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A model for coopetitive games

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Abstract

In the present introductory work we propose an original analytical model of coopetitive game. We shall suggest - after a study of the model - feasible solutions, in a coopetitive perspective, for the interests which drive the players in the game itself.

Keywords. Games and Economics, competition, cooperation, coopetition, normal form games

1 Organization of the paper.

The work is organized as follows:

- section 2 presents the original model of coopetitive game introduced in the literature by D. Carfì;
- section 3 proposes possible solutions concepts for the original model of coopetitive game;
- section 4 provides a first sample of coopetitive game in an intentionally simplified fashion (without direct strategic interactions among players) to emphasize the new role and procedures of coopetition;
- section 5 provides a second sample of coopetitive game, showing possible coopetitive solutions; we propose a linear model, with a direct strategic interactions among players;
- conclusions end up the paper.

The concept of coopetition was essentially devised at micro-economic level for strategic management solutions by Brandenburger and Nalebuff [1995], who suggest, given the competitive paradigm [Porter, 1985], to consider also a cooperative behavior to achieve a win-win outcome for both players.

2 Coopetitive games

2.1 Introduction

In this paper we develop and apply the mathematical model of a *coopetitive game* introduced by David Carfì in [4] and [3]. The idea of coopetitive game is already used, in a mostly intuitive and non-formalized way, in Strategic Management Studies (see for example Brandenburgher and Nalebuff).

2.1.1 The idea.

A coopetitive game is a game in which two or more players (participants) can interact *cooperatively and non-cooperatively at the same time.* Even Brandenburger and Nalebuff, creators of coopetition, did not define, precisely, a *quantitative way to implement coopetition* in the Game Theory context.

The problem to implement the notion of coopetition in Game Theory is summarized in the following question:

- *how do, in normal form games, cooperative and non-cooperative interactions can live together simultaneously, in a Brandenburger-Nalebuff sense?*

In order to explain the above question, consider a classic two-player normal-form gain game $G = (f, >)$ - such a game is a pair in which f is a vector valued function defined on a Cartesian product $E \times F$ with values in the Euclidean plane $\mathbb{R}^2$ and $>$ is the natural strict sup-order of the Euclidean plane itself (the sup-order is indicating that the game, with payoff function f, is a gain game and not a loss game). Let E and F be the strategy sets of the two players in the game G. The two players can choose the respective strategies $x \in E$ and $y \in F$

- cooperatively (exchanging information and making binding agreements);
- not-cooperatively (not exchanging information or exchanging information but without possibility to make binding agreements).

The above two behavioral ways are mutually exclusive, at least in normal-form games:

- the two ways cannot be adopted simultaneously in the model of normal-form game (without using convex probability mixtures, but this is not the way suggested by Brandenburger and Nalebuff in their approach);
- there is no room, in the classic normal form game model, for a simultaneous (non-probabilistic) employment of the two behavioral extremes *cooperation* and *non-cooperation.*

2.1.2 Towards a possible solution.

David Carfì ([4] and [3]) has proposed a manner to pass this *impasse*, according to the idea of coopetition in the sense of Brandenburger and Nalebuff. In a Carfì's coopetitive game model,

- the players of the game have their respective strategy-sets (in which they can choose cooperatively or not cooperatively);
- there is a common strategy set C containing other strategies (possibly of different type with respect to those in the respective classic strategy sets) that *must be chosen cooperatively*;
- the strategy set C can also be structured as a Cartesian product (similarly to the profile strategy space of normal form games), but in any case the strategies belonging to this new set C must be chosen cooperatively.

2.2 The model for n-players

We give in the following the definition of coopetitive game proposed by Carfì (in [4] and [3]).

Definition (of n-player coopetitive game). *Let $E = (E_i)_{i=1}^n$ be a finite n-family of non-empty sets and let C be another non-empty set. We define n-**player coopetitive gain game over the strategy support** (E, C) any pair $G = (f, >)$, where f is a vector function from the Cartesian product ${}^{\times}E \times C$ (here ${}^{\times}E$ denotes the classic strategy-profile space of n-player normal* *form games, i.e. the Cartesian product of the family E) into the n-dimensional Euclidean space $\mathbb{R}^n$ and $>$ is the natural sup-order of this last Euclidean space. The element of the set C will be called* ***cooperative strategies of the game****.*

A particular aspect of our coopetitive game model is that any coopetitive game G determines univocally a family of classic normal-form games and vice versa; so that any coopetitive game could be defined as a family of normal-form games. In what follows we precise this very important aspect of the model.

Definition (the family of normal-form games associated with a coopetitive game). *Let $G = (f, >)$ be a coopetitive game over a strategic support (E, C). And let*

$$g = (g_z)_{z \in C}$$

be the family of classic normal-form games whose member g_z is, for any cooperative strategy z in C, the normal-form game

$$G_z := (f(., z), >),$$

where the payoff function $f(., z)$ is the section

$$f(., z) : \ {}^{\times}E \to \mathbb{R}^n$$

of the function f, defined (as usual) by

$$f(., z)(x) = f(x, z),$$

for every point x in the strategy profile space ${}^{\times}E$. We call the family g (so defined) ***family of normal-form games associated with (or determined by) the game*** *G and we call* ***normal section*** *of the game G any member of the family g.*

We can prove this (obvious) theorem.

Theorem. *The family g of normal-form games associated with a coopetitive game G uniquely determines the game. In more rigorous and complete terms, the correspondence $G \mapsto g$ is a bijection of the space of all coopetitive games - over the strategy support (E, C) - onto the space of all families of normal form games - over the strategy support E - indexed by the set C.*

Proof. This depends totally from the fact that we have the following natural bijection between function spaces:

$$\mathcal{F}({}^{\times}E \times C, \mathbb{R}^n) \to \mathcal{F}(C, \mathcal{F}({}^{\times}E, \mathbb{R}^n)) : f \mapsto (f(., z))_{z \in C},$$

which is a classic result of theory of sets. ■

Thus, the exam of a coopetitive game should be equivalent to the exam of a whole family of normal-form games (in some sense we shall specify).

In this paper we suggest how this latter examination can be conducted and what are the solutions corresponding to the main concepts of solution which are known in the literature for the classic normal-form games, in the case of two-player coopetitive games.

2.3 Two players coopetitive games

In this section we specify the definition and related concepts of two-player coopetitive games; sometimes (for completeness) we shall repeat some definitions of the preceding section.

Definition (of coopetitive game). *Let E, F and C be three nonempty sets. We define* ***two player coopetitive gain game carried by the strategic triple*** *(E, F, C) any pair of the form $G = (f, >)$, where f is a function from the Cartesian product $E \times F \times C$ into the real Euclidean plane $\mathbb{R}^2$ and the binary relation $>$ is the usual sup-order of the Cartesian plane (defined component-wise, for every couple of points p and q, by $p > q$ iff $p_i > q_i$, for each index i).*

Remark (coopetitive games and normal form games). The difference among a two-player normal-form (gain) game and a two player coopetitive (gain) game is the fundamental presence of the third strategy Cartesian-factor C. The presence of this third set C determines a total change of perspective with respect to the usual exam of two-player normal form games, since we now have to consider a normal form game $G(z)$, for every element z of the set C; we have, then, to study an entire ordered family of normal form games in its own totality, and we have to define a new manner to study these kind of game families.

