# Conventional or New? Optimal Investment Allocation across Vintages of Technology

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# Conventional or New? Optimal Investment Allocation across Vintages of Technology [*,†]

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**Abstract**

This paper develops and analyzes a growth model that consists of complementary *long-lived* and *short-lived* vintage-specific capital. As a result of the existence of complementary capital that is vintage compatible but has different longevity, the model generates two distinct investment patterns: (i) if the rate of vintage-specific technological progress is above a threshold–which is the product of long-lived capital's share and the difference in the rates of depreciation–then all new investment is allocated to the capital that embodies the frontier technology; (ii) otherwise, some investment is allocated to obsolete, short-lived capital to exploit the existing stock of obsolete long-lived capital.

The result provides a new explanation for observed investment in obsolete technologies. An important implication of this result is that equipment price-changes do not necessarily reflect the rate of progress, since the prices of obsolete short-lived capital remain the same when the rate of the progress is slow enough (as mentioned in (ii) above). Another implication is that acceleration in the rate of vintage-specific technological progress can cause an abrupt reallocation of investment towards modern capital–consistent with investment booms that are concentrated in certain "high-tech" equipment.

[*]As his last student, I express my deep sorrow for the death of Gary Saxonhouse. This paper is based on a chapter of my Ph.D. dissertation submitted to the University of Michigan. I am grateful to my advisers, John Laitner, Dmitriy Stolyarov, Miles Kimball, Gary Saxonhouse, and Jan Svejnar for their guidance and encouragement. I also thank Hiroshi Ohashi and Tsuyoshi Nakamura for providing their data on steel furnaces, and Yasuyuki Todo, and participants in the seminars at the University of Michigan, Kansai University, Yokohama National University, the University of Tokyo, and Aoyama Gakuin University for their helpful comments.



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# 1 Introduction

When your PC breaks down, would you repair it, or purchase a new one? This study provides answers to this class of investment allocation problem by a conventional vintage growth model with heterogeneous complementary capital. The model delivers two important results: sometimes there is investment in obsolete technologies; and the investment price-changes are proportional to the vintage-specific technological progress *only* when the progress is sufficiently fast. The optimal allocation of investment across vintages depends on the trade-off between the remaining stocks of the complementary obsolete capital and the advancement of the frontier technology.

This paper's model has two key elements: (i) it is a vintage growth model in which a certain technology is built into each unit of capital; and (ii) it has two kinds of complementary capital that have different rates of depreciation.[1] The basic idea behind the model is the following: if one type of complementary capital depreciates more slowly (*long-lived*) than the other (*short-lived*), then investing in short-lived capital with an obsolete technology may be rationalized in order to exploit the existing stock of long-lived capital with its obsolete technology.

The existence of vintage-specific complementary capital in a vintage growth model results in two surprising implications in a steady state. First, if the rate of technological progress is above a threshold–the product of the long-lived capital's share and the difference in the rates of depreciation–then all new investment will concentrate on the two new types of capital. Otherwise, a part of new investment will be allocated to short-lived capital with obsolete vintages as well as to both new capital types, to combine the existing stock of long-lived capital with the obsolete technology.

Second, if the rate of technological progress is below the threshold, then the prices of short-lived obsolete capital remain stable over time even when the rate of progress is positive. This result implies that if production involves vintage-specific complementary capital that has a longer longevity than equipment does, then estimates of the rate of vintage-specific technological progress based on changes in equipment prices[2] may be systematically biased downward.

[1] In this study, "depreciation" solely refers to physical depreciation, and excludes obsolescence, which is explicitly treated as endogenously determined price-changes.

[2] E.g., Gordon (1990), Hulten (1992), Greenwood et al. (1997), and Cummins and Violante (2002) assume a direct relationship between the rate of vintage specific technological progress and the changes in equipment prices based on the theoretical underpinnings of the Solow's vintage growth model.

