The Mean Reversion Stochastic Processes Applications in Risk Management

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Abstract

In this study we investigate using the mean reversion processes in financial risk management, as they provide an good description of stock price fluctuations and market risks. This paper does not aim at being exhaustive, but gives examples for practically implementable models allowing for stylised features in the data. After introducing several widely used the mean reversion processes, we discuss this methods for risk management with Monte Carlo simulations. We also explain how the process can be calibrated based on historical data of Bulgarian 5 year Credit Default Swap and to find out Value at Risk

Key words: Risk Management, Stochastic Processes, Mean Reversion, Monte Carlo Simulation, Calibration interest rate processes,

AMS subject classifications: 60K10; 62P05.

1 Introduction

In risk management it is desirable to grasp the essential statistical features of a time series representing a risk factor to begin a detailed technical analysis of the product or the entity under scrutiny. This pepper will address the following mean reverting processes: Vasicek, CIR, exponential Vasicek.

The rest of this paper is structured as follows. Section 1 discusses a mean reverting behaviour and the appropriate test form mean reversion. Also we test Credit Default Swap (CDS) Bulgarian 5Y index for mean reversion. In section 2 we provide a brief mathematical introduction to the mean reversion processes. We also calibrate some of those processes to the real data. Section Finally, section 4 discusses those models for risk management.

2 The Mean Reverting Behaviour

2.1 Check for mean reversion or stationarity

The presence of an autoregressive (AR) feature can be tested in the data, typically on the returns of a series or on the series itself if this is an interest rate or spread series. In linear processes with normal shocks, this amounts to checking stationarity. If this is present, this can be a feature for mean reversion and a mean reverting process can be adopted as a first assumption for the data. If the AR test is rejected this means that there is no mean reversion, at least under normal shocks and linear models.

2.2 Tests for mean reversion

The mean reversion models can be defined as the property to always revert to a certain constant or time varying level with limited variance around it. This property is true for an AR(1) process if the absolute value of the autoregression coefficient is less than one ($|\alpha| \leq 1$). To be precise, the formulation of the first order autoregressive process AR(1) is:

$$x_{t+1} = \mu + \alpha x_t + \sigma\varepsilon_{t+1} \Rightarrow \triangle x_{t+1} = (1-\alpha)(\frac{\mu}{1-\alpha} - x_t) + \sigma\varepsilon_{t+1} \tag{1}$$

All the mean reverting behaviour in the processes that are introduced in this section is due to an AR(1) feature in the discretised version of the relevant SDE. First it is important to check the validity of the mean reversion assumption. A simple way to do this is to test for stationarity. Since for the AR(1) process $|\alpha| \leq 1$ is also a necessary and sufficient condition for stationarity, testing for mean reversion is equivalent to testing for stationarity. Note that in the special case where $\alpha = 1$, the process behaves like a pure random walk with constant drift, $\triangle x_{t+1} = \mu + \sigma\varepsilon_{t+1}$.

There is a lot of tests available for use for this purpose. Same of them are:

- the Dickey and Fuller (1979) test
- the Augmented DF (ADF) test of Said and Dickey (1984) test;
- the Phillips and Perron. (1988);
- the Variance Ratio (VR) test of Poterba and Summers (1988) and Cochrane (2001);
- the Kwiatkowski, Phillips, Schmidt, and Shin (1992) test.

To illustrate this, the mean reversion of a Credit Default Swap (CDS) Bulgarian 5Y index, is tested using the Augmented Dickey-Fuller test (Said and Dickey (1984). The ADF test for mean reversion is sensitive to outliers. Before testing for mean reversion, the index data have been cleaned by removing the innovations that exceed three times the volatility, as they are considered outliers.

The ADF statistics obtained by the CDS Bulgarian 5Y index is -2.18. The more negative the ADF statistics, the stronger the rejection of the unit root hypothesis. The first and fifth percentiles are -2.58 and -1.95. This means that the test cannot rejects the null hypothesis of unit root at the 0.01 significance level but the test rejects null hypothesis of unit root at the 0.05 significance level. So with confidence level 0.95 we reject the hypothesis that CDS Bulgarian 5Y index is stationarity.