2.4 Terminology and notation

Definitions. *Let $G = (f, >)$ be a two player coopetitive gain game carried by the strategic triple (E, F, C). We will use the following terminologies:*

- *the function f is called the* ***payoff function of the game*** *G;*
- *the first component f_1 of the payoff function f is called* ***payoff function of the first player*** *and analogously the second component f_2 is called* ***payoff function of the second player****;*
- *the set E is said* ***strategy set of the first player*** *and the set F the* ***strategy set of the second player****;*
- *the set C is said the* ***cooperative (or common) strategy set of the two players****;*
- *the Cartesian product $E \times F \times C$ is called the* ***(coopetitive) strategy space of the game*** *G.*

Memento. The first component f_1 of the payoff function f of a coopetitive game G is the function of the strategy space $E \times F \times C$ of the game G into the real line $\mathbb{R}$ defined by the first projection

$$f_1(x, y, z) := \mathrm{pr}_1(f(x, y, z)),$$

for every strategic triple (x, y, z) in $E \times F \times C$; in a similar fashion we proceed for the second component f_2 of the function f.

Interpretation. We have:

- two players, or better an ordered pair $(1, 2)$ of players;
- anyone of the two players has a strategy set in which to choose freely his own strategy;
- the two players can/should *cooperatively* choose strategies z in a third common strategy set C;
- the two players will choose (after the exam of the entire game G) their cooperative strategy z in order to maximize (in some sense we shall define) the vector gain function f.

2.5 Normal form games of a coopetitive game

Let G be a coopetitive game in the sense of above definitions. For any cooperative strategy z selected in the cooperative strategy space C, there is a corresponding normal form gain game

$$G_z = (p(z), >),$$

upon the strategy pair (E, F), where the payoff function $p(z)$ is the section

$$f(., z) : E \times F \to \mathbb{R}^2,$$

of the payoff function f of the coopetitive game - the section is defined, as usual, on the competitive strategy space $E \times F$, by

$$f(., z)(x, y) = f(x, y, z),$$

for every bi-strategy (x, y) in the bi-strategy space $E \times F$.

Let us formalize the concept of game-family associated with a coopetitive game.

Definition (the family associated with a coopetitive game). *Let $G = (f, >)$ be a two player coopetitive gain game carried by the strategic triple (E, F, C). We naturally can associate with the game G a family $g = (g_z)_{z \in C}$ of normal-form games defined by*

$$g_z := G_z = (f(., z), >),$$

for every z in C, which we shall call ***the family of normal-form games associated with the coopetitive game G****.*

Remark. It is clear that with any above family of normal form games

$$g = (g_z)_{z \in C},$$

with $g_z = (f(., z), >)$, we can associate:

- a family of payoff spaces
$$(\mathrm{im} f(., z))_{z \in C},$$
with members in the payoff universe $\mathbb{R}^2$;
- a family of Pareto maximal boundary
$$(\partial^* G_z)_{z \in C},$$
with members contained in the payoff universe $\mathbb{R}^2$;
- a family of suprema
$$(\sup G_z)_{z \in C},$$
with members belonging to the payoff universe $\mathbb{R}^2$;
- a family of Nash zones
$$(\mathcal{N}(G_z))_{z \in C};$$
with members contained in the strategy space $E \times F$;
- a family of conservative bi-values
$$v^{\#} = (v_z^{\#})_{z \in C};$$
in the payoff universe $\mathbb{R}^2$.

And so on, for every meaningful known feature of a normal form game.

Moreover, we can interpret any of the above families as *set-valued paths* in the strategy space $E \times F$ or in the payoff universe $\mathbb{R}^2$.

It is just the study of these induced families which becomes of great interest in the examination of a coopetitive game G and which will enable us to define (or suggest) the various possible solutions of a coopetitive game.

3 Solutions of a coopetitive game

3.1 Introduction

The two players of a coopetitive game G - according to the general economic principles of *monotonicity of preferences* and of non-satiation - should choose the cooperative strategy z in C in order that:

- the reasonable Nash equilibria of the game G_z are f-preferable than the reasonable Nash equilibria in each other game $G_{z'}$;
- the supremum of G_z is greater (in the sense of the usual order of the Cartesian plane) than the supremum of any other game $G_{z'}$;
- the Pareto maximal boundary of G_z is higher than that of any other game $G_{z'}$;
- the Nash bargaining solutions in G_z are f-preferable than those in $G_{z'}$;
- *in general, fixed a common kind of solution for any game G_z, say $S(z)$ the set of these kind of solutions for the game G_z, we can consider the problem to find all the optimal solutions (in the sense of Pareto) of the set valued path S, defined on the cooperative strategy set C. Then, we should face the problem of* ***selection of reasonable Pareto strategies*** *in the set-valued path S via proper selection methods (Nash-bargaining, Kalai-Smorodinsky and so on).*

Moreover, we shall consider the maximal Pareto boundary of the payoff space $\mathrm{im}(f)$ as an appropriate zone for the bargaining solutions.

The payoff function of a two person coopetitive game is (as in the case of normal-form game) a vector valued function with values belonging to the Cartesian plane $\mathbb{R}^2$. We note that in general the above criteria are multi-criteria and so they will generate multi-criteria optimization problems.

In this section we shall define rigorously some kind of solution, for two player coopetitive games, based on a bargaining method, namely a Kalai-Smorodinsky bargaining type. Hence, first of all, we have to precise what kind of bargaining method we are going to use.

3.2 Bargaining problems

In this paper, we shall propose and use the following original *extended* (and quite general) definition of bargaining problem and, consequently, a natural generalization of Kalai-Smorodinsky solution. In the economic literature, several examples of *extended bargaining problems* and *extended Kalai-Smorodinski solutions* are already presented. The essential root of these various extended versions of bargaining problems is the presence of *utopia points* not-directly constructed by the disagreement points and the strategy constraints. Moreover, the Kalai-type solution, of such extended bargaining problems, is always defined as a Pareto maximal point belonging to the segment joining the disagreement point with the utopia point (if any such Pareto point does exist): we shall follow the same way. In order to find suitable new win-win solutions of our realistic coopetitive economic problems, we need such new kind of versatile extensions. For what concerns the existence of our new extended Kalai solutions, for the economic problems we are facing, we remark that conditions of compactness and strict convexity will naturally hold; we remark, otherwise, that, in this paper, we are not interested in proving general or deep mathematical results, but rather to find reasonable solutions for new economic coopetitive context.

