and the possibility to reach it (perhaps, approximately) with a unilateral improvement path. The acyclicity of only the *best response* improvements, however, does not ensure even the mere existence of a Nash equilibrium, even in a compact-continuous two-person game (Kukushkin, 2011, Example 1).

In this paper, we assume that the strategy sets are compact metric spaces, and study what topological conditions should be added to the definition of a Cournot potential in order to ensure the existence of a (pure strategy) Nash equilibrium or, additionally, the possibility to reach a Nash equilibrium in the limit (or as a cluster point) of a best response improvement path. Roughly speaking, we consider two such additional requirements: ω-transitivity, and ω-transitivity plus lower semicontinuity. The first ensures the existence of an equilibrium (as well as "transfinite convergence" to equilibria of all best response improvement paths). The second, the possibility to reach the set of Nash equilibria in the limit of a best response improvement path – an infinitary version of the weak FBRP (Milchtaich, 1996) – and convergence to the set of Nash equilibria of all best response improvement paths in the case of two players.

We do not require the utility functions to be continuous, only assume that the best responses exist everywhere; a well-known sufficient condition for this is the upper semicontinuity of each utility function in own choice. Quite often, the upper hemicontinuity of the best response correspondences also helps; a sufficient condition for that is the upper semicontinuity of each utility function in the total strategy profile and continuity in the choices of others.

While the acyclicity of best response improvements can be shown by *reductio ad absurdum*, as in Theorem 2 of Kandori and Rob (1995) or Theorem 1 of Kukushkin (2004), it is difficult to imagine how the existence of, say, a continuous Cournot potential could be established without producing one explicitly. Fortunately, there are natural classes of strategic games where this is possible; two of them are briefly described in this paper. In the first example, "games with structured utilities," there is even an exact (at least, an ordinal) potential as defined by Monderer and Shapley (1996), hence the behavior of *all* individual improvement paths is rather regular. In the second, "aggregative games," arbitrary improvements may lead nowhere. Sequential Cournot tâtonnement in such games was considered by Jensen (2010), but his results are not directly comparable to ours, see Section 9.5.

In Section 2, the basic definitions are given. Section 3 contains the main "positive" results; Section 4, additional "positive" results which assume the uniqueness of the best responses. In Section 5, we introduce two weaker notions of a potential, which broaden the scope of applications. Sections 6 and 7 present known classes of games where the assumptions of (some of) our theorems are satisfied. Section 8 contains "negative" results, showing the impossibility of easy generalizations. A discussion of various related questions in Section 9 concludes the paper.

2 Preliminaries

Our basic model is a *strategic game with ordinal utilities*. It is defined by a finite set of players N, and strategy sets X_i and ordinal utility functions $u_i\colon X_N \to \mathbb{R}$, where $X_N = \prod_{i\in N} X_i$, for all $i \in N$. We denote $X_{-i} = \prod_{j\in N\setminus\{i\}} X_j$ for each $i \in N$. Given a strategy profile $x_N \in X_N$ and $i \in N$, we denote x_i and x_{-i} its projections to X_i and X_{-i}, respectively; a pair (x_i, x_{-i}) uniquely determines x_N. In the case of $\#N = 2$, we denote $-i$ the partner/rival of player i.

Defining the *best response correspondence* $\mathcal{R}_i\colon X_{-i} \to 2^{X_i}$ for each $i \in N$ in the usual way,

$$\mathcal{R}_i(x_{-i}) := \operatorname*{Argmax}_{x_i\in X_i} u_i(x_i, x_{-i})$$

for every $x_{-i} \in X_{-i}$, we introduce the *best response improvement* relation on X_N ($i \in N$, $y_N, x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{BR}} x_N \rightleftharpoons [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]; \tag{1a}$$

$$y_N \rhd^{\mathrm{BR}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{BR}} x_N]. \tag{1b}$$

Every Nash equilibrium is a maximizer of $\rhd^{\mathrm{BR}}$. If $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$, then every maximizer of $\rhd^{\mathrm{BR}}$ is a Nash equilibrium.

A *Cournot path* is a finite or infinite sequence $\langle x_N^k\rangle_{k=0,1,\ldots}$ such that $x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined. A *Cournot potential* is an irreflexive and transitive binary relation $\succ$ on X_N such that

$$\forall x_N, y_N \in X_N \left[y_N \rhd^{\mathrm{BR}} x_N \Rightarrow y_N \succ x_N\right]. \tag{2}$$

The existence of a Cournot potential is equivalent to the absence of *Cournot cycles*, i.e., Cournot paths $\langle x_N^0, x_N^1, \ldots, x_N^m\rangle$ such that $m > 0$ and $x_N^0 = x_N^m$. If the game is finite, this fact implies that every Cournot path, if continued whenever possible, reaches a Nash equilibrium in a finite number of steps. Example 1 from Kukushkin (2011) shows that a compact-continuous game may admit a Cournot potential and still possess no Nash equilibrium, to say nothing of the convergence of Cournot paths.

Henceforth, we assume that each X_i is a compact metric space and endow X_N with, say, the maximum metric. We do not impose any explicit continuity-style restriction on the utilities; all assumptions are formulated in terms of the best response correspondences. In particular, we assume throughout that $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for every $i \in N$ and $x_{-i} \in X_{-i}$. The upper semicontinuity of u_i in own choice x_i is sufficient for that though by no means necessary. In many results, we assume that each $\mathcal{R}_i$ is upper hemicontinuous. A sufficient condition for that is the upper semicontinuity of u_i in x_N and continuity in x_{-i}.

Let $\succ$ be a binary relation on a metric space X. An *improvement path* of $\succ$ is a (finite or infinite) sequence $\langle x^k\rangle_k$ in X such that $x^{k+1} \succ x^k$ whenever x^{k+1} is defined. Clearly, $\succ$ is a Cournot potential if and only if every Cournot path is an improvement path of $\succ$.

A binary relation $\succ$ on a metric space X is called *ω-transitive* if it is transitive and $x^\omega \succ x^0$ whenever an improvement path $\langle x^k\rangle_{k\in\mathbb{N}}$ of $\succ$ in X converges to $x^\omega \in X$. A relation $\succ$ is *upper semicontinuous* if all its lower contour sets, $\{x \in X \mid y \succ x\}$ ($y \in X$), are open; dually, $\succ$ is *lower semicontinuous* if all its upper contour sets, $\{y \in X \mid y \succ x\}$ ($x \in X$), are open. A relation $\succ$ is *continuous* if its "graph," $\{(x,y) \in X^2 \mid y \succ x\}$, is open. Clearly, every continuous relation is both upper and lower semicontinuous. An upper semicontinuous relation need not be ω-transitive, even if it is transitive; however, an upper semicontinuous Cournot potential can always be extended to an ω-transitive one.

It turns out that two properties of a Cournot potential are most relevant for the questions we address here: ω-transitivity and the conjunction of ω-transitivity and *lower* semicontinuity. These two simple statements are helpful in the following.

Lemma 2.1. *Let $\succ$ be an ω-transitive binary relation on a metric space X. Let $x^\omega \in X$ be a cluster point of an infinite improvement path $\langle x^k \rangle_{k\in\mathbb{N}}$ of $\succ$. Then $x^\omega \succ x^k$ for all $k \in \mathbb{N}$.*

Proof. By the transitivity of $\succ$, we have $x^h \succ x^k$ whenever $h > k$. Since x^ω is a cluster point, there is a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x^{k_h} \to x^\omega$. Since $\langle x^{k_h} \rangle_{h>\bar{h}}$ for each $\bar{h} \in \mathbb{N}$ is also an improvement path of $\succ$ converging to x^ω, we have $x^\omega \succ x^{k_h}$ for all $h \in \mathbb{N}$ by ω-transitivity. For every other $k \in \mathbb{N}$, there is $h \in \mathbb{N}$ such that $k_h > k$, and hence $x^\omega \succ x^{k_h} \succ x^k$. □

Lemma 2.2. *Let Γ be a strategic game where each X_i is a compact metric space. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Let $\langle x^k_N \rangle_{k\in\mathbb{N}}$ be an infinite Cournot path and $X^\omega \subseteq X_N$ be the set of its cluster points. Then*

$$\forall y^\omega_N, x^\omega_N \in X^\omega \left[y^\omega_N \not\succ x^\omega_N \right]. \tag{3}$$

Proof. By Lemma 2.1, we have $x^\omega_N \succ x^k_N$ for each $k \in \mathbb{N}$. If we supposed that $y^\omega_N \succ x^\omega_N$, we would have $x^k_N \succ x^\omega_N$ whenever x^k_N is close enough to y^ω_N by the lower semicontinuity, i.e., a contradiction. □

A (finite or infinite) sequence $\langle x^k_N \rangle_{k=0,1,\dots}$ in X_N *converges to a subset* $Y \subseteq X_N$ if either it is finite and ends at $x^m_N \in Y$ or it is infinite and all its cluster points belong to Y.