There are two interpretations of the combinations of distinct types of complementary capital: components of a system versus the system itself that is integrated by a certain type of intangible capital (e.g., network, organization, process, and scheme); and physical capital versus another certain type of long-lived intangible capital (e.g., computer software, product design, manual). The former should be considered because components of a system typically are shorter-lived than the system itself.[3] The latter should be considered because physical capital, such as engines, always physically wears and/or tears, while certain types of intangible capital, such as product design, or computer software, do not.[4]

In these contexts, the results of the current study can be interpreted as follows. Suppose the CD drive (component/physical capital, short-lived) of your PC crashes for some reason. Then, would you buy a new PC or merely replace the CD drive? If the specifications of a new PC model develop quickly enough, you would purchase a new PC because it has much better features. Or you would replace the CD drive to keep using the existing PC because the remaining system of the PC (system capital, long-lived) and your existing collection of software, which is incompatible with the newest type of PC (intangible capital, long-lived), can be reused with minimal investment in a CD drive. The decision depends on the rate of technological progress, and the importance and remaining size of the long-lived system/intangible capital.

Investment in equipment as short-lived capital is indeed observed in the real economy. For example, many steam locomotives remained in operation after the more efficient diesel locomotive was introduced for commercial demonstration in 1924, and investment in steam locomotives continued for more than 20 years (Felli and Ortalo-Magne (1998), Figures 1 and 5). In this case, short-lived capital will be the locomotives (component/physical capital) and long-lived capital may be the existing network of fuel supply system (system capital), and the existing mechanics' know-how about each type of locomotive (intangible capital).

There are two strands of related literature: models that explain investment in obsolete technologies based on some types of complementarity across capital vintages (Benhabib and Rustichini (1991), Chari and Hopenhayn (1991), Parente (1994), and

[3]For example, Fraumeni (1997) shows that the rate of depreciation of internal combustion engines is 20.6%, while those of ships and boats, farm tractors, and construction tractors are 6.1%, 14.5%, and 16.3%, respectively.

[4]There are other types of intangible capital, such as brand name, which might have higher rates of depreciation than physical capital does. In this case, the roles of two types capital simply switch as discussed in Section 3.

Jovanovic and Nyarko (1996)); and models that have conventional types of vintage growth models, but have investment only in the frontier technology (Solow (1960), Greenwood et al. (1997), and Laitner and Stolyarov (2003)). In contrast to the models in the first group, my model aggregates the economy into the familiar neoclassical growth framework with market values of capital. In contrast to the models in the second group, my model delivers interesting investment patterns across vintages, and makes it possible to study investment distribution at a disaggregated level. Thus, my model features advantages of both groups of literature, providing further possibilities for analyzing sources of economic growth in conventional growth accounting as well as investment patterns across vintages.

The rest of the paper is organized as follows. Section 2 presents the model's framework and a characterization of a steady state. Section 3 discusses applications of the model, and finally, Section 4 concludes the paper.

# 2 The Model

The model has two key elements: (i) it is a vintage growth model in which each vintage of capital works with a separate production function that has a vintage-specific productivity level; and (ii) it has two kinds of vintage compatible capital with different rates of depreciation. Apart from the assumption of capital heterogeneity, all assumptions are essentially identical to those of Solow (1960): the Cobb–Douglas production function, fixed investment rate, competitive markets, perfect foresight / rational expectation, vintage-specific technological progress, and labor that is not vintage specific. This section first details the assumptions, and then develops and analyzes the model.

## 2.1 Setup of The Model

I assume the economy is competitive, and agents have perfect foresight and are rational. Each unit of capital embodies a specific vintage technology. Usage of a vintage technology requires capital goods that are specifically designed for the vintage technology. Let $v \geq T_0$ denote a specific vintage, and assume at time $t$ that vintage $v \leq t$ technology is available for agents.

Each vintage production technology requires three types of inputs: two types of vintage-specific capital, $A$ (*long-lived*) and $B$ (*short-lived*), and vintage-nonspecific labor, $L$. Assume $A$ and $B$ depreciate at the rates $\delta^A$ and $\delta^B$ where $\delta^A \leq \delta^B$. Let a subscript $v$ denote a specific vintage $v$ technology that is embodied in each type of capital; $A_{\underline{v}}$ and $B_{\underline{v}}$ represent the number of units of $A$ and $B$ designed for a specific vintage technology $v$.[5] $L_v$ expresses the amount of labor that is employed for a vintage $v$, although $L$ is not vintage specific.

