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**Abstract**

In this paper we consider the issue of optimal information aggregation for timing decision making. In each period, a decision maker may choose an action which delivers an uncertain payoff, or wait until the next period, in which new information will arrive. The information is provided by a committee of experts. Each member in each period receives a signal correlated to the state. We obtain an optimal rule for aggregating information for each period.

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# 1 Introduction

In this paper we consider the issue of optimal information aggregation for timing decision making. In each period, a decision maker may choose an action which delivers an uncertain payoff, or wait until the next period, in which new information will arrive. The information is provided non-strategically by a committee of experts. The reader may think in the following highly stylized situation: in each period a project may be profitable (good project) or worthless (bad project), but the true state is not known in advance. Instead, each member of a committee of experts in each period receives a signal correlated to the state, which varies along time in a Markovian way. That is, the probability that a project is a good one in some period depends on whether it was a good one in the previous period. When the project is undertaken, the decision maker gets a reward equal to the profits of the project and the game ends. The purpose of this chapter is to obtain an optimal rule for aggregating information for each period.

We build on Ben-Yashar and Nitzan (1997), who consider a committee whose task is to approve or reject projects. In their model, on a periodical basis, each expert receives a signal correlated to the profitability of the project. Basically, the committee faces the same problem every period; thus, Ben-Yashar and Nitzan (1997)'s optimal rule is static. Their model is able to capture situations in which there is no possibility to postpone the execution of the project (either because the opportunity disappears or because in the future new information will not arrive making the waiting worthless for impatient decision-makers). Beside investment decisions of the type "accept or reject", their result is of significance to jury decision making and other political, legal, economical and medical applications in which the choice is dichotomous and cannot be postponed. However, there are many situations in which waiting is possible and it has a value because new information may be coming and more importantly, because project's execution is irreversible (otherwise, the optimal rule would always be undertaking the project and revert if the project turns to be bad). For example, oil drilling can be postponed if a rise in prices is likely to happen. Similarly, an entrepreneur facing uncertain demand may prefer to wait before introducing a new product or brand to the market. Even in some medical treatments, a committee of experts may prefer to wait for the appearance of new symptoms, in order to make a more accurate diagnosis and minimize the risk of choosing a wrong treatment. A monetary policy committee may prefer to wait for the resolution of some uncertainty (the magnitude of a supply or demand shock or of the output gap, or the resolution of a wage bargaining round), before changing the policy instrument. In short, these “accept or wait” situations seem to be almost as pervasive as the “accept or reject” ones mentioned above. The optimal rule derived in this chapter may be suitable in all these settings.

The rest of the paper is organized as follows: section two presents the setup; in section three the optimal rule is derived. I conclude in the last section.

# 2 The model

We consider a decision maker (DM) whose whose task is to make a decision regarding the time of execution of a project, the returns of which depend on a changing environment. At each period, a committee of $N$ experts make a report about the state of the world. $\mathcal{N}$ denotes the set of experts. There are two states of the world: in the high state, the project is profitable, with positive net present value, normalized to $\vartheta = 1$. In the low state, the project is worthless, with $\vartheta = 0$. The state of the world follows a first order Markov process, with transition probabilities $\lambda_1 = \Pr(\vartheta_{t+1} = 1|\vartheta_t = 1)$, $\lambda_0 = \Pr(\vartheta_{t+1} = 1|\vartheta_t = 0)$. Let $\alpha \equiv \Pr(\vartheta_0 = 1)$ denote the prior probability that the project is a good one. In order to focus exclusively on the timing decision, we assume that the net present value is always nonegative, so it is never unprofitable to reject the project. At each $t$, available actions to DM are “undertake the project” ($U$) or “delay decision” ($D$). Let $\mathcal{A} = \{D, U\}$.

The state of the process is not observable. Instead, each expert $i$ receives a private signal $S_t^i$ of the net present value of the project if undertaken at $t$.

**Assumption 1.** $S_t^i$ depends only on the state $\vartheta_t$.

Let $S_t^i \in \{-1, 1\} \forall i \in \mathcal{N}$. For each $i$, denote $\theta_1^i = \Pr(S_t^i = 1|\vartheta_t = 1)$, $\theta_0^i = \Pr(S_t^i = -1|\vartheta_t = 0)$, the precision of the signals. We assume that signals are independent among experts, and that $\theta_0^i, \theta_1^i \geq 1/2$ which means that signals are informative. We may also interpret $S_t^i$ as expert $i$'s assessment of the state of the process at $t$. The model is of limited communication: at each time $t$ each expert only reports the signal $S_t^i$ to the decision maker. Let $S_t \equiv \{S_t^i\}_{i=1}^N$ denote a report profile at time $t$, let $\mathcal{X}$ denote the set of possible report profiles and let $H_t \equiv \{S_\tau\}_{\tau=1}^t$ denote a history of report profiles.