3 Main theorems

Theorem 3.1. *Let each X_i in a strategic game Γ be a compact metric space. Let Γ admit an ω-transitive Cournot potential. Then Γ possesses a (pure strategy) Nash equilibrium.*

Proof. By Theorem 1 from Kukushkin (2008), there exists a maximizer x^0_N of the potential $\succ$ on X_N. By (2), x^0_N is also a maximizer of $\rhd^{\mathrm{BR}}$ on X_N, i.e., a Nash equilibrium. □

Theorem 3.2. *Let Γ be a strategic game where $\#N = 2$, each X_i is a compact metric space, and each $\mathcal{R}_i$ is upper hemicontinuous. Let Γ admit an ω-transitive Cournot potential $\succ$. Let $\langle x^k_N \rangle_{k\in\mathbb{N}}$ be an infinite Cournot path. Then there is a Nash equilibrium among cluster points of the path.*

Proof. We denote $X^\omega \subseteq X_N$ the set of cluster points of $\langle x^k_N \rangle_{k\in\mathbb{N}}$ and pick a maximizer x^ω_N of $\succ$ on X^ω; it exists by Theorem 1 from Kukushkin (2008) since X^ω is compact. As in the proof of Lemma 2.1, we pick a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x^{k_h}_N \to x^\omega_N$. Since N is finite, we may, without restricting generality, assume that $x^{k_h}_i \in \mathcal{R}_i(x^{k_h}_{-i})$ for an $i \in N$ and all h. Then we denote $y^h_N := x^{k_h+1}_N$ ($h \in \mathbb{N}$); clearly, $y^h_N \rhd^{\mathrm{BR}}_{-i} x^{k_h}_N$, and hence $x^{k_h}_i = y^h_i$ and $y^h_{-i} \in \mathcal{R}_{-i}(x^{k_h}_i)$. Without restricting generality, $y^h_N \to y^\omega_N \in X^\omega$; hence $x^\omega_i = y^\omega_i$ and $y^\omega_{-i} \in \mathcal{R}_{-i}(x^\omega_i)$ since $\mathcal{R}_{-i}$ is upper hemicontinuous. Now an assumption that $x^\omega_{-i} \notin \mathcal{R}_{-i}(x^\omega_i)$ would lead to $y^\omega_N \rhd^{\mathrm{BR}}_{-i} x^\omega_N$ and hence $y^\omega_N \succ x^\omega_N$ by (2), contradicting the choice of x^ω_N. Thus, x^ω_N is a Nash equilibrium indeed. $\square$

Remark. The assumption that each $\mathcal{R}_i$ is upper hemicontinuous cannot simply be dropped, see Example 8.1 below. Example 8.5, due to Powell (1973), shows the same for the assumption $\#N = 2$.

Lemma 3.3. *Let a strategic game Γ satisfy all assumptions of Lemma 2.2, and let each $\mathcal{R}_i$ be upper hemicontinuous. Let $\langle x^k_N \rangle_{k\in\mathbb{N}}$ be an infinite Cournot path, $X^\omega \subseteq X_N$ be the set of its cluster points, and $x^\omega_N \in X^\omega$. Then $x^\omega_i \in \mathcal{R}_i(x^\omega_{-i})$ for at least two different players $i \in N$.*

Proof. As in the proof of Theorem 3.2, we pick a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x^{k_h}_N \to x^\omega_N$ and denote $y^h_N := x^{k_h+1}_N$ ($h \in \mathbb{N}$). Without restricting generality, $x^{k_h}_i \in \mathcal{R}_i(x^{k_h}_{-i})$ for an $i \in N$ and all h, $y^h_N \to y^\omega_N \in X^\omega$, and $y^h_N \rhd^{\mathrm{BR}}_j x^{k_h}_N$, hence $y^h_j \in \mathcal{R}_j(y^h_{-j})$, for a $j \in N$ and all h. Note that $i \neq j$ since $x^{k_h}_j \notin \mathcal{R}_j(x^{k_h}_{-j})$, and that $y^\omega_{-j} = x^\omega_{-j}$. By the upper hemicontinuity of $\mathcal{R}_i$ and $\mathcal{R}_j$, we have $x^\omega_i \in \mathcal{R}_i(x^\omega_{-i})$ and $y^\omega_j \in \mathcal{R}_i(y^\omega_{-j})$. Finally, an assumption that $x^\omega_j \notin \mathcal{R}_j(x^\omega_{-j})$ would imply $y^\omega_N \rhd^{\mathrm{BR}} x^\omega_N$, hence $y^\omega_N \succ x^\omega_N$ by (2), contradicting (3). $\square$

Theorem 3.4. *Let Γ be a strategic game where $\#N = 2$, each X_i is a compact metric space, and each $\mathcal{R}_i$ is upper hemicontinuous. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Then every Cournot path converges to the set of Nash equilibria.*

Immediately follows from Lemma 3.3.

Remark. All assumptions are essential here as Examples 8.1, 8.3, and 8.5 show.

Theorem 3.5. *Let each X_i in a strategic game Γ be a compact metric space and each $\mathcal{R}_i$ be upper hemicontinuous. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Then for every $x^0_N \in X_N$ there exists a Cournot path starting at x^0_N and converging to the set of Nash equilibria.*

Proof. Given $x^0_N \in X_N$, we recursively define a Cournot path $\langle x^k_N \rangle_k$. If x^k_N is a Nash equilibrium, the process stops, and we are home. Otherwise, we define $N^*(k) := \{i \in N \mid x^k_i \notin \mathcal{R}_i(x^k_{-i})\}$ and $X^*(k) := \bigcup_{i\in N^*(k)} \big(X_i \times \{x^k_{-i}\}\big)$; $X^*(k)$ is compact. Then we pick a maximizer $x^{k+1}_N = (x^{k+1}_{i(k)}, x^k_{-i(k)})$ of $\succ$ on $X^*(k)$. By (2), we have $x^{k+1}_{i(k)} \in \mathcal{R}(x^k_{-i(k)})$, hence $x^{k+1}_N \rhd^{\mathrm{BR}} x^k_N$, hence $x^{k+1}_N \succ x^k_N$.

Assuming the path infinite, we denote $X^\omega \subseteq X_N$ the set of its cluster points. Supposing, to the contrary, that $x^\omega_i \notin \mathcal{R}_i(x^\omega_{-i})$ for $x^\omega_N \in X^\omega$ and $i \in N$, we pick a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x^{k_h}_N \to x^\omega_N$. Without restricting generality, $x^{k_h}_i \notin \mathcal{R}_i(x^{k_h}_{-i})$, and hence $i \in N^*(k_h)$, for all h. Then we pick $y_N \in X_N$ such that $y_N \rhd^{\mathrm{BR}}_i x^\omega_N$ and hence $y_N \succ x^\omega_N$ by (2). By the lower semicontinuity of $\succ$, we have $(y_i, x^{k_h}_{-i}) \succ x^\omega_N$ for all h large enough. By Lemma 2.1, $(y_i, x^{k_h}_{-i}) \succ x^k_N$ for such h and all k; in particular, $(y_i, x^{k_h}_{-i}) \succ x^{k_h+1}_N$ for all h large enough. Since $(y_i, x^{k_h}_{-i}) \in X^*(k_h)$, this contradicts the choice of $x^{k_h+1}_N$. $\square$

Theorem 3.6. *Let Γ be a strategic game where $\#N = 2$ and each X_i is a compact metric space. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Then for every $x^0_N \in X_N$ there exists a Cournot path starting at x^0_N and converging to the set of Nash equilibria.*

Proof. Given $x_N^0 \in X_N$, we recursively define a Cournot path $\langle x_N^k \rangle_k$ in exactly the same way as in the proof of Theorem 3.5. Assuming the path infinite, we again denote $X^\omega \subseteq X_N$ the set of its cluster points.

Supposing, to the contrary, that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for $x_N^\omega \in X^\omega$ and $i \in N$, we again pick a strictly increasing sequence $\langle k_h \rangle_{h \in \mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$ as well as $y_N \in X_N$ such that $y_N \rhd_i^{\mathrm{BR}} x_N^\omega$ and hence $y_N \succ x_N^\omega \succ x_N^k$ for each $k \in \mathbb{N}$. Without restricting generality, we may assume that either $x_i^{k_h} \notin \mathcal{R}_i(x_{-i}^{k_h})$ for all h, or $x_{-i}^{k_h} \notin \mathcal{R}_{-i}(x_i^{k_h})$ for all h. In the first case, we obtain a contradiction in exactly the same way as in the proof of Theorem 3.5.

In the second case, we notice that $x_N^{k_h} \rhd_i^{\mathrm{BR}} x_N^{k_h-1}$, hence $i \in N^*(k_h - 1)$, for each h. Since $x_{-i}^{k_h-1} = x_{-i}^{k_h}$, we have $(y_i, x_{-i}^{k_h-1}) \succ x_N^{k_h}$ for all $h \in \mathbb{N}$ large enough by the lower semicontinuity of $\succ$, which contradicts the choice of $x_N^{k_h}$ since $(y_i, x_{-i}^{k_h-1}) \in X^*(k_h - 1)$. $\square$

Remark. Example 8.2 shows that the assumption that each $\mathcal{R}_i$ is upper hemicontinuous in Theorem 3.5, as well as the assumption $\#N = 2$ in Theorem 3.6, cannot simply be dropped.

4 Unique best responses

It turns out that the uniqueness of the best responses allows us to dispense with the upper hemicontinuity of $\mathcal{R}_i$ in almost all results of the preceding section, and also obtain a couple of new results without analogs in a more general case. Throughout this section, we use the notation $\mathcal{R}_i(x_{-i}) = \{r_i(x_{-i})\}$ whenever the uniqueness is assumed or established.

Lemma 4.1. *Let Γ be a strategic game where each X_i is a compact metric space. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Let $i \in N$ and $\langle x_N^k \rangle_{k \in \mathbb{N}}$ be such that $x_N^k \to x_N^\omega$, $\#\mathcal{R}_i(x_{-i}^\omega) = 1$, $\mathcal{R}_i(x_{-i}^k) = \{x_i^k\}$ and $x_N^{k+1} \succ x_N^k$ for all $k \in \mathbb{N}$. Then $\mathcal{R}_i(x_{-i}^\omega) = \{x_i^\omega\}$.*

Proof. Suppose the contrary: $y_i := r_i(x_{-i}^\omega) \neq x_i^\omega$. Then $(y_i, x_{-i}^\omega) \rhd_i^{\mathrm{BR}} x_N^\omega$; hence $(y_i, x_{-i}^\omega) \succ x_N^\omega$. By the lower semicontinuity of $\succ$, we have $(y_i, x_{-i}^k) \succ x_N^\omega \succ x_N^k$ for all $k \in \mathbb{N}$ large enough. However, the assumption $x_i^k = r_i(x_{-i}^k)$ implies that $x_i^k = y_i$ or $x_N^k \rhd_i^{\mathrm{BR}} (y_i, x_{-i}^k)$; therefore $x_N^k = (y_i, x_{-i}^k)$ or $x_N^k \succ (y_i, x_{-i}^k)$. $\square$

Theorem 4.2. *Let Γ be a strategic game where $\#N = 2$ and each X_i is a compact metric space. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Let $\langle x_N^k \rangle_{k \in \mathbb{N}}$ be a Cournot path and $X^\omega \subseteq X_N$ be the set of its cluster points. Let $\#\mathcal{R}_i(x_{-i}^k) = 1$ and $\#\mathcal{R}_i(x_{-i}^\omega) = 1$ for all $i \in N$, $k \in \mathbb{N}$, and $x_N^\omega \in X^\omega$. Then $\langle x_N^k \rangle_{k \in \mathbb{N}}$ converges to the set of Nash equilibria.*

Theorem 4.3. *Let each X_i in a strategic game Γ be a compact metric space and let $\#\mathcal{R}_i(x_{-i}) = 1$ for all $i \in N$ and $x_{-i} \in X_{-i}$. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Then for every $x_N^0 \in X_N$ there exists a Cournot path starting at x_N^0 and converging to the set of Nash equilibria.*

In both cases, the proofs virtually repeat those of Theorems 3.4 and 3.5, respectively, only references to the upper hemicontinuity of $\mathcal{R}_i$ should be replaced with those to Lemma 4.1.