Definition (of bargaining problem). *Let S be a subset of the Cartesian plane $\mathbb{R}^2$ and let a and b be two points of the plane with the following properties:*

- *they belong to the small interval containing S, if this interval is defined (indeed, it is well defined if and only if S is bounded and it is precisely the interval $[\inf S, \sup S]_{\leq}$);*
- *they are such that $a < b$;*
- *the intersection*
$$[a, b]_{\leq} \cap \partial^* S,$$
among the interval $[a, b]_{\leq}$ with end points a and b (it is the set of points greater than a and less than b, ***it is not*** *the segment $[a, b]$) and the maximal boundary of S is non-empty.*

In these conditions, we call ***bargaining problem on S corresponding to the pair of extreme points*** *(a, b), the pair*
$$P = (S, (a, b)).$$
Every point in the intersection among the interval $[a, b]_{\leq}$ and the Pareto maximal boundary of S is called ***possible solution of the problem*** *P. Some time the first extreme point of a bargaining problem is called* ***the initial point of the problem*** *(or* ***disagreement point*** *or* ***threat point****) and the second extreme point of a bargaining problem is called* ***utopia point*** *of the problem.*

In the above conditions, when S is convex, the problem P is said to be convex and for this case we can find in the literature many existence results for solutions of P enjoying prescribed properties (Kalai-Smorodinsky solutions, Nash bargaining solutions and so on ...).

Remark. Let S be a subset of the Cartesian plane $\mathbb{R}^2$ and let a and b two points of the plane belonging to the smallest interval containing S and such that $a \leq b$. Assume the Pareto maximal boundary of S be non-empty. If a and b are a lower bound and an upper bound of the maximal Pareto boundary, respectively, then the intersection
$$[a, b]_{\leq} \cap \partial^* S$$
is obviously not empty. In particular, if a and b are the extrema of S (or the extrema of the Pareto boundary $S^* = \partial^* S$) we can consider the following bargaining problem
$$P = (S, (a, b)), \ (\text{or } P = (S^*, (a, b)))$$
and we call this particular problem a *standard bargaining problem on S* (or *standard bargaining problem on the Pareto maximal boundary S^**).

3.3 Kalai solution for bargaining problems

Note the following property.

Property. *If $(S,(a,b))$ is a bargaining problem with $a<b$, then there is at most one point in the intersection*

$$[a,b]\cap \partial^* S,$$

where $[a,b]$ is the ***segment joining the two points*** *a and b.*

Proof. Since if a point p of the segment $[a,b]$ belongs to the Pareto boundary $\partial^* S$, no other point of the segment itself can belong to Pareto boundary, since the segment is a totally ordered subset of the plane (remember that $a<b$). ■

Definition (Kalai-Smorodinsky). *We call* ***Kalai-Smorodinsky solution*** *(or* ***best compromise solution) of the bargaining problem*** *$(S,(a,b))$ the unique point of the intersection*

$$[a,b]\cap \partial^* S,$$

if this intersection is non empty.

So, in the above conditions, the Kalai-Smorodinsky solution k (if it exists) enjoys the following property: there is a real r in $[0,1]$ such that

$$k=a+r(b-a),$$

or

$$k-a=r(b-a),$$

hence

$$\frac{k_2-a_2}{k_1-a_1}=\frac{b_2-a_2}{b_1-a_1},$$

if the above ratios are defined; these last equality is the *characteristic property of Kalai-Smorodinsky solutions.*

We end the subsection with the following definition.

Definition (of Pareto boundary). *We call* ***Pareto boundary*** *every subset M of an ordered space which has only pairwise incomparable elements.*

3.4 Nash (proper) solution of a coopetitive game

Let $N:=\mathcal{N}(G)$ be the union of the Nash-zone family of a coopetitive game G, that is the union of the family $(\mathcal{N}(G_z))_{z\in C}$ of all Nash-zones of the game family $g=(g_z)_{z\in C}$ associated to the coopetitive game G. We call *Nash path of the game G* the multi-valued path

$$z\mapsto \mathcal{N}(G_z)$$

and Nash zone of G the trajectory N of the above multi-path. Let N^* be the Pareto maximal boundary of the Nash zone N. We can consider the bargaining problem

$$P_{\mathcal{N}}=(N^*,\inf(N^*),\sup(N^*)).$$

Definition. *If the above bargaining problem $P_{\mathcal{N}}$ has a Kalai-Smorodinsky solution k, we say that k is the properly coopetitive solution of the coopetitive game G.*

The term "properly coopetitive" is clear:

- *this solution k is determined by cooperation on the common strategy set C and to be selfish (competitive in the Nash sense) on the bi-strategy space $E \times F$.*

3.5 Bargaining solutions of a coopetitive game

It is possible, for coopetitive games, to define other kind of solutions, which are not properly coopetitive, but realistic and sometime affordable. These kind of solutions are, we can say, *super-cooperative*.

Let us show some of these kind of solutions.

Consider a coopetitive game G and

- its Pareto maximal boundary M and the corresponding pair of extrema (a_M, b_M);
- the Nash zone $\mathcal{N}(G)$ of the game in the payoff space and its extrema (a_N, b_N);
- the conservative set-value $G^\#$ (the set of all conservative values of the family g associated with the coopetitive game G) and its extrema $(a^\#, b^\#)$.

We call:

- ***Pareto compromise solution of the game*** G *the best compromise solution (K-S solution) of the problem*
$$(M, (a_M, b_M)),$$
if this solution exists;
- ***Nash-Pareto compromise solution of the game*** G *the best compromise solution of the problem*
$$(M, (b_N, b_M))$$
if this solution exists;
- ***conservative-Pareto compromise solution of the game*** G *the best compromise of the problem*
$$(M, (b^\#, b_M))$$
if this solution exists.

3.6 Transferable utility solutions

Other possible compromises we suggest are the following.

Consider *the transferable utility Pareto boundary M of the coopetitive game G*, that is the set of all points p in the Euclidean plane (universe of payoffs), between the extrema of G, such that their sum
$$+(p) := p_1 + p_2$$
is equal to the maximum value of the addition $+$ of the real line $\mathbb{R}$ over the payoff space $f(E \times F \times C)$ of the game G.

Definition (TU Pareto solution). *We call* ***transferable utility compromise solution of the coopetitive game*** G *the solution of any bargaining problem $(M, (a, b))$, where*

- *a and b are points of the smallest interval containing the payoff space of G*

- *b is a point strongly greater than a;*

- *M is the transferable utility Pareto boundary of the game G;*

- *the points a and b belong to different half-planes determined by M.*

Note that the above fourth axiom is equivalent to require that the segment joining the points a and b intersect M.

3.7 Win-win solutions

In the applications, if the game G has a member G_0 of its family which can be considered as an "initial game" - in the sense that the pre-coopetitive situation is represented by this normal form game G_0 - the aims of our study (following the standard ideas on coopetitive interactions) are

- to "enlarge the pie";

- to obtain a win-win solution with respect to the initial situation.

So that we will choose as a threat point a in TU problem $(M, (a, b))$ the supremum of the initial game G_0.

Definition (of win-win solution). *Let (G, z_0) be a* ***coopetitive game with an initial point****, that is a coopetitive game G with a fixed common strategy z_0 (of its common strategy set C). We call the game G_{z_0} as* ***the initial game of*** *(G, z_0). We call* ***win-win solution of the game*** *(G, z_0) any strategy profile $s = (x, y, z)$ such that the payoff of G at s is strictly greater than the supremum L of the* ***payoff core*** *of the initial game $G(z_0)$.*

Remark 1. The payoff core of a normal form gain game G is the portion of the Pareto maximal boundary G^* of the game which is greater than the conservative bi-value of G.

