No advantageous merging in minimum cost spanning tree problems

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Abstract

In the context of cost sharing in minimum cost spanning tree problems, we introduce a property called *No Advantageous Merging*. This property implies that no group of agents can be better off claiming to be a single node. We show that the sharing rule that assigns to each agent his own connection cost (the Bird rule) satisfies this property. Moreover, we provide a characterization of the Bird rule using No Advantageous Merging.

Keywords: Minimum cost spanning tree problems, cost sharing, Bird rule, No Advantageous Merging.

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1 Introduction

Minimum cost spanning tree problems *(mcstp)* modelize situations where a group of agents, located at different geographical points, want some particular service which can only be provided by a common supplier, or *source.* There are many economic situations that can be modeled in this way, for instance, some houses in a village may want to be connected to a common water source or to a power plant. Other examples include communication networks such as Internet, cable television or telephone.

The agents will be served through connections which entail some cost. However, the agents are not concerned with whether they are connected directly or indirectly to the source.

Hence, the optimal network is a *minimum (cost spanning) tree (mt).* An algorithm for building an *mt* is provided by Prim (1957). But building an *mt* is only a part of the problem. Another important issue is how to divide the cost associated with the *mt* between the agents. Different *rules* give different answers to this question. A *rule* is a mapping that determines, for each specific problem, a division of the amount to be paid by the agents involved in the problem.

Bird (1976) associated a coalitional game with any *mcstp.* In case the *mcstp* has a unique *mt*, he proposed a rule that is known as the *Bird rule.* Granot and Huberman (1981, 1984) studied the *core* and the *nucleolus* of the coalitional game. Sharkey (1995) surveyed most of the literature. Recently, Kar (2002) studied the Shapley value of the game that can be associated with each *mcstp.* Dutta and Kar (2004) proposed a new rule and characterized the Bird rule using a property of restricted consistency. Bergantiños and Vidal-Puga (2004a) defined another rule, φ.

Feltkamp et al. (1994) introduced a rule for *mcstp* called *Equal Remaining Obligations* rule. This rule has been studied in other papers. Branzei et al. (2004) obtained a characterization of the Equal Remaining Obligations rule and Bergantiños and Vidal-Puga (2004b) proved that φ coincides with the Equal Remaining Obligations rule. Moreover, Bergantiños and Vidal-Puga

(2006) proved that φ is the Shapley value of a different coalitional game. More recently, Tijs *et al.* (2004) defined a class of rules called *Obligation Rules*. Moretti *et al.* (2005) studied this class of rules.

Different rules are usually associated with alternative sets of properties that represent ethical or operational principles. The aim of the axiomatic approach is, precisely, to identify each rule with a well-defined set of properties. This helps to understand the nature of the different rules and their applicability. It is therefore important to have alternative characterizations of the same rule, because this allows to have different insights on the principles underlying the rule and on the type of problems for which it might be suitable.

In this paper, we focus on a new property called *No Advantageous Merging (NAM).* The idea behind this property that there exists a planner who wants to construct a network to connect all the agents to a source. In this kind of situations some agents may have incentives to join in advance in order to be treated as a single agent and get advantage. A rule satisfies No Advantageous Merging if the agents have no incentives to do this.

In the general domain of the *mcstp,* No Advantageous Merging is incompatible with *Symmetry.* This implies that we should restrict the domain. However, this new domain does not need to be very restrictive. In particular, in case each *mcstp* has a unique *mt*, the Bird rule satisfies No Advantageous Merging. Moreover, we provide an axiomatic characterization of the Bird rule using No Advantageous Merging.

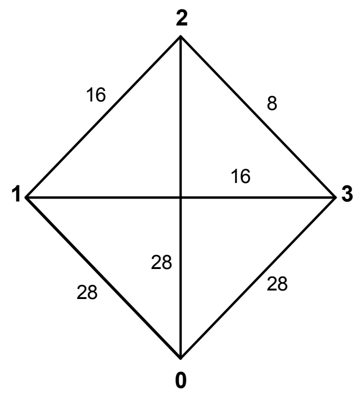
This property is related to others that have been studied in different problems. For instance, a bankruptcy problem describes a situation in which an arbitrator has to allocate a given amount among a group of agents who have claims on it. A property known as No Advantageous Merging has been studied in this framework. This property means that no group of creditors have incentives to pool their claims and to present themselves as a single creditor. We can find a similar example in Social Choice, where *group-strategyproofness* ensures that no subset of agents can gain by reporting false preferences. All these properties have in common that they pretend to avoid agents cheating in order to get advantage.

The paper is organized as follows. In Section 2 we show by an example that No Advantageous Merging is a very strong property in the most general framework. We hence have to restrict ourselves to a smaller class of *mcstp*. In Section 3 we introduce the model. In Section 4 we present the properties used in the characterization. In Section 5 we prove that the Bird rule satisfies these properties and we also present the characterization result. In Section 6 we prove that the properties are independent.

2 An example

Example 2.1 *There are three agents,* $1, 2$ *and* 3. *The connection cost between each agent and the source is 28. The connection cost between agents 1 and 2 and between agents 1 and 3 is 16. The connection cost between agents 2 and 3 is 8.*

This problem is represented in the following figure:

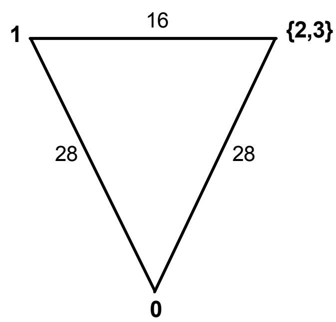


where 0 is the source.

The minimum connection cost is 52 *(there exist more than one possible minimum tree).*

Let x_i *be the assignment that a rule proposes to each agent* i.

Suppose now that the agents in $\{2, 3\}$ *join and act as a single one. The resulting problem can be represented as follows:*

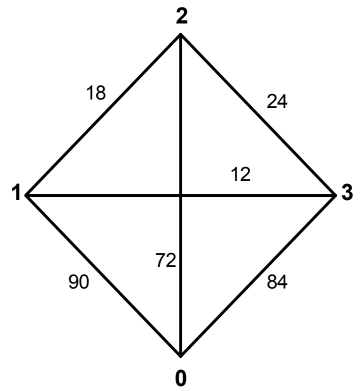


By a symmetry argument, the allocation in this problem should be $(22, 22)$*. Moreover, since the cost of connection between agents* 2 *and* 3 *is* 8*, under No Advantageous Merging the rule should assign them no more than* 30*,* i.e. $x_2 + x_3 \leq 22 + 8$.