We call a Cournot path $\langle x_N^k \rangle_k$ *inclusive* if for each player $i \in N$, there holds $x_i^k \in R_i(x_{-i}^k)$ for some k. A Cournot path $\langle x_N^k \rangle_k$ is *totally inclusive* if, whenever x_N^m is defined, the path $\langle x_N^k \rangle_{k \geq m}$ is inclusive. Thus, a totally inclusive path either is infinite or ends at a Nash equilibrium. We call an infinite Cournot path $\langle x_N^k \rangle_{k \in \mathbb{N}}$ *uniformly inclusive* if there is a natural number $m \in \mathbb{N}$ such that for each $i \in N$ and each $k \in \mathbb{N}$, there is $h \in \mathbb{N}$ such that $k \leq h \leq k + m$ and $x_i^h \in R_i(x_{-i}^h)$. Every infinite Cournot path generated by the sequential tâtonnement process as defined by Moulin (1984, p. 87), see also Jensen (2010, Theorem 2), is uniformly inclusive with $m = \#N - 1$.

Theorem 4.4. *Let Γ be a strategic game where each X_i is a compact metric space. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Let $\langle x_N^k \rangle_{k \in \mathbb{N}}$ be a uniformly inclusive Cournot path and $X^\omega \subseteq X_N$ be the set of its cluster points. Let $\#\mathcal{R}_i(x_{-i}^k) = 1$ and $\#\mathcal{R}_i(x_{-i}^\omega) = 1$ for all $i \in N$, $k \in \mathbb{N}$, and $x_N^\omega \in X^\omega$. Then $\langle x_N^k \rangle_{k \in \mathbb{N}}$ converges to the set of Nash equilibria.*

Proof. Let $x_N^\omega \in X^\omega$; we pick a strictly increasing sequence $\langle k_h \rangle_{h \in \mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$.

Claim 4.4.1. *For each $s \in \mathbb{N}$, the sequence $\langle x_N^{k_h+s} \rangle_h$ converges to x_N^ω.*

Proof of Claim 4.4.1. We argue by induction in s. For $s = 0$, the definition of $\langle k_h \rangle_{h \in \mathbb{N}}$ suffices. The general induction step is identical with the case of $s = 1$. Since X_N is compact and N is finite, for every subsequence $\langle k'_h \rangle_{h \in \mathbb{N}}$ of $\langle k_h \rangle_{h \in \mathbb{N}}$, there are $i \in N$ and a subsequence $\langle k''_h \rangle_{h \in \mathbb{N}}$ of $\langle k'_h \rangle_{h \in \mathbb{N}}$ such that $x_N^{k''_h+1} \to y_N^\omega \in X^\omega$ and $x_N^{k''_h+1} \rhd_i^{\mathrm{BR}} x_N^{k''_h}$. Then $x_i^{k''_h+1} = r_i(x_{-i}^{k''_h+1})$ and $x_{-i}^{k''_h+1} = x_{-i}^{k''_h}$ for all h; hence, $y_{-i}^\omega = x_{-i}^\omega$ and $y_i^\omega = r_i(y_{-i}^\omega) = r_i(x_{-i}^\omega)$ by Lemma 4.1. An assumption that $x_i^\omega \neq y_i^\omega$ would contradict (3); hence $y_N^\omega = x_N^\omega$. Thus, x_N^ω is the unique cluster point of $\langle x_N^{k_h+1} \rangle_{h \in \mathbb{N}}$; therefore, $x_N^{k_h+1} \to x_N^\omega$. □

Let us fix $i \in N$. Since the path is uniformly inclusive, there is an $s \in \{0, \ldots, m\}$ for each $h \in \mathbb{N}$ such that $x_i^{k_h+s} = r_i(x_{-i}^{k_h+s})$. By Claim 4.4.1, for every $\delta > 0$ there is $\bar{h} \in \mathbb{N}$ such that the distance between $x_N^{k_h+s}$ and x_N^ω is less than δ for each $s \in \{0, \ldots, m\}$ and $h > \bar{h}$. Therefore, there is a strictly increasing sequence $\langle k'_h \rangle_{h \in \mathbb{N}}$ such that $x_N^{k'_h} \to x_N^\omega$ and $x_i^{k'_h} = r_i(x_{-i}^{k'_h})$. Now Lemma 4.1 is applicable, implying $x_i^\omega = r_i(x_{-i}^\omega)$. Since $i \in N$ was arbitrary, x_N^ω is a Nash equilibrium. □

Remark. If $\#N = 2$, then every infinite Cournot path is uniformly inclusive (with $m = 1$); Theorem 4.4 in this case becomes identical to Theorem 4.2. The reverse replacement of the uniqueness of the best responses with their upper hemicontinuity in Theorem 4.4 would make it wrong (for $\#N > 2$) as Example 8.5 shows.

Theorem 4.5. *Let Γ be a strategic game where $\#N = 3$ and each X_i is a compact metric space. Let Γ admit an ω-transitive Cournot potential $\succ$ which is also lower semicontinuous. Let $\langle x_N^k \rangle_{k \in \mathbb{N}}$ be an infinite totally inclusive Cournot path and $X^\omega \subseteq X_N$ be the set of its cluster points. Let $\#\mathcal{R}_i(x_{-i}^k) = 1$ and $\#\mathcal{R}_i(x_{-i}^\omega) = 1$ for all $i \in N$, $k \in \mathbb{N}$, and $x_N^\omega \in X^\omega$. Then X^ω contains a Nash equilibrium.*

Proof. For each $i \in N$, we denote X^i the set of $x_N^\omega \in X^\omega$ for which there exists a strictly increasing sequence $\langle k_h \rangle_{h \in \mathbb{N}}$ such that $x_N^{k_h} \to x_N^\omega$ and $x_i^{k_h} = r_i(x_{-i}^{k_h})$ for each $h \in \mathbb{N}$. By Lemma 4.1, $x_i^\omega = r_i(x_{-i}^\omega)$ for every $x_N^\omega \in X^i$. For each pair $I \subset N$, $\#I = 2$, we denote $X^I := \bigcap_{i \in I} X^i$. Whenever $I \neq J$, $X^I \cap X^J$ consists of Nash equilibria. By Lemma 3.3, modified as described after Lemma 4.1, $X^\omega = \bigcup_I X^I$. Since the path is inclusive, $X^i \neq \emptyset$ for each $i \in N$; hence at least two sets X^I are nonempty too.

Assuming, to the contrary, that X^ω contains no Nash equilibrium, we must have $X^I \cap X^J = \emptyset$ for all pairs $I \neq J$. Since each X^I is closed, there are open subsets V^I such that $X^I \subseteq V^I$ for each I and $V^I \cap V^J = \emptyset$ whenever $I \neq J$. Without restricting generality, we may assume that $x_N^k \in \bigcup_I V^I$ for all k. Therefore, there exist $I \neq J$ and a strictly increasing sequence $\langle k_h \rangle_{h\in\mathbb{N}}$ such that $x_N^{k_h} \in V^J$ and $x_N^{k_h+1} \in V^I$ for all $h \in \mathbb{N}$. Again without restricting generality, we may assume that $x_N^{k_h+1} \rhd_i^{\mathrm{BR}} x_N^{k_h}$ for all h and the same $i \in N$ while $x_N^{k_h} \to x_N^\omega \in X^J$ and $x_N^{k_h+1} \to y_N^\omega \in X^I$; clearly, $y_{-i}^\omega = x_{-i}^\omega$. By Lemma 4.1, $y_i^\omega = r_i(y_{-i}^\omega)$; hence $i \in I$. Now if $i \notin J$, we have $y_N^\omega \rhd_i^{\mathrm{BR}} x_N^\omega$; hence $y_N^\omega \succ x_N^\omega$ by (2), contradicting (3). If $i \in I \cap J$, we have $x_i^\omega = r_i(x_{-i}^\omega)$, $y_i^\omega = r_i(y_{-i}^\omega)$, and $y_{-i}^\omega = x_{-i}^\omega$; hence $y_N^\omega = x_N^\omega$, contradicting the assumption $X^I \cap X^J = \emptyset$. □

Remark. Example 8.6 below shows that one could not assert that *every* $x_N^\omega \in X^\omega$ in Theorem 4.5 is a Nash equilibrium. Example 8.7, that even this theorem would be wrong for $\#N > 3$.

5 Weaker concepts

To broaden the scope of applications, we introduce two weaker notions: a "partial Cournot potential" and a "restricted Cournot potential." In the first case, we require (2) to hold only for some pairs $x_N, y_N \in X_N$; in the second, (2) is required to hold for "admissible" best responses only. Both weakenings can be combined, naturally, defining a "partial restricted Cournot potential." The implications of the presence of a Cournot potential in a weaker sense are weaker too, but not very much.

We call a subset $X^0 \subseteq X_N$ *BR-closed* if it satisfies the following conditions.

