Price Calibration of basket default swap: Evidence from Japanese market

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Abstract

The aim of this paper is the price calibration of basket default swap from Japanese market data. The value of this instruments depend on the number of factors including credit rating of the obligors in the basket, recovery rates, intensity of default, basket size and the correlation of obligors in the basket. A fundamental part of the pricing framework is the estimation of the instantaneous default probabilities for each obligor. Because default probabilities depend on the credit quality of the considered obligor, well-calibrated credit curves are a main ingredient for constructing default times. The calibration of credit curves take into account internal information on credit migrations and default history. We refer to Japan Credit Rating Agency to obtain rating transition matrix and cumulative default rates. Default risk is often considered as a rare-event and then, many studies have shown that many distributions have fatter tails than those captured by the normal distribution. Subsequently, the choice of copula and the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives. Joshi and Kainth (2004) introduced an Importance Sampling technique for rare-event that forces a predetermined number of defaults to occur on each path. We consider using Gaussian copula and t-student copula and study their impact on basket credit derivative prices. We will present an application of the Canonical Maximum Likelihood Method (CML) for calibrating t-student copula to Japanese market data.

Keywords: Basket Default Swaps, Credit Curve, Monte Carlo method, Gaussian copula, t-student copula, Japanese market data, CML, Importance Sampling.

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1. Introduction

The credit derivative and structured credit markets have grown very rapidly in size and complexity in recent years. The flourishing world of credit derivatives has in turn spurred huge growth in structured products. The demand for more tailored, tradable, and investment-grade instruments have been an important motivation for developments in structured credit markets. The dominant structured product is basket default swaps and collateralized debt obligation (CDO). The most common type of basket default swaps is the first-to-default swap (FTDS), where the seller compensates the buyer any loss of the principal and also, possibly, the accrued interest of the asset in the reference basket which defaults first. The main difference between (FTDS) and a credit default swap (CDS) is the event causing payout for the contract (in one case, it is the first default of any of a list of names and in the other is default of a single name). A n^{th} to default basket default swap gives protection against the n^{th} default in the underlyings pool of credits. A Collateralized debt obligation (CDO) refers to securitization[1] of pools of assets. A CDO cashflow structure allocates interest income and principal repayments from a collateral pool of different debt instruments to a prioritized collection (tranches) of CDO securities. Following the classification of Tvakoli (2003), a CDO is backed by portfolios of assets that may include a combination of bonds, loans, securitised receivables, asset-backed securities, tranches of other CDO's, or credit derivatives referencing any of the former. Some market practitioners define a CDO as being backed by a portfolio including only bonds. A Collateralized loan obligation (CLO) is a type of CDO that is backed by a portfolio of loans. A Collateralized bond obligation (CBO) is a type of CDO that is backed by a portfolio of bonds issued by a variety of corporate or sovereign obligors.

In order to understand the significance of developments in the Japanese markets, it is useful to review quickly what is happening in the global market. The global market in Credit Derivatives is expected to rise to $33 trillion by the end of 2008 according to a new report to be published by the British Bankers' Association (BBA) at its Credit Derivatives conference[2] . The report, based on a survey of market leaders in credit derivatives, predicts that London will remain one of the world's dominant centres for credit derivatives products According to SIFMA™, Global CDO issuance through the third quarter of 2006, at $322 billion, has exceeded full year 2005 issuance by 20%. Issuance in the third quarter of 2006, at $117.8 billion, also exceeded issuance in the third quarter of last year by 30%. European Securitisation Forum (ESF) Forecasts Issuance to Grow to a New Record of €531 Billion in 2007, Led by residential mortgage-backed securities (RMBS), commercial mortgage-backed securities (CMBS) and CDO. Then, a 16.4 percent growth rate from the €456 billion issued in 2006.
According to Financial Times[3], The explosive growth in credit derivatives is bypassing Japan. It accounts for 5 per cent of global activity, estimates the British Bankers' Association, compared with about 40 per cent for London. There are some hopes that the introduction of risk-weighted Basel II capital rules will stimulate the use of CDSs for hedging purposes. There is some evidence of this: the country's top bank last year issued a handful of balance sheet collateralised debt obligations referencing Japanese names. But do not bank on this starting a virtuous cycle of liquidity. Made-in-Japan CDOs are still more likely to involve overseas names, to pick up bigger spreads.

Actually, more substantial empirical studies are devoted on basket default swap and CDO. The main problem in the pricing of such instruments is modelling the structure of dependency of the default times. Defaults are rarely observed. Copulas can be introduced to model these correlations by using the correlations of corresponding default time. We know that Kendall's tau remains invariant under monotone transformations. This is the foundation of modelling the correlation of credit events by using the correlation of underlying default time via copulas. Li (2000) present a Gaussian copula method for the pricing of first to default swap. Other studies of elliptical copulas with higher tail dependence, such as the t-copula, can be found in Mashal and Naldi (2002). The Marshall-Olkin

[1] According to Tvakoli (2003), securitization has been a means for banks to reduce the size of their balance sheets and to reduce the risk on their balance sheets. This allowed banks to do more business and allowed investors accsess to diversified pools of assets to which they otherwise not have had access.
[2] Thursday September 21,2006
[3] Financial Times, 14.03.2007.

copula is yet another class of copula functions, which stems from the multivariate compound Poisson process. In this model, individual defaults are constructed from a series of independent common shock. Previous work on the use of the Marshall-Olkin copula in the context of credit risk modelling includes Duffie and Pan (2001), Wong (2000), Lindskog and McNeil (2003).Hull & White (2004) develop two procedures to pricing tranches of CDO and n^{th} to default swap. The first procedure involves calculating the probability distribution of the number of defaults by a time T and suited to the situation where companies have equal weight in the portfolio and recovery rates are assumed to be constant. The second involves calculating the probability distribution of the total loss from defaults by time T. Jobst (2002) propose a pricing model that draws expected loan loss of CDO based on parametric bootstrapping through extreme value theory under the impact of asymmetric information. Tavares et al. (2004) present a basket model to deal with the Gaussian copula smile. They combine the copula model (to model the default risk that is driven by the economy) with independent Poisson processes (to model the default risk that is driven by a particular sector and by the company in question).Hull and White (2005) introduce the technique of perfect copulas. Their copula model can be regarded as 'perfect' in that it hits the tranche quotes exactly. The hazard-rate-path probability distribution is the only input about the underlying copula in order to value a CDO. Hull et al. (2005) price CDOs in a Black & Cox (1976) structural model by Monte Carlo simulation. They show that this model yields tranche spreads very similar to the standard Gaussian copula model, indicating that the model is unable to fit senior tranches. They consider two extensions of the model. The first reflects empirical research showing that default correlations are positively dependent on default rates. The second reflects empirical research showing that recovery rates are negatively dependent on default rates. Willemann (2005) extend a well-known structural jump-diffusion model for credit risk to pricing CDO instrument. He shows how the structural jump-diffusion credit risk model of Zhou (1997) can be extended to allow for correlation through innovations to the driving Brownian Motions and through correlation of common jumps in asset value. He provides efficient semi-analytical techniques for the calculation of the portfolio loss distribution. Laurent (2005) consider different pricing models associated with different copulas of default times: Gaussian, Student t, Clayton, Marshall-Olkin, double t. he emphasize the use of stochastic orders to derive some properties of CDO tranche premiums. Totouom & Armstrong (2005) develop a family of dynamic Archimedean copula processes to model the default times. They call for a stochastic process in which the copula defining the defaults amongst the n names is a valid n-copula at any point in time. Burtschell et al. (2005) employ the technique of the double Student-t copula model for the calibration of CDO. They find that this copula model fit better the features to the CDO market in comparison to other models like Gaussian, t-Student, stochastic correlation, Clayton and Marshall-Olkin copulas. Madan (2004) provide details for the pricing of n^{th} to default contracts using the one factor Gaussian and the Clayton copulas. He model the marginal default time densities using Weibull and Frechet families and the joint densities are obtained using the method of copula. Verschuere (2006) present a factor approach combined with copula functions to price tranches of synthetic Collateralized Debt Obligation (CDO) having totally inhomogeneous collateral (the obligors in the CDO pool have different spreads and different notional). Sircar and Zariphopoulou (2006) study the impact of risk aversion on the valuation of basket credit derivatives. They use the technology of utility-indifference pricing in intensity based models of default risk. Abid & Naifar (2007) present a Copula based simulation procedures for pricing basket Credit Derivatives. They argue that not only the choice of copula is an important input, but also the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives

In our paper, we calibrate the price of multi-name credit derivatives such as n^{th} to default swap from Japanese market data. To express dependencies between times of default, Gaussian and student copulas have been considered. We will present an application of the Canonical Maximum Likelihood Method (CML) for calibrating t-student copula. The remainder of this paper is organised as follows: section two describes simulation procedure for pricing basket credit default swap. In section three, we present our data from Japanese market. Section four describes calibration procedure for copulas parameters. Section five present a procedure for calibrating credit curve. Section six calibrate recovery rate. Section seven present an estimation of basket default swap from market data using Monte Carlo simulation. Section nine present an estimation of basket default swap from market data using Importance Sampling technique. Section ten summarizes the findings and concludes.