Proceeding in the same way with coalitions $\{1, 2\}$ *and* $\{1, 3\}$*, we obtain* $x_1 + x_2 \leq 18 + 16$ *and* $x_1 + x_3 \leq 18 + 16$*. However, the minimum connection cost is* 52*, hence we have an incompatibility.*

It may be argued that this is a special example, because there are more than one possible minimum tree. In the literature of *mcstp* it is usual to consider that there exists a unique minimum tree, or even that there are not two arcs with the same cost (see, for instance, Bird (1976) and Dutta and Kar (2004)). We study what happens in this situation.

Example 2.2 *There are three agents,* $1, 2$ *and* 3*. The connection costs between the source and agent* 1 *is 90, between the source and agent* 2 *is 72 and between the source and agent* 3 *is 84. The connection cost between agents* 1 *and* 2 *is* 18*, between agents* 1 *and* 3 *is* 12 *and between agents* 2 *and* 3 *is* 24*. This problem is represented in the following figure:*

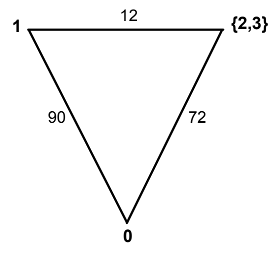


where 0 is the source.

We study the assignment proposed by several rules in the literature. These are given in the following table:

	1	2	3
Shapley value (Kar, 2002)	37	28	37
Bird (1976)	18	72	12
Dutta and Kar (2004)	12	18	72
Nucleolus (Granot and Huberman, 1984)	32	32	38
Bergantiños and Vidal-Puga (2004a)	33	36	33

Assume now that the agents in $\{2,3\}$ *join and act as a single one. The resulting problem can be represented as follows:*



In this case, the proposal given by each of the previous rules is:

	1	$\{2,3\}$
Shapley value (Kar, 2002)	51	33
Bird (1976)	12	72
Dutta and Kar (2004)	72	12
Nucleolus (Granot and Huberman, 1984)	51	33
Bergantiños and Vidal-Puga (2004a)	42	42

The question is: Do these rules satisfy No Advantageous Merging? If we compare the costs that agents 2 and 3 have to pay in both situations, we have

Shapley value (Kar, 2002)	$28 + 37 > 33 + 24$
Bird (1976)	72+12<72+24
Dutta and Kar(2004)	$18 + 72 > 12 + 24$
Nucleolus (Granot and Huberman, 1984)	$32 + 38 > 33 + 24$
Bergantiños and Vidal-Puga (2004a)	$36 + 33 > 42 + 24$

Hence, in this example, only the rule defined by Bird (1976) satisfies No Advantageous Merging. We will prove that this result holds in general.

3 The model

Given a finite set A, we denote the cardinal set of A as 2^A, the cardinality of A as $|A|$ and the set of real $|A|$-tuples whose indices are the elements of A as $\mathbb{R}^A$. Given a function $f : A \to \mathbb{R}$, we denote the set of elements in A that maximize f as $\arg\max_{a\in A}\{f(a)\}$. We define $\arg\min_{a\in A}\{f(a)\}$ analogously.

Let N be a finite set of *agents* who want to be connected to a source. Usually, we denote the set of agents as $N = \{1, 2, ...n\}$. Let $N_0 = N \cup \{0\}$, where 0 is the *source*.

A *cost matrix* on N_0, $C = (c_{ij})_{i,j\in N_0}$ represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$, and $c_{ii} = 0$ for each $i \in N_0$.

We denote the set of all cost matrices on N as $\mathcal{C}^N$. Given two matrices $C, C' \in \mathcal{C}^N$, we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly *mcstp*, is a pair (N_0, C) where N is the set of agents, 0 is the source, and $C \in \mathcal{C}^N$.

A *network* g over N_0 is a subset of $\{(i,j) : i, j \in N_0\}$. The elements of g are called *arcs*. We assume that the arcs are undirected, i.e. (i,j) and (j,i) represent the same arc.

Given a network g and a pair of nodes i and j, a *path* from i to j in g is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$ and $j = i_l$.

A *tree* over $S \subset N_0$ is a network satisfying that for all $i, j \in S$ there exists a unique path from i to j.

Given a network g, we say that two nodes i, j are *connected* in g if there exists a path from i to j in g.

Let $\mathcal{G}^N$ denote the set of all networks over N_0. Let $\mathcal{G}_0^N$ denote the set of all networks over N_0 such that every node in N is connected to the source. Let $\mathcal{T}_0^N$ denote the set of all trees over N_0. Clearly, $\mathcal{T}_0^N \subset \mathcal{G}_0^N \subset \mathcal{G}^N$.

Given $g \in \mathcal{G}^N$, we define the *cost* associated with g in (N_0, C) as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum (cost spanning) tree* for (N_0, C), briefly an *mt*, is a tree $t \in \mathcal{T}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. Given an *mcstp* (N_0, C), an *mt* always exists but it does not need to be unique. We denote the cost associated with any *mt* on (N_0, C) as $m(N_0, C)$.

Given $S \subset N_0$, let C_S denote the matrix C restricted to S. We denote the restriction to S of the *mcstp* (N_0, C) as (S, C_S), and the cost associated with any *mt* on (S, C_S) as $m(S, C_S)$, that is, $m(S, C_S)$ is the cost of connection of the agents of S among themselves. Note that when $0 \notin S$, $m(S, C_S)$ does not include the cost of connection to the source.

Given a tree t, we define the *predecessor set* of a node i in t as $P(i, t) =$

$\{j \in N_0 : j$ is in the unique path from i to the source$\}$. We assume that $i \notin P(i,t)$ and $0 \in P(i,t)$ when $i \neq 0$. For notational convenience, $P(0,t) = \emptyset$. The *geodesic distance* from node i to the source in t is the cardinality of $P(i,t)$. The *immediate predecessor* of agent i in t, denoted as i^0, is the node that comes immediately before i, that is, $i^0 \in P(i,t)$ and $k \in P(i,t)$ implies either $k = i^0$ or $k \in P(i^0,t)$. Note that $P(i^0,t) \subset P(i,t)$ and $P(i,t) \backslash P(i^0,t) = \{i^0\}$. The *follower set* of an agent i in t is the set $F(i,t) = \{j \in N : i \in P(j,t)\}$. The *immediate followers* of agent i in t, denoted as $IF(i,t)$, is the set of agents that come immediately after agent i, that is, $IF(i,t) = \{j \in F(i,t) : j^0 = i\}$.