1. If $y_N \rhd^{\mathrm{BR}} x_N$ and $x_N \in X^0$, then $y_N \in X^0$ too.
2. If $\langle x_N^k \rangle_{k\in\mathbb{N}}$ is an infinite Cournot path, x_N^ω is its cluster point, and $x_N^k \in X^0$ for each k, then $x_N^\omega \in X^0$.

We call an irreflexive and transitive binary relation $\succ$ on X_N a *partial Cournot potential* if there is a BR-closed subset $\emptyset \neq X^0 \subseteq X_N$ such that (2) holds whenever $x_N \in X^0$ (hence $y_N \in X^0$ too).

Theorem 5.1. *Lemmas 2.2 and 3.3, as well as Theorems 3.2, 3.4, 4.2, 4.4 and 4.5 remain valid if the "Cournot potential" in each of them is replaced with "partial Cournot potential" whereas every Cournot path mentioned is contained in X^0. In Lemma 4.1, the conditions that $x_N^k \in X^0$ for all k and $x_N^\omega \in X^0$ should be added too.*

A straightforward proof is omitted. Other results from Section 3 need a more careful treatment.

Theorem 5.2. *Let each X_i in a strategic game Γ be a compact metric space. Let Γ admit an ω-transitive partial Cournot potential such that either X^0 is closed, or the condition $\big[\, [x_N \in X^0 \;\&\; y_N \succ x_N] \Rightarrow y_N \in X^0 \,\big]$ holds. Then Γ possesses a (pure strategy) Nash equilibrium.*

Proof. In the first case, we invoke Theorem 1 from Kukushkin (2008) exactly as in the proof of Theorem 3.1. In the second case, we start with picking an $x_N \in X^0$. If x_N is a maximizer of $\succ$ on X_N, then it is a Nash equilibrium by (2). Otherwise, by the same Theorem 1 from Kukushkin (2008), there is a maximizer y_N of $\succ$ on X_N such that $y_N \succ x_N$. By our assumption, $y_N \in X^0$ and hence is a Nash equilibrium by (2) again. □

Remark. The additional restrictions on X^0 in Theorem 5.2 could be dropped if we required Condition 2 in the definition of a BR-closed subset to hold for transfinite Cournot paths as well.

To extend Theorems 3.5 and 3.6 to games with a partial Cournot potential, more serious modifications are needed. First, we call a subset $X^0 \subseteq X_N$ *BR-accessible* if a finite Cournot path ending in X^0 can be started from every strategy profile in X_N. Generally, a BR-closed subset need not be BR-accessible; therefore, we virtually have to add such an assumption. Even after that, more assumptions are needed. We provide three different sets of such additions. In each case, the proof follows the same scheme as that of Theorem 3.5, but with some modifications in the construction of the "right" Cournot path and in how the final contradiction is obtained.

The following assumption is the most complicated when looked at, but the most convenient to apply:

$$\forall x_N, z_N \in X^0 \, \forall i \in N \, \forall y_i \in X_i \left[[x_i \in \mathcal{R}_i(x_{-i}) \;\&\; (y_i, x_{-i}) \succ z_N] \Rightarrow x_N \succ z_N \right]. \tag{4}$$

The condition may look intolerably artificial, but it is satisfied, e.g., if $\succ$ can be represented by a function $P \colon X_N \to \mathbb{R}$ in the sense that $y_N \succ x_N \iff P(y_N) > P(x_N)$, and if $\mathcal{R}_i(x_{-i}) \subseteq \operatorname{Argmax}_{y_i \in X_i} P(y_i, x_{-i})$ whenever $x_N \in X^0$.

Theorem 5.3. *Let each X_i in a strategic game Γ be a compact metric space and each $\mathcal{R}_i$ be upper hemicontinuous. Let Γ admit an ω-transitive and lower semicontinuous partial Cournot potential $\succ$ with a BR-accessible subset $X^0 \subseteq X_N$ such that the condition (4) holds. Then for every $x_N \in X_N$ there exists a Cournot path starting at x_N and converging to the set of Nash equilibria.*

Proof. Given $x_N \in X_N$, we start with a finite Cournot path ending in X^0. Once there, we recursively define a Cournot path $\langle x_N^k \rangle_k$ in X^0. Having $x_N^k \in X^0$, we define $N^*(k) := \{i \in N \mid x_i^k \notin \mathcal{R}_i(x_{-i}^k)\}$ and $X^*(k) := \bigcup_{i \in N^*(k)} \big(\mathcal{R}_i(x_{-i}^k) \times \{x_{-i}^k\}\big)$. If $N^*(k) = \emptyset$, x_N^k is a Nash equilibrium, and we are already home. Otherwise, $X^*(k)$ is compact and $X^*(k) \subseteq X^0$ since the latter is BR-closed. Then we pick a maximizer $x_N^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k)$ of $\succ$ on $X^*(k)$. By definition, we have $x_{i(k)}^{k+1} \in \mathcal{R}_{i(k)}(x_{-i(k)}^k)$, hence $x_N^{k+1} \rhd^{\mathrm{BR}} x_N^k$, hence $x_N^{k+1} \in X^0$ and $x_N^{k+1} \succ x_N^k$.

Supposing, to the contrary, that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for a cluster point of the path, we pick $y_N \in X_N$ such that $y_N \rhd_i^{\mathrm{BR}} x_N^\omega$; hence $y_N \succ x_N^\omega$ by (2). Exactly as in the proof of Theorem 3.5, we have $(y_i, x_{-i}^{k_h}) \succ x_N^\omega$ and hence $(y_i, x_{-i}^{k_h}) \succ x_N^{k_h+1}$. If $y_i \in \mathcal{R}_i(x_{-i}^{k_h})$, we immediately have a contradiction with the choice of $x_N^{k_h+1}$. Otherwise, we pick a $z_i \in \mathcal{R}_i(x_{-i}^{k_h})$ and obtain the same contradiction, applying (4) with $x_N := (z_i, x_{-i}^{k_h})\,[\in X^*(k_h)]$, y_i as is, and $z_N := x_N^{k_h+1}$. □

Theorem 5.4. *Let each X_i in a strategic game Γ be a compact metric space and each $\mathcal{R}_i$ be upper hemicontinuous. Let Γ admit an ω-transitive and lower semicontinuous partial Cournot potential $\succ$ with a BR-accessible subset $X^0 = \prod_{i \in N} X_i^0 \subseteq X_N$. Then for every $x_N \in X_N$ there exists a Cournot path starting at x_N and converging to the set of Nash equilibria.*

Proof. Exactly as in the proof of Theorem 5.3, we start with a finite Cournot path ending in X^0, and then recursively define a Cournot path $\langle x_N^k \rangle_k$ in X^0, picking a maximizer $x_N^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k)$ of $\succ$ on $X^*(k) := \bigcup_{i \in N^*(k)} \big(\mathcal{R}_i(x_{-i}^k) \times \{x_{-i}^k\}\big)$; clearly, $x_N^{k+1} \in X^0$ and $x_N^{k+1} \succ x_N^k$.

Supposing, to the contrary, that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for a cluster point of the path, we pick $y_N \in X_N$ such that $y_N \rhd_i^{\mathrm{BR}} x_N^\omega$; hence $y_N \in X^0$ since X^0 is BR-absorbing; hence $y_i \in X_i^0$ and hence $(y_i, x_{-i}^{k_h}) \in X^0$ too. Now, if $y_i \in \mathcal{R}_i(x_{-i}^{k_h})$, we immediately have a contradiction with the choice of $x_N^{k_h+1}$. Otherwise, we pick a $z_i \in \mathcal{R}_i(x_{-i}^{k_h})$. Since $(z_i, x_{-i}^{k_h}) \rhd_i^{\mathrm{BR}} (y_i, x_{-i}^{k_h}) \in X^0$, we have $(z_i, x_{-i}^{k_h}) \succ (y_i, x_{-i}^{k_h})$ by (2) therefore, $(z_i, x_{-i}^{k_h}) \succ x_N^{k_h+1}$ with the same contradiction. □

Theorem 5.5. *Let each X_i in a strategic game Γ be a compact metric space and each $\mathcal{R}_i$ be upper hemicontinuous. Let Γ admit an ω-transitive and lower semicontinuous partial Cournot potential $\succ$ with a BR-accessible subset $X^0 \subseteq X_N$ such that the condition $\big[[x_N \in X^0\ \&\ y_N \succ x_N] \Rightarrow y_N \in X^0\big]$ holds. Then for every $x_N \in X_N$ there exists a Cournot path starting at x_N and converging to the set of Nash equilibria.*

Proof. As in the proof of Theorem 5.3, we start with a finite Cournot path ending in X^0. Once there, we recursively define a Cournot path $\langle x_N^k \rangle_k$ in X^0, picking a maximizer $x_N^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k)$ of $\succ$ on $X^*(k) := X^0 \cap \bigcup_{i \in N^*(k)} \big(X_i \times \{x_{-i}^k\}\big)$, which exists for the same reasons as in the proof of Theorem 5.2. Obviously, $x_{i(k)}^{k+1} \in \mathcal{R}(x_{-i(k)}^k)$; hence $x_N^{k+1} \succ x_N^k$ and $x_N^{k+1} \in X^0$. An assumption that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for a cluster point of the path leads to a contradiction with the choice of $x_N^{k_h+1}$ in the same way as in the proof of Theorem 3.5: once $(y_i, x_{-i}^{k_h}) \succ x^\omega$, we must have $(y_i, x_{-i}^{k_h}) \in X^0$ and hence $(y_i, x_{-i}^{k_h}) \in X^*(k_h)$. □

When $\#N = 2$, one would like to drop the upper hemicontinuity assumption in Theorems 5.3 – 5.5 in the same manner as in Theorem 3.6. In the last case, this is done in exactly the same way without any problem. In the other cases, there emerges a problem with the central construction of the Cournot path: the existence of a maximizer of $\succ$ on $X^*(k)$ cannot be taken for granted. The problem can be solved by assuming that, for every $i \in N$ and $x_{-i} \in X_{-i}$, $\mathcal{R}_i(x_{-i})$ is closed in X_i. The assumption is not innocuous, but weaker than the upper hemicontinuity. Then $X^*(k)$ will be compact and hence Theorem 1 from Kukushkin (2008) will be applicable. In the case of Theorem 5.4, a broader assumption will do: for each $i \in N$, either X_i^0 or every $\mathcal{R}_i(x_{-i})$ ($x_{-i} \in X_{-i}$) is closed in X_i. Then $X^*(k)$, defined as the product of either $X_i^0 \times \{x_{-i}^k\}$ or $\mathcal{R}_i(x_{-i}^k) \times \{x_{-i}^k\}$, will be compact and every maximizer of $\succ$ on $X^*(k)$ will belong to X^0.