Remark 2. From an applicative point of view, the above requirement (to be strictly greater than L) is very strong. More realistically, we can consider as win-win solutions those strategy profiles which are strictly greater than any reasonable solution of the initial game G_{z_0}.

Remark 3. Strictly speaking, a win-win solution could be not Pareto efficient: it is a situation in which the players both gain with respect to an initial condition (and this is exactly the idea we follow in the rigorous definition given above).

Remark 4. In particular, observe that, if the collective payoff function

$$^+(f) = f_1 + f_2$$

has a maximum (on the strategy profile space S) strictly greater than the collective payoff L_1+L_2 at the supremum L of the payoff core of the game G_{z_0}, the portion $M(> L)$ of Transferable Utility Pareto boundary M which is greater than L is non-void and it is a segment. So that we can choose as a threat point a in our problem $(M, (a, b))$ the supremum L of the payoff core of the initial game G_0 *to obtain some compromise solution.*

3.7.1 Standard win-win solution.

A natural choice for the utopia point b is the supremum of the portion $M_{\geq a}$ of the transferable utility Pareto boundary M which is upon (greater than) this point a:

$$M_{\geq a} = \{m \in M : m \geq a\}.$$

3.7.2 Non standard win-win solution.

Another kind of solution can be obtained by choosing b as the supremum of the portion of M that is bounded between the minimum and maximum value of that player i that gains more in the coopetitive interaction, in the sense that

$$\max(\mathrm{pr}_i(\mathrm{im} f)) - \max(\mathrm{pr}_i(\mathrm{im} f_0)) > \max(\mathrm{pr}_{3-i}(\mathrm{im} f)) - \max(\mathrm{pr}_{3-i}(\mathrm{im} f_0)).$$

3.7.3 Final general remarks.

In the development of a coopetitive game, we consider:

- a first virtual phase, in which the two players make a binding agreement on what cooperative strategy z should be selected from the cooperative set C, in order to respect their own rationality.
- then, a second virtual phase, in which the two players choose their strategies forming the profile (x, y) to implement in the game $G(z)$.

Now, in the second phase of our coopetitive game G we consider the following 4 possibilities:

1. the two players are non-cooperative in the second phase and they do or do not exchange info, but the players choose (in any case) Nash equilibrium strategies for the game $G(z)$; in this case, for some rationality reason, the two players have devised that the chosen equilibrium is the better equilibrium choice in the entire game G; we have here only one binding agreement in the entire development of the game;
2. the two players are cooperative also in the second phase and they make a binding agreement in order to choose a Pareto payoff on the coopetitive Pareto boundary; in this case we need two binding agreements in the entire development of the game;
3. the two players are cooperative also in the second phase and they make two binding agreements, in order to reach the Pareto payoff (on the coopetitive Pareto boundary) with maximum collective gain (first agreement) and to share the collective gain according to a certain subdivision (second agreement); in this case we need three binding agreements in the entire development of the game;
4. the two players are non-cooperative in the second phase (and they do or do not exchange information), the player choose (in any case) Nash equilibrium strategies; the two players have devised that the chosen equilibrium is the equilibrium with maximum collective gain and they make only one binding agreement to share the collective gain according to a certain subdivision; in this case we need two binding agreements in the entire development of the game.

4 Dynamics

Consider a coopetitive game $f : S \times C \to \mathbb{R}^n$.

The function

$$\Phi : C \to \mathrm{Diff}(M) : z \mapsto \Phi(z),$$

where

$$\Phi(z) : f(., z_0)(S) \to f(., z)(S) : f(x, z_0) \mapsto f(x, z),$$

is well defined if, for every point P in $f(., z_0)$, we have

$$f(x, z) = f(x', z),$$

for any $z \in C$ and any $x, x' \in S$ such that $f(x, z_0) = f(x', z_0) = P$.

Moreover, if C is the real interval $[a, b]$, note that, for every x_0 in the strategy space S, the curve

$$f(x_0, .) : C \to \mathbb{R}^2 : z \mapsto f(x_0, z)$$

is smooth and well defined, we call it the payoff evolution of the initial strategy x_0. In general, we cannot consider this evolution as an orbit of the initial payoff $f(x_0, a)$, but, if we define the z-state spaces

$$M_z = \{(x, X) \in S \times f(., z)(S) : X = f(x, z)\}$$

and more general the fiber-space

$$\mathcal{F} = (\mathcal{E}, C, \rho),$$

where $\mathcal{E}$ is the disjoint union of the family M, that is

$$\mathcal{E} = \{(z, x, X) \in C \times S \times f(S) : X = f(x, z)\},$$

and the projection is the map

$$\rho : \mathcal{E} \to C : (z, x, X) \mapsto z,$$

we have the evolution of the element (z_0, x_0, X) into the fibration $\mathcal{E}$, defined by

$$\gamma(x_0, X) : C \to \mathcal{E} : z \mapsto (z, x_0, f(x_0, z)).$$

5 First example

5.1 Payoff function of the game

We consider a coopetitive gain game with payoff function given by

$$\begin{aligned} f(x, y, z) &= (x + 1/(x + 1) - z, (1 + m)y + (1 + n)z) = \\ &= (x + 1/(x + 1), (1 + m)y) + z(-1, 1 + n) \end{aligned}$$

for every x, y, z in $[0, 1]$.

5.2 Study of the game $G = (f, >)$

Note that, fixed a cooperative strategy z in U, the section game $G(z) = (p(z), >)$ with payoff function $p(z)$, defined on the square $U \times U$ by

$$p(z)(x, y) = f(x, y, z),$$

is the translation of the game $G(0)$ by the "cooperative" vector

$$v(z) = z(-1, 1 + n),$$

so that we can study the initial game $G(0)$ and then we can translate the various informations of the game $G(0)$ by the vector $v(z)$.

So, let us consider the initial game $G(0)$. The strategy square $S = U^2$ of $G(0)$ has vertices 0_2, e_1, 1_2 and e_2, where 0_2 is the origin, e_1 is the first canonical vector $(1, 0)$, 1_2 is the sum of the two canonical vectors $(1, 1)$ and e_2 is the second canonical vector $(0, 1)$.

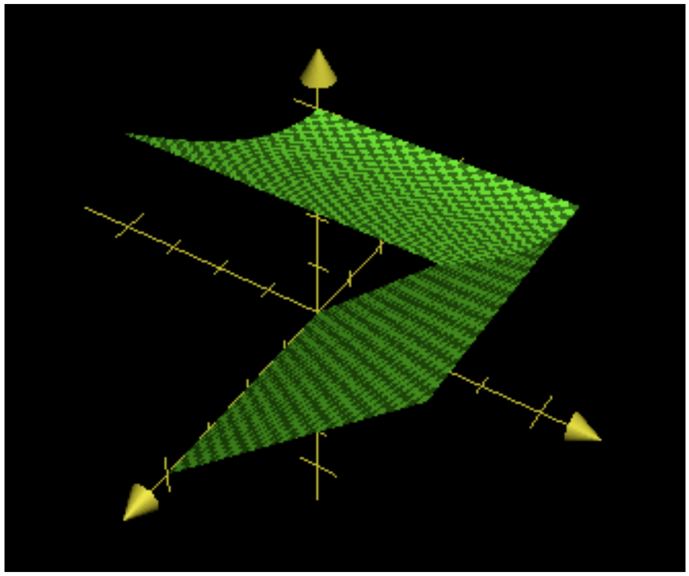

Figure 1: 3D representation of the initial game $(f(.,0),<)$.

