Should the Neoclassical growth model include the saving flow in the Utility function?

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Abstract

Despite 'joy of giving models' have been extensively examined in the literature, the Ramsey growth model has never been explored under the assumption of a direct preference for bequeathing savings that are reinvested. This assumption implies a Utility function depending on both consumption and savings, which may also be motivated as one that captures a direct preference for thriftiness or wealth accumulation arguably involved. The resulting growth model generalizes those accounting for the capitalist spirit as Zou (1994), and shows that the restrictive standard one is perhaps not the actual optimized version of the Solow model.

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"The difficulty lies not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every corner of our minds."

J.M.Keynes, New York 1987.

1.Introduction

It goes to our mind almost systematically, that the flow of saving should never be included in the Utility function along with consumption, to solve the inter-temporal maximization problem of the consumer. Two reasons can be stated precisely and may sound a bit obvious only at first glance. The first one which is to consider savings as having no intrinsic value for the individual, can arguably be contested in some cases. The second, a technical redundancy deduced from standard conceptions, can be shown to be misleading, especially in models of accumulation. The goal of this paper is to criticize the systematic neutrality of a Utility effect of savings in the literature, and to report the relevant implications in the Ramsey growth model where specifically, such neutrality is perhaps inappropriate.

To contest the first reason for excluding savings from Utility in the Ramsey context, it might not be necessary to recall the capitalist spirit literature motivating that individuals derive direct Utility from wealth accumulation for the status; see for example Weber (1930), Zou (1994), Bakshi and Chen (1996), or Carroll (2000). Indeed, by introducing the saving flow in the Utility function, not only is the continual seek for a higher capital and status captured, but also a direct preference for thriftiness which could be involved if the individual worries about the necessity to renovate the depreciated capital, or to accumulate sufficient wealth for the growing household.[1]

In life-cycle (LC) or overlapping generations (OLG) models, the neutrality of this direct preference for thriftiness or wealth accumulation is systematically constrained every period, such that the individual saves only to defer own consumption in the future when 'profitable'. Because of this particularity, alternative models accounting for the bequest motive have been proposed in the literature. One type includes for example Barro (1974) or Laitner (1992), where successive generations working for only one period value the level of Utility of their children. Another consists in "joy of giving models" like for example Andreoni (1990), or more recently Dynan, Skinner and Zeldes (2002), where individuals gain satisfaction from bequeathing a part their

[1] This idea can be related to the modern literature of anticipatory feelings, like for example Kuznitz, Kandel and Fos (2008). The individual is supposed conscious and can be affected in the present by future situations, even under a deterministic context. In our case, the thrifty individual obtains 'comfort' when retaining a part of income, by avoiding anxiety.

lifetime income to the children. Although the second type has been extensively examined in the literature, the altruistic Ramsey model has never been explored under the assumption of a direct preference for bequeathing savings that are reinvested. It could be supposed for example, that each generation working at period t cares about saving for renovating or increasing the capital left to the children at t+1, before retiring or eventually dying. From this conceptual viewpoint, a direct preference for saving finds a very strong motivation in that framework, aside from a plausible preference for thriftiness or wealth accumulation.

Technically, it may have seemed redundant at first glance to introduce explicitly this preference in the model, because the time preference rate already measures the relative preference for current consumption, and the dynamics is already such that savings and accumulation occur when the interest rate exceeds this time preference rate. However, the time discounter is precisely defined by Samuelson (1937) as a degree of impatience that reflects the relative preference for current well-being[2], and the standard dynamics is specific to the lifetime utility function which corresponds by construction to one of a LC model in an infinite-horizon case. Assuming a direct preference for saving or accumulating generates a more general dynamics, which also extends those of models accounting for the capitalist spirit as Zou (1994). Furthermore, it allows to recover logically the basic static properties of the exogenous Solow version.

Similarly to growth models involving absolute wealth in Utility as Zou (1994), or relative wealth as Corneo and Jeanne (2001), the presented model allows to invest more than in the standard version by modulating the preference for wealth accumulation. This property is known as a plausible way to contrast with the contested lower boundary condition and convergence theorem of the traditional theory. It implies for instance that necessary and sufficient conditions required to meet the golden rule of accumulation can be specified. Another similarity that might also be important to report, is that the effect of the natural growth rate of workers on the steady-state level of capital per capita appears confirmed. Those common results could eventually be interpreted as two steps already made towards a reconciliation with Solow's static properties of the steady-state.

In contrast with this previous literature however, the proposed preference function generates a slower transition, and offers the possibility to reach also lower steady-state levels of capital per

[2] The love of wealth for its own sake (comfort, status, etc.) is independent from the degree of impatience. Precisely, given Utility depends now on total income, the time discounter could eventually reflect the relative preference for current income.

capita than in the standard model by investing less. Aside from allowing to recover a total coherence with the basic exogenous version, this second property might complement explanations of cross-country differences, and for instance, reconcile more empirical growth facts of developing countries with optimal growth theory.

The proposed Ramsey model is not presented as an augmented version of the standard one only, or as a generalization of capitalist spirit growth models, but as being perhaps an appropriate formulation that could have been missed because of an eventual misunderstanding of the preference for saving. Given such properties of the results indeed, nothing really precludes the possibility that the standard model assumes an unsuitable Utility function to optimize the Solow version. Precisely, it could for example be necessary to clarify first the goal of the individual in the exogenous model, before deciding which preferences would be more appropriate. Does this individual want to consume or to accumulate wealth, when choosing a constant saving rate $s \in (0,1)$? If the answer is 'both', which of those two alternatives is most preferred by this 'saver'?

The remainder is organized as follows. Section 2 presents conditions under which it might be relevant to introduce savings in Utility. Section 3 shows that applying this type of preferences in the Ramsey model appears to generate more consistent results than in the standard case. Section 4 concentrates on a discussion of the transition through a comparative analysis, and Section 5 concludes.

2. A Further Formulation of Preferences

It is highly important to justify in details why individuals could derive direct Utility from saving. The first section presents what is meant by a direct preference for thriftiness and wealth accumulation. The second concerns 'joy of bequeathing'.

A. Preference for Thriftiness and Wealth

According to its generic definition, the concept of Utility which measures satisfaction or welfare is of course not limited to consumption and leisure; examples in the literature are indeed gifts, cash-holding or wealth. As soon as a particular individual endowed of personal preferences gains satisfaction from actions decided among a set of alternatives, then such actions could eventually be arguments of a Utility function of this individual. For example, an individual could be endowed of intrinsic preferences for consuming and saving. The question about why this individual appreciates to be thrifty, is out of the microeconomic field and relies completely on personal reasons it takes as given. Perhaps, it is not necessarily a Utility function including a saving effect that needs to be justified, but eventually the assumption that all individuals can never feel (instantaneously) better by saving instead of consuming; one evidence that indeed contradicts this assumption is at least miserliness (under an extreme case.)

More particularly, some reasons could be exposed to defend the relevance of controlling a saving (or investment) utility effect in neoclassical growth models.

The conceptual logic of the Solow model is to involve a thrifty agent who saves each time a part of income to accumulate or renovate capital in the long run. It could be plausible to consider that saving (or investing) causes this individual to feel immediately better because, (i) it removes bad anticipatory feelings like anxiety or sadness if future wealth appears insufficient for some reasons (welfare of children, renovation of capital, job loss, illness, retirement, bequests, etc.); (ii) enhancing wealth is a way to achieve a higher social status, as argued for example by Weber (1930) who defines this desire as the spirit of capitalism.

In real life indeed, reasons why individuals decide to save or invest are not necessarily economic. Admittedly, some of the reasons are either psychological or sociological. It may not be surprising therefore if the traditional theory involving an agent who derives Utility from the act of consuming only, appears limited when applied to real data. For instance, Kenneth Arrow also said in 1988:

> "The dominant paradigm was that people essentially saved to spend in their lifetime. But today the evidence seems to be accumulating that this hypothesis is not true, and everybody seems to agree that you cannot explain savings solely on a life cycle basis […] So to conclude, I think that the key thing when it comes to the relationship between economics and sociology is the willingness to look at new kinds of data, like in savings. I think that once you do that, you are automatically going to be forced to consider social elements. Just ask different questions, and I think you are going to be forced into considering and drawing upon sociology."
>
> (Stanford University, April 1988).