Assume each vintage-specific production function has the Cobb-Douglas form,

$$Y_v = q_{\underline{v}} A_{\underline{v}}^{\alpha} B_{\underline{v}}^{\beta} L_v^{1-\alpha-\beta}, \tag{1}$$

where $Y_v$ is output from vintage $v$ technology, $q_{\underline{v}}$ is vintage-specific technology level that is monotonically increasing and piecewise-continuous in $v$, and $\alpha$ and $\beta$ are constant shares of two capital types.[6] Assume that output is homogeneous and keeps a constant physical unit over time.

Each physical unit of capital can be used for investment in each physical unit of either type of capital or consumption. I assume that a fixed portion of aggregate output is used for investment, and investment is irreversible,

$$\begin{aligned} \sigma Y &= I^A + I^B \\ &= \int_{T_0}^{t} I_{\underline{v}}^A \, dv + A_{\underline{t}} + \int_{T_0}^{t} I_{\underline{v}}^B \, dv + B_{\underline{t}}, \end{aligned} \tag{2}$$

where aggregate output $Y$ is defined by,

$$Y = \int_{T_0}^{t} Y_v \, dv. \tag{3}$$

Note that investment consists of the part for existing technologies and the part for the frontier technology.[7] Further, note that the prices of capital types in units of output should satisfy $P_{\underline{v}}^A \in [0,1]$ and $P_{\underline{v}}^B \in [0,1]$ since each type of capital is freely disposable, and investment in capital types with existing vintage technology is possible.

The setup of the model based on the straightforward neoclassical assumptions

[5] Throughout the paper, I underline vintage subscripts for vintage specific amounts in order to distinguish them from vintage non-specific amounts.

[6] In the model presented here, I omit Hicks-Neutral technological progress that affects all vintages of production, since the omission does not change the main point of the result. Chapter 3 in Aruga (2006) shows the case when the neutral technological progress is also considered.

[7] Different notations are used to distinguish the mass of investment in the frontier technology ($X_{\underline{t}}$) from the distribution of investment in existing technologies ($I_{\underline{v}}^X$).

turns out to be crucial in applying it to various types of economic activities that are discussed in Section 3.

## 2.2 Vintage Aggregation

In this subsection, I derive the aggregate production function that summarizes the allocation of the two types of capital across vintages. The result hinges on the assumption of a competitive market and becomes key in characterizing the steady state in the following section.

**Lemma 1** (Vintage Aggregation). *(i) Define aggregate capital types,*

$$A \equiv \int_{T_0}^{t} \frac{MPA_{\underline{v}}}{MPA_{\underline{t}}} A_{\underline{v}} \, dv, \tag{4}$$

$$B \equiv \int_{T_0}^{t} \frac{MPB_{\underline{v}}}{MPB_{\underline{t}}} B_{\underline{v}} \, dv, \tag{5}$$

*and aggregate labor,*

$$L \equiv \int_{T_0}^{t} L_v \, dv, \tag{6}$$

*where $MPA_{\underline{v}}$ and $MPB_{\underline{v}}$ are marginal products of $A$ and $B$ with vintage $v$ at time $t$.*

*Then, aggregate output and inputs have the relationship,*

$$Y = q_{\underline{t}} A^{\alpha} B^{\beta} L^{1-\alpha-\beta}. \tag{7}$$

*(ii) Furthermore, define aggregate consolidated capital,*

$$J \equiv \int_{T_0}^{t} J_{\underline{v}} \, dv, \tag{8}$$

*where vintage consolidated capital is defined by,*

$$J_{\underline{v}} \equiv \left[ q_{\underline{v}} A_{\underline{v}}^{\alpha} B_{\underline{v}}^{\beta} \right]^{\frac{1}{\alpha+\beta}} . \tag{9}$$

*Then, the consolidated capital and aggregate inputs have the relationship,*

$$J = \left[ q_{\underline{t}} A^{\alpha} B^{\beta} \right]^{\frac{1}{\alpha+\beta}} , \tag{10}$$

*and the labor and output allocations across vintages are given by,*

$$L_v = \frac{J_v}{J} L, \tag{11}$$

$$Y_v = \frac{J_v}{J} Y. \tag{12}$$

*Proof:* See Appendix A.1.1.