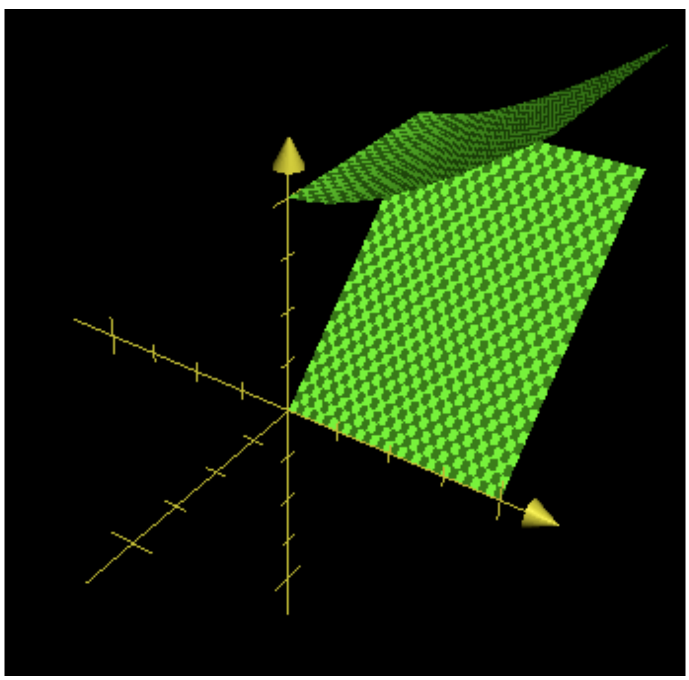

Figure 2: 3D representation of the initial game $(f(.,0),<)$.

5.3 Topological Boundary of the payoff space of G_0

In order to determine the Pareto boundary of the payoff space, we shall use the technics introduced by D. Carfi in [2]. We have

$$p_0(x, y) = (x + 1/(x + 1), (1 + m)y),$$

for every x, y in $[0, 1]$. The transformation of the side $[0, e_1]$ is the trace of the (parametric) curve $c : U \to \mathbb{R}^2$ defined by

$$c(x) = f(x, 0, 0) = (x + 1/(x + 1), 0),$$

that is the segment

$$[f(0), f(e_1)] = [(1, 0), (3/2, 0)].$$

The transformation of the segment $[0, e_2]$ is the trace of the curve $c : U \to \mathbb{R}^2$ defined by

$$c(y) = f(0, y, 0) = (1, (1 + m)y),$$

that is the segment

$$[f(0), f(e_2)] = [(1, 0), (1, 1 + m)].$$

The transformation of the segment $[e_1, 1_2]$ is the trace of the curve $c : U \to \mathbb{R}^2$ defined by

$$c(y) = f(1, y, 0) = (1 + 1/2, (1 + m)y),$$

that is the segment

$$[f(e_1), f(1_2)] = [(3/2, 0), (3/2, 1 + m)].$$

Critical zone of $G(0)$. The Critical zone of the game $G(0)$ is empty. Indeed the Jacobian matrix is

$$J_f(x, y) = \begin{pmatrix} 1 + (1 + x)^{-2} & 0 \\ 0 & 1 + m \end{pmatrix},$$

which is invertible for every x, y in U.

Payoff space of the game $G(0)$. So, the payoff space of the game $G(0)$ is the transformation of topological boundary of the strategic square, that is the rectangle with vertices $f(0, 0)$, $f(e_1)$, $f(1, 1)$ and $f(e_2)$.

Nash equilibria. The unique Nash equilibrium is the bistrategy $(1, 1)$. Indeed,

$$1 + (1 + x)^{-2} > 0$$

so the function f_1 is increasing with respect to the first argument and analogously

$$1 + m > 0$$

so that the Nash equilibrium is $(1, 1)$.

5.4 The payoff space of the coopetitive game G

The image of the payoff function f, is the union of the family of payoff spaces

$$(\mathrm{im} p_z)_{z \in C},$$

that is the convex envelope of the union of the image $p_0(S)$ (S is the square $U \times U$) and of its translation by the vector $v(1)$, namely the payoff space $p_1(S)$: the image of f is an hexagon with vertices $f(0, 0)$, $f(e_1)$, $f(1, 1)$ and their translations by $v(1)$.

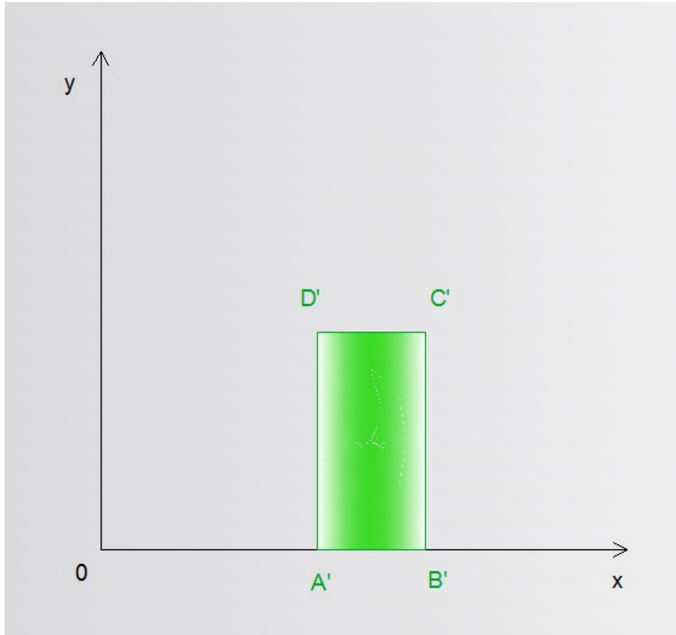

Figure 3: Initial payoff space of the game $(f, <)$.

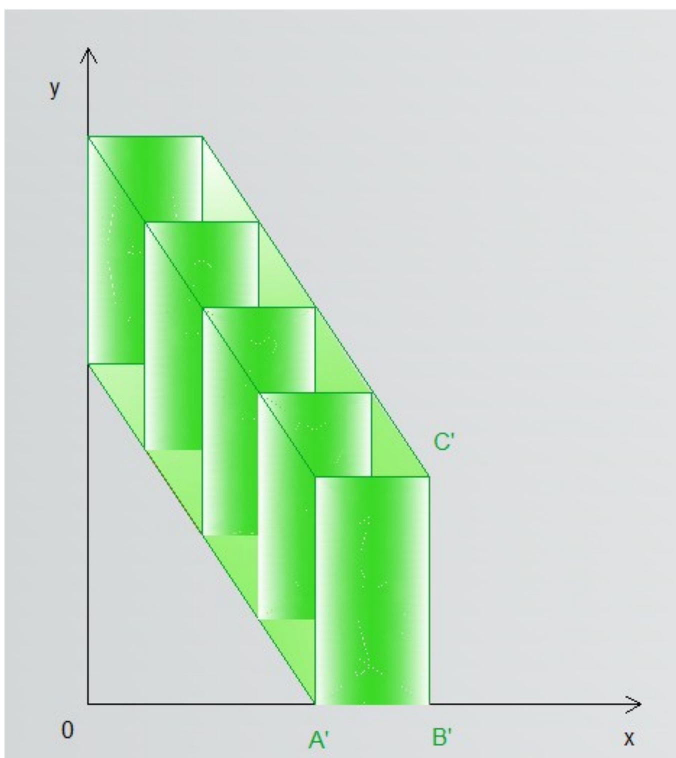

Figure 4: Payoff space of the game $(f, <)$.