2. Pricing of Basket Credit Default Swaps

The most common type of basket default swaps is the first-to-default swap (FTDS), where the seller compensates the buyer any loss of the principal and also, possibly, the accrued interest of the asset in the reference basket which defaults first. The main difference between (FTDS) and a credit default swap (CDS) is the event causing payout for the contract (in one case, it is the first default of any of a list of names and in the other is default of a single name). A n^{th} to default basket default swap gives protection against the n^{th} default in the underlying pool of credits.

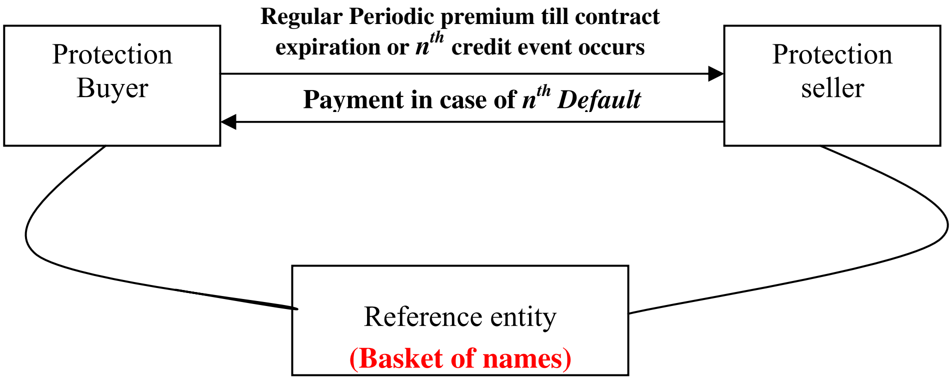

Figure 1: *n^{th} to Default Basket*

The valuation of a basket default swap comes down to the calculation of relevant default probabilities. If defaults of the reference entities are independent, a closed-form formula for such probabilities can be derived. If defaults of the reference entities are not independent, then the calculations of first-to-default or nth-to-default probabilities are more difficult. Generally, closed-form formulae are not available and Monte Carlo simulation is used.

Suppose a basket of credit default swap with the following characteristics:
V: The total value of the basket. $V = \sum_{i=1}^{N} A_i$ with A is the notional amount of each contract.
N: The number of contracts in the basket.
δ : The payment frequency. $\delta = 1$ for annual payment frequency.
T: The maturity date of the basket.
n: The basket seniority. n=1 for first to default basket, n=2 for second to default basket.
s: The Fair price of the basket.

According to Galiani (2003), the risk neutral price of the n^{th} to default basket swap is computed by equating the expected value of the discounted premium payment leg (fixed cash flow to be paid till contract expiration T or n^{th} credit event occurs) with the expected value of the discounted default leg (contingent payment in case of default), under the equivalent martingale measure P^*. Under this measure, the price processes of any tradeable security, discounted by the money market account, are P^*-martingales with respect to some filtration.
The premium legs are paid as long as the underlying credit has not defaulted until the maturity of the contract. The present value of the premium leg of the n^{th} to default basket default swap can be computed as follows:

$$E(\mathrm{PL}_n) = E\left[p\delta\beta(t_i)\sum_{i=1}^{K} \mathrm{I}_{\{\tau^n > t\}} \right] \tag{2.1}$$

Where ($p = f.A$) the premium leg as function as the fair spread of the contract (f) as a fraction of is notional amount (A) in basis point, $t_i, i \in \{1,...,k\}$ are the payment dates either until T or until $\tau < T$ in case of default, δ_i is the frequency of payment and $\beta(t)$ is the discount factor.

Let $F_{(n)}(t) = \mathrm{P}^*\left(\tau^n \le t\right)$ be the distribution function of τ^n, then we can rewrite the premium leg as:

$$\mathrm{E}(\mathrm{PL}_n) = \mathrm{p}\delta\beta(\mathrm{t})\sum_{i=1}^{K}\left[1 - \mathrm{F}^n(\mathrm{t}_i)\right] \tag{2.2}$$

The second part for pricing n^{th} to default swap is the default leg $\mathrm{E}[\mathrm{DL}_n]$. The default leg can be expressed as the difference between the expected discounted default payment $E[DP_n]$ and the expected discounted accrued premium $E[AP_n]$. Then, $\mathrm{E}[\mathrm{DL}_n] = E[DP_n] - E[AP_n]$.
With:

$$E[DP_n] = E\left[A.\left(1 - R^n\right)\beta\left(\tau^n\right)\sum_{j=1}^{N}\mathrm{I}_{\{\tau^n \le T\}}\right]$$

$$= A\sum_{j=1}^{N}\left(1 - R^n\right)\int_0^T \beta\left(\tau^n\right)F_{(n)}^{n^{th}=j}(dt) \tag{2.3}$$

We notice that $F_{(n)}^{n^{th}=j}(t)$ is the distribution function of n^{th} basket default relative to the j^{th} defaulter for allowing different recovery rates for the obligors.

$$E[AP_n] = E\left[p\sum_{i=1}^{K}\left(\frac{\tau^n - t_{i-1}}{t_i - t_{i-1}}\delta\right)\beta\left(\tau^n\right)\mathrm{I}_{\{t_{i-1} < \tau^n \le t_i\}}\right]$$

$$= p\sum_{i=1}^{K}\int_{t_{i-1}}^{t_i}\frac{v - t_{i-1}}{t_i - t_{i-1}}\delta\beta(v)F_{(n)}(dv) \tag{2.4}$$

The fair spread of the basket default swap is given by:

$$s = \frac{E[DL_n]}{E[PL_n] + E[AP_n]} = \frac{\sum_{j=1}^{N}\left(1 - R^n\right)\int_0^T \beta\left(\tau^n\right)F_{(n)}^{n^{th}=j}(dt)}{\sum_{i=1}^{K}\delta\beta(t)\left[1 - F^n(t_i)\right] + \int_{t_{i-1}}^{t_i}\frac{v - t_{i-1}}{t_i - t_{i-1}}\delta\beta(v)F_{(n)}(dv)} \tag{2.5}$$

From equation (2.5), we notice that the fair price depend on default probabilities, the structure of dependency between default times and the market perception of the loss severity given default through the recovery rate. The calibration of basket credit default swap fair spread from Japan market data contain the following steps:

- Step 1: Calibrate the parameters of the copula functions (Gaussian copula and t-student copula) chosen for modelling the structure of dependency among the names in the basket. The calibration of the parameters are described in§ 4.2.

- Step 2: Calibrate credit curve for each names in the basket as described in § 5.

- Step 3: Calibrate recovery rate for each names in the basket as described in § 6.

- Step 4: Generate correlated default times concisely with the estimated parameters as in § 7.

3. Data description

Tokyo Financial Exchange (TFX) distributes credit default swap Reference Rates with collaboration with many financial institutions[4]. The indicative rates provided by the designated financial institutions are the rates (premium) of the following standardized credit default swap contracts:

- 5 year maturity ;
- Amount of 500 million yen for notional principle;
- 3 credit events such as "bankruptcy", "failure to pay", and "restructuring (old restructuring)";
- A Physical Settlement is required in case that any credit event occurs.

We refer to Japan Credit Rating Agency (JCR) to obtain rating and industry sector of each reference entity. Daily stock prices of each name are obtained from Tokyo Stock Exchange.
We construct an example of basket default swap with five names of different rating and industry sector. Similarly, we set the notation used throughout this paper:

Basket default swap	
Initial Par Value $V = \sum_{i=1}^{N} A_i$ A is the notional amount of the contract	2500 million yen
Number of obligors: N	5
Start date: t_0	26/01/2007
Maturity date: T	26/01/2012
Payment frequency: $\delta = 1$	Annual payment
Fair price	s

Table 1: *Basket default swap description*

Name	Notional Principle (YEN)	5y CDS spread	Industry sector	Rating
East Japan Railway Company (1)	500 million yen	4,86	Transport-Rail	AAA
Bridgestone Corporation (2)	500 million yen	6,72	Rubber-Tires	AA
Konica Minolta Holdings, Inc. (3)	500 million yen	18,00	Photo Equipment&Supplies	A
Sanyo Electric Co., Ltd. (4)	500 million yen	101,84	Electric Products-Misc	BBB
Japan Airlines Corporation (5)	500 million yen	233,71	Airlines	BBB

Table 2: *Basket default swap at January, 26th 2007*

[4] 15 Financial institutions, which support to distribute CDS Reference Rates for the infrastructure development of credit default swap market (Barclays Capital Japan Limited, Bear Stearns (Japan) Ltd.,...).