Many research works in various fields of the literature have incorporated factors affecting saving decisions or wealth accumulation in preference functions; for example, uncertainty, bequests, gifts, and more recently, social status and anticipatory feelings. Notable works follow respectively Yaari (1965), Barro (1974), Andreoni (1990), Zou (1994), and Kuznitz, Kandel and Fos (2008).

In the neoclassical growth framework, a direct preference for the stock of capital has been introduced by Zou (1994) to formalize the spirit of capitalism of Max Weber[3], and proved to be relevant for the explanation of several empirical growth facts. In contrast, Corneo and Jeanne (2001) have proposed to account for the relative amount of wealth instead, by suggesting that individuals care about their relative position in a society. In the presented paper, an alternative way to formalize the continual seek for a higher status is proposed through a saving (or investment) effect in the flow of Utility, meaning that individuals value the growing of wealth (or capital) each time, rather than the absolute or relative stock itself.

Aside from the status motive or anticipatory feelings, earlier literatures have already considered the possibility for individuals to derive Utility from gifts or bequests. It might be worth discussing implications of ways to characterize the intergenerational altruism in the Ramsey growth model.

B. Joy of Bequeathing

Traditional inter-temporal maximization problems of the consumer might be separated into two major categories. The first one includes LC or OLG models as developed by Modigliani and Brumberg (1954), Modigliani (1986), Diamond (1965), etc. The second is constituted of altruistic or inter-dependent generations' models as in Barro (1974), Laitner (1992), Yaari (1965), etc.

In the first case, each generation derives utility only from own lifetime consumption and dissaves, or 'plans' to dissave, entirely during retirement. The second case mainly includes models where life-cycle utility can also, either be affected by the welfare of children, or by the act of giving or bequeathing (which gave the name of 'joy-of-giving' or 'warm glow' to this literature).

Recall now the conceptual approach of the altruistic Utility proposed by Barro (1974) that serves as a basis to the standard Ramsey model. Assuming each generation lives for only two periods, utility of individuals born at time t is expressed as:

[3] More precisely, Zou (1994) recalls a specification introduced initially by Kurz (1968), and motivates strongly its relevance. The love of wealth for its own sake is explained by the notoriety, power, and influence it procures.

(1) $$V_t = U_t + \gamma V_{t+1},$$

where V_t denotes total Utility of parents, U_t is the Utility from life-cycle consumption (of both periods), V_{t+1} is the total Utility of children, and γ is the discounting factor that measures the degree of altruism. Substituting recursively all Utilities up to generation *T*, gives the following dynastic function:

(2) $$V_t = \sum_{j=t}^{T-1} \gamma^{j-t} U_t + \gamma^{T-t} V_T,$$

where the term $\gamma^{T-t} V_T$ vanishes as *T* tends to infinity. In other words, the resulting specification simplifies by construction to one of a LC model in the infinite horizon case. Under this assumption, the optimal rule which governs saving decisions is such that individuals save to defer consumption only when profitable; ie. only when interest opportunities are more valuable than current consumption. In the standard Ramsey model particularly, properties of the production technology are such that the agent finds optimal to save and to accumulate up to an equilibrium level when starting from below, and to disaccumulate when departing from above. Recalling that the time discounting factor is technically bounded downward by the natural growth rate in that framework, the inevitable consequence is that the household is 'condemned to poverty' if marginal productivity of capital, as the unique driving force of the accumulation process, happens unfortunately to be low. Many research works in endogenous growth theory have tried to escape this constraining rule or 'lower boundary condition' by assuming different production technologies, but as suggested by Zou (1994), it seems reasonable to revise preferences as well, and to admit that individuals could still decide to invest more and grow under such unfavorable conditions.

An alternative conceptual approach to the lifetime Utility of an altruistic household, is one which separates Utility from giving, and Utility from the well-being of children which depends on their own total income at time *t+1*. For instance, the previous Utility of a generation born at time t could be expressed as:

(3) $$V_t = U_t(c_t, s_t) + \gamma V_{t+1},$$

where c_t is consumption and s_t is the bequest. This intergenerational 'transfer' of income might be a convenient way to measure the degree of altruism in the Ramsey framework, alternatively to the time discounting factor only. In that case, the bequest reflects the desire to renovate or to increase the capital stock left to the children at time *t+1*, before retiring or eventually dying. Interestingly, this idea may apply to various examples in real life. Aside from business or private concerns indeed, the natural environment could also be assimilated as a productive capital for which (insufficient) investments and efforts are made today, in order to preserve it for the children and future generations.

3. A Further Formulation of the Ramsey Growth Model

The first section presents the model which involves an interesting application of the Pontryagin's Maximum Principle. The second discusses the steady-state and Golden Rule.

A. The Optimal Control Program

At time *t*, a representative generation composed of L workers cares about consuming and reinvesting a part of income produced. The instantaneous preference function is defined by $U_t(c_t, s_t)$, where c_t denotes per capita consumption at t, s_t denotes per capita savings (or bequests), and $\theta \in (0,1)$ is a proportion which measures the degree of preference for consumption over savings. The budget constraint of this representative agent is given by $c_t + s_t = y_t$, where y_t denotes income per worker. The production function is supposed of the Cobb-Douglas form with constant returns to scale, such that $y_t = f(k_t) = k_t^\alpha$, where k_t denotes capital per worker, and $\alpha \in (0,1)$. The dynastic Utility function (after substitution of s_t) is denoted by $V\{c_t, k_t\}$. It is maximized over an infinite horizon subject to a dynamic constraint of capital accumulation (by the social planner):

(4) $$\underset{c_t}{\text{Max}}\; V\{c_t, k_t\} = \int_{t_0}^{\infty} U_t(c_t, k_t)\, e^{(n-\beta)t} dt,$$

(5) $$\textit{s.t.} \qquad \dot{k}_t = f(k_t) - c_t - (n + \delta)k_t,$$

where n is the natural growth rate of workers, β is the usual degree of impatience and δ is the rate of depreciation of capital. We impose the usual restriction $\beta > n$ to ensure a feasible interior solution to the problem.

Some interesting steps of the resolution are worth presenting. For the sake of concreteness, we assume a Cobb-Douglas Utility function $U_t(c_t, s_t) = c_t^{\theta} s_t^{1-\theta}$ and its log-transformation $U_t(c_t, s_t) = \theta \ln(c_t) + (1-\theta)\ln(s_t)$.

Considering the first specification for example, the Hamiltonian is given by:

(6) $H_t(c_t, s_t) = c_t^{\theta}[f(k_t) - c_t]^{1-\theta} e^{(n-\beta)t} + \lambda_t e^{(n-\beta)t}[f(k_t) - c_t - (n+\delta)k_t],$

Following the Pontryagin's Maximum principle, the first order condition implies $\partial U_t / \partial c_t = U_{c_t} = \lambda_t$, which means that the static maximization of H_t is satisfied under the following conditions:

$$\begin{cases} c_t = \theta f(k_t) & if \;\; \lambda_t = 0 \\ c_t \in [0; \theta f(k_t)[& if \;\; \lambda_t > 0 \end{cases}$$

The Pontryagin's method requires to substitute a maximizing condition in the expressions that serve to derive the dynamic equation of the control. In our case, the homothetic property of the instantaneous Utility function allows us to define a convenient maximizing condition by letting $a \in (0,1)$ such that[4]:

$$\lambda_t = \left(\frac{c_t}{s_t}\right)^{\theta} \left(\frac{1-a}{a}\right) \geq 0 \;\; \text{and} \;\; \frac{c_t}{s_t} = \frac{a\theta}{1-a\theta}$$

Letting $f_k = \frac{\partial f(k_t)}{\partial k_t}$, the second condition implies:

(7) $$(1-\theta)c_t^{\theta} s_t^{-\theta} f_k + \lambda_t(f_k - \beta - \delta) = -\dot{\lambda}_t,$$

[4] An excellent reference explaining substitutions of Pontryagin's maximizers is H.Schättler and U.Ledzewicz (2012), p.96. Technical details about this condition are exposed in Appendix (Figure A1 illustrates this condition)

A standard resolution that involves the substitution of equation (5) and static equilibrium values of the co-state and ratio defined above, leads to the dynamic equation of consumption $\dot{c}_t(c_t, k_t, a)$. To simplify notations, we will omit the arguments.