3 The Mean Reverting Models

3.1 The Vasicek Model

The Vasicek model, owing its name to Vasicek (1977), is one of the earliest stochastic models of the shortterm interest rate. It assumes that the instantaneous spot rate (or "short rate") follows an Ornstein-Uhlenbeck process with constant coefficients under the statistical or objective measure used for historical estimation:

$$dx_t = \alpha(\theta - x_t)dt + \sigma dW_t \tag{2}$$

with α, θ and σ positive and dW_t is a standard Wiener process. The Vasicek model has a major shortcoming that is the non null probability of negative rates. This is an unrealistic property for the modelling of positive entities like interest rates or credit spreads. The explicit solution to the SDE (2) between any two instants s and t, with $0 < s < t$, can be easily obtained from the solution to the Ornstein-Uhlenbeck SDE, namely:

$$x_t = \theta(1 - e^{(-\alpha(t-s))}) + x_s e^{-\alpha(t-s)} + \sigma e^{-\alpha t} \int_s^t e^{\alpha t} dW_u \tag{3}$$

The discrete time version of this equation, on a time grid $t_0, t_1, t_2...$ with (assume constant for simplicity) time step $\triangle t = t_i - t_{i-1}$ is:

$$x(t_i) = c + bx(t_{i-1}) + \delta\varepsilon(t_i) \tag{4}$$

where the coefficients are:

$$c = \theta(1 - e^{-\alpha\triangle t}) \tag{5}$$

$$b = e^{-\alpha\triangle t} \tag{6}$$

W_t is a Gaussian white noise N (0, 1). The volatility of the innovations can be deduced by the Ito lemma

$$\delta = \sigma\sqrt{(1 - e^{-2\alpha\triangle t})/2\alpha} \tag{7}$$

To calibrate the Vasicek model the discrete form of the process is used, Equation (2). The coefficients c, b and δ are calibrated using the equation (4). The calibration process is simply an OLS regression of the time series $x(t_i)$ on its lagged form. The OLS regression provides the maximum likelihood estimator for the parameters c, b and δ. By resolving the three equations system one gets the following α, θ and σ parameters:

(8) $$\alpha = -ln(b)/\triangle t$$

(9) $$\theta = c/(1-b)$$

(10) $$\sigma = \delta/\sqrt{(b^2-1)\triangle t/2ln(b)}$$

Using (8), (9) and (10) one obtains the estimators for CDS Bulgarian 5Y index for the parameters α, θ and σ. Table 2

3.2 The Exponential Vasicek Model

To address the positivity of interest rates and ensuring their distribution to have fatter tails, it is possible to transform a basic Vasicek process y(t) in a new positive process x(t). The most common transformations to obtain a positive number x from a possibly negative number y are the square and the exponential functions. Therefore, if one assumes y to follow a Vasicek process,

(11) $$dy_t = \alpha(\theta - y_t)dt + \sigma dW_t$$

then one may transform y. If one takes the square to ensure positivity and define, then by Ito's formula x satisfies:

(12) $$dx_t = (B + A\sqrt{x_t} - Cx_t)dt + v\sqrt{x_t}dW_t$$

for suitable constants A,B,C and v. Ito's formula this time reads, for under the objective measure we will have the following

(13) $$dx_t = \alpha x_t(m - log(x_t))dt + \sigma x_t dW_t$$

where α and σ are positive, and a similar parametrisation can be used under the risk neutral measure.

(14) $$m = \theta + \frac{\sigma^2}{2\alpha}$$

Also The explicit solution of the Exponential Vasicek equation is then, between two any instants $0 < s < t$:

(15) $$x_t = \exp(\theta(1 - e^{(-\alpha(t-s))}) + log(x_s)e^{-\alpha(t-s)} + \sigma e^{-\alpha t}\int_s^t e^{\alpha t}dW_u).$$

The OLS regression of the log-spread time series on its first lags (Equation (4)) gives the coefficients c, b and ?. Then the ?, ? and ? parameters in Equations (12) are deduced using equations (8), (9) and (10). So with similar way we could estimate the parameters α, θ and σ.