### Timing

The sequence of events after DM selects $a_t = D$ is the following one: (i) the state changes ($\vartheta_t \to \vartheta_{t+1}$) according to the transition probabilities $\lambda_1$and $\lambda_0$; (ii) experts independently observe signals $S^i_{t+1}$ and report them to DM, who uses the profile $S_{t+1}$ to update the information used for decision making, and selects $a_{t+1}$. If $a_{t+1} = U$ is selected, a reward equal to $\vartheta_{t+1}$ is received.

DM's task at each time is to select an action $a_t \in \mathcal{A}$ based on the information available at $t$, mainly the history of report profiles $H_t$. A decision rule $f_t$ at time $t$ is a function that maps every report at time $t$ to the action set $\mathcal{A} = \{U, D\}$, $f_t : \mathcal{X}^t \to \mathcal{A}$. The problem addressed in the present paper is to find an optimal rule for each $t$.[1] Conditional probability that net present value at period $t$ is $\vartheta_t = 1$ is denoted $p_t = \Pr(\vartheta_t = 1|H_t)$. If the state transits from $\vartheta_{t-1}$ to $\vartheta_t$, DM observes $S_t$ and updates the probability $p_{t-1} \to p_t$ using Bayes' rule.

Let $g(H_t|\vartheta_t = j)$ be the conditional probability density function of the random variable $H_t$. Assume that a prior $\tilde{p}_t = \Pr(\vartheta_t = 1|H_{t-1})$ is known and let $\Pr(S_t = S|\vartheta_t = j) \equiv R_{jS}$. We define the function $T(H_t) = S_t$, that is, $T$ picks the last signal from the history $H_t$ of signals. Regard $\vartheta_t$ as a parameter that takes values in the parameter space $\{0, 1\}$. $\vartheta_t$ is unobserved, but a history of signals $H_t = \{S_1, ..., S_t\}$ is observed and it is available for making inferences relating to the value of $\vartheta_t$. If in order to be able to compute the posterior distribution of $\vartheta_t$ from any prior distribution, only $T(H_t)$ is needed, then $T$ is a sufficient statistic[2].

**Lemma 1.** *$T(H_t) = S_t$ is a sufficient statistic for the family $\{g(\cdot|\vartheta_t = j)\}_{j=0,1}$.*

*Proof.* If $\tilde{p}_t$ characterizes the prior distribution for $\vartheta_t$ ($\tilde{p}_t = \Pr(\vartheta_t = 1|H_{t-1})$), then the posterior distribution $p_t = \Pr(\vartheta_t = 1|H_t)$ is, by Bayes rule,

$$\begin{aligned}\Pr(\vartheta_t = 1|H_t) &= \Pr(\vartheta_t = 1|S_t, H_{t-1}) = \frac{\Pr(S_t = S|\vartheta_t = 1, H_{t-1}) \times \Pr(\vartheta_t = 1|H_{t-1})}{\Pr(S_t = S|H_{t-1})} \\ &= \frac{\Pr(S_t = S|\vartheta_t = 1) \times \Pr(\vartheta_t = 1|H_{t-1})}{\Pr(S_t = S|H_{t-1})}\end{aligned}$$

where the last equality results from assumption 1. The prior is $\Pr(\vartheta_t = 1|H_{t-1}) = \tilde{p}_t$. Straight-

[1] A more general rule should map every history of reports to the action set. Due to the Markov assumption made above, restricting the domain of the function $f_t$ to the current report involves no loss in generality, as it is shown below.

[2] See for example De Groot (1970).

forward computation gives $\Pr(S_t = S|H_{t-1}) = R_{1S}\tilde{p}_t + R_{0S}(1-\tilde{p}_t)$ so

$$\Pr(\vartheta_t = 1|H_t) = \frac{R_{1S}\tilde{p}_t}{R_{1S}\tilde{p}_t + R_{0S}(1-\tilde{p}_t)}.$$

