An interest rate model with Markov chain volatility level

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Abstract

We consider a two factor interest rate model, where the volatility level follows continuous time finite state Markov chain. We derive the close form solution of bond price that involves fundamental matrix.

Key words: interest rate model, bond price close form solution, Markov chain volatility level

1 Introduction

We consider a two factor interest rate model, where the volatility level follows continuous time finite state Markov chain. We derive the close form solution of bond price that involves fundamental matrix.

This paper derive its motivation from the following papers Elliott et al. (2002) and Elliott et al. (1999) in which a hidden Markov model (HMM) with mean reverting characteristic as a model for time series, in particular the series of interest rate.

2 The Markov chain model for the volatility structure

We consider the following modification of the Vasicek model, where the interest rate $r = (r_t : 0 < t \leq T)$ is described by the stochastic differential equation:

$$dr_t = (\alpha - \beta r_t)dt + \sigma_t dW_t \tag{1}$$

where α and β are positive constants, $W = \{W_t : 0 < t \leq T\}$ is a Wiener process independent of σ_t. We consider the situation where the process r_t is observed and inference are to be made about the process σ_t and other parameters.

We assume that there is an underlying probability space (Ω, F, P). Under this assumptions we defined a n-state continuous time Markov chain $X = \{X_i : 0 \leq t \leq T$ that is indentical to σ_t after a transformation of the state space. We assume P is risk-neutral measure.

We choose the state space for X to take the set $\{\epsilon_1, ...\epsilon_n\}$ of unit vector in R^n. That is $\epsilon_i = (0, 0, ...1, 0...0)'$ where the i-th component is unity and zero otherwise. Then it could be written:

$$\sigma_t = \sigma(X_t) = \langle \sigma, X_t \rangle \tag{2}$$

for the vector $\sigma = (\sigma_1, ...\sigma_n) \in R^n, \sigma_i > 0$, where $\langle \sigma, X_t \rangle$ denotes inner product of vectors σ and X_t. The vectors of probabilities p_i is given by $p_t = E[X_t]$ and $p_t = (p_t^1, p_t^2, ...p_t^n)'$.

Let $A = \{A_t : 0 \leq t \leq T\}$ be a family of transition matrix from the continuous Markov chain X, so that p_t satisfies the forward Kolmogorov equation $dp_t/dt = A_t p_t$ which gives the initial probability vector p_0. So that the transition matrix A_t determines the dynamic of the level of σ as described in (2). A_t is defined as following Q-matrix

$$A_t = (a_{ij(t)}), \quad a_{ij}(t) \geq 0, \quad i \neq j$$

and

$$\sum_{j=1}^{n} a_{ij}(t) = 0, \quad 1 \leq i \leq n. \tag{3}$$

So, the interest rate is conditionally Gaussian conditioned on the independent path of the Markov chain, which described σ_t.

3 Deriving the closed form solution of bond price

We consider the interest rate process which is given by:

$$dr_t = (\alpha - \beta r_t)dt + \sigma_t dW_t \tag{4}$$

We suppose σ_t is described by the Markov chain as above so

$$\sigma_t = \sigma(X_t) = \langle \sigma, X_t \rangle \tag{5}$$

From Elliott et al (1994), we have that the dynamics of X are given by

$$X_t = X_0 + \int_o^t A_u X_u du + M_t \tag{6}$$

where M is a martingale with respect to the filtration generated by X. Then

$$dX_t = A_t X_t dt + dM_t \tag{7}$$

As P is the risk-neutral measure, the price at time $t \leq T$ of zero-coupon bond with maturity T is

$$P(t,T) = E[e^{(-\int_t^T r_u du)|\mathcal{F}_t}]$$

$$\begin{aligned} r_t &= e^{-\beta t}\left(r_0 + \int_0^t e^{\beta u}\alpha du + \int_0^t e^{\beta u}\sigma_u dW_u\right) \\ &= e^{-\beta t}\left(r_0 + \int_0^t e^{\beta u}\alpha du + \int_0^t e^{\beta u}\langle \sigma, X_u\rangle dW_u\right) \end{aligned}$$