3.3 The CIR Model

he model proposed by Cox, Ingersoll, and Ross (1985b) was introduced in the context of the general equilibrium model of the same authors Cox, Ingersoll, and Ross (1985a). By choosing an appropriate market price of risk, the xt process dynamics can be written with the same structure under both the objective measure and risk neutral measure. Under the objective measure one can write

$$dy_t = \alpha(\theta - x_t)dt + \sigma sqrt x_t dW_t \tag{16}$$

with α, θ, σ and dW_t is a standard Wiener process

The simulation of the CIR process can be done using a recursive discrete version of Equation (64) at discretisation times t_i :

$$dx(t_i) = \alpha\theta\triangle t + (1 - \alpha\triangle t x(t_{i-1})) + \sigma\sqrt(x(t_{i-1}\triangle t)\varepsilon_t \tag{17}$$

with ε $M(0,1)$. Alternative positivity preserving schemes are discussed, for example in Brigo and Mercurio (2006). For simulation purposes it is more precise but also more computationally expensive to draw the simulation directly from the following exact conditional distribution of the CIR process:

$$f_{CIR}(x_{t_i}|x_{t_{i-1}}) = ce^{-u-v}(\frac{v}{u})^{q/2}I_q(2\sqrt(uv)) \tag{18}$$

$$c = \frac{2\alpha}{\sigma(1 - e^{-\alpha\triangle t}} \tag{19}$$

$$u = cx_{t_i}e^{-\alpha\triangle t} \tag{20}$$

$$v = cx_{t_{i-1}} \tag{21}$$

$$q = \frac{2\alpha\theta}{\sigma^2} - 1 \tag{22}$$

with I_q the modified Bessel function of the first kind of order q and $triangle t = t_i - t_{i-1}$. The process x follows a noncentral chi-squared distribution $x(t_i)|x(t_{i-1}$ $X^2(2cx(t_{i-1}$ $2, 2u)$. To calibrate this model the Maximum Likelihood Method is applied:

$$\arg \max_{\alpha>0,\theta>0,\sigma>0} \log(\ell) \tag{23}$$

The Likelihood can be decomposed, as for general Markov processes, as a product of transition likelihoods:

$$\ell = \prod f_{CIR}(x(t_i)|x(t_{i-1}), \alpha, \theta, \sigma) \tag{24}$$

The calibration is done by Maximum Likelihood. To speed up the calibration one uses as initial guess for α the AR(1) coefficient of the Vasicek regression (Equation (49)) and then infer θ and σ using the first two moments of the process. The long-term mean of the CIR process is θ and its long-term variance is $\theta\sigma^2/(2\alpha)$. Then:

$$\alpha_0 = -\log(b)/\triangle t \tag{25}$$

$$\theta_0 = E(x_t) \tag{26}$$

$$\sigma = \sqrt{2\alpha_0 Var(x_t)/\theta_0}. \tag{27}$$

3.4 Risk management

Once the process has been chosen and calibrated to historical data, typically through regression or maximum likelihood estimation, we may use the process to simulate the risk factor over a given time horizon and build future scenarios for the portfolio under examination. On the terminal simulated distribution of the portfolio one may then single out several risk measures. This report does not focus on the risk measures themselves but on the stochastic processes estimation preceding the simulation of the risk factors.

Throughout this paper we use the estimated parameters of CIR model in table2 to estimate the values for VaR to Bulgarian 5 year CDS indexes. The density of the simulating paths based on CIR model is close to the empirical density. The tail behavior of probability distributions and reveals that the CIR model density represents the fat tails of empirical data better than the standard Geometric Brownian Motion process.

Value at Risk is shown in Table 3. In case more than one model is suited to the task, one can analyze risks with different models and compare the outputs as a way to reduce model risk. The survey could be extended not only to single time series. Correlation or more generally dependence across risk factors, leading to multivariate processes modeling, will be addressed in future work.

Table 1 Augmented Dickey-Fuller test

e Instrument	Number of observations	Augmented Dickey-Fuller test	0.01 significance level	0.05 significance level	0.1 significance level
CDS Bulgarian 5Y index	510	-2.1827	-2.5836	-1.9573	-1.6311
US 10 years Treasury Yields	1820	-5.4229	-2.5836	-1.9573	-1.6311

Table 2.1 Calibrating parameters Vasicek Model

Instrument	α	θ	σ
CDS Bulgarian 5Y index	0.0066	274.65	13.9159
US 10 years Treasury Yields	0.0034	3.6706	0.0634

Table 2.2 Calibrating parameters CIR Model

Instrument	α	θ	σ
CDS Bulgarian 5Y index	0.0010	6.0714	0.7211
US 10 years Treasury Yields	0.0023	3.7518	0.0334

Table 3 Value at Risks of a Bulgarian 5y CDS

	percentile	percentile
	0.99	0.95
Simulated GBM model	-0.082	-0.059
Simulated VaR Vasicek model	-0.092	-0.065
Simulated VaR CIR model	-0.128	-0.072

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