4. Calibration of copulas

4.1. Stylised facts about Copula functions

The copula function links the univariate margins with their full multivariate distribution. It presents a useful tool when modelling non Gaussian data since the Pearson's correlation coefficient is adapted for linear dependence and normal distribution. One appealing feature of a copula function is that the margins do not depend on the choice of the dependency structure and then, we can model and estimate the structure of dependency and the margins separately. Copula functions are getting more and more popular credit correlation modelling due to its simplicity and fast computation. Embrechts, et al (1999) clarified many issues concerning dependence and its relationship to correlation, especially in financial data such as market crashes, credit crises. According to Gennheimer (2002) there are several reasons why copulas are such an attractive tool for modelling dependence:
1. They provide us with a powerful tool for building a large number of multivariate models and are extremely useful in the Monte Carlo simulation of dependent risk factors.
2. They allow us to overcome the fallacies and dangers of approaches to dependence that focus only on correlation.
3. They provide a way of studying scale-free measures of dependence.
4. They express dependence on a quantile scale, which we will find is useful for describing the dependence of extreme outcomes.

Frey & McNeil (2002) analyse default correlation and the pricing and hedging of credit sensitive instruments with copulas functions. Jouanin et al (2001) address the problem of incorporating default dependency in intensity-based credit risk models by using copulas functions to model the joint distribution of the default times. Abid & Naifar (2005, 2006) study the impact of Stock returns volatility of reference entities on credit default swap rates using Archimedean copula.
For n uniform random variables $u_1, u_2, ..., u_n$, the joint distribution function C is defined as:

$$C(u_1, u_2, ..., u_n, \theta) = \Pr[U_1 \le u_1, U_2 \le u_2, ..., U_n \le u_n] \quad (4.1)$$

With θ is the dependence parameter.
We present the following definition for the bivariate case: A copula function is the restriction to $[0,1]^2$ of a continuous bivariate distribution function whose margins are uniform on $[0,1]$. A (bivariate) copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies the boundary conditions:

$$C(t,0) = C(0,t) = 0 \text{ and } C(t,1) = C(1,t) = t \text{ for } t \in [0,1] \quad (4.2)$$

Similarly, copula satisfies the 2-increasing property:

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0 \quad (4.3)$$

For all u_1, u_2, v_1, v_2 in $[0,1]$ and $u_1 \le u_2$ and $v_1 \le v_2$.
A copula is symmetric if:

$$C(u, v) = C(v, u) \text{ for all } (u, v) \text{ in } [0,1]^2 \text{ and is asymmetric otherwise.} \quad (4.4)$$

Sklar (1959) shows the importance of copulas as a universal tool for studying multivariate distributions[5]. By definition, applying the cumulative distribution function (CDF) to a random variable (r.v.) results in a r.v. that is uniform on the interval [0,1]. Let X a random variable with continuous distribution function F_X, $F_X(X)$ is uniformly distributed on the interval [0,1]. This result is known as the probability integral transformation theorem and present many statistical procedures. With this result in hand, we may introduce the copula using basic statistical theory. In particular, the copula C for (X,Y) is just the joint distribution function for the random couple $F_X(X)$, $F_Y(Y)$ provided F_X and F_Y are continuous.

[5] The original definition of copula is given by Sklar (1959) and the Sklar's theorem is considered as the most important theorem about copula functions. The problem of obtaining a joint distribution is reduced to selecting the appropriate copula.

The previous representation is called canonical representation of the distribution. Thus, copulas link joint distribution functions to their margins. Then, in continuous distribution, the problem of obtaining the joint distribution has reduced to selecting the appropriate copula. We can build multidimensional distributions with different marginals.
Copula functions allow us to separate the structure of dependency between default times into two parts: the first part is the specification of the marginal distribution function (the distribution of default time of each obligor. The second part is the choice of the appropriate copula which describes the structure of dependency between default times.
Numerous copulas can be found in the literature (see Nelson (1999) and Joe (1997)). The most commonly applied copula function (especially in finance modelling) is the Gaussian copula[6]. This could be justified by the fact that the multivariate normal distribution has two appealing characteristics: first, their marginal distributions are normal and second, it can be fully described by their marginal distribution and a variance-covariance matrix. For univariate margins $F_1,\ldots,F_n$ which are Gaussians, the dependence structure among the margins is described by a unique normal copula function. Let $X_1,\ldots,X_n$ be random variables which are standard normal distributed with means $\mu_1,\ldots,\mu_n$, standard deviations $\sigma_1,\ldots,\sigma_n$ and correlation matrix Σ. Then, the distribution function $C_\Sigma(u_1,\ldots,u_n)$ of the random variables $U_i = \Phi\left(\frac{X_i - \mu_i}{\sigma_i}\right), i \in \{1,\ldots,n\}$ is a Gaussian copula with correlation matrix Σ. $\Phi(.)$ denotes the cumulative univariate standard normal distribution function.

The Gaussian copula can be written as:

$$C_\Sigma(u_1,\ldots,u_n) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Sigma}} \int_{-\infty}^{\Phi^{-1}(u_1)} \ldots \int_{-\infty}^{\Phi^{-1}(u_n)} \exp\left(-\frac{1}{2}(\vec{v}-\vec{\mu})^T \Sigma^{-1}(\vec{v}-\vec{\mu})\right) dv_1 \ldots dv_n \tag{4.5}$$

With Φ^{-1} is the inverse of the standard univariate Gaussian distribution function.
By differentiating equation (2.5) with respect to $u_1,\ldots,u_n$, we obtain the density of the Gaussian copula:

$$C_\Sigma(u_1,\ldots,u_n) = \frac{1}{\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}(v_1,\ldots,v_n)\left(\Sigma^{-1} - I\right)\begin{pmatrix} v_1 \\ . \\ . \\ . \\ v_n \end{pmatrix}\right); \text{With } v_i = \Phi^{-1}(u_i), i \in \{1,\ldots,n\}. \tag{4.6}$$

If we use a Gaussian copula, we preserve the underlying distribution of the individual random variables but the joint distribution is like a multidimensional Gaussian. This naturally assigns very little weight to the tails. In reality, we find that within the financial markets, tail events occur much more frequently. So we would like a joint distribution which has fatter tails but preserves the same (bell shaped, non-skewed) characteristics of the Gaussian, hence we use the t-Student copula.

$$t_\nu(x) = \int_{-\infty}^{x} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\Pi}} \left(1 + \frac{s^2}{\nu}\right)^{-\frac{\nu+1}{2}} ds\,. \tag{4.7}$$

The t-student copula with the correlation matrix Σ and ν degrees of freedom is presented as follow:

$$C_{\nu,\Sigma}(u_1,\ldots,u_n) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \ldots \int_{-\infty}^{t_\nu^{-1}(u_n)} k_{\nu,\Sigma}\left(1 + \frac{1}{\nu}\left(\vec{v} - \vec{u}\right)^T \sum^{-1}\left(\vec{v} - \vec{u}\right)\right)^{-\frac{\nu+n}{2}} dv_1 \ldots dv_2 \tag{4.8}$$

[6] Credit Metrics ™ and KMV model implicitly incorporate copula functions based on the multivariate Gaussian distribution of asset value process.

With:

$$k_{\nu,\Sigma} = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\Pi\nu)^n \det(\Sigma)}} . \tag{4.9}$$

4.2. Estimating parameters of copulas

The Gaussian copula is completely determined by the knowledge of the correlation matrix $\sum$ and the parameters involved are simple to estimate. To simulate random variables from Gaussian copula, it is enough to simulate a vector from the standard multivariate normal distribution with correlation matrix $\sum$ and then to transform this vector through a univariate cumulative distributions function so that you can obtain a vector from the chosen copula. The matrix $\sum$ positive definite can be easily determined with the cholesky decomposition in order to calculate a matrix A such as $AA^T = \sum$
The following algorithm generate random variates $(u_1,...,u_n)$ which are determination of correlated uniform variates on [0,1] from the Gaussian copula with the correlation matrix Σ:

- Find the Cholesky decomposition[7] A of the correlation matrix Σ, such that $\Sigma = A \cdot A^T$;
- Simulate n independent standard normal random variates $Z = (z_1, z_2,..., z_n)^T$;
- Set $x = A \cdot Z$;
- Set x back to an n-dimensional vector u of uniform variates on $[0,1]$ by computing $u = \Phi(x)$. The vector u is a random variate from the n-dimensional Gaussian copula C_Σ.