Given correspondences $\mathcal{R}_i^*\colon X_{-i} \to 2^{X_i}$ such that

$$\emptyset \neq \mathcal{R}_i^*(x_{-i}) \subseteq \mathcal{R}_i(x_{-i}) \tag{5}$$

for every $i \in N$ and $x_{-i} \in X_{-i}$ ("admissible best responses"), we define the *admissible best response improvement* relation $\rhd^{\mathrm{ABR}}$ on X_N by replacing (1) with

$$y_N \rhd_i^{\mathrm{ABR}} x_N \rightleftharpoons [y_{-i} = x_{-i}\ \&\ x_i \notin \mathcal{R}_i(x_{-i})\ \&\ y_i \in \mathcal{R}_i^*(x_{-i})];$$
$$y_N \rhd^{\mathrm{ABR}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{ABR}} x_N].$$

We call an irreflexive and transitive binary relation $\succ$ on X_N a *restricted Cournot potential* if there are correspondences $\mathcal{R}_i^*\colon X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ such that (2) holds for $\rhd^{\mathrm{ABR}}$. A Cournot path is *admissible* if $x_N^{k+1} \rhd^{\mathrm{ABR}} x_N^k$ for each relevant k.

Remark. If we define admissible best responses by $\mathcal{R}_i^*(x_{-i}) := \mathcal{R}_i(x_{-i})$ for all $i \in N$ and $x_{-i} \in X_{-i}$, then every restricted Cournot potential is just a Cournot potential. When all best responses are single-valued, as in Section 4, there is no other way to define admissible best responses.

Theorem 5.6. *Lemmas 2.2 and 3.3, as well as Theorems 3.1, 3.2, and 3.4, remain valid if the "Cournot potential" in each of them is replaced with "restricted Cournot potential," the assumptions on $\mathcal{R}_i$ are shifted onto $\mathcal{R}_i^*$, and only admissible Cournot paths are allowed.*

A straightforward proof is omitted.

To extend Theorems 3.5 and 3.6 to games with a restricted Cournot potential, we need an additional assumption, somewhat similar to (4):

$$\forall x_N, z_N \in X_N \, \forall i \in N \, \forall y_i \in X_i \left[[x_i \in \mathcal{R}_i^*(x_{-i}) \;\&\; (y_i, x_{-i}) \succ z_N] \Rightarrow x_N \succ z_N \right]. \tag{6}$$

The assumption is satisfied, e.g., if $\succ$ can be represented by a function $P\colon X_N \to \mathbb{R}$ in the sense that $y_N \succ x_N \iff P(y_N) > P(x_N)$, and if $\mathcal{R}_i^*(x_{-i}) \subseteq \mathrm{Argmax}_{y_i \in X_i} P(y_i, x_{-i}) \subseteq \mathcal{R}_i(x_{-i})$.

Theorem 5.7. *Let each X_i in a strategic game Γ be a compact metric space. Let Γ admit an ω-transitive and lower semicontinuous restricted Cournot potential $\succ$ such that each $\mathcal{R}_i^*$ is upper hemicontinuous and (6) is satisfied. Then for every $x_N^0 \in X_N$ there exists an admissible Cournot path starting at x_N^0 and converging to the set of Nash equilibria.*

Proof. Given $x_N^0 \in X_N$, we recursively define a Cournot path $\langle x_N^k \rangle_k$ in X^0 in a way similar to the proofs of Theorems 3.5 and 5.3. Given $x_N^k \in X^0$, we define $N^*(k) := \{i \in N \mid x_i^k \notin \mathcal{R}_i(x_{-i}^k)\}$ and $X^*(k) := \bigcup_{i \in N^*(k)} \big(\mathcal{R}_i^*(x_{-i}^k) \times \{x_{-i}^k\}\big)$. If $N^*(k) = \emptyset$, x_N^k is a Nash equilibrium, and we are already home. Otherwise, we pick a maximizer $x_N^{k+1} = (x_{i(k)}^{k+1}, x_{-i(k)}^k)$ of $\succ$ on $X^*(k)$, which exists because $X^*(k)$ is compact. By definition, we have $x_{i(k)}^{k+1} \in \mathcal{R}_{i(k)}^*(x_{-i(k)}^k)$, hence $x_N^{k+1} \rhd^{\mathrm{ABR}} x_N^k$, hence $x_N^{k+1} \succ x_N^k$.

Supposing, to the contrary, that $x_i^\omega \notin \mathcal{R}_i(x_{-i}^\omega)$ for a cluster point of the path, we argue similarly to the proof of Theorem 5.3. First, we pick y_N for which $y_N \rhd_i^{\mathrm{ABR}} x_N^\omega$, obtaining $y_N \succ x_N^\omega$ and hence $(y_i, x_{-i}^{k_h}) \succ x_N^{k_h+1}$. Second, if $y_i \in \mathcal{R}_i^*(x_{-i}^{k_h})$, then $(y_i, x_{-i}^{k_h}) \in X^*(k_h)$ and we immediately have a contradiction with the choice of $x_N^{k_h+1}$. Otherwise, we pick a $z_i \in \mathcal{R}_i^*(x_{-i}^{k_h})$ and obtain the same contradiction, applying (6) with $x_N := (z_i, x_{-i}^{k_h})\,[\in X^*(k_h)]$, y_i as is, and $z_N := x_N^{k_h+1}$. □

When $\#N = 2$, we can drop the upper hemicontinuity assumption in Theorem 5.7, demanding instead that every $\mathcal{R}_i^*(x_{-i}^{k_h})$ is closed in X_i.

6 Games with structured utilities

A *game with structured utilities* (and additive aggregation) may have an arbitrary finite set of players N and arbitrary sets of strategies whereas the utility functions satisfy certain structural requirements. There is a set A of *processes* and a finite subset $\Upsilon^i \subseteq \mathrm{A}$ of processes where each player $i \in N$ participates (given exogenously). With every $\alpha \in \mathrm{A}$, an *intermediate utility function* is associated,

$\varphi_\alpha \colon X_{N(\alpha)} \to \mathbb{R}$, where $N(\alpha) = \{i \in N \mid \alpha \in \Upsilon^i\}$. The "ultimate" utility functions of the players are built of the intermediate utilities:

$$u_i(x_N) := \sum_{\alpha \in \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}), \tag{7}$$

where $i \in N$ and $x_N \in X_N$. Defining $P \colon X_N \to \mathbb{R}$ by

$$P(x_N) := \sum_{\alpha \in \mathrm{A}} \varphi_\alpha(x_{N(\alpha)}), \tag{8}$$

we immediately see that P is an exact potential (Monderer and Shapley, 1996):

$$P(x_N) = \sum_{\alpha \in \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}) + \sum_{\alpha \in \mathrm{A} \setminus \Upsilon^i} \varphi_\alpha(x_{N(\alpha)}) = u_i(x_N) + Q_i(x_{-i})$$

for all $i \in N$ and $x_N \in X_N$; clearly, it is a Cournot potential as well. If all functions φ_α are continuous, then P is continuous too. If we additionally assume, e.g., each set X_i to be convex and each function φ_α strictly concave, then the results of Section 4 become applicable.

Remark. A strategic game admits an exact potential if and only if it can be represented as a game with structured utilities and additive aggregation rule (7), see Kukushkin (2007, Theorem 5).

Utility functions satisfying (7) can be found in so called "network transmission games," see, e.g., Facchinei et al. (2011) and references therein, which are somewhat similar to Rosenthal's (1973) congestion games, but do not belong to the class. There is a directed graph with the set of links E; each player $i \in N$ is assigned a path $\pi_i \subseteq E$ in the graph (between a source and a target) and sends a *flow* $x_i \in [0, b_i] \subset \mathbb{R}$ along the path, getting a reward $w_i(x_i)$ depending on her flow and bearing costs $\sum_{e \in \pi_i} c_e(\sum_{j \colon e \in \pi_j} x_j)$ depending on the total flow through each link in π_i. Setting $\mathrm{A} := E \cup N$, $\Upsilon^i := \pi_i \cup \{i\}$, $\varphi_i(x_i) := w_i(x_i)$ and $\varphi_e(x_{N(e)}) := -c_e(\sum_{j \in N(e)} x_j)$ for each $e \in E$, we see that (7) holds for each player.

The notion of a partial Cournot potential is handy in the case of a Cournot oligopoly with a linear inverse demand function (Monderer and Shapley, 1996). The strategies are $X_i := [0, K_i]$; the utilities, $u_i(x_N) := x_i \cdot \max\{a - b \cdot \sum_{j \in N} x_j, 0\} - C_i(x_i)$, where $a, b > 0$ and $C \colon \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and such that $C(0) = 0$. Denoting $X^0 := \{x_N \in X_N \mid a - b \cdot \sum_{i \in N} x_i \geq 0\}$ and assuming $X^0 \neq \emptyset$ (otherwise, there is nothing to discuss here), we easily check that X^0 is both BR-closed and BR-accessible, and the function

$$P(x_N) := \sum_{i \in N} \left[a x_i - C_i(x_i)\right] - b \cdot \sum_{i \in N} \sum_{j \in N} x_i x_j$$

represents a partial Cournot potential satisfying (4) on X^0. Thus, Theorems 5.1 and 5.3 are applicable.