Let Π_N denote the set of all orders in N. Given $\pi \in \Pi_N$, let P_i^π denote the set of elements in N which come before i in the order given by π, *i.e.*

$$P_i^\pi := \{j \in N : \pi(j) < \pi(i)\}.$$

There are several algorithms in the literature to construct an *mt.* Prim (1957) provides such an algorithm: Sequentially, the agents connect to the source. At each stage, the cheapest arc between the connected and the unconnected agents is added. Formally, Prim's algorithm is defined as follows:

Let $S_g^0 = \{0\}$ and $g^0 = \emptyset$.

Stage 1: Take an arc $(0,i)$ such that $c_{0i} = \min\limits_{j \in N}\{c_{0j}\}$. Now, $S_g^1 = \{0,i\}$ and $g^1 = \{(0,i)\}$.

Stage p: Assume we have defined $S_g^{p-1} \subset N_0$ and $g^{p-1} \in \mathcal{G}^N$. We now define S_g^p and g^p. Take an arc (i,j), $i \in S_g^{p-1}$, $j \in N_0 \backslash S_g^{p-1}$, such that $c_{ij} = \min\limits_{k \in S_g^{p-1}, l \in N_0 \backslash S_g^{p-1}} \{c_{kl}\}$. Now $S_g^p = S_g^{p-1} \cup \{i\}$ and $g^p = g^{p-1} \cup \{(i,j)\}$.

This process finishes in n stages. We say that g^n is a tree obtained via Prim's algorithm.

This algorithm provides an *mt*, but not necessarily unique (when the minimizer arc is not unique). Moreover, each *mt* can be obtained via Prim's algorithm.

A *(cost allocation) rule* is a function ϕ that assigns to each *mcstp* (N_0, C) a vector $\phi(N_0, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} \phi_i(N_0, C) = m(N_0, C)$, where

$\phi_i(N_0, C)$ represents the cost assigned to agent i.

Notice that we implicitly assume that the agents build an *mt*.

Bird (1976) introduced a rule which is defined through Prim's algorithm. He assumed that there is a unique *mt*.

Given an *mcstp* (N_0, C) and an *mt* $t = \{(i^0, i)\}_{i \in N}$ in (N_0, C), the *Bird rule (B)* is defined as:

$$B_i^t(N_0, C) = c_{i^0 i}$$

for each $i \in N$.

The idea of the Bird rule is quite simple: The agents connect to the source following Prim's algorithm and each agent pays the cost of his connection.

Finally, we define a concept introduced by Norde *et al.* (2004) that will be used in some of the proofs.

Given $S \subset N_0$, we say that $i, j \in S$, $i \neq j$ are (C, S)-*connected* if there exists a path g from i to j satisfying that $c_{kl} = 0$ for all $(k, l) \in g$.

We say that $S \subset N_0$ is a *C-component* if two conditions hold: First, for all $i, j \in S$, i and j are (C, S)-*connected.* Second, S is maximal, *i.e.* if $S \subsetneq T$, there exist $i, j \in T$, $i \neq j$ such that i and j are not (C, T)-*connected.*

The set of *C-components* constitutes a partition of N_0.

4 Properties

Before introducing the properties of the rules, we define the domain restriction on the set of permissible cost matrices that will be used. This restriction is necessary because of the incompatibility presented in Section 2.

Definition 4.1 $\mathcal{D} := \{C \in \mathcal{C}^N$: *no two edges with positive cost have the same value*$\}$.

Remark 4.1 *Dutta and Kar (2004) define two different domain restrictions. These definitions are the following:*

$C^1 := \{C \in \mathcal{C}^N : C$ *induces a unique* mt *for all* $N\}$,

$C^2 := \{C \in C^1$: *no two edges of the unique mt have the same cost*$\}$.

Recall that Bird (1976) defines the *Bird rule* when there is a unique *mt.* Over $\mathcal{D}$, there might exist several *mt.* In the next Proposition, we prove that even though this is true, the *Bird rule*'s assignment is the same for all of them.

Proposition 4.1 *Let $C \in \mathcal{D}$ and let t, t' be two different* mt *on (N_0, C). Then, $B_i^t(N_0, C) = B_i^{t'}(N_0, C)$ for all $i \in N$.*

Proof. We will construct t and t' following Prim's algorithm.

Since $C \in \mathcal{D}$, both trees will have the same arcs until one agent connects to the source with null cost. In that case, more than one arc with the same cost may exist. Assume that it happens in stage p. Hence, $t^{p-1} = t'^{p-1}$ and $S_t^{p-1} = S_{t'}^{p-1}$. By definition of the Bird rule, $B_i^t(N_0, C) = B_i^{t'}(N_0, C)$ for all $i \in S_t^{p-1} = S_{t'}^{p-1}$.

Since we have found an arc with null cost, we have a non-trivial *C-component,* say $S \subset N_0$. Since all the agents in S, but the first one, connect with null cost, whatever the order of connection of the agents from the *C-component,* each of them should pay zero under t and t'. Hence $B_i^t(N_0, C) = B_i^{t'}(N_0, C)$ for all $i \in S$.

When all the agents in S are connected to the source, the following arc that connects to the source, if any, will have positive cost. Hence, the domain $\mathcal{D}$ requires that the arcs formed in both trees, t and t' will be the same again until a new *C-component* appears.

The procedure for the rest of the *C-components* is analogous because all the agents in the *C-component* (but the first one) connect with null cost, and they pay zero. ■

We now introduce different properties of the rules.

Definition 4.2 *A rule ϕ satisfies* Core Selection (CS) *if for all* mcstp (N_0, C) *and all $S \subset N$, $\sum_{i \in S} \phi_i(N_0, C) \leq m(S_0, C_{S_0})$.*

This property says that no group of agents can be better off constructing their own network instead of paying what the rule ϕ proposes to them.

Before moving on to the next property, we introduce the concept of an *extreme null point.*

Definition 4.3 *Given an* mcstp (N_0, C) *and an mt* t *in* (N_0, C)*, we say that* $i \in N$ *is an* extreme point *in* t *if* $F(i,t) = \emptyset$*.*

Definition 4.4 *Given an* mcstp (N_0, C) *we say that* $i \in N$ *is an* Extreme Null Point (ENP) *if it is an extreme point in all the mt in* (N_0, C) *and moreover* $c_{i^0 i} = 0$*.*

Dutta and Kar (2004) defined the concept of extreme point. They argued that since node i is an extreme point, this node is of no use to the rest of the network since no node is connected to the source through node i.