The resulting differential system is:

$$(8) \quad \begin{cases} \dot{c}_t = \frac{1-a\theta}{1-\theta}[(2-a\theta-\theta)f_k - (1-a)(\delta+\beta)]c_t - a\theta(n+\delta)f_k k_t, \\ \\ \dot{k}_t = f(k_t) - c_t - (n+\delta)k_t. \end{cases}$$

In clear, there can be multiple dynamical systems depending on the value of $a \in (0,1)$, which satisfy the first necessary conditions. This means that among all admissible values of a, or λ_t as explained by Schättler and Ledzewicz (2012), it remains to determine which one(s) maximize(s) $V_t\{c_t^*, k_t^*\}$. In that sense, we may now define a as being a choice parameter associated to a set of controlled trajectories, and a^* as being the rational choice associated to the optimal one which maximizes total welfare.

Contrary to the differential equation of consumption (8), the one of the log-transformed Utility can admit the value of $\theta = 1$, in which case it simplifies to the standard Ramsey equation. Its expression is given by:

$$(9) \quad \frac{\dot{c}_t}{c_t} = \frac{1-a\theta}{1+a\theta(a-2)}[2a(1-\theta)f_k + (1-a)(f_k - \beta - \delta)] - \frac{a(1-\theta)}{1+a\theta(a-2)}(n+\delta)\alpha,$$

The graphical resolution of the dynamical systems shows that for any $a \in (0,1)$, there exists a unique saddle path in each case leading to a steady state equilibrium denoted by $\{c_T^*(a), k_T^*(a)\}$. The phase diagram corresponding to the multiplicative Cobb Douglas case (figure 1), shows that the parameter a affects only the convexity of the '$\dot{c}_t = 0$ locus'[5]. The level of consumption increases to the left of this locus, and decreases to the right. For the case of the log-Utility (figure 2), the phase diagram is identical to the standard one except that the vertical '$\dot{c}_t = 0$ locus' depends now on the parameter a.

[5] It can indeed be shown that the second derivative of the '$\dot{c}_t = 0$ locus' with respect to k_t is strictly positive for $a < 1$, and tends to zero when a tends to 1.

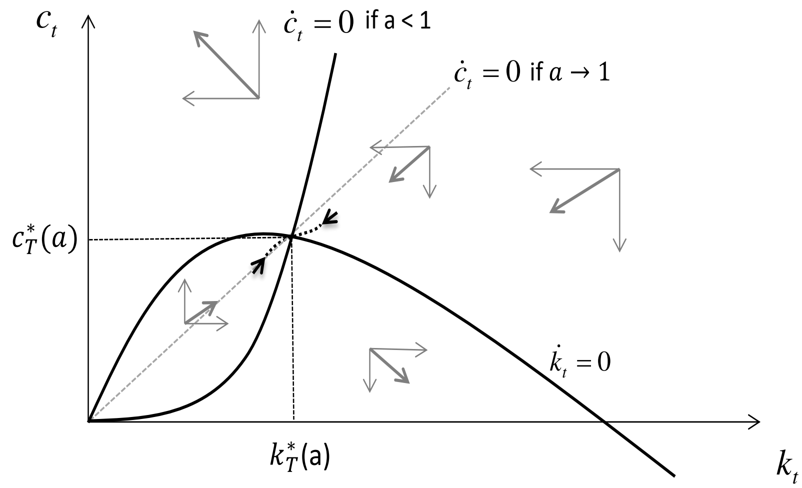

Figure 1. Phase Diagram for the Multiplicative Cobb-Douglas Case

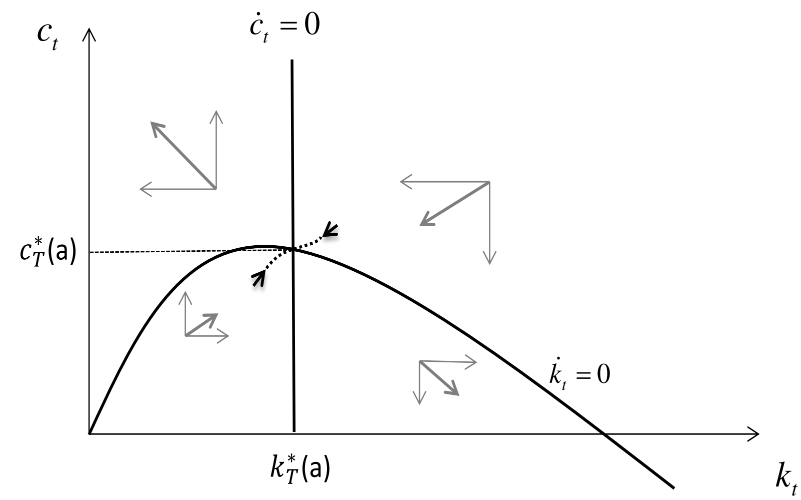

Figure 2. Phase Diagram for the Additive Cobb-Douglas Case

Let $\{c_t^*(a), k_t^*(a)\}$ denote any equilibrium (or saddle path) solution that leads to the steady-state at $t = T$. It can easily be noticed that the transversality condition $\lim_{t\to\infty} \lambda_t(a) k_t^*(a) e^{(n-\beta)t} = 0$, is fulfilled at equilibrium if $\beta > n$ because:

$$\begin{cases} \lim_{t\to\infty} \lambda_t(a) = U_{c_T^*(a)}, \ \text{from the first necessary condition,} \\ \\ \lim_{t\to\infty} k_t^*(a) e^{(n-\beta)t} = k_T^*(a). \lim_{t\to\infty} e^{(n-\beta)t} = 0, \ \text{ if } \beta > n. \end{cases}$$

Let the control set be defined by:

$$Z = \{c_t^*(a) \in \mathbb{R}^{++} \, / \, c_t^*(a) < \, y_t^*(a) \; \forall \, a \in (0,1)\},$$

such that total welfare $V_t\{c_t^*(a), k_t^*(a)\}$ is restricted to the set of real numbers. We can now proceed to a formal definition of a feasible optimal solution to the problem.

PROPERTY 1: An admissible controlled trajectory $\{c_t^*(a^*), k_t^*(a^*)\}$ *satisfying the necessary Pontryagin's conditions* $\forall \, a^* \in (0,1)$, $c_t^*(a^*) \in Z$ *and* $k_t^*(a^*) \in \mathbb{R}^{++}$, *is an optimal controlled trajectory if and only if* $V_t\{c_t^*(a), k_t^*(a)\} \leq V_t\{c_t^*(a^*), k_t^*(a^*)\} \quad \forall \, a \in (0,1) \, / \, a \neq a^*$, $c_t^*(a) \in Z$, $k_t^*(a) \in \mathbb{R}^{++}$.

An important result to keep in mind, is that the rational choice of the optimal trajectory is determinant for the terminal steady-state $\{c_T^*(a^*), k_T^*(a^*)\}$ reached in the long run.

B. Steady-states and Golden Rule

Contrary to the multiplicative Cobb Douglas Utility function, the log form allows to derive a steady-state solution analytically, which is:

$$k_T^*(a) = \left[\frac{(1-a\theta)(1-2a\theta+a)\alpha}{a(1-\theta)(n+\delta)\alpha+(1-a\theta)(1-a)(\beta+\delta)}\right]^{\frac{1}{(1-\alpha)}}. \tag{10}$$

For any non-corner point $(\theta, a) \in (0,1) \times (0,1)$, the steady-state level of capital (or income) per worker is a decreasing function of the natural growth rate. Hence, as soon as individuals are assumed to gain some Utility from accumulating or saving, the negative impact of population growth known from the basic exogenous Solow model is recovered.