The t-student copula is defined by two parameters: the correlation matrix $\sum$ and the number of degrees of freedom ν. To simulate random variates from the t-Student copula $C_{\nu,\Sigma}$ with the correlation matrix Σ and ν degrees of freedom, we can use the following algorithm:

- Find the Cholesky decomposition A of the correlation matrix Σ, such that $\Sigma = A \cdot A^T$;
- Simulate n independent standard normal random variates $Z = (z_1, z_2,..., z_n)^T$;
- simulate a random variate, s, from χ^2_ν distribution, independent of Z;
- Set $y = A \cdot Z$;
- Set $x = y\sqrt{\frac{\nu}{s}}$;
- Set x back to an n-dimensional vector u of uniform variates on $[0,1]$ by computing $u = t_\nu(x)$. The vector u is a random variate from the n-dimensional t-Student copula $C_{\nu,\Sigma}$.

In terms of the appropriate choice for the number of degrees of freedom, it is often necessary to carry out some statistical tests with historical data to ascertain how fat we require the tails to be. Many woks

[7] A symmetric and positive definite matrix can be efficiently decomposed into a lower and upper triangular matrix. For a given matrix, this is achieved by the LU decomposition which factorizes A=LU. If A satisfies the above criteria, one can decompose more efficiently into $A = LL^T$, where L (which can be seen as the ``matrix square root" of A) is a lower triangular matrix with positive diagonal elements. L is called the Cholesky triangle.

explain how to calibrate t-student copula to real market data (Mashal and Zeevi (2003), Romano (2002), Meneguzzo and Vecchiato (2002)…). According to Galiani (2003), we find three methods:

- *Exact Maximum Likelihood Method (EML)*: The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. Unlike the case of Gaussian copula, the calibration of t-student copula via the EML methode requires simultaneous estimation of the parameters of the margins and the parameters related to the dependence structure.

- *The Inference for Margins Method (IFM)*: This method has emerged as the preferred fully parametric method because it is close to EML in approach and is easier to implement. Joe and Xu (1996) present an approach that consists of estimating univariate parameters from separately maximizing univariate likelihoods, and then estimating dependence parameters from separate bivariate likelihoods or from a multivariate likelihood. The jackknife method is proposed for obtaining standard errors of the parameters and functions of the parameters. Kim et al (2007) compare the maximum likelihood (ML) and IFM methods with a semi-parametric method that treats the univariate marginal distributions as unknown functions. They find that, in terms of statistical computations and data analysis, the semi-parametric method is better than ML and IFM methods when the marginal distributions are unknown which is almost always the case in practice.

- *The Canonical Maximum Likelihood Method (CML)*: Both EML and IFM require the parametric form of the univariate margins. The CML method does not require any prior assumption on the distribution form of the margins. Genest et al (1995) proposed a semi-parametric procedure for estimating the dependence parameters in a family of multivariate distributions when one does not want to specify any parametric model to describe the marginal distribution. This procedure consists of transforming the marginal observations into uniformly distributed vectors using the empirical distribution function. Then, the copula parameters are estimated by maximisation of a Pseudo log-likelihood function.

The CML tend to estimate the unknown parametric marginals $F_n(.)$ for $n = 1,...,N$ with the empirical distribution functions $\hat{F}_n(.)$ defined as: $\hat{F}_n(.) = \frac{1}{T}\sum_{t=1}^{T} I_{\{X_{nt} \le .\}}$ for $n = 1,...,N$ and $\{X_{nt} \le .\}$ represents the indicator function. The CML method is composed with two steps:

- Step1: Transformation of the data set $X = \left(X_{1t}, X_{2t},..., X_{Nt}\right)_{t=1}^{T}$ into uniform variates, using the empirical marginal distribution. Then, for $t = 1,...,T$;

$$\hat{u}_t = \left(\hat{u}_1^t, \hat{u}_2^t,..., \hat{u}_N^t\right) = \left[\hat{F}_1(X_{1t}), \hat{F}_2(X_{2t}),..., \hat{F}(X_{Nt})\right]$$

- Step2: Estimation of the vector of the copula parameters α as follow:

$$\hat{\alpha}_{CML} = \arg\max_{\alpha} \sum_{t=1}^{T} \operatorname{Ln} C\left(\hat{u}_1^t,..., \hat{u}_N^t; \alpha\right)$$

In our paper, we present an application of the CML method for calibrating the parameter of the t-student copula. Bouyé et al (2000) present an approach based on a recursive optimization procedure for the correlation matrix. We use the approach proposed by Mashal & Zeevi (2002) based on the rank correlation estimator given by the Kendall tau. Schweizer & Wolff (1981) show that two standard nonparametric rank correlation (Kendall's tau and Spearman Rho) can be expressed solely in terms of the copula function. They are nonparametric measures of dependence since they are independent of the margins.

The Kendall's tau (τ) for two random variables X and Y is the probability of concordance minus the probability of discordance. Suppose that (X,Y) and (X*,Y*) are two independent realizations of a joint distribution: $\tau = 4 \iint_{[0,1]^2} C(u_1, u_2)\, dC(u_1, u_2) - 1$

$$= \text{Prob}\left\{(X^* - X)(Y^* - Y) \geq 0\right\} - \text{Prob}\left\{(X^* - X)(Y^* - Y) < 0\right\}$$

For simplicity, it is assumed that the marginal distributions are continuous. Following Genest & MacKay (1986), Kendall tau verifies the following properties:

- $-1 \leq \tau \leq 1$
- τ is invariant under monotone transformations: if f and g are monotone increasing or decreasing functions, then $\tau(f(X), g(Y)) = \tau(X, Y)$
- $\tau = 0$ if X and Y are independent (but not conversely).

According to Lindskog et al (2001), the following theorem is presented:

Theorem 1: Let $X \sim E_N(\nu, \Sigma, \rho)$. X_i and X_j are continuous for $i, j \in \{1,2,...N\}$, then:

$$\tau(X_i, X_j) = \frac{2}{\pi} \arcsin R_{ij}. \qquad (4.2.1)$$

Where $E_N(\nu, \Sigma, \rho)$ denote the N-dimensional elliptical distribution with parameters (ν, Σ, ρ), $\tau(X_i, X_j)$ is The Kendall's tau and R_{ij} denote the Pearson correlation coefficient for the random variables (X_i, X_j).

As mentioned in table 2, our basket credit default swap is composed with five names. The calibration procedures contain three steps:

-Step1: Transform the initial stock prices data X into a set of uniform variates $\hat{U}$ using the empirical marginal transformation.

-Step 2: Estimate the correlation matrix R^{CML} from equation (4.2.1).

- Step 3: Extract the CML Estimator of the degrees of freedom by maximizing the log likelihood function of the t-student copula: $\nu^{CML} = \arg\max_{\nu \in (2,\infty]} \sum_{t=1}^{T} Log C^{Student}\left(\hat{u}_1^t, \hat{u}_2^t, ..., \hat{u}_N^t, R^{CML}, \nu\right)$.

The above method has computational advantages over other methods. This method does not require matrix inversion and therefore has the advantage of being numerically stable in the presence of close-to-singular correlation matrices.

The existing literature on default correlations can be divided into two approaches: the structural approach that models default correlations through companies' assets values; and the reduced-form approach that models default correlations through default intensities. Elizalde (2005) distinguishs three different approaches to model default correlation in the literature of intensity credit risk modelling. The first approach introduces correlation in the firms' default intensities making them dependent on a set of common variables and on a firm specific factor (conditionally independent defaults models). The second approach default dependencies arise from direct links between firms (Contagion models). The third model default correlation makes with copula functions. Li (2000) points out that CreditMetrics uses a bivariate normal copula function with asset correlation as the correlation factor.Patel & Pereira (2005) extract common or latent factors that drive companies default correlations using a factor-analytical technique. The results indicate that the common factors, which capture the overall state of the economy, explain default correlations. In KMV methodology, the model and the correlation matrix is computed on asset returns but proxied by equity returns on CreditMetrics. The correlation model in CreditMetrics provides estimates of equity correlations instead of asset correlations.. Then, it has become market practice to use equity correlation as a proxy for asset correlation.