We argue the same for every extreme null point. Moreover, since i connects to the source with null cost, it does not increase the total cost of the network. Hence, we can consider that node i is not beneficial for the rest of agents but neither is a problem for them. So, it seems reasonable that the allocation of the rest of the agents does not change if he connects to the source.

The property that we define states that if agent i is an *ENP*, no agent j will pay a different cost in order to include agent i in the network. Formally:

Definition 4.5 *A rule* ϕ *satisfies* Independence of Extreme Null Points (IENP) *if for all mcstp* (N_0, C) *and all ENP* $i \in N$*,*

$$\phi_j(N_0 \backslash \{i\}, C_{N_0 \backslash \{i\}}) = \phi_j(N_0, C)$$

for all $j \in N \backslash \{i\}$*.*

This property implies that each *ENP* pays zero.

Remark 4.2 *This property is similar to the one defined by Derks and Haller (1999) called "Null Player Out", which requires that a null player (that is, an individual whose contribution to any coalition is zero) does not influence the utility allocation within the rest of the society*[1]*.*

[1] Hamiache (2006) uses the term "Independence of Irrelevant Players".

We next consider the possibility that a group of agents $A \subset N$ joins in advance to be treated as a single node $\alpha \in A$.

The result is a new problem, called *reduced problem*, where the cost of connection between some node i in $N_0 \backslash A$ and α is the minimal connection cost between node i and the agents in A. The rest of the costs remain as in the initial problem. Formally,

Definition 4.6 *Given an mcstp* (N_0, C) *and* $\alpha \in A \subset N$, *the* reduced problem $(N_0^{A\alpha}, C^{A\alpha})$ *is defined as* $N^{A\alpha} = (N \backslash A) \cup \{\alpha\}$, $c_{ij}^{A\alpha} = c_{ij}$ *for all* $i, j \in N_0 \backslash A$, *and* $c_{i\alpha}^{A\alpha} = \min_{j \in A}\{c_{ij}\}$ *for all* $i \in N_0 \backslash A$.

We introduce a new property in *mcstp*:

Definition 4.7 *Given* $C \in \mathcal{D}$ *a rule* ϕ *satisfies* No Advantageous Merging (NAM) *if*

$$\sum_{i \in A} \phi_i(N_0, C) \leq \phi_\alpha(N_0^{A\alpha}, C^{A\alpha}) + m(A, C_A)$$

for all $\alpha \in A \subset N$.

This property asserts that no group of agents might have any incentive to join in advance, assuming the cost, to be treated as a single agent.

In the next proposition we state that in the reduced problem the matrix $C^{A\alpha}$ is well-defined over $\mathcal{D}$.

Proposition 4.2 *If* $C \in \mathcal{D}$, *then* $C^{A\alpha} \in \mathcal{D}$ *for all* $\alpha \in A \subset N$.

The proof is straightforward and we omit it.

5 The main result

In this section we present a characterization of the Bird rule in $\mathcal{D}$. It is the only rule satisfying *CS*, *IENP* and *NAM.* First, in Proposition 5.1, we prove that the *Bird rule* satisfies the mentioned properties.

Proposition 5.1 *Over the domain $\mathcal{D}$, the Bird rule satisfies CS, IENP and NAM.*

Proof. B satisfies *CS.* See Bird (1976, page 340).

B satisfies *IENP.* It is straightforward.

B satisfies *NAM:*

Let $t = \{(i^0, i)\}_{i \in N}$ be an *mt* in the problem (N_0, C). Let $\alpha \in A \subset N$.

Consider now (A, C_A) and let τ be an *mt* in (A, C_A), *i.e.* $c(\tau) = m(A, C_A)$.

Let i^* be the first agent in A that connects to the source following Prim's algorithm in t (*i.e.* $S_t^p \cap A = \{i^*\}$ and $S_t^{p-1} \cap A = \emptyset$). Note that $i^{*0} \in N_0 \backslash A$.

Given t, we construct a new network in the problem (N_0, C). Let

$$t^* = (t \backslash \{(i^0, i)\}_{i \in A}) \cup \{(i^{*0}, i^*)\} \cup \tau.$$

It is straightforward to check that t^* is a tree in (N_0, C).

Since t is an *mt* in this problem, we have that $c(t) \leq c(t^*)$. Hence,

$$c(t) \leq c(t) - \sum_{i \in A} c_{i^0 i} + c_{i^{*0} i^*} + c(\tau)$$

that is,

$$\sum_{i \in A} c_{i^0 i} \leq c_{i^{*0} i^*} + c(\tau) = c_{i^{*0} i^*} + m(A, C_A).$$

Since i^* is the first agent in A to be connected, $c_{i^{*0} i^*} = \min\limits_{j \in A} \{c_{i^{*0} j}\} = c^{A\alpha}_{i^{*0} \alpha}$. Hence, we can rewrite the above expression as:

$$\sum_{i \in A} c_{i^0 i} \leq c^{A\alpha}_{i^{*0} \alpha} + m(A, C_A).$$

It is clear that, following Prim's algorithm, we can construct an *mt* t' in $(N_0^{A\alpha}, C^{A\alpha})$ such that $(i^{*0}, \alpha) \in t'$. Hence, by definition of the Bird rule,

$$\sum_{i \in A} B_i(N_0, C) \leq B_\alpha(N_0^{A\alpha}, C^{A\alpha}) + m(A, C_A).$$

■

Now we present a characterization of the Bird rule.

Theorem 5.1 *Over the domain $\mathcal{D}$, a rule ϕ satisfies NAM, CS, and IENP if and only if $\phi = B$.*

Proof. Let $t = \{(i^0, i)\}_{i \in N}$ be an *mt* in the problem (N_0, C). Let ϕ be a rule satisfying *NAM*, *CS* and *IENP*.

We will prove that $\phi_i(N_0, C) = c_{i^0 i}$ for all $i \in N$ by induction on the geodesic distance of agent $i \in N$ to the source in t. Let $\sigma(i)$ denote such a geodesic distance.

For $\sigma(i) = 0$ the result is trivial.

Assume $\phi_i(N_0, C) = c_{i^0 i}$ for all $i \in N$ such that $\sigma(i) < l$.

Now, we will prove that the result is true for any $i \in N$ such that $\sigma(i) = l$.

Let $S = P(i, t) \backslash \{0\}$. Notice that $m(S_0 \cup \{i\}, C_{S_0 \cup \{i\}}) = \sum_{j \in S} c_{j^0 j} + c_{i^0 i}$.