Interestingly enough, the aggregate production function across vintages has the same form as (1) with frontier technology level $q_{\underline{t}}$ and the aggregate inputs defined as (4) - (6).

Further note that if returns on capital of each type of capital are independent of respective vintages, (4) and (5) simply show the total values of the capital types in units of frontier vintage capital types. Note that I can derive the aggregate production function and determine the allocation of labor across vintages without knowing prices of capital types.[8]

## 2.3 Steady State

This section analyzes the steady state property of the model. As in other studies, the steady state analysis as an approximation provides significant implications about the existence of the vintage compatible complementary capital.

I define the steady state of interest as follows.

**Definition 1** (Steady State)**.** In a steady state, all the quantities grow at constant rates, and the real interest rate is constant.

Throughout the steady state analysis, I assume vintage-specific technological change $\hat{q}_{\underline{t}}$, and labor growth $\hat{L}$ are constant.

Solow (1960)'s vintage growth model speculates that all new investment concentrates on the capital that has the newest available vintage. In his model, this speculation is allowed because the marginal product of capital with the frontier vintage technology is always higher than any obsolete vintage capital because vintage-nonspecific labor can be freely reallocated to the frontier production technology.

This is not necessarily the case in the current model, however. The key is the existence of vintage compatible vintage-specific capital with different longevity. Sup-

[8]The aggregate production function can be expressed as, $Y = J^{\alpha+\beta} L^{1-\alpha-\beta}$, which has the same form as Solow (1960). $J$ stands for Solow's *Jelly Capital*.

pose that, initially, the allocation of long-lived and short-lived capital with a specific vintage $v$ is optimal such that the prices of two capital types are the same. Then, over time, the existing stock of long-lived capital becomes relatively abundant compared to that of short-lived capital without investment. In this case, if long-lived capital is important enough in production and lasts much longer than short-lived capital, and the rate of the vintage-specific technological progress is slow enough, then investment in the obsolete short-lived capital may become more attractive than that in new short-lived capital. The possibility of investment in obsolete vintage capital complicates the characterization of investment patterns and price distribution across vintages and capital types.

In order to systematically solve the problem, I now define four possible investment schemes of available specific vintage production, $v < t$.

**Definition 2** (Investment Scheme)**.** Define the four investment schemes such that if the production with an existing vintage technology, $v \leq t$, is:

(i) in scheme (a), there is positive continuous investment only in $A_{\underline{v}}$;

(ii) in scheme (b), there is positive continuous investment only in $B_{\underline{v}}$;

(iii) in scheme (c), there is no positive continuous investment in either $A_{\underline{v}}$ or $B_{\underline{v}}$;

(iv) in scheme (d), there is positive continuous investment in both $A_{\underline{v}}$ and $B_{\underline{v}}$.

Using Definition 1 and 2, I can characterize investment schemes across vintages in a steady state as following proposition.

**Proposition 1** (Investment Scheme)**.** *In a steady state, the investment scheme is unique across vintages, and:*

*(i) if $\hat{q} \geq \alpha(\delta^B - \delta^A)$, then the investment scheme is (c) $\forall\ v \leq t$, and firms invest only in the two capital types with the frontier technology;*

*(ii) otherwise, the investment scheme is (b) $\forall\ v \leq t$, and firms invest in short-lived capital with obsolete technologies in addition to both capital types with the frontier technology.*

*Proof:* See Appendix A.1.2.