5.5 Pareto maximal boundary of payoff space of G

The Pareto sup-boundary of the coopetitive payoffspace $f(S)$ is the segment $[P', Q']$, where $P' = f(1,1)$ and

$$Q' = P' + v(1).$$

Possibility of global growth. It is important to note that the absolute slope of the Pareto (coopetitive) boundary is $1+n$. Thus the collective payoff $f_1 + f_2$ of the game is not constant on the Pareto boundary and, therefore, the game implies the possibility of a global growth.

Trivial bargaining solutions. The Nash bargaining solution on the segment $[P', Q']$ with respect to the infimum of the Pareto boundary and the Kalai-Smorodinsky bargaining solution on the segment $[P', Q']$, with respect to the infimum and the supremum of the Pareto boundary, coincide with the medium point of the segment $[P', Q']$. This solution is not acceptable from the first player point of view, it is collectively better than the supremum of G_0 but it is disadvantageous for first player (it suffers a loss!): this solution can be thought as a rebalancing solution but it is not realistically implementable.

5.6 Transferable utility solution

In this coopetitive context it is more convenient to adopt a transferable utility solution, indeed:

- the point of maximum collective gain on the whole of the coopetitive payoff space is the point
$$Q' = (1/2, 2+m+n).$$

5.6.1 Rebalancing win-win best compromise solution

Thus we propose a rebalancing win-win kind of coopetitive solution, as it follows (in the case $m = 0$):

1. we consider the portion s of transferable utility Pareto boundary
$$M := (0, 5/2+n) + \mathbb{R}(1, -1),$$
obtained by intersecting M itself with the strip determined (spanned by convexifying) by the straight lines $e_2 + \mathbb{R}e_1$ and
$$(2+n)e_2 + \mathbb{R}e_1,$$
these are the straight lines of maximum gain for the second player in games $G(0)$ and G respectively.

2. we consider the Kalai-Smorodinsky segment s' of vertices $(3/2, 1)$ - supremum of the game $G(0)$ - and the supremum of the segment s.

3. our best payoff coopetitive compromise is the unique point K in the intersection of segments s and s', that is the best compromise solution of the bargaining problem
$$(s, (\sup G_0, \sup s)).$$

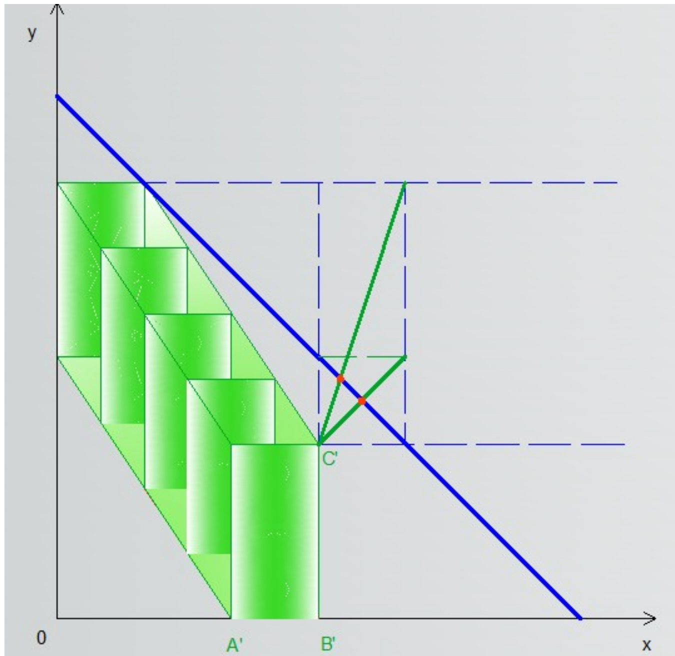

Figure 5: Two Kalai win-win solutions of the game $(f, <)$, represented with $n = 1/2$.

5.7 Win-win solution

This best payoff coopetitive compromise K represents a win-win solution with respect to the initial supremum $(3/2, 1)$. So that, as we repeatedly said, *also first player can increase its initial profit from coopetition.*

Win-win strategy procedure. The win-win payoff K can be obtained (by chance) in a properly coopetitive fashion in the following way:

- 1) the two players agree on the cooperative strategy 1 of the common set C;
- 2) the two players implement their respective Nash strategies of game $G(1)$; the unique Nash equilibrium of $G(1)$ is the bistrategy $(1, 1)$;
- 3) finally, they share the "social pie"

$$5/2 + n = (f_1 + f_2)(1, 1, 1),$$

in a **cooperative fashion** (by contract) according to the decomposition K.

6 The second example

6.0.1 Main Strategic assumptions.

We assume that:

- any real number x, belonging to the interval $E := [0, 3]$, represents a possible strategy of first player;

- any real number y, in the same interval $F := E$, represents a possible strategy of the second player;
- any real number z, again in the interval $C = [0, 2]$, can be a possible cooperative strategy of the two players.

6.1 Payoff function of the game

We consider a coopetitive gain game with payoff function $f : S \to \mathbb{R}^2$, given by

$$\begin{aligned} f(x, y, z) &= (2 + x - y/3 - z, 2 - 2x/3 + (1 + m)y + (1 + n)z) = \\ &= (2, 2) + (x - y/3, -2x/3 + (1 + m)y) + z(-1, 1 + n), \end{aligned}$$

for every (x, y, z) in $S := [0, 3]^2 \times [0, 2]$.

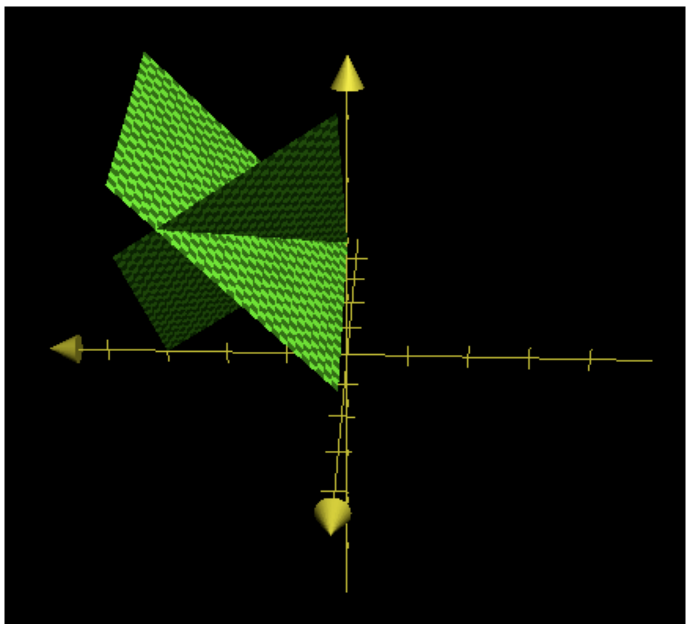

Figure 6: 3D representation of $(f, <)$.