Under *CS*,

$$\sum_{j \in S} \phi_j(N_0, C) + \phi_i(N_0, C) \leq m(S_0 \cup \{i\}, C_{S_0 \cup \{i\}}) = \sum_{j \in S} c_{j^0 j} + c_{i^0 i}.$$

Under the induction hypothesis, $\phi_j(N_0, C) = c_{j^0 j}$ for all $j \in S$. Hence,

$$\phi_i(N_0, C) \leq c_{i^0 i}.$$

Now we prove that $c_{i^0 i} \leq \phi_i(N_0, C)$.

Let $F_i = F(i, t)$ and let $\eta = |IF(i, t)|$. This means that there are η agents connected directly to agent i in the *mt*, maybe $\eta = 0$.

Consider a new problem $(N_0^\epsilon, C^\epsilon)$ similar to (N_0, C), but adding an imperfect substitute for agent i. Formally, $N_0^\epsilon = N_0 \cup \{\alpha\}$ with $\epsilon > 0$ sufficiently small, $c^\epsilon_{jj'} = c_{jj'}$ for all $j, j' \in N_0$, $c^\epsilon_{i\alpha} = 0$, $c^\epsilon_{j\alpha} = c_{ji} + \epsilon_j$, $\epsilon_j \leq \epsilon$ for all $j \in IF(i, t)$, and $c^\epsilon_{j\alpha}$ large enough for all $j \in N_0 \backslash (IF(i, t) \cup \{i\})$. Under these conditions, $t^\epsilon := t \cup \{(i, i^*)\}$ is an *mt* in $(N_0^\epsilon, C^\epsilon)$, $c(t^\epsilon) = c(t)$, and α is an ENP in $(N_0^\epsilon, C^\epsilon)$.

Assume the agents in $F_i \cup \{\alpha\}$ join to be treated as a single node α. That is, consider the reduced problem[2] $(N_0^{\epsilon A\alpha}, C^{\epsilon A\alpha})$ with $A = F_i \cup \{\alpha\}$.

[2]We write $N_0^{\epsilon A\alpha}$ instead of $(N^\epsilon)_0^{A\alpha}$ and $C^{\epsilon A\alpha}$ instead of $(C^\epsilon)^{A\alpha}$.

By definition, $c_{i\alpha}^{\epsilon A\alpha} = \min_{j\in A}\{c_{ij}^{\epsilon}\}$, hence $c_{i\alpha}^{\epsilon A\alpha} \leq c_{i\alpha}^{\epsilon}$. Since $c_{i\alpha}^{\epsilon} = 0$, we have $c_{i\alpha}^{\epsilon A\alpha} = 0$.

It is straightforward to check that $t' = \{(j^0, j)\}_{j\in N\backslash F_i} \cup \{(i,\alpha)\}$ is an *mt* in $(N_0^{A\alpha}, C^{A\alpha})$.

Since ϕ satisfies *NAM*,

$$\sum_{j\in A} \phi_j(N_0^{\epsilon}, C^{\epsilon}) \leq \phi_{\alpha}(N_0^{\epsilon A\alpha}, C^{\epsilon A\alpha}) + m(A, C_A^{\epsilon}).$$

Under *IENP,* $\phi_{\alpha}(N_0^{\epsilon A\alpha}, C^{\epsilon A\alpha}) = 0$. Hence,

$$\sum_{j\in A} \phi_j(N_0^{\epsilon}, C^{\epsilon}) \leq m(A, C_A^{\epsilon}). \tag{1}$$

We study both terms.

Let $K = N\backslash F_i$. Note that $i \in K$ and $K = N^{\epsilon}\backslash A$.

Claim I $\sum_{j\in A} \phi_j(N_0^{\epsilon}, C^{\epsilon}) = m(N_0, C) - \sum_{j\in K\backslash\{i\}} \phi_j(N_0^{\epsilon}, C^{\epsilon}) - \phi_i(N_0^{\epsilon}, C^{\epsilon})$.

In the problem $(N_0^{\epsilon}, C^{\epsilon})$,

$$m(N_0^{\epsilon}, C^{\epsilon}) = \sum_{j\in K\backslash\{i\}} \phi_j(N_0^{\epsilon}, C^{\epsilon}) + \phi_i(N_0^{\epsilon}, C^{\epsilon}) + \sum_{j\in A} \phi_j(N_0^{\epsilon}, C^{\epsilon}).$$

Clearly, $m(N_0^{\epsilon}, C^{\epsilon}) = m(N_0, C)$ and hence the result.

Claim II $m\left(A, C_A^{\epsilon}\right) \leq m(N_0, C) - m\left(K_0\backslash\{i\}, C_{K_0\backslash\{i\}}^{\epsilon}\right) - c_{i^0 i} + \eta\epsilon$.

Let $IF_i = IF(i,t)$. By definition of C^{ϵ}, for an ϵ sufficiently small, we can construct an *mt* τ on (A, C_A^{ϵ}) such that the immediate followers of i in t connect to α and the rest of agents in A connect to the same nodes as in t, *i.e.* $\tau = \{(j^0, j)\}_{j\in F_i\backslash IF_i} \cup \{(\alpha, j)\}_{j\in IF_i}$ is an *mt* on (A, C_A^{ϵ}).

Hence, $m(A, C_A^{\epsilon}) = \sum_{j\in F_i\backslash IF_i} c_{j^0 j}^{\epsilon} + \sum_{j\in IF_i} c_{\alpha j}^{\epsilon}$.