The equity prices used in this paper is from Tokyo Stock Exchange. It consist of 754 daily observation Spanned from January, 02 th 2004 to January 25 th 2007. As mentioned in table 2, our basket credit default swap is composed with five names. The calibration procedures contain three steps:

-Step1: Transform the initial stock prices data X into a set of uniform variates $\hat{U}$ using the empirical marginal transformation.

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From equation (4.2.1), we obtain the following correlation matrix :

1	0.34197	0.17727	0.19895	0.17839
0.34197	1	0.31106	0.33717	0.21573
0.17727	0.31106	1	0.36427	0.25319
0.19895	0.33717	0.36427	1	0.29069
0.17839	0.21573	0.25319	0.29069	1

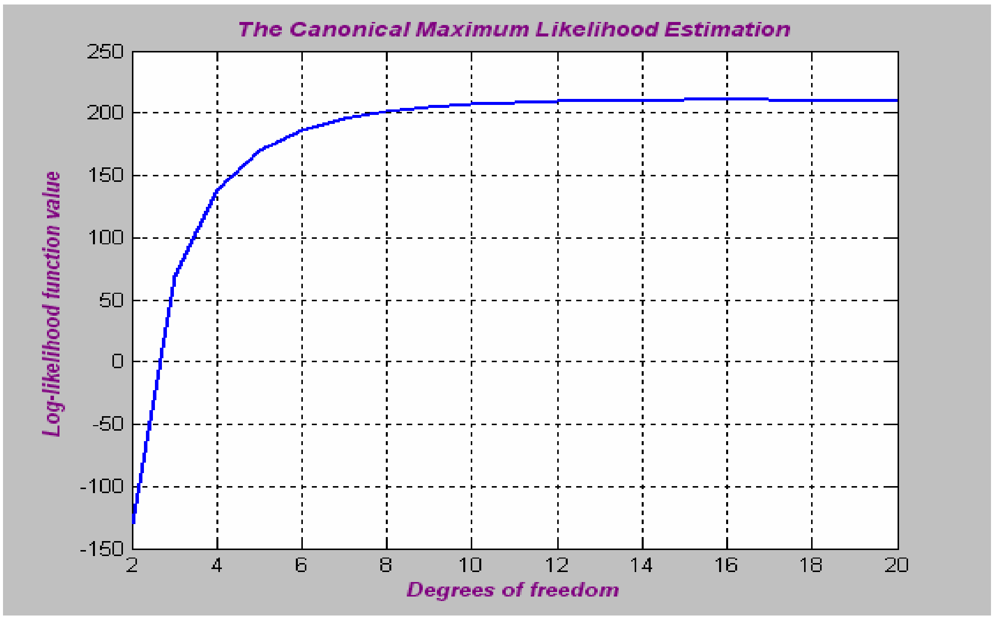

Figure 2 : Log-likelihood function of the t-student copula density as a function of degrees of freedom

The degrees of freedom is then $\nu = 10$.

5. Calibration of credit curves

In the basket default swap pricing, one of the fundamentally important issues is the estimation of default probabilities and correlations. Because default probabilities depend on the credit quality of the considered obligor, well-calibrated credit curves are a main ingredient for constructing default times. The calibration of credit curves take into account internal information on credit migrations and default history. There are dozens of rating agencies that provide yearly rating migration and default data for corporations and governments. From this data a rating transition matrix can be constructed. Such a matrix gives transition probabilities for a corporation migrating, in one year, from one rating level to another. Of particular interest is the transition probability to default status[8]. To calibrate credit curve, we will introduce the concept of hazard function. In the following, we present some mathematical

[8] The method used by Standard & Poor's to estimate credit curves involves two stages. The first stage is the estimation of the probabilities of transitions between different ratings (transition matrix). The second stage is the repeated application of this matrix to determine the credit curves. In both cases, rating transitions are assumed to follow a Markov process, in which transition probabilities are constant over time, and do not depend on the previous rating of the firm.

background for modelling default times. We assume a probability space $\left(\Omega, \Im, \{\Im_t\}_{t\geq 0}, \mathrm{P}\right)$, where Ω is the underlying probability space containing all possible events over a finite time horizon. $\Im$ is a σ-field representing the collection of all events. $\{\Im_t\}_{t\geq 0}$ is a filtration that carries information with times and P is a probability measure[9]. The pricing is assumed under no arbitrage and then, P is risk-neutral measure.

Default time τ_R for each R-rated obligor $R = \{1,..., K\}$ should be a random variable and the event of default should be known for everybody at any times because we assume a perfect market with a free flow of information. Default is a stopping time τ with respect to the filtration: $\{\tau < t\} \subset \Im_t$, $\forall t \geq 0$. Let $F_R(t) = \mathrm{P}(\tau_R \leq t)$ be the distribution function and $f_R(t)$ is the density function of stopping time, the hazard rate h_R of τ_R or intensity process is defined such that we have for the probability of default for R-rated obligor until time ($t + \Delta t$) given survival till t is given by:

$$h_R(t) = \frac{f_R(t)}{1 - F_R(t)} = \frac{1}{1 - F_R(t)} \times \frac{dF_R(t)}{dt} = \lim_{\Delta \to 0} \frac{P(t < \tau \leq t + \Delta t / \tau_R > t)}{\Delta t} \tag{5.1}$$

With $dF_R = (t < \tau_R \leq t + dt, \tau_R > t)$, $1 - F_R(t) = P(\tau_R > t) = S_R(t)$ is called the survival function for R-rated obligor that gives the probability that a security will attain age t .The hazard rate function gives the instantaneous default probability for a security that has attained age t .The marginal survival distributions for R-rated obligor $S_R(t_i)$ is assumed to be smooth and strictly decreasing, this can be written as:

$$S_R(t_i) = 1 - F_R(t_i) = e^{-\int_0^t h_R(s)ds} \tag{5.2}$$

Then
$$F_R(t_i) = 1 - e^{-\int_0^t h_R(s)ds} \tag{5.3}$$

$h_R(.)$ is the default intensity process for R-rated obligor. The default times τ_R are defined:

$$\tau_i := \inf\left\{ t \geq 0 : \int_0^t h_R(s)ds \geq \theta_R \right\} \tag{5.4}$$

Where θ_R has an exponential distribution with unit intensity[10].

Let $Q=(q_{i,j},\ i,j = 1,2,\ldots, k)$ be a one year rating transition matrix with rating grades R_1, …, R_{k-1} and default state D, where $q_{i,j}$ is the probability that an obligor with current rating R_i will migrate to rating R_j in one year.

The aim of this section is to calibrate a credit curve for each rating, where a credit curve for R is a mapping: $t \mapsto P(t, R) = \mathrm{P}[R \to D \text{ in time t}]$, with R the rating grades R_1, …, R_{k-1} and default state D. Where $\to$ denotes migration from a rating state to another. Then, $P(t, R)$ is the probability that an obligor with a current rating defaults with the next t years. Suppose that a firm's rating follows a time homogeneous Markov process model[11] with the transition probability matrix (as have with Jarrow et al (1997) and other authors). Jarrow et al (1997) model default and transition probabilities by using a

[9] According to Bielecki et al (2005), the probability space is endowed with a filtration $\Im = H \vee F$, where the filtration H carries information about evolutions of credit events, such as changes in credit ratings of perspective credit names and F is some reference filtration.

[10] The exponential distribution is used to model Poisson processes, which are situations in which an object initially in state A can change to state B with constant probability per unit time λ. The time at which the state actually changes is described by an exponential random variable with parameter λ

[11] A process with the Markov property is called a Markov process. A stochastic process has the Markov property if the conditional probability distribution of future states of the process, given all past and the present state, depends only upon the present state.

discrete time, time-homogenous Markov chain on a finite state space S={1,......,K}. The state space S represents the different rating classes. It seems to be common market practice to model default probability term structures via Markov chain techniques.

Given the one-year K × K transition matrix M, we are interested in finding a generator matrix such as $M = \exp(Q)$. Dealing with the question if it exist a generator matrix, the following theorem Noris (1998) can be used:

Theorem 2: If the transition matrix M=($m_{i,j,}$ i,j = 1,2,…, k) is strictly diagonal dominant, ie: $p_{ii} > 0.5$ for every I, then the log-expansion

$$\widetilde{Q}_n = \sum_{K=1}^{n} (-1)^{K+1 \frac{(M-I)^s}{K}}, n \in \mathrm{N}$$

converge to a matrix $\widetilde{Q} = \widetilde{q}_{ij}$, $i, j = 1,..., K$. satisfying:

- $\sum_{j=1}^{K} \widetilde{q}_{ij} = 0$ for every i=1,…K.
- $\exp\left(\widetilde{Q}\right) = M$.