The threshold $\alpha(\delta^B - \delta^A)$ is the product of long-lived capital's share and the difference in the rates of depreciation. The economic intuition is the following: investment

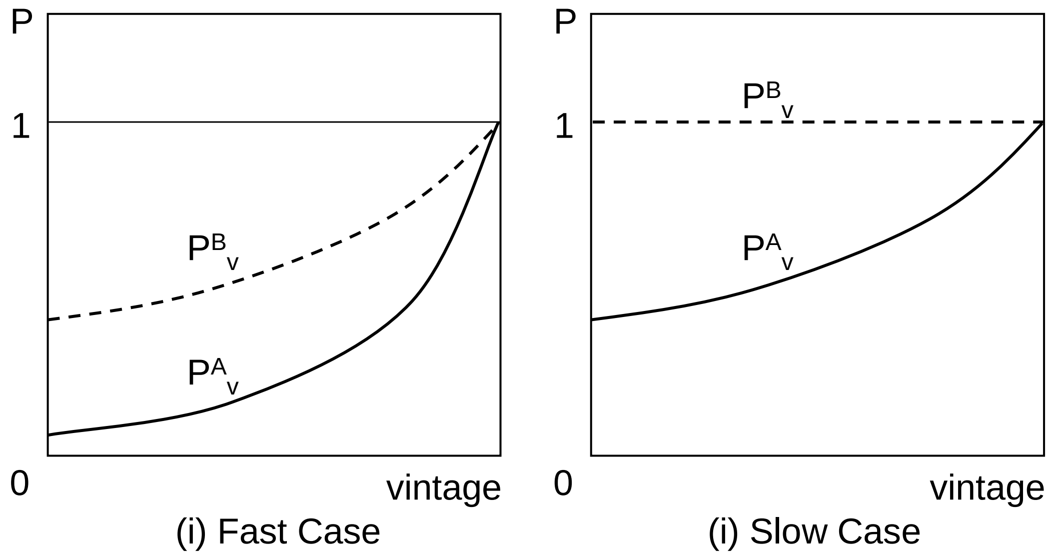


Figure 1: Prices of capital across vintages when: (i) $\hat{q} > \alpha(\delta^B - \delta^A)$, investment scheme is (c); and (ii) $\hat{q} < \alpha(\delta^B - \delta^A)$, investment scheme is (b).

in obsolete short-lived capital is more likely to occur when $\alpha$ is large where long-lived capital is relatively more important; and when $\delta^B - \delta^A$ is large where the difference in the remaining stocks of existing technology more quickly raises the marginal product of short-lived capital.

Figure 1 shows the price distributions of the two capital types in the two steady states in Proposition 1. In the (i) fast case where $\hat{q} > \alpha(\delta^B - \delta^A)$, prices of both capital types of a specific vintage fall as the vintage becomes obsolete because their marginal products do not exceed those of frontier capital types without investment. On the other hand, in the (ii) slow case where $\hat{q} < \alpha(\delta^B - \delta^A)$, the prices of short-lived capital across vintages hold because marginal products of obsolete short-lived capital without investment are higher than those of the newest capital types, and thus investment in obsolete vintage short-lived capital occurs.

Now, I characterize the allocation of the two capital types across vintages in a steady state as follows.

**Proposition 2** (Steady State)**.** *Define the aggregate effective labor,*

$$N \equiv q_{\underline{t}}^{1/(1-\alpha-\beta)} L, \tag{13}$$

*and use lower case letters to express the effective labor amounts:* $a = A/N$, *and*