By definition, $c_{j^0 j}^{\epsilon} = c_{j^0 j}$ for all $j \in F_i\backslash IF_i$ and $c_{\alpha j}^{\epsilon} \leq c_{ij} + \epsilon$ for all $j \in IF_i$. Since $\eta = |IF_i|$, we have

$$m(A, C_A^{\epsilon}) \leq \sum_{j\in F_i} c_{j^0 j} + \eta\epsilon. \tag{2}$$

Consider now $(F_i \cup \{i\}, C_{F_i \cup \{i\}})$. It is straightforward to check that we can construct an *mt* t^* on $(F_i \cup \{i\}, C_{F_i \cup \{i\}})$ such that each agent in F_i connects to the same nodes as in t, *i.e.* $t^* = \{(j^0, j)\}_{j \in F_i}$ is an *mt* on $(F_i \cup \{i\}, C_{F_i \cup \{i\}})$.
Hence, $m(F_i \cup \{i\}, C_{F_i \cup \{i\}}) = \sum\limits_{j \in F_i} c_{j^0 j}$.
Replacing this expression in (2), we have

$$m(A, C_A^\epsilon) \leq m(F_i \cup \{i\}, C_{F_i \cup \{i\}}) + \eta\epsilon.$$

On the other hand, since no agent in K connects to the source through agent i in t, we have

$$m(N_0, C) = m(K_0 \backslash \{i\}, C_{K_0 \backslash \{i\}}) + c_{i^0 i} + m(F_i \cup \{i\}, C_{F_i \cup \{i\}}).$$

Combining the last two expressions:

$$\begin{aligned} m(A, C_A^\epsilon) &\leq m(N_0, C) - m(K_0 \backslash \{i\}, C_{K_0 \backslash \{i\}}) - c_{i^0 i} + \eta\epsilon \\ &= m(N_0, C) - m(K_0 \backslash \{i\}, C^\epsilon_{K_0 \backslash \{i\}}) - c_{i^0 i} + \eta\epsilon. \end{aligned}$$

Applying Claim I and Claim II in (1) we deduce

$$m(K_0 \backslash \{i\}, C^\epsilon_{K_0 \backslash \{i\}}) - \sum_{j \in K \backslash \{i\}} \phi_j(N_0^\epsilon, C^\epsilon) + c_{i^0 i} - \eta\epsilon \leq \phi_i\left(N_0^\epsilon, C^\epsilon\right).$$

Since ϕ satisfies *CS*,

$$c_{i^0 i} - \eta\epsilon \leq \phi_i\left(N_0^\epsilon, C^\epsilon\right).$$

Under *IENP*, $\phi_i\left(N_0^\epsilon, C^\epsilon\right) = \phi_i\left(N_0, C\right)$. Thus, $c_{i^0 i} - \eta\epsilon \leq \phi_i\left(N_0, C\right)$. But $\phi_i(N_0, C)$ does not depend on ϵ. Hence, $c_{i^0 i} \leq \phi_i(N_0, C)$. ■

6 Independence of the axioms

In this section we show that the three axioms used in Theorem 5.1 are independent.

The following rule assigns to each agent half of the cost of his adjacent arcs in the *mt*. Moreover, the agents that connect directly to the source pay the entire connection cost with the source. Formally,

$$\phi_i^1(N_0, C) := \begin{cases} c_{0i} + \frac{1}{2} \sum\limits_{j \in IF(i,t)} c_{ij} & \text{if } i^0 = 0 \\ \frac{1}{2} c_{i^0 i} + \frac{1}{2} \sum\limits_{j \in IF(i,t)} c_{ij} & \text{if } i^0 \neq 0 \end{cases}$$

for all $C \in \mathcal{D}$ and $i \in N$.

This rule satisfies a stronger property than *NAM*, which states that if a group of agents A join in advance in order to be treated as a single node, no agent in $N \backslash A$ gets worse off in the reduced problem. Formally,

Definition 6.1 *Let $C \in \mathcal{D}$ and $\alpha \in A \subset N$. A rule ϕ satisfies* Strong No Advantageous Merging (SNAM) *if*

$$\phi_i \left(N_0^{A\alpha}, C^{A\alpha} \right) \leq \phi_i \left(N_0, C \right)$$

for all $i \in N \backslash A$.

It is not difficult to check that *SNAM* implies *NAM*.

Even though this property is defined for any $A \subset N$, we can restrict ourselves to the case $|A| = 2$. The reason is that any rule that satisfies $SNAM$ for $|A| = 2$ will also satisfy $SNAM$ for every $|A| > 2$.

Hence, it is useful to study the reduced problem when $A = \{\alpha, \beta\}$. Given an *mt* t in (N_0, C), we can construct an *mt* in $\left(N_0^{A\alpha}, C^{A\alpha} \right)$ by simply deleting the most expensive arc in the path that joins α and β, as shown in Figure 1.

This result is formally stated in the next lemma:

Lemma 6.1 *Let (N_0, C) be an mcstp and let t be an mt in (N_0, C). Given $A = \{\alpha, \beta\} \subset N$, let $\tau_{\alpha\beta}$ be the path that connects α and β in t. Let $\hat{t} \colon = t \backslash \{(k,l)\}$ for some $(k,l) \in \arg\max_{(i,j) \in \tau_{\alpha\beta}} c_{ij}$.*

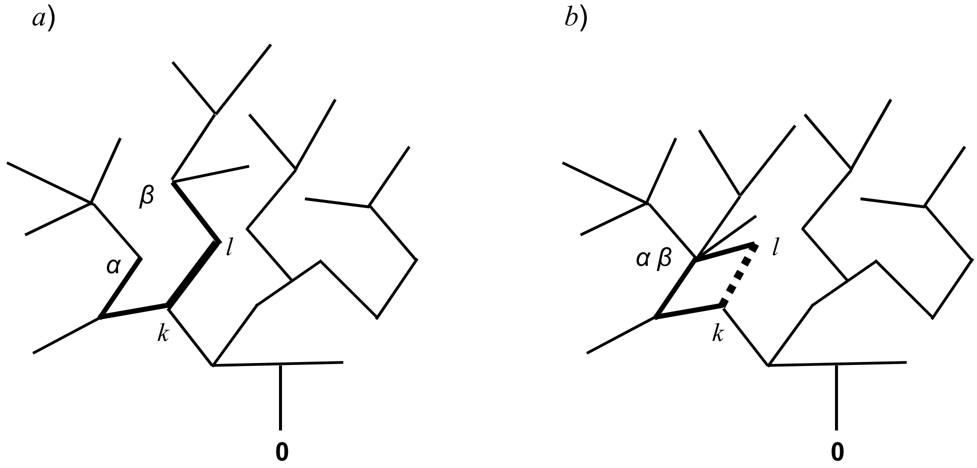


Figure 1: Figure 1a) represents an *mt* in (N_0, C). The most expensive arc in the path that connects α and β is (k,l). In Figure 1b) nodes α and β join and the most expensive arc is removed. The resulting tree is an *mt* in the reduced problem.

The network

$$t^{A\alpha} := \left(\hat{t} \backslash \{(\beta,i)\}_{(\beta,i)\in\hat{t}}\right) \cup \{(\alpha,i)\}_{(\beta,i)\in\hat{t}}$$

is an mt on $\left(N_0^{A\alpha}, C^{A\alpha}\right)$.