There might be two reasons why this result should actually be defended. First, it seems reasonable to expect that an optimized version of the Solow model, which goal focuses only on producing an endogenous saving or investment rate along the transition, should preferably preserve plausible static properties of the steady-state equilibrium.

Second, it is well known that 'wealth-dilution' is one effect of population growth that has been widely admitted and confirmed empirically; for example, Mankiw, Romer and Weil (1992) might be one of the most influential contribution in that sense. It seems therefore counterfactual to admit that the negative impact of high fertility is entirely compensated by an increase in the investment-output ratio, as predicted by the standard Ramsey model.

Another important static property that plays a crucial role for the dynamics is the one of the golden rule from Phelps (1961). The Ramsey growth model, as it is used in most macroeconomic studies, is characterized by a steady-state level of capital per worker that remains always lower than the consumption maximizing level (and hence, than the over-accumulation one as well). As explained previously, this constitutes one of the reasons why alternative capitalist spirit models have been proposed in the literature, following for example Zou (1994) and Corneo and Jeanne (2001). Before presenting necessary and sufficient conditions for a golden rule steady-state, it may be preferable to present first a more general property that concerns any 'terminal' or constrained steady-state.

PROPERTY 2: Among all feasible controlled trajectories $\{c_t^*(\theta,a), k_t^*(\theta,a)\} \in Z \times \mathbb{R}^{++}$, *converging to a particular steady-state solution* $\{\overline{c_T^*}, \overline{k_T^*}\}$, *there exists a preference-choice couple* $(\theta^*, a^*) \in (0,1) \times (0,1)$ *compatible with an optimizing behavior; ie. which satisfies:* $V_t\{c_t^*(\theta,a), k_t^*(\theta,a)\} \leq V_t\{c_t^*(\theta^*,a^*), k_t^*(\theta^*,a^*)\}, \forall\ (\theta,a) \in (0,1) \times (0,1)\ /\ (\theta,a) \neq (\theta^*,a^*)$

This property resulting directly from the resolution might be viewed as a reciprocal of the first one. Indeed, a steady-state dynamics is an optimal controlled trajectory as soon as there is no other ways to reach the same dynamics with a higher total welfare. Suppose for example that the constrained terminal state corresponds to the solution of the Solow model:

$$\overline{k_T^*} = \left(\frac{\overline{s_T^*}}{n+\delta}\right)^{\frac{1}{1-\alpha}} \quad \text{where} \quad f_{\overline{k_T^*}} = \frac{(n+\delta)\alpha}{\overline{s_T^*}} \quad \text{and} \quad \overline{s_T^*} \in (0,1)$$

The determination of θ a posteriori for a given $\overline{s_T^*}$, requires to solve a quadratic equation in θ, which implies two admissible sets (θ_i, a_i), $\forall$ *i=1,2*. In some cases, restrictions imposed allow to eliminate one of the sets entirely, so that a unique underlying combination of parameters (θ_i^*, a_i^*) that maximizes total welfare can be identified; it is for example the case for relatively high or low steady-states. Under cases where both sets offer potential candidates however, the terminal state assumed can possibly be generated by two different optimal trajectories (or two different rational behaviors). In such a multiple equilibrium context, the goodness of fit to real data may lastly decide.

The golden rule steady-state which maximizes consumption is reached if $\overline{s_T^*} = \alpha$. For the multiplicative Cobb-Douglas form, the following condition must hold:

(11) $$\left[1 + \frac{1-\theta}{(1-a\theta)X} a\theta(n+\delta)\right]^{-1} = \alpha,$$

where $X = (2 - a\theta - \theta)(n+\delta) - (1-a)(\delta+\beta)$, and for the log-Utility case,

(12) $$\frac{1-a\theta}{a(1-\theta)}\left[1 + a(1-2\theta) - (1-a)\frac{\beta+\delta}{n+\delta}\right] = \alpha.$$

4. Numerical Analysis of the Dynamics

This section presents results of numerical simulations. The analysis focuses on the additive log Utility function with no loss of generality concerning the Cobb Douglas form. Following the first property, the first part analyzes the dynamics of the optimal path. The second addresses the question of recovering this path starting from a given steady-state. It also compares the dynamics with those of standard growth models and the one proposed by Zou (1994).

A. General Properties of the Dynamics

The question that comes first is to know how total welfare changes with respect to the parameter a, which has been defined previously as reflecting the choice of an admissible trajectory made by the individual. The next interesting step is to understand how the variables behave along the optimal path[6].

Numerical simulations show that for a preference parameter θ that tends to one, the optimal value of a decreases and the maximum total welfare tends to stabilize for $a < a^*$. In figure 3 for example, when $\theta = 0.95$, the maximum total welfare attains $V^* = 241.62$ at $a^* = 0.71$, and remains almost constant for $a \leq 0.71$ (precisely, it decreases slowly until $V = 241.36$ at $a =$

[6] Numerical computations of saddle paths are made according to the shooting method. A solution a^* is considered sufficiently accurate if in its neighborhood, changes in total welfare become relatively negligible. In this part, parameters kept constant are assigned the following values: $\alpha = 0.3, \beta = 0.02, n = 0, \delta = 0.05, k_0 = 1$. For convenience, total welfare is calculated with re-scaled variables (c_t and s_t are both multiplied by 100).

0.01). As θ decreases, the value of a^* increases but at a much lower rate; for instance $a^* = 0.76$ when $\theta = 0.85$ and $a^* = 0.78$ when $\theta = 0.2$. This figure shows also that the choice of the right equilibrium co-state (or value of a) matters much more for lower values of θ. For example, the size of the welfare gain from an admissible path to the optimal one can exceed 26% in the case where $\theta = 0.2$.

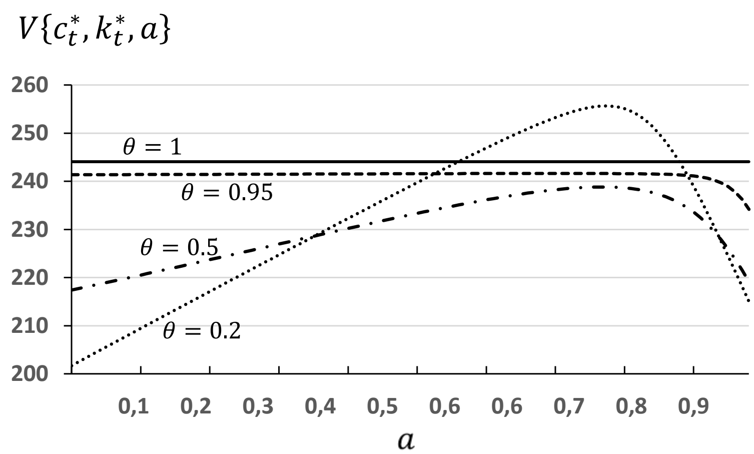

Figure 3. Total Welfare by Admissible Path

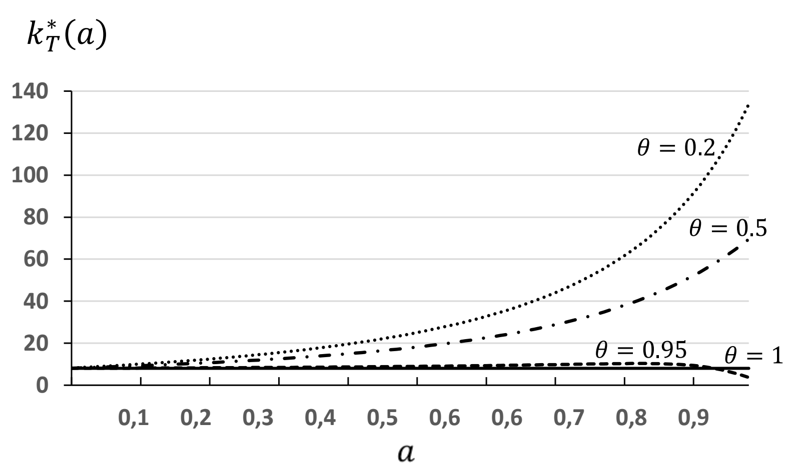

Figure 4. Steady-state Per Capita Capital by Admissible Path

Another result that figure 3 illustrates, is that individuals with different degrees of preference for saving (or bequeathing) face different levels of total welfare. When θ remains relatively high, total welfare is lower than in the standard case where individuals value consumption only. As θ

decreases below a certain cutoff value, total welfare tends to increase back until reaching higher values than in the standard model. Besides the fact that the steady-state saving rate for such values of θ seems unreasonable, as indicated by figure 6, this parameter describing fixed preferences is conceptually not to be 'selected' so that total welfare or even consumption is maximized[7].