Given continuous and strictly increasing mappings $\lambda_i \colon \mathbb{R} \to \mathbb{R}$, we may extend this approach further, replacing (7) with

$$u_i(x_N) = \lambda_i\big(\sum_{\alpha \in \Upsilon^i} \varphi_\alpha(x_{N(\alpha)})\big), \tag{9}$$

for all $i \in N$ and $x_N \in X_N$. Then $P(x_N) := \sum_{\alpha\in\mathrm{A}} \varphi_\alpha(x_{N(\alpha)})$ is an ordinal potential, hence a continuous Cournot potential again. This trick works, e.g., for the Cournot oligopoly with identical linear cost functions (Monderer and Shapley, 1996; Kukushkin, 1994): $X_i := [0, K_i]$; $u_i(x_N) := x_i \cdot Q(\sum_{j\in N} x_j) - c \cdot x_i$, where $Q\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $c \geq 0$. Assuming $\{x_N \in X_N \mid Q(\sum_{i\in N} x_i) - c > 0\} \neq \emptyset$ (otherwise, there is nothing to discuss here again), we may pick $\varepsilon > 0$ such that $X^0 := \{x_N \in X_N \mid \big[Q(\sum_{i\in N} x_i) - c\big] \cdot \prod_{i\in N} x_i > \varepsilon\} \neq \emptyset$. It is easily checked now that X^0 is both BR-closed and BR-accessible, and the function

$$P(x_N) := \big[Q(\sum_{i\in N} x_i) - c\big] \cdot \prod_{i\in N} x_i$$

represents a partial Cournot potential on X^0 [the logarithm of P is of the form (8)]. Thus, Theorems 5.1 and 5.5 are applicable.

A partial Cournot potential for the voluntary provision of a public good with Cobb-Douglas utilities can be constructed very similarly.

7 Aggregative games with the single crossing conditions

A rather general (though not the most general imaginable) definition of an aggregative game sounds as follows: each X_i is a compact subset of $\mathbb{R}$, and there are mappings $\sigma_i\colon X_{-i} \to \mathbb{R}$ such that

$$u_i(x_N) = U_i(\sigma_i(x_{-i}), x_i)$$

for all $i \in N$ and $x_N \in X_N$. For each $i \in N$, we denote $S_i := \sigma_i(X_{-i}) \subset \mathbb{R}$, and redefine the best response correspondence:

$$R_i(s_i) := \operatorname*{Argmax}_{x_i\in X_i} U_i(s_i, x_i).$$

Our assumption $\mathcal{R}_i(x_{-i}) \neq \emptyset$ is equivalent to $R_i(s_i) \neq \emptyset$ for each $s_i \in S_i$.

We also assume that each player's best responses are increasing in s_i (in a rather strong sense):

$$[s_i' > s_i \;\&\; x_i' \in R_i(s_i') \;\&\; x_i \in R_i(s_i)] \Rightarrow x_i' \geq x_i \tag{10}$$

for all $i \in N$ and $s_i', s_i \in S_i$. The following *strict single crossing* condition (Milgrom and Shannon, 1994) is sufficient for (10):

$$[x_i' > x_i \;\&\; s_i' > s_i \;\&\; U_i(s_i, x_i') \geq U_i(s_i, x_i)] \Rightarrow U_i(s_i', x_i') > U_i(s_i', x_i) \tag{11}$$

for all $i \in N$, $x_i', x_i \in X_i$, and $s_i', s_i \in S_i$.

If each σ_i is increasing in each x_j, then the existence of a Nash equilibrium (but not the acyclicity of the best response improvements) immediately follows from Tarski's fixed point theorem. Novshek (1985) was the first to notice that the existence also obtains in the case of $\sigma_i(x_{-i}) = -\sum_{j\neq i} x_j$; this fact has nothing to do with Tarski's theorem. Kukushkin (2004) proved the impossibility of Cournot cycles in both Novshek' case and when $\sigma_i(x_{-i}) = \sum_{j\neq i} x_j$. Dubey et al. (2006) modified a trick invented by Huang (2002) for different purposes, providing a tool for the construction of a continuous partial Cournot potential. A rather broad class of aggregative games where the trick works is described in

Jensen (2010); the class may be the broadest possible although it is unclear how such a claim could be proven. (The technical assumptions of Jensen's main theorem, however, should have been much stronger.)

We describe the trick in some details for a case of intermediate generality (Kukushkin, 2005), sufficient for many applications in economics. Let

$$\sigma_i(x_{-i}) = \sum_{j \neq i} a_{ij} x_j,$$

with $a_{ij} = a_{ji} \in \mathbb{R}$ for all $i \neq j$. Assuming that each best response correspondence R_i is upper hemicontinuous and satisfies (10), the approach of Huang-Dubey-et-al. recommends the following steps. First, we pick a selection r_i from each R_i, e.g., $r_i(s_i) := \min R_i(s_i)$ for every $s_i \in S_i$; then we extend r_i to the whole closed interval $[\min S_i, \max S_i]$ preserving its monotonicity; finally, we define

$$P(x_N) := \sum_{i \in N} \Big[-x_i \cdot \max S_i + \int_{\min S_i}^{\max S_i} \min\{x_i, r_i(s_i)\}\, ds_i \Big] + \frac{1}{2} \Big[\sum_{i,j \in N\ i \neq j} a_{ij} \cdot x_i \cdot x_j \Big].$$

Straightforward calculations show that $P(y_i, x_{-i}) \geq P(x_N)$ whenever $y_i \in R_i(\sigma_i(x_{-i}))$, and that $P(y_N) > P(x_N)$ whenever $y_N \rhd_i^{\mathrm{BR}} x_N$ and $x_i \in X_i^0 := \bigcup_{s_i \in S_i} R_i(s_i)$. Therefore, P represents a continuous partial Cournot potential satisfying (2) on $X^0 := \prod_{i \in N} X_i^0$. Thus, Theorems 5.1, 5.3, and 5.4 are applicable.

Remark. When $a_{ij} \geq 0$ for all $j \neq i$, we have a game with strategic complementarity; when $a_{ij} \leq 0$ for all $j \neq i$, a game with strategic substitutability. A more general situation with coefficients of both signs is also possible.

The following *weak single crossing* condition (Shannon, 1995),

$$[x_i' > x_i \;\&\; s_i' > s_i \;\&\; U_i(s_i, x_i') > U_i(s_i, x_i)] \Rightarrow U_i(s_i', x_i') \geq U_i(s_i', x_i)$$

for all $i \in N$, $x_i', x_i \in X_i$, and $s_i', s_i \in S_i$, ensures the monotonicity of R_i in a rather weak sense:

$$[s_i' > s_i \;\&\; x_i' \in R_i(s_i') \;\&\; x_i \in R_i(s_i)] \Rightarrow [\min\{x_i', x_i\} \in R_i(s_i) \;\text{ or }\; \max\{x_i', x_i\} \in R_i(s_i')] \tag{12}$$

for all $i \in N$ and $s_i', s_i \in S_i$. Since every $R_i(s_i)$ is compact, (12) implies, by Theorem 3.2 from Veinott (1989), the existence of an increasing selection r_i from R_i. Defining R_i^* by the closure of the graph of r_i, we immediately see that R_i^* is upper hemicontinuous and satisfies both (5) and (10). In other words, if the best responses are upper hemicontinuous and increasing in the sense of (12), while aggregation rules σ_i belong to the class described in Jensen (2010), i.e., allow the Huang-Dubey-et-al. trick to work, then the game admits a partial restricted Cournot potential satisfying both (4) and (6). Therefore, Nash equilibria exist and can be reached with admissible Cournot paths from anywhere.

8 "Counterexamples"

This section consists of examples showing the impossibility of easy generalizations.

Example 8.1. Let us consider a game where $N := \{1, 2\}$, $X_i := [0, 1] \cup \{2\}$, and the preferences are defined by these utility functions:

$$u_i(x_N) := \begin{cases} \min\{2x_i - x_{-i}, -2x_i + x_{-i} + 2\}, & x_N \in [0,1] \times [0,1]; \\ 1, & x_i = 2,\ x_{-i} \in [0,1[; \\ 2, & x_i = 2,\ x_{-i} = 1; \\ x_i, & x_{-i} = 2. \end{cases}$$

Each utility function u_i is upper semicontinuous in x_N and continuous in x_i; the only discontinuity in x_{-i} happens when $x_{-i} = 1$ and $x_i = 2$. The best responses are easy to compute:

$$\mathcal{R}_i(x_{-i}) = \begin{cases} \{2\}, & x_{-i} \in \{1, 2\}; \\ \{2, x_{-i}/2 + 1/2\}, & x_{-i} \in [0, 1[. \end{cases}$$

There is a unique Nash equilibrium, $(2, 2)$.

To define a Cournot potential, we introduce an auxiliary function on $\mathbb{R}^2$: $\psi(x, y) := \min\{x,\ -x + y + 1\}$. Then we define a continuous function on X_N:

$$P(x_N) := \begin{cases} \max\{\psi(x_1, x_2), \psi(x_2, x_1)\}, & x_N \in [0,1] \times [0,1]; \\ 2 + \min_i x_i, & \text{otherwise.} \end{cases}$$

Claim 8.1.1. *If $y_N \rhd^{\mathrm{BR}} x_N$, then $P(y_N) > P(x_N)$, i.e., P represents a Cournot potential.*

Proof of Claim 8.1.1. Let $y_N \rhd^{\mathrm{BR}}_i x_N$; if $x_{-i} = 2$, we are home immediately. Let $x_{-i} \in [0, 1]$; hence $x_i \in [0, 1]$ too, and hence $P(x_N) \leq 1$. If $y_i = 2$, then $P(y_N) \geq 2 > P(x_N)$. Let $y_i \in [0, 1]$; then $y_i = x_{-i}/2 + 1/2 > x_{-i}$. We have $\psi(y_i, x_{-i}) = y_i > x_{-i} \geq \psi(x_{-i}, y_i)$ and hence $P(y_N) = y_i$. Meanwhile, $\psi(x_{-i}, x_i) \leq x_{-i} < y_i$ and $\psi(x_i, x_{-i}) < y_i = \max_{z_i \in [0,1]} \psi(z_i, x_{-i})$; therefore, $P(y_N) > P(x_N)$ again. □

Since $\mathcal{R}_i$ are not upper hemicontinuous, neither Theorem 3.2, nor Theorem 3.4 is applicable here. Indeed, a Cournot path converging to $(1, 1)$, which is not an equilibrium, can be started from every strategy profile in $[0, 1[\times [0, 1[$. On the other hand, Theorem 3.6 *is* applicable; actually, the unique Nash equilibrium can be reached from every strategy profile after, at most, two best response improvements.