Proof. *Consider the mcstp* (N_0, C') *defined as* $c'_{ij} = c_{ij}$ *for all* $\{i,j\} \neq \{\alpha,\beta\}$ *and* $c'_{\alpha\beta} = c_{kl}$. *It is straightforward to check that* t *is also an mt on* (N_0, C') *(see for example the proof of Proposition 2.2iii in Aarts and Driessen (1993)).*

Since the cost of the arc (α,β) *does not affect the definition of* $\left(N^{A\alpha}, C^{A\alpha}\right)$, *both* $C^{A\alpha}$ *and* $C'^{A\alpha}$ *coincide. Hence, it is enough to prove that* $t^{A\alpha}$ *is an mt on* $\left(N^{A\alpha}, C'^{A\alpha}\right)$. *We proceed by a contradiction argument. Assume there exists a tree* t^* *on* $\left(N^{A\alpha}, C'^{A\alpha}\right)$ *such that* $c\left(t^*, C'^{A\alpha}\right) < c\left(t^{A\alpha}, C'^{A\alpha}\right)$. *The counterpart of* t^* *in* (N_0, C') *is defined as follows. Let*

$$O_\beta := \left\{(\alpha,i) \in t^* : c'^{A\alpha}_{\alpha i} = c'_{\beta i}\right\}$$

be the set of arcs in t^* *that would be adjacent to* β *(the rest of the arcs*

$(\alpha, i) \in t^*$ *satisfy* $c'^{A\alpha}_{\alpha i} = c'_{\alpha i}$*). We define the following tree in* (N_0, C')*:*

$$t' := (t^* \backslash O_\beta) \cup \{(\alpha, \beta)\} \cup \{(\beta, i)\}_{(\alpha,i)\in O_\beta}.$$

To see that t' *is indeed a tree in* (N_0, C')*, notice that it has exactly* n *arcs (*$n-1$ *arcs from* t^* *plus* (α, β)*) and all of the nodes in* N *are connected to the source: those that connect to the source through* O_β *in* t^* *will now connect first to* β *and then to* α *through* (α, β)*.*

We will prove that $c(t', C') < c(t, C')$*, which is a contradiction because* t *is an mt on* (N_0, C')*. Notice that* $c'^{A\alpha}_{ij} = c'_{ij}$ *for all* $(i, j) \in t^* \backslash O_\beta$*. Thus,*

$$c(t^* \backslash O_\beta, C') = c\left(t^* \backslash O_\beta, C'^{A\alpha}\right).$$

Hence,

$$\begin{aligned} c(t', C') &= c(t^* \backslash O_\beta, C') + c'_{\alpha\beta} + \sum_{(\alpha,i)\in O_\beta} c'_{\beta i} \\ &= c\left(t^* \backslash O_\beta, C'^{A\alpha}\right) + c'_{\alpha\beta} + \sum_{(\alpha,i)\in O_\beta} c'^{A\alpha}_{\alpha i} \\ &= c\left(t^*, C'^{A\alpha}\right) + c'_{\alpha\beta} < c\left(t^{A\alpha}, C'^{A\alpha}\right) + c'_{\alpha\beta} \leq c(t, C') \end{aligned}$$

where the last inequality comes from

$$\begin{aligned} c\left(t^{A\alpha}, C'^{A\alpha}\right) &= c\left(\hat{t} \backslash \{(\beta, i)\}_{(\beta,i)\in\hat{t}}, C'^{A\alpha}\right) + \sum_{(\beta,i)\in\hat{t}} c'^{A\alpha}_{\alpha i} \\ &\leq c\left(\hat{t} \backslash \{(\beta, i)\}_{(\beta,i)\in\hat{t}}, C'\right) + \sum_{(\beta,i)\in\hat{t}} c'_{\beta i} \\ &= c\left(\hat{t}, C'\right) = c(t, C') - c'_{kl} = c(t, C') - c'_{\alpha\beta}. \end{aligned}$$

■

Proposition 6.1 *Over the domain* $\mathcal{D}$*,* ϕ^1 *satisfies* SNAM *and* IENP*.*

Proof. *Let* $C \in \mathcal{D}$ *and let* t *be an* mt *in* (N_0, C)*.*

It is straightforward to check that ϕ^1 *satisfies* IENP.

We prove that ϕ^1 *satisfies* SNAM.

We can assume that $|A| = 2$*. Let* $A = \{\alpha, \beta\} \subset N$*.*

Under Lemma 6.1, $t^{A\alpha}$ *is an mt in* $\left(N_0^{A\alpha}, C^{A\alpha}\right)$.

We prove that no agent $i \in N^{A\alpha} \backslash \{\alpha\}$ *is worse off in the reduced problem than in* (N_0, C).

Let $(i,j) \in t^{A\alpha}$, $i \neq \alpha, \beta$. *We have two cases:*

- *If* $j = \alpha$, *by definition of* $t^{A\alpha}$, $(i,\alpha) \in t$ *or* $(i,\beta) \in t$. *By definition,* $c_{i\alpha}^{A\alpha} = \min\{c_{i\alpha}, c_{i\beta}\}$. *Hence, whatever agent* i *pays in the reduced problem for* (i,α) *is not more than what he pays in* (N_0, C) *for* (i,α) *or* (i,β).

- *If* $j \neq \alpha$, *by definition of* $t^{A\alpha}$, $(i,j) \in t$. *Moreover,* $c_{ij}^{A\alpha} = c_{ij}$. *Hence, whatever agent* i *pays in the reduced problem for* (i,j) *is the same as what he pays in* (N_0, C) *for* (i,j).

■

This rule violates *CS*. Consider the *mcstp* (N_0, C) with $N = \{1,2\}$, $c_{01} = 10$, $c_{02} = 15$, and $c_{12} = 6$. The unique *mt* is $t = \{(0,1),(1,2)\}$. In this case, $\phi_1^1(N_0, C) = 13 > 10 = m(\{0,1\}, C_{\{0,1\}})$.

Now we present a rule that satisfies *CS* and *SNAM* (and hence *NAM*) but does not satisfy *IENP*. First, we order the agents following their cost of direct link to the source. Second, we assign to each agent the cost of the most expensive arc from him to a source, taking into account that the agents that precede him in the order are considered as sources. We define the rule formally as follows:

Given an *mt* t on (N_0, C) and $i, j \in N$, let τ_{ij} denote the path that connects i and j in t. Let $\pi \in \Pi_N$ such that $\pi(i) < \pi(j) \Rightarrow c_{0i} \leq c_{0j}$. The rule ϕ^2 is defined as follows:

$$\phi_i^2(N_0, C) := \min_{j \in P_i^\pi \cup \{0\}} \max\{c_{kl} : (k,l) \in \tau_{ij}\}.$$

This rule appears in Bergantiños and Vidal-Puga (2004a, after Corollary 4.1) with a different formulation.