The convergence $\widetilde{Q}_n \to \widetilde{Q}$ is is geometrically fast and defines a K × K matrix having row-sums of zero and satisfying $M = \exp(\widetilde{Q})$.

The generator of a time-continuous Markov chain is given by a so called Q-matrix $= q_{ij}$ satisfying the following properties:

- $\sum_{j=1}^{K} q_{ij} = 0$, for evey i=1,…K. (1)
- $0 \le -q_{ii} < \infty$, for evey i=1,…K (2)
- $q_{ij} \ge 0$, for all I,j=1,…K with $i \ne j$ (3)

Theorem 3: The following two properties are equivalent for a matrix $Q \in \Re^{K \times K}$:
-Q satisfies properties (1),(2) and (3)
-exp(tQ) is a stochastic matrix for every $t \ge 0$, where exp(.) denotes the matrix exponential

The construction of credit curves which are compatible with table 1is presented on the following steps:
- Step 1: Compute the log-expansion $\widetilde{Q} = \widetilde{q}_{ij}$ of the one-year rating transition matrix M;
- Step 2: The calibration of a generator Q-matrix based on the log-expansion of M and a so-called diagonal adjustment (see Kreinin &Sidelnikova (2001));
- Step 3: The approximation of the original matrix M by exp(Q), see Bluhm (2003), Bluhm & overbeck (2006). The error is: $\left\| M - \exp(Q) \right\|_2 = \sqrt{\sum_{i,j=1}^{K} \left(m_{ij} - (\exp(Q))_{ij} \right)^2}$
- Step 4: The credit curves can be read-off from the collection of matrices $\exp(tQ)_{t \ge 0}$ by looking at the default columns. Otherwise, we can generate default probability term structures based on the continuous time-homogeneous Markov chain generated by Q via:

$$P(t, R) = (\exp(tQ))_{row(R),K} \quad (\mathrm{t} \ge 0)$$

Where row (R) denotes the transition matrix row corresponding to the given rating R

In the present paper, we use Japanese data and then we refer to Japan Credit Rating Agency to obtain rating transition matrix and cumulative default rates[12]. We use one year rating transition matrix to computes either marginal or cumulative default probabilities over one or multiple time steps. Marginal default probabilities are computed by applying recursively the standard Markov chain over the different transition states.[13] The one-year ratings transition matrix provided by Japan Credit Rating Agency is presented on table 1. Credit migration matrices are said to be diagonally dominant, meaning that most of the probability mass resides along the diagonal, most of the time there is no migration.

	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB	BBB-	BB+	BB	BB-	B+	B	B-	CCC or lower	D
AAA	95.41	4.59	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA+	3.64	91.44	4.92	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA	0.30	2.73	84.41	8.77	2.85	0.71	0.24	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
AA-	0.00	0.36	3.72	84.23	10.42	1.26	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A+	0.00	0.00	0.00	5.72	83.68	9.15	0.66	0.54	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.24
A	0.00	0.00	0.19	0.08	5.46	84.65	8.22	1.34	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
A-	0.00	0.00	0.13	0.00	0.25	7.58	84.33	5.99	1.54	0.13	0.00	0.00	0.04	0.00	0.00	0.00	0.00	0.00
BBB+	0.00	0.00	0.00	0.00	0.00	0.62	9.91	78.25	10.38	0.73	0.02	0.00	0.00	0.04	0.06	0.00	0.00	0.00
BBB	0.00	0.00	0.00	0.00	0.00	0.00	0.71	4.99	85.70	7.51	0.71	0.17	0.00	0.00	0.06	0.00	0.00	0.16
BBB-	0.00	0.00	0.00	0.00	0.00	0.00	0.01	1.12	7.85	80.96	7.58	1.88	0.18	0.20	0.05	0.08	0.00	0.09
BB+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.57	12.74	70.25	10.82	3.56	0.00	0.00	0.21	0.43	0.43
BB	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.90	10.48	67.62	10.00	3.49	2.06	1.59	0.00	2.86
BB-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	15.07	11.64	34.25	26.03	0.68	0.00	6.85	5.48
B+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	12.37	4.12	14.43	16.49	47.42	0.00	0.00	5.15	0.00
B	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	69.84	0.00	6.35	23.81
B-	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	68.18	4.55	27.27
CCC or lower	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Table 3: One-year rating transition matrix (for period January 1995-decembre 2005)

Transition matrix presented in table 3 is reduced into another smaller sub-Transition Matrix. Rating classes are aggregated into groups and then scales them to meet the 100% probability criteria.

RATING	AAA	AA	A	BBB	BB	B	CCC	DEFAULT
AAA	95,410%	4,590%	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%
AA	1,313%	93,527%	4,423%	0,657%	0,080%	0,000%	0,000%	0,000%
A	0,000%	2,014%	93,433%	3,159%	1,316%	0,000%	0,000%	0,079%
BBB	0,000%	0,000%	3,750%	92,491%	3,513%	0,163%	0,000%	0,083%
BB	0,000%	0,000%	0,000%	5,403%	77,894%	11,353%	2,427%	2,923%
B	0,000%	0,000%	0,000%	4,124%	11,681%	61,817%	5,350%	17,028%
CCC	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%	0,000%	100,000%

Table 4: Modified Transition Matrix

Table 4 indicates one-year ratings migration probabilities from modified Transition Matrix. For example, a BBB rated bond has a 3,513% probability of being downgraded to a BB+ rating by the end of one year. To use a ratings transition matrix to calculate the one year default probabilities, we simply take the default probabilities indicated in the last column and ascribe them to bonds of the corresponding credit ratings. For example, with this approach, we would ascribe an BBBrated bond a 0,083% probability of default within one year.Transition matrices can be used to compute credit curves over multiple horizon. This is done by transforming the transition probabilities into marginal default probabilities, which are then converted into survival or Hazard rates or expected default probabilities. Table 5 present cumulative default probabilities for each rating class.

[12] "JCR Rating Matrices and Cumulative Default Rates", Rating Perspective, April 2006.

[13] We use The Risksvr™ calculation engine.

RATING	1	2	3	4	5	6	7	8
AAA	0,000%	0,000%	0,000%	0,001%	0,004%	0,009%	0,017%	0,030%
AA	0,000%	0,006%	0,024%	0,058%	0,113%	0,193%	0,299%	0,435%
A	0,079%	0,194%	0,395%	0,691%	1,078%	1,551%	2,099%	2,715%
BBB	0,083%	0,294%	0,753%	1,449%	2,353%	3,428%	4,637%	5,950%
BB	2,923%	9,565%	16,591%	23,316%	29,446%	34,891%	39,659%	43,805%
B	17,028%	33,249%	44,061%	51,585%	57,050%	61,181%	64,416%	67,022%
CCC	100,000%	100,000%	100,000%	100,000%	100,000%	100,000%	100,000%	100,000%

Table 5: Cumulative Default Probabilities

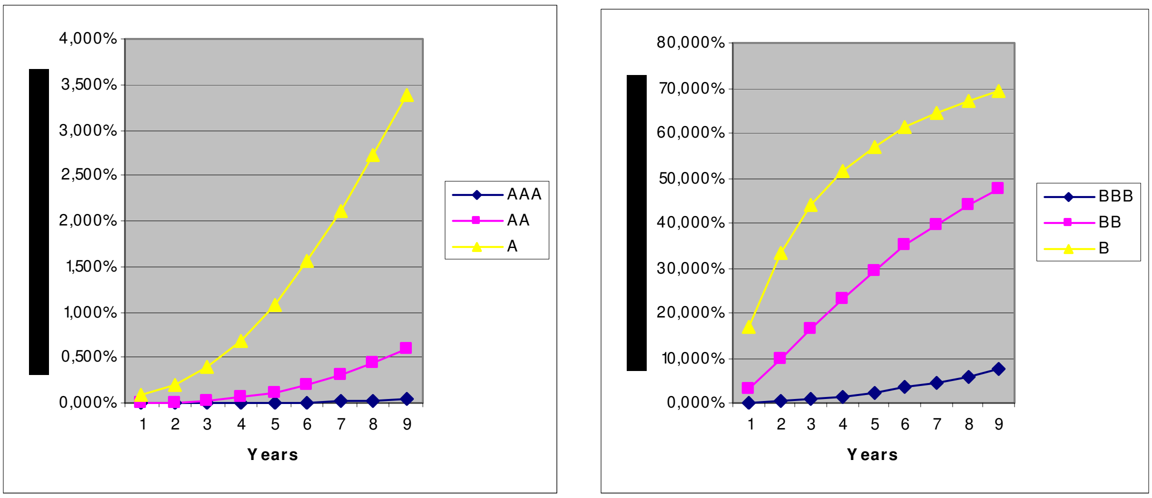

Figure 3: Calibrated curves $P(t, R)_{t \geq 0}$

Figure 1 shows a chart of our credit curves just constructed from rating transition matrix obtained from Japan Credit Rating Agency. We notice that credit curves assigned to non-investment grade[14] have a tendency to slow-down their growth, because conditional on having survived for some time the chances for further survival get better. However, for investment grade ratings (AAA, AA, A), we notice the opposite effect. After the calibration of the credit curves, we can obtain the cumulative default probabilities for any names over any time interval [0,t]. Then, we can define the marginal distribution of default times.