If we knew the trajectory of σ_u, or $X_u, u \leq T$ then by the Hull-White result (1990),

$$P(t,T,X) = E\left[e^{(-\int_t^T r_u du)|\mathcal{F}_t}\right] = e^{-r_t C(t,T) - A(t,T)} \tag{8}$$

where r_t is the solution to the equation (4) and terminal conditions $C(T,T) = 0$ and also $A(T,T) = 0$.

$$C(t,T) = \int_t^T e^{-\int_t^s b(v)dv} ds$$

and

$$A(t,T) = \int_t^T \left(a(s)C(s,T) - \frac{1}{2}\sigma^2(t)C^2(s,T) \right) ds,$$

so in our case where α and β are constants $C(t,T)$ and $A(t,T)$ get the expressions

$$C(t,T) = \frac{e^{\beta t}}{\beta} \left[e^{-\beta t} - e^{-\beta T} \right] \tag{9}$$

$$A(t,T) = \int_t^T \left(\frac{\alpha e^{\beta s}}{\beta} [e^{-\beta s} - e^{-\beta T}] - \frac{1}{2}\sigma_s^2 \frac{[e^{-\beta s} - e^{-\beta T}]^2}{\beta^2} e^{2\beta s} \right) ds \tag{10}$$

Evaluating the first part of (11) we get

$$\begin{aligned} A(t,T) &= \int_t^T \left(-\tfrac{1}{2}\sigma_s^2 \tfrac{[e^{-\beta s} - e^{-\beta T}]^2}{\beta^2} e^{2\beta s} \right) ds \\ &+ \frac{\alpha}{\beta}(T-t) - \frac{\alpha}{\beta^2} + \frac{\alpha}{\beta^2} e^{-\beta(T-t)}, \end{aligned} \tag{11}$$

Now let evaluate the integral of (13)

$$\begin{aligned} \int_t^T \left(-\frac{1}{2}\sigma_s^2 \frac{[e^{-\beta s} - e^{-\beta T}]^2}{\beta^2} e^{2\beta s} \right) ds &= \int_t^T \left(-\tfrac{1}{2}\sigma_s^2 \left[\tfrac{1-e^{-\beta(T-s)}}{\beta} \right]^2 \right) ds = \\ &= -\frac{1}{2} \int_t^T \langle X_s, \sigma \rangle^2 \left[\frac{1-e^{-\beta(T-s)}}{\beta} \right]^2 ds = \\ &= -\frac{1}{2} \int_t^T \langle X_s, \phi_s \rangle^2 ds \end{aligned}$$

where $\phi_s = \frac{1-e^{-\beta(T-s)}}{\beta}\sigma$

$$C(t,T) = \frac{e^{\beta t}}{\beta} \left[1 - e^{-\beta(T-t)} \right]$$

$$\begin{aligned} A(t,T) &= F(t,T) + B(t,T) = \\ &= \frac{\alpha}{\beta}(T-t) - \frac{\alpha}{\beta^2} + \frac{\alpha}{\beta^2} e^{-\beta(T-t)} - \frac{1}{2} \int_t^T \langle X_s, \phi_s \rangle^2 ds \end{aligned}$$

we aim to obtain a closed form solution of the bond price. So far, we have

$$\begin{aligned} P(t,T,X) &= e^{-r_t C(t,T) - A(t,T)} = e^{-r_t C(t,T) - F(t,T) - B(t,T)} = \\ &= e^{-r_t C(t,T)} e^{-F(t,T)} e^{\frac{1}{2} \int_t^T \langle X_s, \phi_s \rangle^2 ds} \end{aligned} \tag{12}$$

where

All that remains to be done is the evaluation of the expectation of $e^{\frac{1}{2}\int_t^T \langle X_s,\phi_s\rangle^2 ds}$, where ϕ_s is deterministic for $s \le T$. So let define

$$Z_{t,s} := e^{\frac{1}{2}\int_t^T \langle X_s,\phi_s\rangle^2 ds}$$

and from Elliott et al (1999)