6.2 Study of the second game $G = (f, >)$

Note that, fixed a cooperative strategy z in $2U$, the section game $G(z) = (p(z), >)$ with payoff function $p(z)$, defined on the square E^2 by

$$p(z)(x, y) := f(x, y, z),$$

is the translation of the game $G(0)$ by the "cooperative" vector

$$v(z) = z(-1, 1 + n),$$

so that, we can study the initial game $G(0)$ and then we can translate the various informations of the game $G(0)$ by the vectors $v(z)$, to obtain the corresponding information for the game $G(z)$.

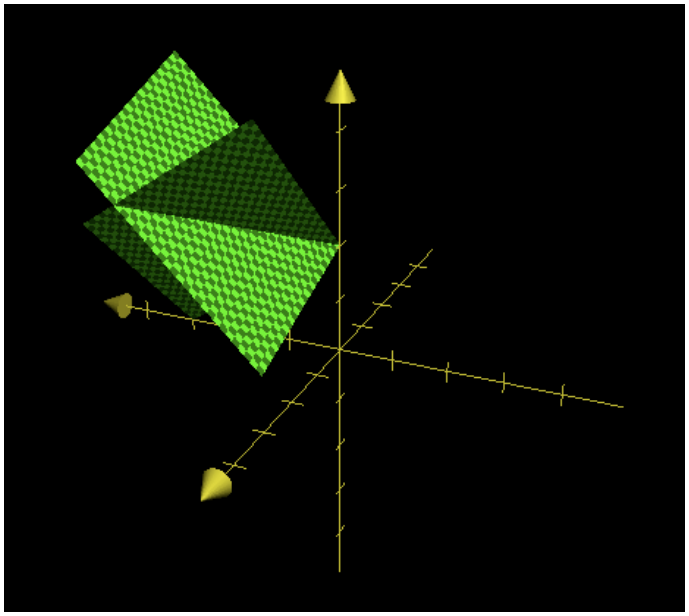

Figure 7: 3D representation of $(f, <)$.

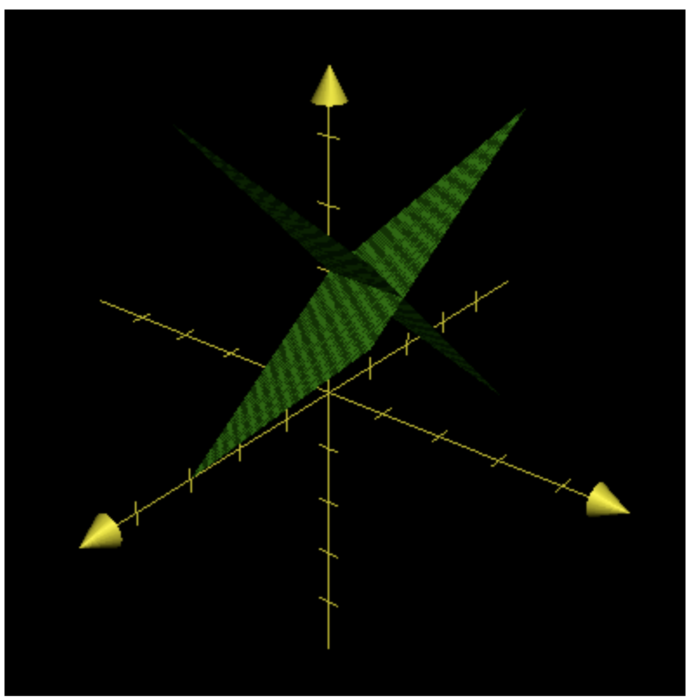

Figure 8: 3D representation of $(f, <)$.

So, let us consider the initial game $G(0)$. The strategy square E^2 of $G(0)$ has vertices 0_2, $3e_1$, 3_2 and $3e_2$, where 0_2 is the origin of the plane $\mathbb{R}^2$, e_1 is the first canonical vector $(1,0)$, 3_2 is the vectors $(3,3)$ and e_2 is the second canonical vector.

6.3 Topological Boundary of the payoff space of G_0

In order to determine the the payoff space of the linear game it is sufficient to transform the four vertices of the strategy square (the game is an affine invertible game), the critical zone is empty.

6.3.1 Payoff space of the game $G(0)$.

So, the payoff space of the game $G(0)$ is the transformation of the topological boundary of the strategy square, that is the parallelogram with vertices $f(0,0)$, $f(3e_1)$, $f(3,3)$ and $f(3e_2)$. As we show in the below figure 9.

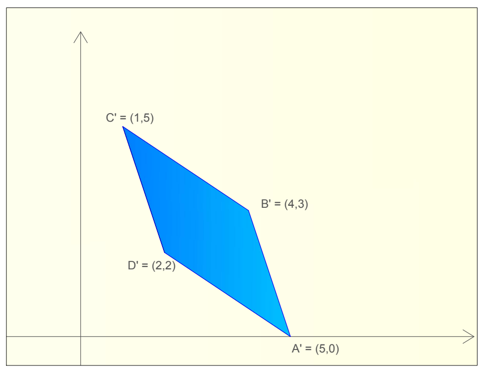

Figure 9: Initial payoff space of the game $(f, <)$.

6.3.2 Nash equilibria.

The unique Nash equilibrium is the bistrategy $(3,3)$. Indeed, the function f_1 is linear increasing with respect to the first argument and analogously the function f_2 is linear and increasing with respect to the second argument.

6.4 The payoff space of the coopetitive game G

The image of the payoff function f, is the union of the family of payoff spaces

$$(\mathrm{im} p_z)_{z \in C},$$

that is the convex envelope of the union of the image $p_0(E^2)$ and of its translation by the vector $v(2)$, namely the payoff space $p_2(E^2)$: the image of f is an hexagon with vertices $f(0,0)$, $f(3e_1)$, $f(3,3)$ and their translations by $v(2)$. As we show below.

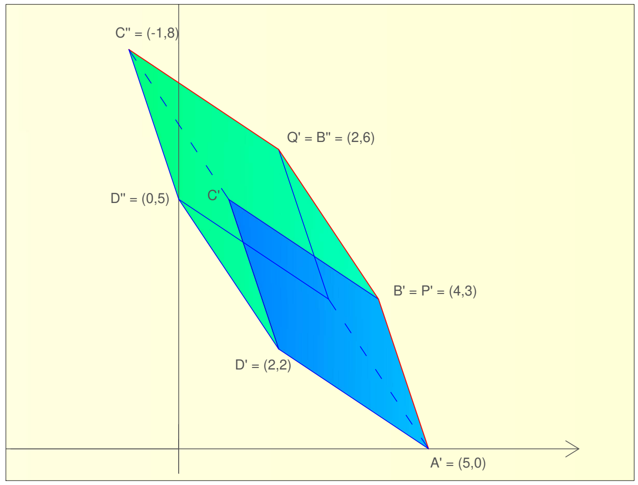

Figure 10: Payoff space of the game $(f, <)$.