In figure 4, it is interesting to notice that accounting for supplementary (social) motives for saving, does not always mean a higher steady-state capital per capita than in the standard model. For values of θ that are close to one (0.95 in our example), there are some admissible trajectories (for high values of a) which lead to lower steady-state capital per capita than in the standard case. This interesting remark will be developed later.

Concerning transitions towards higher-steady-states, figure 5 shows that (unconstrained) optimal trajectories exhibit a faster growth when preferences for saving for social reasons become more intense. Interestingly, although the speed of growth increases with such intense preferences, thrifty individuals appear less sensitive to variations of the interest rate compared to those who care about consumption only. For instance, figure 6 shows that the magnitude of the variation of the saving rate along the optimal path increases with θ.

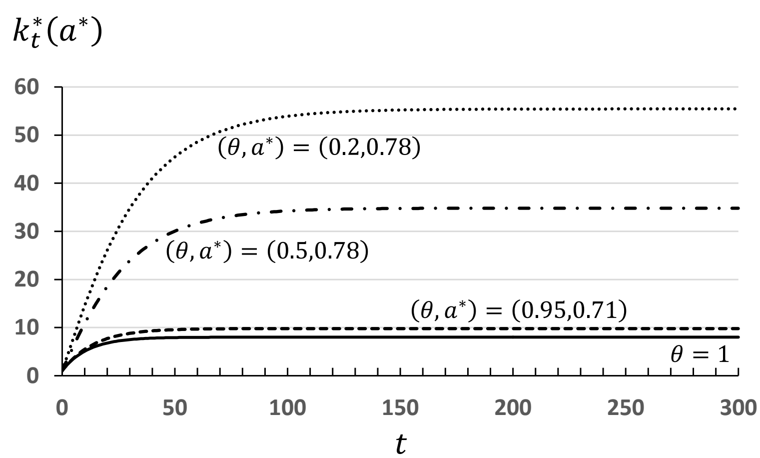

Figure 5. Equilibrium Path of Capital Per Capita

[7] The golden rule of capital accumulation is indeed not necessarily what individuals prefer.

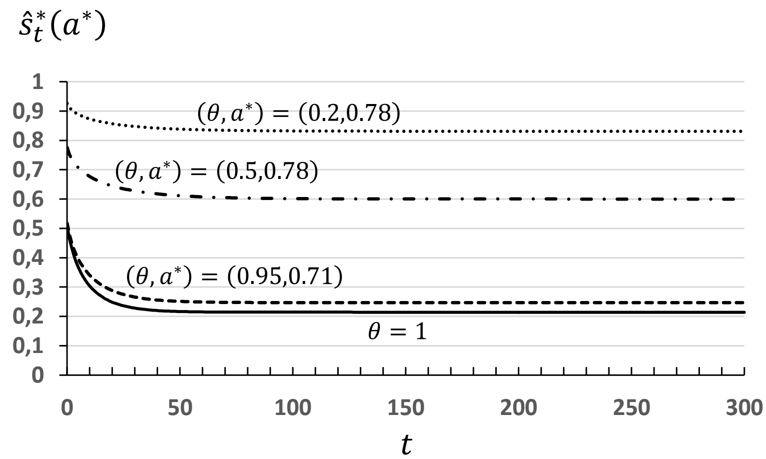

Figure 6. Equilibrium Path of the Saving Rate

B. Dynamics in Constrained Optimization Cases

Suppose for example that individuals are endowed of preferences for consuming and bequeathing such that future generations benefit from the golden rule steady-state at time *T*, where consumption is maximized.

It is well known that for this steady-state to be possible under the standard altruistic case, the flow of Utility must be augmented to include a direct preference for wealth (or status) as for example in Zou (1994). A general specification widespread in this literature is:

(13) $$W\{c_t, k_t\} = \int_{t_0}^{\infty} [u_t(c_t) + v_t(k_t)]\, e^{(n-\beta)t} dt,$$

where $v_t(k_t)$ represents the part of utility derived from the capital stock, with $v_k > 0$ and $v_{kk} < 0$. Preserving same notations as in the presented paper, the resolution of the program under this assumption leads to the following rule:

(14) $$f_{k_T} = \delta + \beta - \frac{v_k}{u_c},$$

Hence, given the ratio $^{v_k}/_{u_c}$ is always positive, the steady-state capital per capita in such models will always be greater than in the standard one. The alternative Utility function proposed in this paper generates a more general version of the neoclassical growth model by contrasting this result.

For a convenient comparative analysis, suppose the flow of Utility of the capitalist spirit model of Zou (1994) is given by:

(15) $$u_t(c_t) + v_t(k_t) = ln(c_t) + \gamma ln(k_t), \text{ where } \gamma \in [0, \infty).$$

For $n = 0$, it can be shown that the steady-state capital per capita is given by:

(16) $$k_T = \left(\frac{\alpha+\gamma}{(\gamma+1)\delta+\beta}\right)^{\frac{1}{1-\alpha}},$$

and is increasing in γ. The value of this parameter can be easily deduced such as to meet the golden rule steady-state.

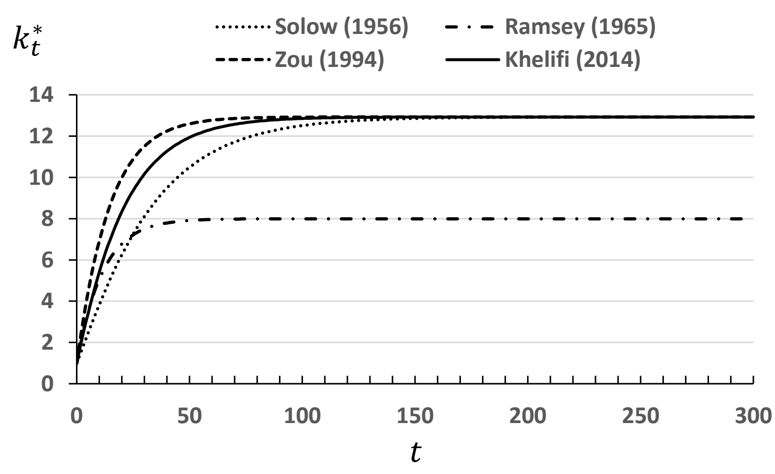

Figure 7. Equilibrium Path of Capital per Capita

Notes: Common parameters in each model are assigned the following values: $\alpha = 0.3, \beta = 0.02, n = 0, \delta = 0.05, k_0 = 1$. The steady-state saving rate associated to the golden rule is therefore 0.3. In the presented model, the optimal pair (θ, a^*) is deduced accordingly and equals (0.901,0.85) and in the model of Zou (1994), $\gamma = 0.171$. The standard Ramsey results have been included in each figure to compare the dynamics; the steady-state level of the saving rate $\hat{s}_T^*$ equals 0.21 in that model.