We can construct a default probability curve from given default swap rates. Such a method is often called bootstrapping[15]. As we know, the price of single name credit default swap can be affected by marginal probability of default. Hull(2002) mentioned that the credit default swap market is so liquid that we can use the credit default swap spread data to calibrate the default intensity. According to Choudhury (2006), historical evidence suggests that the credit default swap market can anticipate rating downgrades. Take the example of El Paso Corporation[16], the cost of five-year CDS protection more than doubled between September 12 and September 23, 2002 from 575 basis points to 1250 basis points. Two month later, El Paso Corporation was downgraded from Baa3 to B2. Then, we can construct a default probability curve from default swap rates. Galiani (2003) provide a practical application for extracting the term structure of instantaneous default probabilities. In the corporate

[14] Ratings between BB and Default are called speculative grade or high yield
[15] Bootstrapping, iteratively stepping through market quotes to determine the underlying curve, is widely used as the standard method of estimating interest rate curves.
[16] El Paso Corporation provides natural gas and related energy products in a safe, efficient, and dependable manner. It is one of North America's largest independent natural gas producers.

CDS market, trading has been concentrated largely in the five-year maturity contract. In our case, we have only five year credit default swap rates. Then, we can't estimate the instantaneous default probabilities from CDS market data.

6. Determination of recovery rates

Market participants on correlation dependent credit derivatives should not just evaluate default probabilities and correlations but also assess recovery if default occurs. The relationship between recovery rates and default rates has been neglected in pricing basket credit derivatives models, as most of them focused adopted static loss assumptions, treating the recovery rate as a constant parameter. Jarrow (2001) presents a new methodology for for implicit estimation of a liquidity premium, the recovery rate, and the default probabilities using debt and equity prices. Recovery rates and default probability are correlated and depend on the state of the economy. Merrick (2001) introduces a joint implied parameter approach to sovereign bond pricing to extract market assumptions about both the expected recovery ratio and the default probability term structure by applying a consistent valuation framework to a cross-section of market prices on outstanding bond issues. Navarro (2005) applies the model presented by Merrick Jr. (2001) to estimate both the default recovery rates and the implied default probabilities of the Argentinean Sovereign Bonds during the crisis which took place in December 2001. Sironi et al (2002) shown that recovery rates and default rates are negatively related: higher default rates are associated with lower recovery rates and lower default rates are correlated with higher recovery rates. Moody's Special Comment (2003) shows that recovery rates may be impacted by quality and type of assets, life of assets and their ability to continue generating revenues. Recovery rates also vary from industry to industry. Guha (2002) examined a unique dataset of defaulted bond prices of predominantly US-based issuers who at the time of the initial default event had several publicly traded bonds outstanding. The author find that the amount recovered in default follows a recovery face value form. Altman et al (2003) presents a detailed review of the recovery rate, more specifically, its relationship with the probability of default of an obligor. Altman et al (2005) analyzes the association between default and recovery rates on credit assets and seeks to empirically explain this critical relationship. We examine recovery rates on corporate bond defaults over the period 1982–2002. They explain recovery rates by specifying rather straightforward linear, logarithmic, and logistic regression models. They find that bond default rates can explain about 51% of the variation in the annual recovery rate with the level of default rates (with the linear model) and 60% with the logarithmic relationship. Andritzky (2004) calculates implied recovery rates and implied default probabilities in a risk neutral setting for Argentine US-Dollar Eurobonds during the Argentine crisis from 2000 to 2002. Pan & Singleton (2006) explore the nature of default arrival and recovery implicit in the term structures of sovereign CDS spreads. They show in CDS markets where recovery is a fraction of face value and can be separately identified through the information contained in the term structure of CDS spreads. Bakshi et al (2006) explains how risk-neutral recovery rates are related to the density of the physical recovery, and counterparty risk aversion. They derive defaultable coupon bond prices under broad recovery specifications: the bondholders recover a fraction of the face value of the bond, a fraction of the present value of face and a fraction proportional to the pre-default market value. Schneider et al (2007) analyse the recovery value of the reference asset in a credit event and its determinants. In a regression analysis, they observe that 30% of the variance in the loss given default is explained by rating and industry. Significant negative coefficients for the Industrials and Oil & Gas sectors support the intuition that obligors with substantial tangible assets are expected to recover more in default.

In our paper, we work with average bond recovery rates by year prior to default as reported by Moody's special comment (2007) because the lack of information's from Japan Credit Rating Agency.

RATING (JCR)	RATING (Moody's)	Recovery rates (5 years prior to default)
AAA	Aaa	74.1%
AA	Aa	41.6%
A	A	48.4%
BBB	Baa	43.9%
BB	Ba	44.2%
B	B	42.3%
CCC	CCC	34.7

Table 6: Average bond recovery rates by year prior to default 1982-2006.

7. Computing Times to Default

In this section, we present some mathematical background for modelling joint default times. We assume a probability space $\left(\Omega, \Im, \{\Im_t\}_{t\geq 0}, \mathrm{P}\right)$, where Ω is the underlying probability space containing all possible events over a finite time horizon. $\Im$ is a σ-field representing the collection of all events. $\{\Im_t\}_{t\geq 0}$ is a filtration that carries information with times and P is a probability measure[17]. The pricing is assumed under no arbitrage and then, P is risk-neutral measure.

Default time τ_i for each obligor $i = \{1,...,n\}$ should be a random variable and the event of default should be known for everybody at any times because we assume a perfect market with a free flow of information[18]. Default is a stopping time τ with respect to the filtration: $\{\tau < t\} \subset \Im_t$, $\forall t \geq 0$. Let $F(t) = \mathrm{P}(\tau \leq t)$ be the distribution function and $f(t)$ is the density function of stopping time, the hazard rate h of τ or intensity process is defined such that we have for the probability of default until time (t + Δt) given survival till t is given by:

$$\mathrm{h}(\mathrm{t}) = \frac{\mathrm{f}(\mathrm{t})}{1-\mathrm{F}(\mathrm{t})} = \frac{1}{1-\mathrm{F}(\mathrm{t})} \times \frac{\mathrm{dF}(\mathrm{t})}{\mathrm{dt}} = \lim_{\Delta \to 0} \frac{\mathrm{P}(\mathrm{t} < \tau \leq \mathrm{t} + \Delta \mathrm{t} / \tau > \mathrm{t})}{\Delta \mathrm{t}} \tag{7.1}$$

With $dF = (t < \tau \leq t + dt, \tau > t)$, $1 - F(t) = P(\tau > t) = S(t)$ is called the survival function that gives the probability that a security will attain age t .the hazard rate function gives the instantaneous default probability for a security that has attained age t .The marginal survival distributions $S_i(t_i)$ is assumed to be smooth and strictly decreasing, this can be written as:

$$\mathrm{S_i}(\mathrm{t_i}) = 1 - \mathrm{F_i}(\mathrm{t_i}) = \mathrm{e}^{-\int_0^{\mathrm{t}} \mathrm{h_i}(\mathrm{s})\mathrm{ds}} \tag{7.2}$$

Then
$$F_i(t_i) = 1 - e^{-\int_0^t h_i(s)ds} \tag{7.3}$$

$h_i(.)$ is the default intensity process for entity i. The default times τ_i are defined:

$$\tau_i := \inf\left\{ t \geq 0 : \int_0^t h_i(s)ds \geq \theta_i \right\} \tag{7.4}$$

[17] According to Bielecki et al (2005), the probability space is endowed with a filtration $\Im = H \vee F$, where the filtration H carries information about evolutions of credit events, such as changes in credit ratings of perspective credit names and F is some reference filtration.