$$X_{t,s} = X_t + \int_t^s AX_v dv + M_t$$

The process ZX following

$$d(Z_{t,s}X_{t,s}) = Z_{t,s}dX_{t,s} + X_{t,s}dZ_{t,s}$$

Having next two representations

$$dZ_{t,x} = -\frac{1}{2}\langle X_s,\phi_s\rangle^2 Z_{t,s}ds$$

$$dX_{t,x} = AX_{t,x}ds + dM_s$$

Then the process XZ have following representation

$$\begin{aligned} d(Z_{t,s}X_{t,s}) &= Z_{t,s}[AX_{t,x}ds + dM_s] + X_{t,s}[-\tfrac{1}{2}\langle X_s,\phi_s\rangle^2 Z_{t,s}ds] = \\ &= Z_{t,s}AX_{t,x}ds + Z_{t,s}dM_s - X_{t,s}\frac{1}{2}\langle X_s,\phi_s\rangle^2 Z_{t,s}ds = \\ &= AX_{t,x}Z_{t,s}ds + Z_{t,s}dM_s - \frac{1}{2}\langle X_s,\phi_s\rangle^2 X_{t,s}Z_{t,s}ds \end{aligned}$$

or into integral form

$$\begin{aligned} Z_{t,s}X_{t,s} &= Z_{t,t}X_{t,t} + \textstyle\int_t^T AZ_{t,s}X_{t,x}ds + \int_t^T Z_{t,s}dM_s \\ &- \int_t^T \frac{1}{2}\langle X_s,\phi_s\rangle^2 Z_{t,s}X_{t,s}ds \end{aligned}$$

We have $dZ_{t,t} = e^{\int_t^t \frac{1}{2}\langle X_s,\phi_s\rangle^2 ds} = e^0 = 1$, so

$$\begin{aligned} Z_{t,s}X_{t,s} &= X_{t,t} + \textstyle\int_t^T AZ_{t,s}X_{t,x}ds + \int_t^T Z_{t,s}dM_s \\ &- \int_t^T \frac{1}{2}\langle X_s,\phi_s\rangle^2 Z_{t,s}X_{t,s}ds \end{aligned}$$

Taking expectation of $Z_{t,s}X_{t,s}$ and considering martingale property of M etc $E[\int_t^T Z_{t,s}dM_s] = 0$ we have

$$\begin{aligned} E[Z_{t,s}X_{t,s}] &= E[X_{t,t}] + E\left[\textstyle\int_t^T AZ_{t,s}X_{t,x}ds\right] - E\left[\int_t^T \frac{1}{2}\langle X_s,\phi_s\rangle^2 Z_{t,s}X_{t,s}ds\right] \\ &= X_t + \int_t^T AE\left[Z_{t,s}X_{t,x}\right]ds - \frac{1}{2}\int_t^T E\left[\langle X_s,\phi_s\rangle^2 Z_{t,s}X_{t,s}\right]ds \end{aligned}$$

Considering that $\langle X_s, \phi_s\rangle^2 Z_{t,s}X_{t,s} = \langle X_s, \phi_s\rangle\langle X_s, \phi_s\rangle Z_{t,s}X_{t,s}$ and $\langle X_s, \phi_s\rangle^2 = [\sum_{i=1}^n \langle X_{t,s}, \epsilon_i\rangle S_i(t,s)]^2 = [\langle X_{t,s}, \epsilon_1\rangle S_1(t,s)]^2 + [\langle X_{t,s}, \epsilon_2\rangle S_2(t,s)]^2 + \ldots [\langle X_{t,s}, \epsilon_n\rangle S_n(t,s)]^2 = \langle X_{t,s}, \epsilon_1\rangle^2 S_1(t,s)^2 + \langle X_{t,s}, \epsilon_2\rangle^2 S_2(t,s)^2 + \ldots\langle X_{t,s}, \epsilon_n\rangle^2 S_n(t,s)^2 = \sum_{i=1}^n \langle X_{t,s}, \epsilon_i\rangle^2 S_i(t,s)^2$ so finally we have

$$\langle X_s, \phi_s\rangle^2 = \sum_{i=1}^n \langle X_{t,s}, \epsilon_i\rangle^2 S_i(t,s)^2$$

then

$$\langle X_s, \phi_s\rangle^2 Z_{t,s}X_{t,s} = \Psi(t,s)Z_{t,s}X_{t,s}$$

where $\Psi(t,s)$ is matrix with $(S_1^2(t,s), S_2^2(t,s), \ldots S_n^2(t,s))$ on the diagonal.