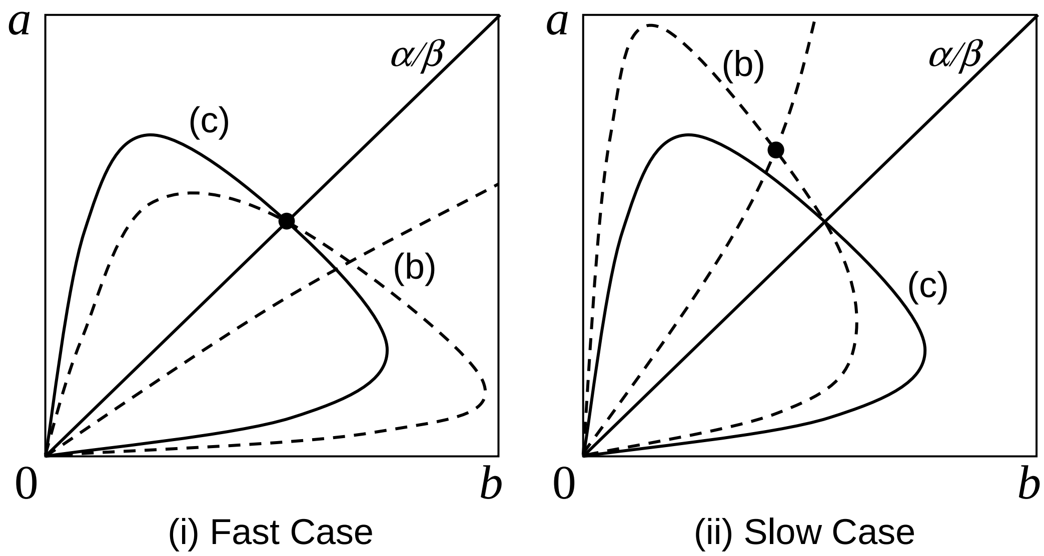


Figure 2: Relationship between $a$ and $b$ implied from (14)–upward sloping curve–and (15)–circular curve.

$b = B/N$.

*Then, in a steady state, $a$ and $b$ have a relationship from the profit maximization conditions,*

$$\beta a^{\alpha} b^{\beta-1} - \alpha a^{\alpha-1} b^{\beta} = \begin{cases} \delta^B - \left[\delta^A + \frac{\hat{q}}{\alpha}\right] & \text{(b), or} \\ 0 & \text{(c),} \end{cases} \tag{14}$$

*and a condition from the laws of motion,*

$$\sigma a^{\alpha} b^{\beta} = \begin{cases} \left[\delta^A + \frac{\hat{q}}{\alpha}\right] a + \delta^B b + \hat{N}[a+b] & \text{(b), or} \\ \left[\frac{\hat{q}+\alpha\delta^A+\beta\delta^B}{\alpha+\beta} + \hat{N}\right][a+b] & \text{(c),} \end{cases} \tag{15}$$

*depending on the investment scheme, and there are unique, constant, and stable steady state values of $a$ and $b$ that satisfies conditions (14) and (15).*

*Proof:* See Appendix A.1.3.

A steady state that started at $T_0 = -\infty$ are characterized in Appendix A.2.

Figure 2 shows possible relationships of $a$ and $b$ implied by (14) and (15) when (i) fast case where $\hat{q} > \alpha(\delta^B - \delta^A)$ and (ii) slow case where $\hat{q} < \alpha(\delta^B - \delta^A)$, respectively. In both cases, dashed lines show the relationships of $a$ and $b$ when the investment scheme is (b), and solid lines show those when the scheme is (c). The solid circles show the values of $a$ and $b$ in the two cases of steady state.

Table 1: Properties of two cases of steady state.

| Steady state | (i) Fast | (ii) Slow |
|---|---|---|
| Technological progress ($\hat{q}$) | $> \alpha(\delta^B - \delta^A)$ | $< \alpha(\delta^B - \delta^A)$ |
| Investment Scheme | (c) | (b) |
| Investment | Frontier only | Frontier and Obsolete $B$ |
| $\hat{P}^A_{\underline{v}}$ | $-[\hat{q} + \beta(\delta^B - \delta^A)]/(\alpha + \beta)$ | $-\hat{q}/\alpha$ |
| $\hat{P}^B_{\underline{v}}$ | $-[\hat{q} - \alpha(\delta^B - \delta^A)]/(\alpha + \beta)$ | 0 (Remains 1) |
| $A_{\underline{v}}/B_{\underline{v}}$ | $> \alpha/\beta$ | $> \alpha/\beta$ |
| $[P^A_{\underline{v}} A_{\underline{v}}]/[P^B_{\underline{v}} B_{\underline{v}}]$ | $\alpha/\beta$ | $> \alpha/\beta$ |
| $A/B$ | $\alpha/\beta$ | $> \alpha/\beta$ |
| Diffusion | Fast | Slow |