Thus, in order to compute $\Pr(\vartheta_t = 1|H_t)$ from the prior $\tilde{p}_t$, only the value of $S_t$ is needed, and $T(H_t) = S_t$ is a sufficient statistic for the family $\{g(\cdot|\vartheta_t = j)\}_{j=0,1}$. Finally, note that $\tilde{p}_t = \lambda_0(1-p_{t-1}) + \lambda_1 p_{t-1} \equiv \Gamma(p_{t-1})$. Then,

$$p_t = \frac{R_{1S}\Gamma(p_{t-1})}{R_{1S}\Gamma(p_{t-1}) + R_{0S}(1-\Gamma(p_{t-1}))}$$

which only depends on $p_{t-1}$ and $S$. ∎

From lemma 1, it results that in order to calculate $p_t$, the only pertinent information that DM uses is $S_t$ and $p_{t-1}$, so history $H_{t-1}$ does not provide more information than $p_{t-1}$We refer to $p_t$ as the information state at period $t$. Let $\gamma(S,p) \equiv R_{0S}(1-\Gamma(p)) + R_{1S}\Gamma(p)$ and $\Phi(S,p) \equiv \frac{R_{1S}\Gamma(p)}{R_{0S}(1-\Gamma(p))+R_{1S}\Gamma(p)}$. $\gamma(S,p)$ is the likelihood of $S$ and $\Phi(S,p)$ is the bayesian update of $p$, made after the observation of $S$. At period $t$, given that the project has not been undertaken yet, expected net present value of undertaking the project is $p_t$, and if the project is not undertaken, expected net present value is $\beta\sum_{S\in\mathcal{X}}\gamma(S_t,p_t)V_{t+1}(\Phi(S,p_t))$ where $0<\beta<1$ is DM's discount factor. Using definitions above, functional equation associated to DM's problem is

$$\begin{aligned} V_t(p) &= \max\left\{p, \beta\sum_{S\in\mathcal{X}}\gamma(S,p)V_{t+1}(\Phi(S,p))\right\}, t = 1,...,T-1 \qquad (1) \\ V_T(p) &= p \end{aligned}$$

A solution to problem (1) maps each possible value of $p_t$, to an action $a \in \{U, D\}$, for each $t$.

# 3 Optimal aggregation of information

Let $\mathcal{U}_t$ denote the subset of $[0,1]$ for which the optimal action at $t$ is $a_t^* = U$, that is, $\mathcal{U}_t = \{p \in [0,1] : V_t(p) = p\}$. The following result characterizes the solution to problem (1). Its proof is provided in the appendix.

**Proposition 1.**

$(i)$ $1 \in \mathcal{U}_t \forall t$

$(ii)\, \mathcal{U}_t$ *is convex*

$(iii)\, \mathcal{U}_1 \subset \mathcal{U}_2 \subset ... \subset \mathcal{U}_t \subset ... \subset \mathcal{U}_T = [0, 1]$

From Proposition 1, each $\mathcal{U}_t$ has the form $[p_t^*, 1]$. Then, there exist threshold values $\{p_t^*\}_{t=1}^T$ such that, for each $t$, the following policy is optimal:

$$a_t^* = \begin{cases} D & \text{if} \quad p_t < p_t^* \\ U & \text{if} \quad p_t \geq p_t^* \end{cases}$$

Assume that at $t-1$ the decision has been to delay and denote with $f_t^*$ an optimal decision rule, and let $-1$ correspond to decision to delay ($D$) and $+1$ correspond to decision to undertake ($U$).

**Theorem 1.**

$$f_t^*(S) = \text{sign}\left(\sum_{i=1}^N w_i x_i(S) + b_t\right)$$

*where*

$$\text{sign}(a) = \begin{cases} 1 \textit{ if } a \geq 0 \\ -1 \textit{ if } a < 0 \end{cases} ; \qquad w_i = \tfrac{1}{2}\left(\ln \tfrac{\theta_i^0}{1-\theta_i^0} + \ln \tfrac{\theta_i^1}{1-\theta_i^1}\right);$$

$$x_i(S) = \begin{cases} 1 \textit{ if } S^i = 1 \\ -1 \textit{ if } S^i = -1 \end{cases} ; \qquad b_t(S) = \xi_t + \phi_t + \psi;$$

$$\xi_t = \ln \tfrac{\Gamma(p_{t-1})}{1-\Gamma(p_{t-1})}; \qquad \phi_t = \ln \tfrac{1-p_t^*}{p_t^*};$$

$$\psi = \tfrac{1}{2}\textstyle\sum_{i=1}^N \left(\ln \tfrac{\theta_i^1}{\theta_i^0} + \ln \tfrac{1-\theta_i^1}{1-\theta_i^0}\right).$$

*and* $p_{t-1}$ *is computed recursively as*

$$p_{t-1} = \frac{R_{1S_{t-1}}\Gamma(p_{t-2})}{R_{1S_{t-1}}\Gamma(p_{t-2}) + R_{0S_{t-1}}(1-\Gamma(p_{t-2}))}, \qquad p_0 = \alpha,$$