Example 8.2. Let us consider a game where $N := \{1, 2, 3\}$, $X_1 := X_2 := [0, 1]$, $X_3 := \{0, 1\}$, and the preferences are defined by these utility functions: $u_3(x_N) := 1$ if $x_N = (1, 1, 1)$, $u_3(x_N) := 0$ otherwise, whereas for $i \in \{1, 2\}$, $u_i(x_N) := \min\{2x_i - x_{3-i}, -2x_i + x_{3-i} + 2\}$. Both functions u_1, u_2 are continuous in x_N; u_3 is upper semicontinuous in x_N and continuous in x_3. The best responses are easy to compute: $\mathcal{R}_i(x_{-i}) = \{x_{3-i}/2 + 1/2\}$ for $i = 1, 2$, $\mathcal{R}_3(x_{-3}) = \{1\}$ if $x_{-3} = (1, 1)$, and $\mathcal{R}_3(x_{-3}) = X_3$ otherwise. There is a unique Nash equilibrium, $(1, 1, 1)$.

To define a Cournot potential, we use the same auxiliary function on $\mathbb{R}^2$: $\psi(x, y) := \min\{x, -x + y + 1\}$ and define a continuous function on X_N by $P(x_N) := \max\{\psi(x_1, x_2),\ \psi(x_2, x_1)\} + x_3$.

Claim 8.2.1. *If $y_N \rhd^{\mathrm{BR}} x_N$, then $P(y_N) > P(x_N)$, i.e., P represents a Cournot potential.*

Proof of Claim 8.2.1. For players 1 or 2, the argument is the same as in the proof of Claim 8.1.1, one only has to consider fewer cases. The situation $y_N \rhd_3^{\mathrm{BR}} x_N$ is only possible when $y_{-3} = x_{-3} = (1,1)$, $x_3 = 0$ and $y_3 = 1$. $\square$

Every Cournot path started from $[0,1[\times[0,1[\times\{0\}$ converges to $(1,1,0)$, which is not an equilibrium. Thus, the upper hemicontinuity assumption in Theorem 3.5, as well as the assumption $\#N = 2$ in Theorem 3.6, are essential.

Example 8.3. In a plane with polar coordinates (ρ,φ) ($\rho \geq 0$, $0 \leq \varphi < 2\pi$), we define a compact subset

$$X := \bigl\{(\rho,\varphi) \mid 1 \leq \rho \leq 2\bigr\}$$

and a mapping $f\colon X \to X$ by

$$f(\rho,\varphi) := \begin{cases} \bigl(1, \min\{3\varphi/2, \pi + \varphi/2\}\bigr), & \rho = 1; \\ \bigl((\rho+1)/2, \min\{3\varphi/2, \pi + \varphi/2\} \oplus \pi/[1 - \log_2(\rho - 1)]\bigr), & \rho > 1; \end{cases}$$

where $\oplus$ denotes addition modulo 2π. Clearly, f is continuous and $(1,0)$ is its unique fixed point. Defining $X^0 := \{(\rho,\varphi) \in X \mid \rho = 1\}$ and $X^* := X \setminus X^0$, we immediately see that $f^k(x)$ converges to $(1,0)$ whenever $x \in X^0$ and to X^0 whenever $x \in X^*$.

Now we define a strategic game: $N := \{1,2\}$, $X_1 := X_2 := X$, $u_i(x_N) := -d(x_i, f(x_{-i}))$, where d denotes distance in the plane. Both utilities are continuous; the best responses are unique, $\mathcal{R}_i(x_{-i}) = \{f(x_{-i})\}$. The strategy profile $((1,0),(1,0))$ is a unique Nash equilibrium.

Then we define a function $P\colon X \times X \to \mathbb{R}$ in this way:

$$P(x_1,x_2) := \begin{cases} 0, & \rho_1 = \rho_2 = 1 \;\&\; \varphi_1 = \varphi_2 = 0; \\ \min_i \varphi_i + \max_i u_i(x_N) - 2\pi, & \rho_1 = \rho_2 = 1 \;\&\; \max_i \varphi_i > 0; \\ \min_i (1 - \rho_i) + \max_i u_i(x_N) - 2\pi, & \text{otherwise.} \end{cases}$$

The function is upper semicontinuous, but not continuous.

Claim 8.3.1. *If $x'_N \rhd^{\mathrm{BR}} x_N$, then $P(x'_N) > P(x_N)$, i.e., P represents a Cournot potential.*

Proof of Claim 8.3.1. Let $x'_{-i} = x_{-i}$ and $x'_i = f(x_{-i}) \neq x_i$, hence $u_i(x'_N) = 0 \geq u_{-i}(x'_N)$. If $x_{-i} = (1,0)$, then $P(x_N) < 0 = P(x'_N)$ and we are home.

Let $\rho_{-i} = 1$ and $\varphi_{-i} > 0$. Then $\rho'_i = 1$ and $\varphi'_i > \varphi_{-i}$, hence $P(x'_N) = \varphi_{-i} - 2\pi$. If $\rho_i > 1$, then $P(x_N) < -2\pi < P(x'_N)$. If $\rho_i = 1$, then we consider two alternatives. If $\varphi_i \geq \varphi_{-i}$, then $\max_i u_i(x_N) < 0$, hence $P(x_N) < \varphi_{-i} - 2\pi = P(x'_N)$; if $\varphi_i < \varphi_{-i}$, then $P(x_N) \leq \varphi_i - 2\pi < P(x'_N)$.

Finally, let $\rho_{-i} > 1$. Then $P(x'_N) = 1 - \rho_{-i} - 2\pi$. If $\rho_i \leq \rho_{-i}$, then $x_{-i} \neq f(x_i)$, hence $P(x_N) < 1 - \rho_{-i} - 2\pi = P(x'_N)$. If $\rho_i > \rho_{-i}$, then $P(x_N) \leq 1 - \rho_i - 2\pi < P(x'_N)$. $\square$

We see that the assumptions of Theorem 3.2 are satisfied. Moreover, the potential is upper semicontinuous, and the best responses are single-valued. Meanwhile, every Cournot path started from $X^* \times X^*$ has an infinite number of cluster points besides the unique equilibrium, i.e., does not converge to the set of equilibria. Thus, the *lower* semicontinuity of the potential in Theorem 3.4 is essential.

Example 8.4. We consider a modification of Example 8.3 with the same subset X

$$X := \{(\rho, \varphi) \mid 1 \leq \rho \leq 2\}$$

of the plane with polar coordinates and a different continuous mapping $f\colon X \to X$,

$$f(\rho, \varphi) := \begin{cases} (\rho, \varphi), & \rho = 1, \\ \big((\rho+1)/2, \varphi \oplus \pi/[1 - \log_2(\rho - 1)]\big), & \rho > 1, \end{cases}$$

where $\oplus$ again denotes addition modulo 2π. Defining $X^0 := \{(\rho, \varphi) \in X \mid \rho = 1\}$ and $X^* := X \setminus X^0$, we immediately see that $f(x) = x$ whenever $x \in X^0$, and $f^k(x)$ converges to X^0 whenever $x \in X^*$.

Now we define a strategic game in exactly the same way as in Example 8.3: $N := \{1, 2\}$, $X_1 := X_2 := X$, $u_i(x_N) := -d(x_i, f(x_{-i}))$, where d denotes distance in the plane. Again, both utilities are continuous; the best responses are unique, $\mathcal{R}_i(x_{-i}) = \{f(x_{-i})\}$. The set of Nash equilibria of the game is $\{x_N \in X^0 \times X^0 \mid x_1 = x_2\}$.

Then we define a continuous function $P\colon X \times X \to \mathbb{R}$ by

$$P(x_N) := \min_i (1 - \rho_i) + \max_i u_i(x_N).$$

An argument similar to the proof of Claim 8.3.1, but even simpler, shows that P represents a Cournot potential. Meanwhile, the set of cluster points of any Cournot path started from $X^* \times X^*$ is the whole set of Nash equilibria of the game. We see that the assumptions of Theorem 3.4, even Theorem 4.4, do not ensure the convergence of every Cournot path to *a* Nash equilibrium.

The following example is essentially due to Powell (1973).

Example 8.5. Let us consider a game where $N := \{1, 2, 3\}$, $X_i := [-2, 2]$, and the preferences of each player are defined by the same continuous utility function:

$$u(x_N) := \sum_{i,j \in N,\, i \neq j} x_i \cdot x_j/2 - \sum_{i \in N} \big[\max\{x_i - 1, 0, -1 - x_i\}\big]^2.$$

Clearly, u is an exact potential of the game; hence it represents a continuous Cournot potential. Note that the game belongs to the class considered in Section 7 with $\sigma_i(x_{-i}) := \sum_{j \neq i} x_j$; the strict single crossing condition (11) is easy to check. Note also that u is concave in each x_i.

The best responses are easy to compute; given $i \in N$ and $x_{-i} \in X_{-i}$, we denote $s_i := \sum_{j \neq i} x_j$.

$$\mathcal{R}_i(x_{-i}) = \begin{cases} \{2\}, & s_i \geq 2; \\ \{1 + s_i/2\}, & 0 < s_i \leq 2; \\ [-1, 1], & s_i = 0; \\ \{-1 + s_i/2\}, & -2 \leq s_i < 0; \\ \{-2\}, & s_i \leq -2. \end{cases}$$

There are two Nash equilibria maximizing the utility/potential: $(2, 2, 2)$ and $(-2, -2, -2)$. $(0, 0, 0)$ is also a Nash equilibrium.