6.5 Pareto maximal boundary of the payoff space of G

The Pareto sup-boundary of the coopetitive payoff space $f(S)$ is the union of the segments $[A', B']$, $[P', Q']$ and $[Q', C'']$, where $P' = f(3, 3, 0)$ and

$$Q' = P' + v(2).$$

6.5.1 Possibility of global growth.

It is important to note that the absolute slopes of the segments $[A', B']$, $[P', Q']$ of the Pareto (coopetitive) boundary are strictly greater than 1. Thus the collective payoff $f_1 + f_2$ of the game is not constant on the Pareto boundary and, therefore, the game implies the possibility of a transferable utility global growth.

6.5.2 Trivial bargaining solutions.

The Nash bargaining solution on the entire payoff space, with respect to the infimum of the Pareto boundary and the Kalai-Smorodinsky bargaining solution, with respect to the infimum and the supremum of the Pareto boundary, are not acceptable for first player: they are collectively (TU) better than the Nash payoff of G_0 but they are disadvantageous for the first player (it suffers a loss!): these solutions could be thought as rebalancing solutions, but they are not realistically implementable.

6.6 Transferable utility solutions

In this coopetitive context it is more convenient to adopt a transferable utility solution, indeed:

- the point of maximum collective gain on the whole of the coopetitive payoff space is the point $Q' = (2, 6)$.

6.6.1 Rebalancing win-win solution relative to maximum gain for the second player in G

Thus we propose a rebalancing win-win coopetitive solution relative to maximum gain for the second player in G, as it follows (in the case $m = 0$):

1. we consider the portion s of transferable utility Pareto boundary
$$M := Q' + \mathbb{R}(1, -1),$$
obtained by intersecting M itself with the strip determined (spanned by convexifying) by the straight lines $P' + \mathbb{R}e_1$ and $C'' + \mathbb{R}e_1$, *these are the straight lines of Nash gain for the second player in the initial game* $G(0)$ *and of maximum gain for the second player in* G, *respectively*.

2. we consider the Kalai-Smorodinsky segment s' of vertices B' - Nash payoff of the game $G(0)$ - and the supremum of the segment s.

3. our best payoff rebalancing coopetitive compromise is the unique point K in the intersection of segments s and s', that is the best compromise solution of the bargaining problem $(s, (B', \sup s))$.

Figure 11 below shows the above extended Kalai-Smorodinsky solution K and the Kalai-Smorodinsky solution K' of the classic bargaining problem (M, B'). It is evident that the distribution K is a rebalancing solution in favor of the second player with respect to the classic solution K'.

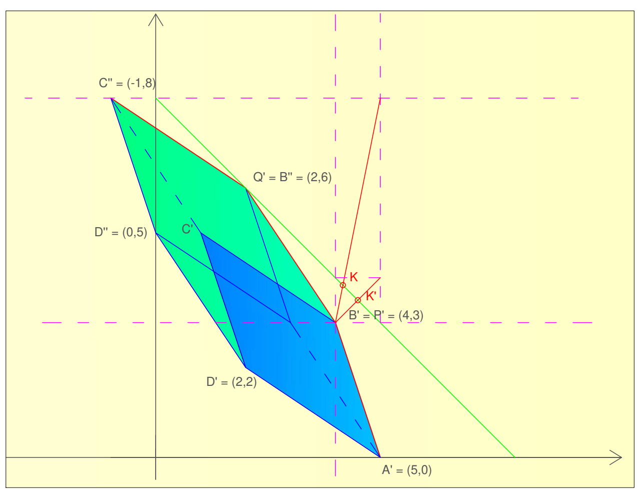

Figure 11: Two Kalai win-win solutions of the game $(f, <)$, represented with $n = 1/2$.

6.6.2 Rebalancing win-win solution relative to maximum Nash gain for the second player

We propose here a more realistic rebalancing win-win coopetitive solution relative to maximum Nash gain for the second player in G, as it follows (again in the case $m = 0$):

1. we consider the portion s of transferable utility Pareto boundary

$$M := Q' + \mathbb{R}(1, -1),$$

   obtained by intersecting M itself with the strip determined (spanned by convexifying) by the straight lines $P' + \mathbb{R}e_1$ and $Q' + \mathbb{R}e_1$, *these are the straight lines of Nash gain for the second player in the initial game $G(0)$ and of maximum Nash gain for the second player in G, respectively.*

2. we consider the Kalai-Smorodinsky segment s' of vertices B' - Nash payoff of the game $G(0)$ - and the supremum of the segment s.

3. our best payoff rebalancing coopetitive compromise is the unique point K in the intersection of segments s and s', that is the best compromise solution of the bargaining problem $(s, (B', \sup s))$.

Figure 12 below shows the above extended Kalai-Smorodinsky solution K and the Kalai-Smorodinsky solution K' of the classic bargaining problem (M, B'). The new distribution K is a rebalancing solution in favor of the second player, more realistic than the previous rebalancing solution.

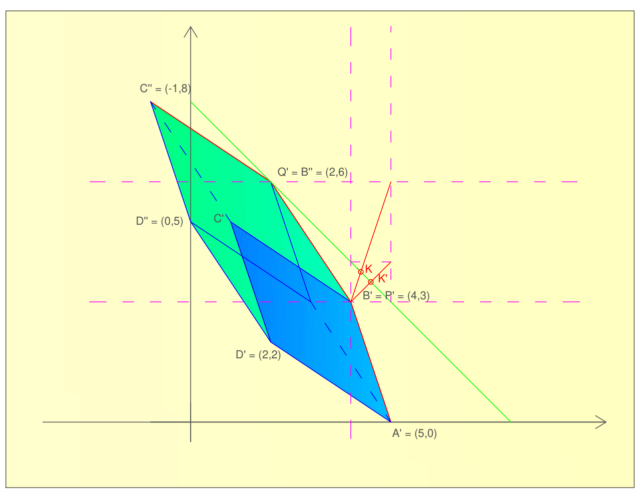

Figure 12: Two Kalai win-win solutions of the game $(f, <)$, represented with $n = 1/2$.

6.7 Win-win solution

The payoff extended Kalai-Smorodinsky solutions K represent win-win solutions, with respect to the initial Nash gain B'. So that, as we repeatedly said, *also first player can increase its initial profit from coopetition.*

6.7.1 Win-win strategy procedure.

The win-win payoff K can be obtained in a **properly transferable utility coopetitive fashion**, as it follows:

- 1) the two players agree on the cooperative strategy 2 of the common set C;
- 2) the two players implement their respective Nash strategies in the game $G(2)$, so competing *à la Nash*; the unique Nash equilibrium of the game $G(2)$ is the bistrategy $(3, 3)$;
- 3) finally, they share the "social pie"

$$(f_1 + f_2)(3, 3, 2),$$

  in a **transferable utility cooperative fashion** (by binding contract) according to the decomposition K.

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