*with*

$$R_{1S_{t-1}} = \Pr(S_{t-1}|\vartheta_{t-1} = 1), \qquad R_{0S_{t-1}} = \Pr(S_{t-1}|\vartheta_{t-1} = 0).$$

*Proof.* DM uses the profile $S$ to update the state $p_{t-1}$ to $p_t = \Phi(S, p_{t-1})$. Then, decision is to undertake if $\Phi(S, p_{t-1}) \geq p_t^*$, that is, if

$$\frac{R_{1S}\Gamma(p_{t-1})}{R_{0S}(1-\Gamma(p_{t-1})) + R_{1S}\Gamma(p_{t-1})} \geq p_t^*, \tag{2}$$

which taking logs and using definitions of $\phi_t$ and $\xi_t$ above, can be expressed as $\ln\frac{R_{1S}}{R_{0S}} + \phi_t + \xi_t \geq 0$. Denoting by $1(S)$ the subset of experts that report $S_i = 1$, and by $-1(S)$ the subset of experts that report $S_i = -1$, when the profile is $S$, it is straightforward to show that the log likelihood ratio is

$$\ln\frac{R_{1S}}{R_{0S}} = \sum_{i \in 1(S)} \ln\frac{\theta_1^i}{\left(1-\theta_0^i\right)} + \sum_{i \in -1(S)} \ln\frac{\left(1-\theta_1^i\right)}{\theta_0^i}.$$

Using definitions of $w_i$, $x_i(S)$, and $\psi$ above, we get $\ln\frac{R_{1S}}{R_{0S}} = \sum_{i=1}^N w_i x_i(S) + \psi$, so condition (2) becomes $\sum_{i=1}^N w_i x_i(S) + \psi + \phi_t + \xi_t \geq 0$,or equivalently $\operatorname{sign}\left(\sum_{i=1}^N w_i x_i(S) + b_t\right) = 1$. By a similar procedure, it is shown that

$$\Phi(S, p_{t-1}) < p_t^* \Leftrightarrow \operatorname{sign}\left(\sum_{i=1}^N w_i x_i(S) + b_t\right) = -1.$$

∎

# 4 Conclusions

In this paper we used a simple model to show that in time-varying environments with an unobservable state, a decision maker that faces irreversible costs of making decisions, and assigns a positive value to the option to wait, should use a time varying information aggregation rule in order to determine the optimal period to undertake a project. If the timing decision is decentralized to a common interest committee (that is, a committee in which every member has the same ex-post payoff) who decides using a quota rule, then the committee will be better off using a time varying quota. If the rule aggregates information optimally, then there exists an equilibrium in which each member votes sincerely, that is, according to the private signal. (McLennan (1998), Theorem 1). The optimal quota rule can be expressed as a weighted majority rule with a time varying bias component.

# Appendix

## Proof of Proposition 1

The proof requires the following lemmata, which are standard in the literature of partially observable Markov processes. (for example, Bertsekas (1995), Degroot (2004), Smallwood and Sondik (1973).)

**Lemma 2.** *(i) $V_t(p) \leq 1$ for every $t$; (ii) for every $p$, $V_t(p)$ is non increasing in $t$;*

*Proof.* (i) By assumption, the NPV of the project is $1$ or $0$ and is perceived only in the period in which action $U$ is selected. Thus, $V_t(p) \leq 1$. (ii) Note that

$$\begin{aligned} V_{T-1}(p) &= \max\left\{p, \beta \sum_{S\in\mathcal{X}} \gamma(S,p)\,\Phi(S,p)\right\} \\ &= \max\left\{p, \beta \sum_{S\in\mathcal{X}} R_{1S}\Gamma(p)\right\} = \max\{p, \beta\Gamma(p)\} \geq p = V_T(p) \end{aligned}$$

where the first equality follows because optimal action at period $T$ is to undertake the project ($a_T^* = U$) and then, $V_T(p) = p$, the second equality follows from the expressions for $\gamma(S,p)$ and $\Phi(S,p)$ and the third equality follows because $\sum_{S\in\mathcal{X}} R_{1S} = \sum_{S\in\mathcal{X}} \Pr(S \mid NPV_t = 1) = 1$. Suppose that $V_t(p) \geq V_{t+1}(p)$ for some $t$ and for all $p \in [0,1]$. To complete the proof, note that

$$\begin{aligned} V_{t-1}(p) &= \max\left\{p, \beta \sum_{S\in\mathcal{X}} \gamma(S,p)\,V_t(\Phi(S,p))\right\} \\ &\geq \max\left\{p, \beta \sum_{S\in\mathcal{X}} \gamma(S,p)\,V_{t+1}(\Phi(S,p))\right\} = V_t(p) \end{aligned}$$

where the inequality follows from the induction hypothesis. ∎

**Lemma 3.** *Let $V_{t-1}^D(p) = \beta \sum_{S\in\mathcal{X}} \gamma(S,p)\,V_t(\Phi(S,p))$ and suppose that $V_t(p)$ is convex. Then $V_{t-1}^D(p)$ is also convex.*