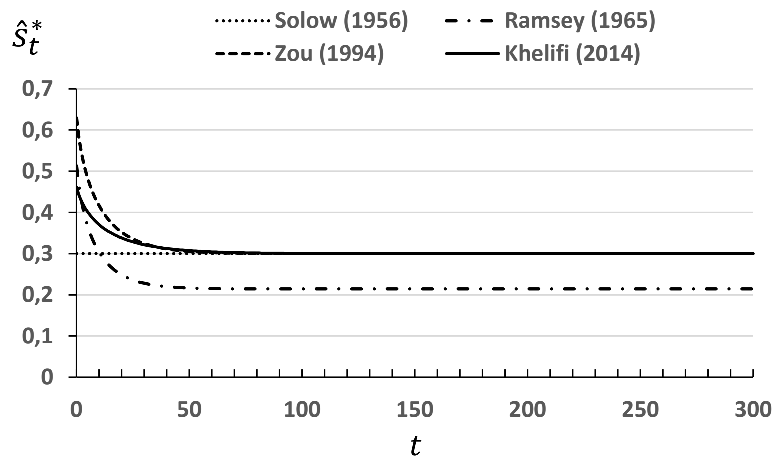

Figure 8. Equilibrium Path of the Saving Rate

Notes: Common parameters in each model are assigned the following values: $\alpha = 0.3, \beta = 0.02, n = 0, \delta = 0.05, k_0 = 1$. The steady-state saving rate associated to the golden rule is therefore 0.3

The slowest transition towards the golden rule steady-state in figure 7 and 8, is unsurprisingly the Solow one where individuals do not benefit from highest returns initially. In all other cases, an optimal decision implies a faster speed of growth which differs depending on the type of preferences assumed. It appears clearly that the model of Zou (1994) generates the faster transition. In other words, the stock of capital in the Utility function, compared to the unconsumed part of income, affects more intensively the willingness to accumulate.

Clearly, it is not possible a priori, or theoretically, to determine which one of those augmented versions of the Ramsey model ensures a superior fit to empirical data. However, every common parameters equal, the proposed model offers the possibility to adjust a slower dynamics towards the same steady-state.

An additional particularity of the proposed model shown by figure 4, is to allow for some admissible controlled trajectories towards lower steady-states as well. Departing from the standard model, this figure indicates that the steady-state level of capital per capita increases with the degree of preference for saving $(1-\theta)$ and with the choice parameter a, except for few cases where $(1-\theta)$ is relatively low. Figures 9 and 10 present an example of transition towards a lower steady-state. For a 'terminal' saving rate $\hat{s}_T^*$ of 0.15 (versus 0.21 in the standard model), two different optimal trajectories are possible. In both cases, the speed of growth remains logically greater than in the Solow model. For the case where the value of θ is higher, the corresponding value of 'a' leading to the specified steady-state with the maximum welfare is also higher and very close to one. Interestingly, the difference between the values of a in each case is such that the trajectory of the individual endowed of the higher preference for consumption θ, appears faster than the one who values wealth accumulation more intensively. In that case, the thriftiest individual is again less sensitive to variations of capital returns.

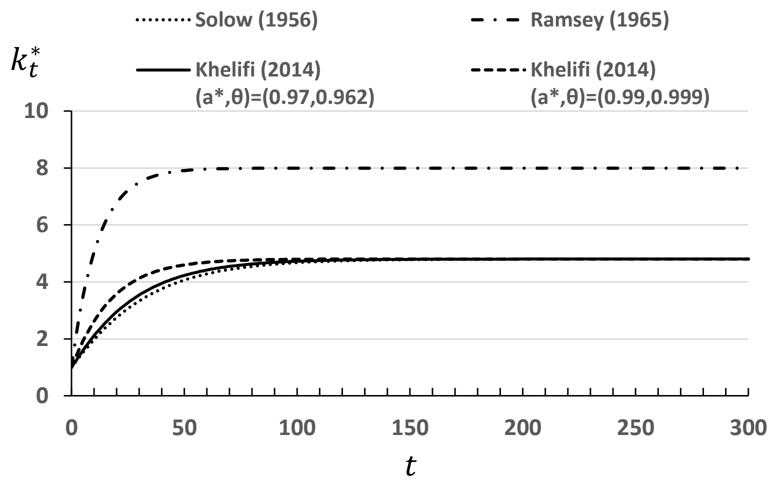

Figure 9. Equilibrium Path of Capital Per Capita

Notes: Common parameters in each model are assigned the following values: $\alpha = 0.3, \beta = 0.02, n = 0, \delta = 0.05, k_0 = 1$. The steady-state saving rate in the Solow model and in the presented one equals 0.15. Two solutions (θ_i, a_i^*) are compatible with an optimizing behavior towards this steady-state, (0.999,0.99) and (0.962,0.97). The standard Ramsey results are reported for comparisons (in this model, $\hat{s}_T^* = 0.21$).

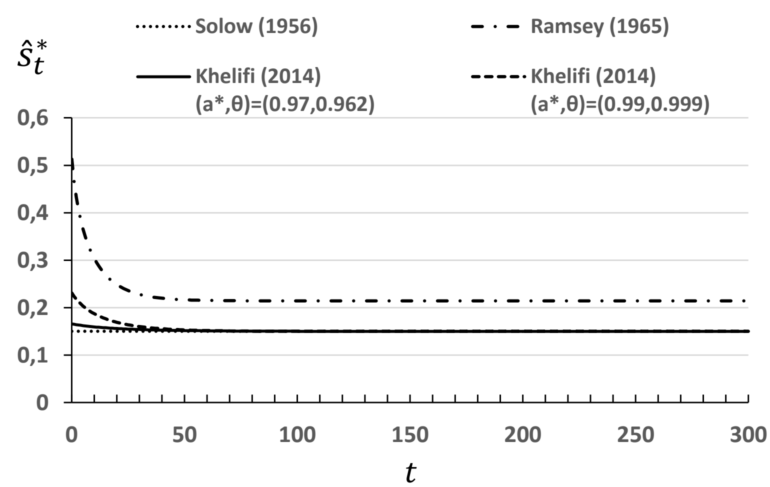

Figure 10. Equilibrium Path of the Saving Rate

Notes: Common parameters in each model are assigned the follow values: $\alpha = 0.3, \beta = 0.02, n = 0, \delta = 0.05, k_0 = 1$. The steady-state saving rate in the Solow model and in the presented one equals 0.15

The constrained optimization case should perhaps not be interpreted as one where the representative household targets the terminal steady-state to reach. Instead, it should simply be considered as a convenient way to recover the utility function under the assumption of rationality from the observed empirical data. It might eventually be more useful in cases of relatively low steady-states like in developing countries, where households can be exposed to constraints regarding their saving or investment capacities.

5. Conclusion

Is the individual in the Solow model a 'consumption-lover' with a saving rate of 50%? Is the preference for accumulating implicitly involved in growth models, and should it be taken into account independently from personal motives (power, influence, thriftiness, miserliness, gifts, bequests, etc.)?

No matter what answers are, the standard model generates a steady-state level of capital per capita bounded by the golden rule level, where the resulting equilibrium saving rate reaches a maximum value of 'α' (roughly estimated at 30%), when the time preference rate tends to its lower bound level n. Another apparent weakness of this model, is that it predicts the counterfactual result of a wealth-dilution effect totally absorbed by a reduction in the consumption share.

Capitalist spirit growth models as proposed for example by Zou (1994), offer a reasonable way to contrast these constraining and counterfactual properties. However, the love of wealth accumulation as formalized in such models, allows to extend the set of possible dynamics to efficient and over-accumulation ones only, so that explanations of low GDP levels remain limited to the time preference rate essentially. If adjusting the time preference rate in the standard model has been inappropriate to explain differences among 'rich countries', and if controlling some additional willingness to invest has proved necessary, then it seems reasonable to admit that same extensions are required for low GDP countries. The proposed model offers the possibility to expand the set of steady-states on both sides of the standard version, so that eventual poverty traps characterized by insufficient investment can also be reconciled in a same way with optimal growth theory.