[18] Friewald (2004) presents two well-known algorithms to simulate default time: the compensator method and the inverse-CDF method.

Where θ_i has an exponential distribution with unit intensity[19].

After the calibration of the credit curves, we can obtain the cumulative default probabilities for any names over any time interval [0,t]. Then, we can define the marginal distribution of default times.
Given a credit curve $P(t,R)_{t\geq 0}$, of an asset or obligor with a Rating R up to a given time period t, there exists a unique default times distribution for R-rated obligors.
Effects regarding correlation are particularly strong for some of the most recent innovations in credit markets, namely single-tranche CDOs and basket default swap. We use the index i to refer to the i-th obligor. The rating assigned to an obligor i will be denoted by $R(i)$ and assume a portfolio of n obligors. Consequently, the joint distribution of default times τ_i :

$$F(t_1,...,t_n) = P(\tau_{R(1)} \leq t_1,...,\tau_{R(n)} \leq t_n) \tag{7.5}$$

The joint survival time distribution is given by:

$$S(t_1,...,t_n) = P(\tau_{R(1)} > t_1,...,\tau_{R(n)} > t_n) \tag{7.6}$$

$$= C\left(e^{\left(-\int_0^t h_1(s)ds\right)}, e^{\left(-\int_0^t h_2(s)ds\right)},..., e^{\left(-\int_0^t h_n(s)ds\right)} \right) \tag{7.7}$$

For deterministic intensities, this framework converges to Li (2000) model[20] .The times of default τ_i are defined as the first time the default countdown processes:

$\lambda_i(t) := e^{\left(-\int_0^t h_i(s)ds\right)}$ reach the level of the trigger variables U_i :

$$\tau_i := \inf\{t \geq 0 : \lambda_i(t) \leq U_i\} \tag{7.8}$$

The choice of a dependence structure between default times drives the prices of basket default swaps and CDO tranches. Copulas functions allow us to separate the problem of modelling the default times into two parts: first, the specification of the marginal distribution functions and second, the choice of a suitable copula which describes the dependence structure between the default times. Then, the marginal distributions together with the choice of a suitable copula are sufficient to specify the full joint distribution of the default times.
The benefits for using copulas to modelling joint default times:
- Maintains input correlation matrix reasonably well.
- Distribution-free approach.
- Can be employed in simulation procedures.
- Allows for various dependence structures (including tail dependence).
- Generates an exact joint distribution.

To simulate the joint default times for the five names, we present the following steps:

Step 1: simulate N-dimensional vector of correlated uniform random variates from a copula (C_Σ or $C_{v,\Sigma}$) as described in Annex 1. The correlation matrix and the appropriate number of degrees of freedom are estimated from § 2.2.

[19] The exponential distribution is used to model Poisson processes, which are situations in which an object initially in state A can change to state B with constant probability per unit time λ. The time at which the state actually changes is described by an exponential random variable with parameter λ.
[20] Li (2000) presents a simple and computationally inexpensive algorithm for simulating correlated defaults. His methodology builds on the implicit assumption that the multivariate distribution of default times and the multivariate distribution of asset returns share the same dependence structure, which he assumes to be Gaussian and is therefore fully characterised by a correlation matrix.

Step 2: Translate the corresponding uniform variates into default time for each obligors. The default dates can now be derived from the uniform random variates. They are given by : $\tau_i = -\frac{Ln(u_i)}{\lambda}$.

8. Pricing under Monte Carlo simulations

We have an example of basket credit default swap with five names of different rating and industry sector. We try to price the 5 year basket credit default swaps. The default event is triggered by the nth default in the basket. The seller of the basket credit default swap will face the default payment upon the nth default, and the buyer will pay the spread price until nth default or until maturity T. Let ($\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$) denote the default order. The fair spread of the basket default swap is given by equation (2.5). It is not easy to get the distribution of τ_n so that it is difficult to calculate the expectations in all the legs. Monte Carlo simulation can be used to get the fair spread of the nth basket default swap.

Pricing a nth to default basket default swap under Gaussian and t-student copula using Monte Carlo simulations can be presented as the following steps:

Step 1: simulate N-dimensional vector of correlated uniform random variates from the corresponding copula with respect to correlation matrix and hazard rates
Step 2: Translate the corresponding uniform variates into default time for each obligors.
Step 3: Sort the credits with respect to their default time τ_i and determine the nth default time τ^n. For first to default swap, we find the first default time $\tau^* = \min_i \tau_i$.
Step 4: Based on specific realization of τ^n Determine the present value of the premium leg $E[PL_n]$.
Step 5: Determine the present value of the default leg $E[DL_n]$.
Step 6: Repeat all steps above until the required number of scenarios has been simulated and the sample average fair spread of the nth to default basket swap as described in the equation (2.5).

The fair prices of the basket default swap as described in § 2.1 under Gaussian an t-student Copula and Monte Carlo simulation are presented as follows:

Fair price (Gaussian copula)	Fair price (t-student copula
First to default	First to default
327.6322	286.1316

Table 7: Fair spread of first to default swap under Monte Carlo simulation

The higher the quality of the obligors the less likely are the defaults, then, a basket of high quality credit will be cheaper than a basket of low quality credit.We notice that the premium computed under Gaussian and t-student copulas are different. Burtschell et al (2005) find that Gaussian and t-student copulas lead to quite similar premium for first to default swap when they change the number of names in the basket. Similarly, the differences are minor when they change the rank of default

9. Pricing under Importance sampling

Variance reduction has always been a central issue in Monte Carlo experiments. Importance sampling is a variance reduction technique that can be used in the Monte Carlo simulation. The idea behind Importance sampling is that certain values of the input random variables in a simulation have more impact on the parameter being estimated than others. If these "important" values are emphasized by sampling more frequently, then the estimator variance can be reduced. In the Monte Carlo simulation procedures, a path will only result in a default payoff if the n th default occurs before the maturity.

Joshi & Kainth (2004) apply importance sampling to the pricing of nth to default credit swaps within the Li model and obtain stable and sizeable speed ups. They show that Monte Carlo simulations in the Li model can be slow to converge and present procedures for accelerating the computation of prices and sensitivities to hazard rates. Glasserman &Li (2007) develop importance sampling (IS) procedures for rare-event simulation for credit risk measurement. They focus on the normal copula model originally associated with J.P. Morgan's CreditMetrics system. Dependence between obligors is captured through a multivariate normal vector of latent variables; a particular obligor defaults if its associated latent variable crosses some threshold. Glassserman &Juneja (2006) have considered the problem of simultaneous estimation of the probabilities of multiple rare events. Successful applications of importance sampling for rare event simulation typically focus on the probability of a single rare event. As a way of demonstrating the effectiveness of an importance sampling technique, the probability of interest is often embedded in a sequence of probabilities decreasing to zero. They shows that Importance sampling based on exponential twisting produces asymptotically efficient estimates of rare event probabilities in a wide range of problems. Abid & Naifar (2007) argue the choice of copula and the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives. They present copulas based simulations procedures for basket credit derivatives under Monte Carlo and Importance Sampling simulations.

Fair price (Gaussian copula)	Fair price (t-student copula
First to default	First to default
316.1415	312.7362

Table 8: Fair spread of first to default swap under importance sampling simulation

We notice that the spread of first to default swap change with the structure of dependency and the simulation techniques. Then, the choice of copula and the choice of procedures for rare-event simulation govern, also, the pricing of basket credit derivatives.

10. Conclusion

The creation, calibration and pricing of basket credit derivatives raise many technical questions and issues: How to obtain a realistic migration matrix? How we can calibrate credit curve and recovery rate for each obligor? What simulation procedures used for rare-event simulation?.
In this paper, we present a methodology for price calibration of basket default swap from Japanese market data. A fundamental part of the pricing framework is the estimation of default probabilities, recovery rate and the structure of dependency. We present a copula based simulation procedure for pricing basket default swap and we calibrate parameters from market data. We refer to Japan Credit

Rating Agency to obtain rating transition matrix and cumulative default rates. We consider using Gaussian copula and t-student copula and study their impact on basket credit derivative prices. We will present an application of the Canonical Maximum Likelihood Method (CML) for calibrating t-student copula. The basic problem of using simulation is that defaults are rare events, and a large number of simulation paths are usually required to achieve a sufficient sampling of the probability space. Subsequently, the choice of copula and the choice of procedures for rare-event simulation govern the pricing of basket credit derivatives. After the calibration of the parameters, we estimate the fair price of first to default swap under Monte Carlo an importance sampling simulations and Gaussian and student copulas.

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