Fixing an arbitrary $\delta \in]0, 1/4[$, we consider a sequential Cournot path starting at $x_N^0 := (1 + 4\delta, -1 - 2\delta, 1 + \delta)$: $x_N^1 = (-1 - \delta/2, -1 - 2\delta, 1 + \delta)$; $x_N^2 = (-1 - \delta/2, 1 + \delta/4, 1 + \delta)$; $x_N^3 = (-1 - \delta/2, 1 + \delta/4, -1 - \delta/8)$; $x_N^4 = (1 + \delta/16, 1 + \delta/4, -1 - \delta/8)$; $x_N^5 = (1 + \delta/16, -1 - \delta/32, -1 - \delta/8)$; $x_N^6 = (1 + \delta/16, -1 - \delta/32, 1 + \delta/64)$. Comparing x_N^0 and x_N^6, we see how the path will continue *ad infinitum*. Thus, it has six cluster points: $(1, -1, 1)$, $(-1, -1, 1)$, $(-1, 1, 1)$, $(-1, 1, -1)$, $(1, 1, -1)$, and $(1, -1, -1)$, none of which is an equilibrium.

We see that Theorem 3.4 cannot be extended to $n > 2$, while Theorems 4.4 and 4.5 are wrong without the uniqueness of the best responses.

Example 8.6. Having in mind the same set $X := \{(\rho, \varphi) \mid 1 \leq \rho \leq 2\}$ and mapping $f\colon X \to X$ as in Example 8.4, we consider a strategic game with $N := \{1, 2, 3\}$, $X_1 := X_2 := X$, $X_3 := [0, \pi]$, and a common utility function of all players

$$u_i(x_N) := P(x_N) := -\min\{d(x_1, f(x_2)), d(x_2, f(x_1))\} + \min\{1 - \rho_1, 1 - \rho_2\} + \\ x_3 \cdot \big(\psi(\varphi_1, \varphi_2) - |x_3 - \psi(\varphi_1, \varphi_2)|\big),$$

where

$$\psi(\varphi_1, \varphi_2) := \max\{\min\{\varphi_1, \varphi_2, 2\pi - \varphi_1, 2\pi - \varphi_2\} - \Delta, 0\},$$

and $0 < \Delta < \pi/2$. Clearly, P is continuous, and represents a Cournot potential of the game.

When $x_3 = 0$, the best responses of players 1 and 2 are the same as in Example 8.4: $\mathcal{R}_1(x_{-1})\Big|_{x_3=0} = \{f(x_2)\}$ and $\mathcal{R}_2(x_{-2})\Big|_{x_3=0} = \{f(x_1)\}$. Therefore, the set of cluster points of any Cournot path such that $x_3^0 = 0$, $\rho_1^0 > \rho_2^0 > 1$, $x_N^{2k+1} \rhd_1^{\mathrm{BR}} x_N^{2k}$ and $x_N^{2k+2} \rhd_2^{\mathrm{BR}} x_N^{2k+1}$ is $\{x_N \in X_N \mid \rho_1 = \rho_2 = 1,\ \varphi_1 = \varphi_2,\ x_3 = 0\}$. The utility function $u_3(x_N)$ is piecemeal-quadratic in x_3 and it is easily checked that it strictly increases when $x_3 \leq \psi(\varphi_1, \varphi_2)$ and strictly decreases when $x_3 \geq \psi(\varphi_1, \varphi_2)$; therefore, $\mathcal{R}_3(x_1, x_2) = \{\psi(\varphi_1, \varphi_2)\}$.

Thus, we see that the best responses along the Cournot path, as well as at every cluster point, are unique. Moreover, the path is totally inclusive since $x_3^k \in \mathcal{R}_3(x_{-3}^k)$ whenever $\psi(\varphi_1^k, \varphi_2^k) = 0$, which happens an infinite number of times. However, not every cluster point is an equilibrium (only those points where $\psi(\varphi_1, \varphi_2) = 0$). In other words, Theorem 3.4 cannot be extended to $\#N > 2$ even under the uniqueness of the best responses.

Example 8.7. Let us add one more player to Example 8.6, and one more additive term to the common utility function. Thus, $N := \{1, 2, 3, 4\}$, $X_1 := X_2 := X$, $X_3 := X_4 := [0, \pi]$, and

$$u_i(x_N) := P(x_N) := -\min\{d(x_1, f(x_2)), d(x_2, f(x_1))\} + \min\{1 - \rho_1, 1 - \rho_2\} + \\ x_3 \cdot \big(\psi(\varphi_1, \varphi_2) - |x_3 - \psi(\varphi_1, \varphi_2)|\big) + x_4 \cdot \big(\psi^*(\varphi_1, \varphi_2) - |x_4 - \psi^*(\varphi_1, \varphi_2)|\big),$$

where X, f, ψ and Δ are the same as in Example 8.6, whereas

$$\psi^*(\varphi_1, \varphi_2) := \max\{\min\{|\varphi_1 - \pi|, |\varphi_2 - \pi|\} - \Delta, 0\}.$$

Again, P is continuous and represents a Cournot potential of the game.

When $x_3 = x_4 = 0$, the best responses of players 1 and 2 are again the same as in Example 8.4: $\mathcal{R}_1(x_{-1})\Big|_{x_3=x_4=0} = \{f(x_2)\}$ and $\mathcal{R}_2(x_{-2})\Big|_{x_3=x_4=0} = \{f(x_1)\}$. Therefore, every Cournot path such that

$x_3^0 = x_4^0 = 0$, $\rho_1^0 > \rho_2^0 > 1$, $x_N^{2k+1} \rhd_1^{\mathrm{BR}} x_N^{2k}$ and $x_N^{2k+2} \rhd_2^{\mathrm{BR}} x_N^{2k+1}$ has the same set $\{x_N \in X_N \mid \rho_1 = \rho_2 = 1,\ \varphi_1 = \varphi_2,\ x_3 = x_4 = 0\}$ as the set of cluster points. Similarly to the preceding example, $\mathcal{R}_3(x_{-3}) = \{\psi(\varphi_1, \varphi_2)\}$ and $\mathcal{R}_4(x_{-4}) = \{\psi^*(\varphi_1, \varphi_2)\}$.

Again, the best responses along the totally inclusive Cournot path, as well as at every cluster point, are unique. However, there is no equilibrium among the cluster points of the path because the equalities $\psi(\varphi_1, \varphi_2) = 0$ and $\psi^*(\varphi_1, \varphi_2) = 0$ are incompatible. In other words, Theorem 4.5 would be just wrong for $\# N > 3$.

9 Concluding remarks

9.1. ω-transitivity of a Cournot potential alone ensures the "transfinite convergence" of every Cournot path to Nash equilibria; a formal exposition can be found in Kukushkin (2010). The concept might seem exotic, but there is something to it. If, e.g., we replace all $X_i = [-2, 2]$ in Example 8.5 with arbitrary finite subsets, retaining the same common utility function, then every Cournot path will reach an equilibrium in a finite number of steps. Therefore, one can argue that the problem illustrated by the example is just an artefact of the suboptimal way to introduce infinity: the behavior of transfinite dynamics is much closer to what happens in a finite model.

9.2. It is worth stressing once again: None of the results of this paper needs a *numeric* potential; moreover, in each of the "counterexamples" in Section 8, there is a numeric potential, which does not help. The upper semicontinuity of a Cournot potential does not ensure any better properties of best response dynamics than just ω-transitivity. And a continuous Cournot potential also seems not to have any advantage over an ω-transitive and lower semicontinuous one.

9.3. Similarly to the preceding remark, a "best-response potential" (Voorneveld, 2000) gives no advantage over a Cournot potential as defined here. The only difference is that the *continuity* of a best-response potential implies the upper hemicontinuity of all best response correspondences, hence Theorem 3.5 absorbs Theorem 3.6, and Examples 8.1 or 8.2 become impossible.

9.4. If the Cournot potential in Lemma 4.1 is *continuous*, then each r_i will be continuous too. Under the actual assumptions of the lemma such a claim would be wrong.

9.5. Theorem 2 of Jensen (2010) is neither weaker, nor stronger than any result of this paper. It establishes the convergence of sequential Cournot tâtonnement to Nash equilibria under an assumption concerning paths were the players consecutively replace one best response with another. The assumption is automatically satisfied if all best responses are single-valued, in which case our Theorem 4.4 is a bit stronger. It is worth noting that Example 8.2 shows Jensen's theorem to be, strictly speaking, wrong (upper semiconinuity of utility functions is not enough).

9.6. If we modify the constructions of Section 6, replacing the sum in (7) with the minimum, cf. Germeier and Vatel' (1974), then the leximin ordering on X_N will be a potential in the sense of (2) for coalition improvements, hence a Cournot potential as well. Since the ordering is not lower semicontinuous, our main results are inapplicable even though no counterexample is known. Funnily, aggregative games of Section 7 with $\sigma_i(x_{-i}) = \min_{j \neq i} x_j$ for all $i \in N$ or $\sigma_i(x_{-i}) = -\min_{j \neq i} x_j$ for all $i \in N$ also admit ω-transitive Cournot potentials. And the existence of a lower semicontinuous (partial) Cournot potential in every such game also remains neither proven, nor disproved so far.

9.7. The Cournot path leading nowhere in Example 8.5 needs a carefully chosen initial point. It does not matter here since the only objective of the example is to demonstrate the invalidity of straightforward extensions of Theorems 3.4 and 4.4. Powell (1973) also provides a more complicated example where such paths can be started from every point in an open subset.

9.8. Most likely, Example 8.6 can be modified so that only one cluster point will be an equilibrium (Δ in the definition of ψ should depend on ρ_1 and ρ_2). However, this is hardly important for anything. A much more intriguing question is whether Theorem 3.4 would still be wrong for $\#N > 2$, and Theorem 4.5 for $\#N > 3$, if all best responses were assumed to be single-valued *everywhere*. The constructions of Examples 8.6 and 8.7 seem not to allow an appropriate modification.

9.9. Everything in this paper is about games with ordinal preferences. For applications of the idea of potential games to the best responses in the context of cardinal utilities, see, e.g., Morris and Ui (2004).

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