*Proof.* Let $\xi$ and $v \in [0,1]$ and let $p = \mu\xi + (1-\mu)v$, $0 < \mu < 1$; we need to show that $V_{t-1}^D(p) \leq \mu V_{t-1}^D(\xi) + (1-\mu)\,V_{t-1}^D(\nu)$; note that

$$\gamma(S,p) = \gamma(S, \mu\xi + (1-\mu)v) = \mu\gamma(S,\xi) + (1-\mu)\gamma(S,v)$$

and

$$\begin{aligned}
\Phi(S,p) &= \frac{R_{1S}\Gamma(\mu\xi+(1-\mu)v)}{\gamma(S,\mu\xi+(1-\mu)v)} \\
&= \frac{\mu\gamma(S,\xi)}{\mu\gamma(S,\xi)+(1-\mu)\gamma(S,\nu)}\Phi(S,\xi)+\frac{(1-\mu)\gamma(S,\nu)}{\mu\gamma(S,\xi)+(1-\mu)\gamma(S,\nu)}\Phi(S,\nu)
\end{aligned}$$

It follows from the convexity of $V_t(p)$ that

$$\gamma(S,p)V_t(\Phi(S,p)) \leq \mu\gamma(S,\xi)V_t(\Phi(S,\xi))+(1-\mu)\gamma(S,\nu)V_t(\Phi(S,\nu))$$

hence

$$\begin{aligned}
V_{t-1}^D(p) &= \beta\sum_{S\in\mathcal{X}}\gamma(S,p)V_t(\Phi(S,p)) \\
&\leq \mu\beta\sum_{S\in\mathcal{X}}\gamma(S,\xi)V_t(\Phi(S,\xi))+(1-\mu)\beta\sum_{S\in\mathcal{X}}\gamma(S,\nu)V_t(\Phi(S,\nu)) \\
&= \mu V_{t-1}^D(\xi)+(1-\mu)V_{t-1}^D(\nu)
\end{aligned}$$

thus $V_{t-1}^D(p)$ is convex. ∎

**Lemma 4.** *$V_t(p)$ is convex for $t=1,...,T$.*

*Proof.* We proceed by induction. Suppose that $V_t^D(p)$ is convex, $t\leq T-1$. Then, $V_t(p)\equiv\max\{p,V_t^D(p)\}$ is convex since it is the maximum of two convex functions, and $V_{t-1}^D(p)\equiv\beta\,\mathrm{E}_S V_t(\Phi(p))$ is also convex (Lemma 3). This implies that $V_{t-1}(p)\equiv\max\{p,V_{t-1}^D(p)\}$ is convex in $p$. To complete the proof, note that $V_T^D(p)=0$ and $V_T^U(p)=p$ are linear functions, so $V_T(p)\equiv\max\{p,V_T^D(p)\}$ is convex, and by Lemma 3, $V_{T-1}^D(p)$ is also convex. ∎

*Proof of Proposition 1.* (i) $1\in\mathcal{U}_T$ because $V_T(1)=1$. Then, $V_t(1)=1\ \forall t<T$ because for each $p$, $V_t(p)$ is non increasing in $t$ (Lemma 2) and for each $t$, is bounded above by $1$ (Lemma 2). We conclude that $1\in\mathcal{U}_t\forall t$. (ii) Suppose $p$ and $p'$ are in $\mathcal{U}_t$. Let $p''=\nu p+(1-\nu)p'$ for some $\nu\in(0,1)$. Then, $V_t(p'')\leq\nu V_t(p)+(1-\nu)V_t(p')=\nu p+(1-\nu)p'=p''$ where the inequality is a result of convexity of value function (Lemma 4) and the first equality results because $p$ and $p'$ belong to $\mathcal{U}_t$. But, by definition of the value function, $V_t(p'')\geq p''$ so we conclude that $V_t(p'')=p''$ and thus $p''\in\mathcal{U}_t$. (iii) Suppose $p\in\mathcal{U}_t$. Then $p=V_t(p)\geq V_{t+1}(p)$ (Lemma 2) and by definition of the value function, $V_{t+1}(p)\geq p$ . Thus $p=V_{t+1}(p)$, so $p\in\mathcal{U}_{t+1}$. ∎

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