Consider the following example:

Let (N_0, C) such that $N = \{1, 2, 3, 4\}$ and

$c_{01} = 0$, $c_{02} = 0$, $c_{03} = 20$, $c_{04} = 40$

$c_{12} = 8$, $c_{13} = 10$, $c_{14} = 5$

$c_{23} = 25$, $c_{24} = 30$, $c_{34} = 2$.

The *mt* is $t = \{(0, 1), (0, 2), (1, 4), (4, 3)\}$.

In this case, there are two admissible orders: $\pi^1 = (1, 2, 3, 4)$ and $\pi^2 = (2, 1, 3, 4)$. In both orders, $\phi^2(N_0, C) = (0, 0, 5, 2)$.

We can also see that this rule is different from the Bird rule.

As ϕ^1 does, ϕ^2 satisfies *SNAM* and hence *NAM*. We prove this result in the next proposition.

Proposition 6.2 *Over the domain $\mathcal{D}$, ϕ^2 satisfies* SNAM *and* CS.

Proof. *First we prove that ϕ^2 satisfies* CS*:*

Bergantiños and Vidal-Puga (2004a, after Corollary 4.1) prove that this rule satisfies Population Monotonicity (PM), and moreover PM implies CS.

We now prove in a intuitive way that ϕ^2 satisfies SNAM*:*

Let $C \in \mathcal{D}$ and let t be an mt *on (N_0, C).*

We can assume that $|A| = 2$. Let $A = \{\alpha, \beta\} \subset N$.

Under Lemma 6.1, $t^{A\alpha}$ is a mt *in $(N_0^{A\alpha}, C^{A\alpha})$.*

We prove that every agent $i \in N \backslash A$ is not worse off in the reduced problem than in (N_0, C).

Let Π' be the set of orders over $N^{A\alpha}$, let $\pi' \in \Pi'$ such that $\pi'(i) < \pi'(j) \Rightarrow c_{0i}^{A\alpha} \leq c_{0j}^{A\alpha}$.

Note that by definition, the order π' coincides with the order π in the sense that the order is preserved in both problems. Formally,

- *If $i, j \notin A$, then $\pi(i) < \pi(j)$ iff $\pi'(i) < \pi'(j)$.*
- *For the agents in A, if the first agent in A is in position p in π, then α will be in position p in π'.*

Let $i \in N \backslash A$. We need to prove that $\phi_i^2 \left(N_0^{A\alpha}, C^{A\alpha}\right) \leq \phi_i^2 \left(N_0, C\right)$.

Let $j \in P_i^\pi$. It is enough to prove that the maximum cost of the path between i and j in $t^{A\alpha}$ (we take α when $j = \beta$) is not more than the maximum cost of the path between i and j in t. Let (k,l) be the most expensive arc in the path from α to β in t. We distinguish two cases:

- *If (k,l) does not belong to the path from i to j in t, the path from i to j in $t^{A\alpha}$ is the same. Hence, the maximum cost of the path from i to j is the same in t as in $t^{A\alpha}$.*
- *If (k,l) belongs to the path from i to j in t, the cost from i to j in $t^{A\alpha}$ is not more because we have removed the most expensive arc of the path from α to β.*

■

Since ϕ^2 satisfies *NAM* and *CS*, and it is different from the Bird rule it is clear that ϕ^2 violates *IENP*.

Finally, we define a rule, ϕ^3, that satisfies *CS* and *IENP* but does not satisfy *NAM*.

This rule is similar to the rule proposed by Dutta and Kar (2004). However, ϕ^3 assigns zero cost to the *ENP*'s and the assignment of the rest of the agents does not depend on the *ENP*'s.

Formally:

Let $\Omega := \{i \in N : i \text{ is an } \textit{Extreme Null Point} \text{ in } (N_0, C)\}$

Consider the following algorithm:

Let $S^0 = \{0\}$, $t^0 = \emptyset$, $p^0 = 0$. Let $S_c^0 = N_0 \backslash S^0$.

Step 1: Choose an ordered pair (a_1^0, a_1) such that $(a_1^0, a_1) \in \arg \min_{\substack{(i,j)\in S^0 \times S_c^0 \\ j \notin \Omega}} c_{ij}$.

Define $p^1 = \max(p^0, c_{a_1^0 a_1})$, $S^1 = S^0 \cup \{a_1\}$, $t^1 = t^0 \cup \{(a_1^0, a_1)\}$, $S_c^1 = N_0 \backslash S^1$.

Step k: Assume we have defined p^{k-1}, S^k, t^{k-1} and S_c^k. Take an ordered pair $(a_k^0, a_k) \in \arg \min_{\substack{(i,j)\in S^{k-1} \times S_c^{k-1} \\ j \notin \Omega}} c_{ij}$. Now, $S^k = S^{k-1} \cup \{a_k\}$, $t^k = t^{k-1} \cup \{(a_k^0, a_k)\}$, $p^k = \max(p^{k-1}, c_{a_k^0 a_k})$ and $S_c^k = N_0 \backslash S^k$.

Also,

$$\phi^3_{a_{k-1}}(N_0, C) = \min(p^{k-1}, c_{a^0_k a_k}). \tag{3}$$

The algorithm finishes at step $m = |N| \setminus |\Omega|$. We define:

$$\phi^3_{a_m}(N_0, C) = t^m \tag{4}$$

and

$$\phi^3_i(N_0, C) = 0 \text{ for all } i \in \Omega. \tag{5}$$

The rule ϕ^3 is described by equations (3),(4) and (5).

In case that the set of *ENP* is empty, ϕ^3 coincides with the rule proposed by Dutta and Kar (2004).

To see that ϕ^3 does not satisfy *NAM* we can consider Example 2.2. In this example, the assignment proposed by ϕ^3 coincides with the assignment proposed by the rule presented by Dutta and Kar (2004) because there are no *ENP*'s.

Proposition 6.3 *Over the domain $\mathcal{D}$, ϕ^3 satisfies* CS *and* IENP*.*

Proof. *It is clear that ϕ^3 satisfies* IENP*. On the other hand, the proof that ϕ^3 satisfies CS is similar to the proof that the rule proposed by Dutta and Kar (2004) satisfies CS (see Dutta and Kar (2004, Theorem 1)) and we omit it.*

■

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