$$E[Z_{t,s}X_{t,s}] = X_t + \int_t^T AE[Z_{t,s}X_{t,s}]ds - \frac{1}{2}\int_t^T \Psi(t,s)E[Z_{t,s}X_{t,s}]ds$$

Let define $\widehat{z} := E[Z_{t,s}X_{t,s}]$ then

$$\widehat{z}_{t,s} = X_t + \int_t^T A\widehat{z}_{t,s}ds - \frac{1}{2}\int_t^T \Psi(t,s)\widehat{z}_{t,s}ds$$

$$\widehat{z}_{t,s} = X_t + \int_t^T \left(A - \frac{1}{2}\Psi(t,s)\right)\widehat{z}_{t,s}ds$$

Let $M = A - \frac{1}{2}\Psi$ then

$$\widehat{z}_{t,s} = X_t + \int_t^T M\widehat{z}_{t,s}ds \tag{13}$$

so differential equation which we should solve is

$$\frac{d\widehat{z}_{t,s}}{ds} = M(t,s)\widehat{z}_{t,s} \tag{14}$$

with initial condition $X_t = \widehat{z}_{t,t}$

For the expectation term in (12) $e^{\frac{1}{2}\int_t^T \langle X_s termin(12), \phi_s\rangle^2 ds}$ we have the following, using $\langle X_T, \mathbf{1}\rangle$

$$E\left[\exp\left(-\int_t^T \langle X_v, \phi_v\rangle dv\right)|\mathcal{F}_t\right]$$

$$= E\left[\exp\left(-\int_t^T \langle X_v, \phi_v\rangle dv\right)\langle X_T, \mathbf{1}\rangle|\mathcal{F}_t\right]$$

$$= \left\langle E\left[\exp\left(-\int_t^T \langle X_v, \phi_v\rangle dv\right)X_T|\mathcal{F}_t\right], \mathbf{1}\right\rangle$$

$$= \langle E\left[Z_{t,T}X_T|\mathcal{F}_t\right], \mathbf{1}\rangle$$

$$= \langle\widehat{z}_{t.T}, \mathbf{1}\rangle$$

4 The fundamental matrix solution

Suppose $S(t,u)$ is a matrix value function satisfying certain conditions. Then the linear matrix differential equation

$$\frac{d}{du}\Phi(t,u) = S(t,u)\Phi(t,u), \qquad \Phi(t,t) = I$$

has a unique solution defined for $0 \leq t \leq u < \infty$ Hale (1969).

I is nxn matrix. For each $u \geq t \geq 0$ the matrix $\Phi(t,u)$ is nonsingular. Applying this result to the case where $S(t,u) = M(t,v)$. Then $\widehat{z}_{t,T} = \Phi(t,T)X_t$, so

$$E\left[\exp\left(-\int_t^T \langle X_v, \phi_v\rangle dv\right)|\mathcal{F}_t\right] = \langle \widehat{z}_{t,T}, \mathbf{1}\rangle = \langle \Phi(t,T)X_t, \mathbf{1}\rangle. \tag{15}$$

Then the closed form solution of price of zero coupon bond is given by

$$P(t,T,X) = e^{(-r_t C(t,T) - F(t,T))}\langle \Phi(t,T)X_t, \mathbf{1}\rangle. \tag{16}$$

where

$$C(t,T) = \frac{e^{\beta r}}{\beta}(e^{-\beta t} - e^{-\beta T})$$

$$F(t,T) = \frac{\alpha}{\beta}(T-t) - \frac{\alpha}{\beta^2} + \frac{\alpha}{\beta^2}e^{-\beta(T-t)}$$

So those equation is a closed form solution for the bond price we are looking for.

5 Conclusion

This paper contributes to the development of interest rate model with Markov chain property of volatility structure.

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