Several issues that have been investigated in the literature on the basis of the standard Ramsey growth structure, might be worth revisiting within such a more general framework that sounds arguably more realistic. An additional advantage that might motivate the use of this model particularly, concerns the context of an open-economy. Indeed, it is well-known that the standard dynamics presents the counterfactual result of a consumption and wealth that tend to zero in an open-economy where the time preference rate exceeds the world interest rate; see for example Barro and Sala-i-Martin (1995, chapter 3). A recent suggestion from Hof and Wirl (2008) to overcome this problem, is to augment the Utility function by including absolute or relative wealth as an argument. Besides the fact that this type of preferences might still be restrictive, as explained in this paper, the initial level of capital must be greater than a lower bound level to ensure a possible accumulation towards a steady-state where both capital and consumption are greater than zero. Such a condition is not required in the proposed model which offers a simpler framework.

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APPENDIX 1: RESOLUTION

For online publication.

The social planner solves:

$$\underset{c_t}{\text{Max}}\ V\{c_t, k_t\} = \int_{t_0}^{\infty} U_t(c_t, k_t)\, e^{(n-\beta)t} dt$$
$$s.t. \qquad \dot{k_t} = f(k_t) - c_t - (n+\delta)k_t$$

Resolution for the multiplicative Cobb Douglas case:

$$H_t(c_t, s_t) = c_t^{\theta}[f(k_t - c_t)]^{1-\theta} e^{(n-\beta)t} + \quad \lambda_t e^{(n-\beta)t}[f(k_t) - c_t - (n+\delta)k_t]$$

The necessary conditions:

$i)\quad H(t, \lambda_0, \lambda_t, k_t, c_t) = \max_{\nu \epsilon (0;\theta y_t)} H(t, \lambda_0, \lambda_t, k_t, \nu)$

$ii)\quad \frac{\partial H}{\partial k}(t, \lambda_0, \lambda_t, k_t, c_t) = \frac{\partial[\lambda_t e^{(n-\beta)t}]}{\partial t}$

$iii)\quad \frac{\partial H}{\partial \lambda}(t, \lambda_0, \lambda_t, k_t, c_t) = \frac{dk_t}{dt} = \dot{k_t}$

$iv)\quad \lim_{t\to\infty} \lambda_t e^{(n-\beta)t} k_t = 0$

The first order condition implies: $\partial U_t / \partial c_t = U_{c_t} = \lambda_t$. This static maximizing condition is to be substituted in the dynamical expressions that serve to derive the differential equation of the control. With $\lambda_t \geq 0$ and U_t homothetic, we can express an explicit condition in a convenient way by letting $a \in (0,1)$ such that: $\lambda_t = \left(\frac{c_t}{s_t}\right)^{\theta}\left(\frac{1-a}{a}\right) \geq 0$ and $\frac{c_t}{s_t} = \frac{a\theta}{1-a\theta}$

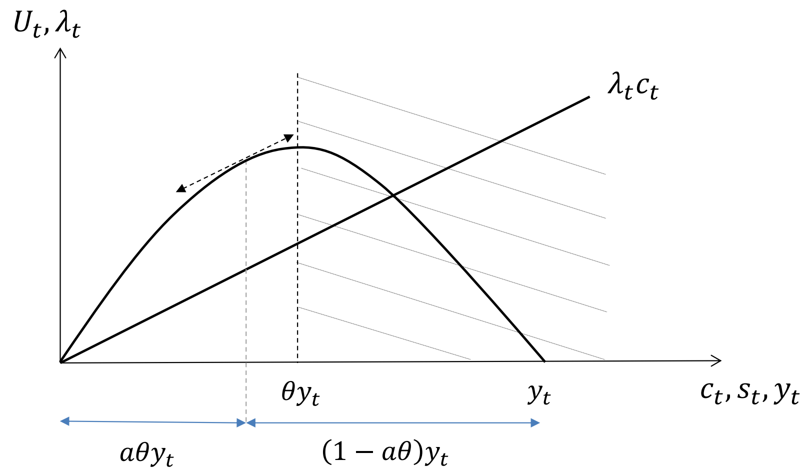

Figure A1. Static Equilibrium Condition in the Multiplicative Cobb-Douglas Case

As explained by Schättler and Ledzewicz (2012) p.96, the substitution of the necessary condition for a static maximization corresponds to a weak formulation of the Pontryagin's Maximum Principle. Just like in the standard Ramsey model, it is the necessary condition that is substituted in our case (for instance, the equilibrium co-state and ratio). This same resolution allows a direct confrontation of both versions.

ii) $(1-\theta)\left(\frac{c_t}{s_t}\right)^{\theta} f' + \lambda_t[f' - \delta - \beta] = -\dot{\lambda}_t$

$$(1-\theta)\left(\frac{c_t}{s_t}\right)^{\theta} f' + \left(\frac{1-a}{a}\right)\left(\frac{c_t}{s_t}\right)^{\theta} [f' - \delta - \beta] = -\dot{\lambda}_t$$

$$\left(\frac{c_t}{s_t}\right)^{\theta}\left[\left((1-\theta) + \frac{1-a}{a}\right) f' - \frac{1-a}{a}(\delta+\beta)\right] = -\dot{\lambda}_t$$

Differentiating totally condition *(i)*:

$$\frac{\partial U_c}{\partial c_t}\dot{c}_t + \frac{\partial U_c}{\partial k_t}\dot{k}_t = \dot{\lambda}_t$$

$$\theta(1-\theta)\left(\frac{c_t}{s_t}\right)^{\theta}\frac{1}{c_t}\left[\left(-\frac{s_t}{c_t} - \frac{c_t}{s_t} - 2\right)\dot{c}_t + f'\left(1 + \frac{c_t}{s_t}\right)\dot{k}_t\right] = \dot{\lambda}_t$$

Constraining the static maximizing condition by substituting λ_t means that the associated equilibrium ratio $\frac{c_t}{s_t}$ defined above can be constrained as well. The expression simplifies to:

$$\theta(1-\theta)\left(\frac{c_t}{s_t}\right)^{\theta}\frac{1}{c_t}\left[\left(-\frac{1}{a\theta(1-a\theta)}\right)\dot{c}_t + f'\frac{1}{1-a\theta}\dot{k}_t\right] = \dot{\lambda}_t$$

Combining this expression with condition *(ii)* leads to:

$$\frac{\theta(1-\theta)}{a\theta(1-a\theta)}\frac{\dot{c}_t}{c_t} = f'\frac{\theta(1-\theta)}{1-a\theta}\dot{k}_t\frac{1}{c_t} + \left[\frac{1-a\theta}{a}f' - \frac{1-a}{a}(\beta+\delta)\right]$$

$$-\frac{\theta(1-\theta)}{a\theta(1-a\theta)}\dot{c}_t + \frac{\theta(1-\theta)}{(1-a\theta)}f'\dot{k}_t = -\left[\frac{1-a\theta}{a}f' - \frac{1-a}{a}(\beta+\delta)\right]c_t$$

$$f'\left[\frac{1-a\theta}{a}c_t + \frac{\theta(1-\theta)}{1-a\theta}\dot{k}_t\right] - \frac{1-a}{a}(\delta+\beta)c_t = \frac{(1-\theta)}{a(1-a\theta)}\dot{c}_t$$

Introducing condition *(iii)*:

$$f'\left[(1-a\theta)c_t + \frac{(1-\theta)a\theta}{1-a\theta}y_t - \frac{(1-\theta)a\theta}{1-a\theta}c_t - \frac{(1-\theta)a\theta}{1-a\theta}(n+\delta)k_t\right] - (1-a)(\delta+\beta)c_t = \frac{(1-\theta)}{(1-a\theta)}\dot{c}_t$$

$$f'\left[c_t\left[(1-a\theta) + \frac{(1-\theta)(1-a\theta)}{1-a\theta}\right] - \frac{(1-\theta)a\theta}{1-a\theta}(n+\delta)k_t\right] - (1-a)(\delta+\beta)c_t = \frac{(1-\theta)}{(1-a\theta)}\dot{c}_t$$

The resulting dynamical system is:

$$\begin{cases} \dot{c}_t(c_t,k_t) = \frac{1-a\theta}{1-\theta}\left[(2-a\theta-\theta)f' - (1-a)(\delta+\beta)\right]c_t - a\theta(n+\delta)f'k_t \\ \\ \dot{k}_t(c_t,k_t) = f(k_t) - c_t - (n+\delta)k_t \end{cases}$$

Resolution for the additive Cobb Douglas case:

$\partial U_t/\partial c_t = 0$ implies:

$$\frac{\theta}{c_t} - \frac{1-\theta}{y_t - c_t} = \lambda_t$$

$$\frac{\theta y_t}{c_t(y_t-c_t)} - \frac{c_t}{c_t(y_t-c_t)} = \lambda_t$$

Defining $\frac{c_t}{s_t} = \frac{a\theta}{1-a\theta}$ where $c_t = a\theta y_t \in \left]0, \theta y_t\right[$,

$$\frac{a\theta y_t}{ac_t(y_t - c_t)} - \frac{a\theta}{c_t(1 - a\theta)} = \lambda_t$$

$$\frac{1}{a(1 - a\theta)y_t} - \frac{1}{(1 - a\theta)y_t} = \lambda_t$$

$$\lambda_t = \frac{1 - a}{a(1 - a\theta)y_t}$$

Differentiating totally the first order condition gives:

$$\left[-\frac{\theta}{c_t^2} - \frac{1 - \theta}{(y_t - c_t)^2}\right]\dot{c}_t + \frac{(1 - \theta)f'(k_t)}{(y_t - c_t)^2}\dot{k}_t = \dot{\lambda}_t\,,$$

The first term in brackets can for example be expressed more conveniently :

$$\left[-\frac{\theta}{c_t^2} - \frac{1 - \theta}{(y_t - c_t)^2}\right] = -\frac{\theta(a\theta y_t)^2}{c_t^2(a\theta y_t)^2} - \frac{(1 - \theta)[(1 - a\theta)y_t]^2}{(y_t - c_t)^2[(1 - a\theta)y_t]^2} = -\frac{\theta}{(a\theta y_t)^2} - \frac{(1 - \theta)}{[(1 - a\theta)y_t]^2}$$

Following same simplifying tricks, the differentiated condition can be expressed as:

$$\frac{-\theta(a^2\theta - 2a\theta + 1)}{(a\theta)^2(1 - a\theta)^2 y_t^2}\dot{c}_t + \frac{(1 - \theta)f'}{(1 - a\theta)^2 y_t^2}\dot{k}_t = \frac{-(1 - \theta)f'}{(1 - a\theta)y_t} - \frac{(1 - a)}{a(1 - a\theta)y_t}(f' - \delta - \beta)$$

$$\frac{-(a^2\theta - 2a\theta + 1)}{a(1 - a\theta)}\frac{\dot{c}_t}{c_t} + (1 - \theta)f'\left[1 - \frac{(n + \delta)k_t}{(1 - a\theta)y_t}\right] = -(1 - \theta)f' - \frac{1 - a}{a}(f' - \delta - \beta)$$

$$\frac{-(a^2\theta - 2a\theta + 1)}{a(1 - a\theta)}\frac{\dot{c}_t}{c_t} = f'\left[-2(1 - \theta) - \frac{1 - a}{a} + \frac{(1 - \theta)(n + \delta)k_t}{(1 - a\theta)y_t}\right] + \frac{1 - a}{a}(\beta + \delta)$$

The resulting system is therefore:

$$\begin{cases} \dfrac{\dot{c}_t}{c_t} = \dfrac{1 - a\theta}{1 + a\theta(a - 2)}[2a(1 - \theta)f_k + (1 - a)(f_k - \beta - \delta)] - \dfrac{a(1 - \theta)}{1 + a\theta(a - 2)}(n + \delta)\alpha \\ \dot{k}_t = f(k_t) - c_t - (n + \delta)k_t \end{cases}$$

APPENDIX 2: SIMULATION *(Not for Publication)*
Coded on R

The main functions that have been used for the numerical experiments are available in two separate word files:"Generalized Ramsey (2014)-Constrained-Log Utility" and "Generalized Ramsey (2014)-Direct-Log Utility". In each file, two main functions are designed for the calculation of the optimal path. Given it could take time to iterate on values of the parameter $a \in (0,1)$, calculations have been separated into two steps.

Concerning the computation of the saddle path, the number of iterations and the length of the vectors (variable "len") should be increased for more precision (it becomes necessary when θ is low, like for example 0.2). Indeed, when the steady-state is very high, the quality of the convergence is affected. The shooting method is based on the fact that divergence occurs in two possible ways; if c(0) is too low, the dynamics diverges in the South-East, and if c(0) is too high, in the North-West.

i) The first function called "localize($\alpha, \delta, n, \beta, \hat{s}^*, k_0$)" gives a good idea of the solution quickly (< 2 minutes). It calculates the total welfare (from t = 0 to 300) produced by saddle path solutions for 10 values of the parameter *a* (from *a=1* to *0.1*). It selects the solution that maximizes welfare and exports the dynamics in a CSV file (per default on "My Documents"). In the constrained case, the saving rate enters as an argument. For example executing the following lines:

```
localize1 <- localize(0.3,0.05,0,0.02,0.3,1)
localize1
```

displays:

	param_a	root_teta1	tot_welfare1	root_teta2	tot_welfare2
[1,]	0.99	1.000142	0	0.85642353	237.0250
[2,]	0.90	1.024210	0	0.89801213	239.3556
[3,]	0.80	1.118228	0	0.89427204	239.4766
[4,]	0.70	1.266442	0	0.86212898	238.4923
[5,]	0.60	1.471755	0	0.81157829	236.9400
[6,]	0.50	1.762348	0	0.73765246	234.6945
[7,]	0.40	2.200000	0	0.62500000	231.2965
[8,]	0.30	2.930664	0	0.43600285	225.6182
[9,]	0.20	4.393092	0	0.05690752	214.2522
[10,]	0.10	8.781785	0	-1.08178467	0.0000

Given the solution lies betwwen a = 0.9 and a = 0.7, the second function takes a last argument "max_a" to iterate more precisely starting from the upper bound selected (0.9 in this example). For example typing:

```
calcul1 <- calcul(0.3,0.05,0,0.02,0.3,1,0.9)
calcul1
```

displays:

	param_a	root_teta1	tot_welfare1	root_teta2	tot_welfare2
[1,]	0.89	1.030502	0	0.8998355	239.4675
[2,]	0.88	1.037585	0	0.9010512	239.5501
[3,]	0.87	1.045428	0	0.9016989	239.6059
[4,]	0.86	1.053988	0	0.9018263	239.6380
[5,]	0.85	1.063222	0	0.9014840	239.6491
[6,]	0.84	1.073089	0	0.9007205	239.6422
[7,]	0.83	1.083552	0	0.8995809	239.6195
[8,]	0.82	1.094578	0	0.8981045	239.5832
[9,]	0.81	1.106144	0	0.8963253	239.5351
[10,]	0.80	1.118228	0	0.8942720	239.4766
[11,]	0.79	1.130817	0	0.8919682	239.4088
[12,]	0.78	1.143900	0	0.8894332	239.3328
[13,]	0.77	1.157473	0	0.8866825	239.2494
[14,]	0.76	1.171535	0	0.8837285	239.1590
[15,]	0.75	1.186086	0	0.8805808	239.0622
[16,]	0.74	1.201132	0	0.8772468	238.9594
[17,]	0.73	1.216679	0	0.8737317	238.8509
[18,]	0.72	1.232739	0	0.8700391	238.7367
[19,]	0.71	1.249322	0	0.8661713	238.6172

Each function outputs a file containing the dynamics for the maximum welfare. The unconstrained solution on the other file works exactly the same. It is the preference parameter which enters as an argument in that case.