(14) (c) is a straight line from the origin with the slope of $\alpha/\beta$ in both cases. On the other hand, (14) (b) is a convex curve from the origin above the straight line in the slow case, while it is a concave curve below the straight line in the fast case. (15) is a circular curve that goes through the origin. The curves of (15) (b) and (c) intersect at a point where $a > 0$ and $b > 0$, and $a/b = \alpha/\beta$. The slope of (15) (b) is flatter than that of (c) at the intersection in the slow case, while it is steeper in the fast case.[9]

## 2.4 Properties of the Two Types of Steady State

Table 1 summarizes the properties of the two cases of steady state. In the fast case, the investment schemes of all the available vintages are (c); all new investment is allocated to the frontier technology capital types, $A_{\underline{t}}$ and $B_{\underline{t}}$, and the ratio of those is always the same as the ratio of capital's shares, $A_{\underline{t}}/B_{\underline{t}} = \alpha/\beta$. This case is expressed as the intersection of the solid curves labeled (c) in Figure 2 (i).

In this case, both prices of two capital types of a specific vintage decline exponentially over time. The prices of short-lived capital are higher than those of long-lived capital with the same vintages because short-lived capital of that vintage becomes relatively scarce compared to long-lived capital of that vintage over time. This is because their depreciation rates differ and there is no investment in vintage capital

[9]The disembodied heterogeneous capital model in Chapter 2 of Aruga (2006) is a special case of the model with (14) (b) and $\hat{q} = 0$. In the current model, the difference in the rates of depreciation is canceled in the scheme (c), and extra term $-\hat{q}/\alpha$ for (b) show up because of the embodiment assumption.

types.

As $\hat{q}$ goes up, the allocations of two capital types and labor skew toward the newest technology. Although there is a difference in the rates of physical depreciation, the ratio of aggregate amounts of them, $a/b$, and that of market values of their vintage, $[P^A_{\underline{v}} A_{\underline{v}}]/[P^B_{\underline{v}} B_{\underline{v}}]$, keep $\alpha/\beta$ even when $\hat{q}$ changes. The reason is that prices of vintage capital types adjust such that they cancel the difference in their rates of depreciation. Indeed, the total depreciation–the sum of obsolescence and physical depreciation–is $[\hat{q} + \alpha\delta^A + \beta\delta^B]/[\alpha + \beta]$ for both capital types in the fast case.

Laitner and Stolyarov (2003)'s model is a special case of the fast case. They assume a single rate of depreciation, $\delta^A = \delta^B$, which assures $\alpha(\delta^B - \delta^A) = 0 \leq \hat{q}$ as long as the rate of technological progress is positive. The current model shows, however, that even when rates of depreciation differ, similar results to those in their model are observed with some sets of parameters. This is because when technological change is fast enough, the economy does not care about obsolete technology, and instead focuses on the frontier technology. This results in investment only in capital types with the frontier technology, which is similar to Laitner and Stolyarov (2003).

The slow case is considerably different from the fast case and the existing models that have investment only in the frontier technology. In this case, investment is not only allocated to the frontier technology capital types, $A_{\underline{t}}$ and $B_{\underline{t}}$, but also to existing short-lived capital with obsolete vintages, $B_{\underline{v}}\, \forall\, v < t$. The ratio of investment in the frontier capital types, $A_{\underline{t}}/B_{\underline{t}}$, is identical to the aggregate amounts, $A/B$. This steady state is expressed as the intersection of the dashed curves labeled (b) in Figure 2 (ii). The ratio $a/b$ is larger than $\alpha/\beta$ as in the figure.

Prices of short-lived capital of all vintages are the same since the marginal product of obsolete short-lived capital exceeds that of new capital types without investment. This is because a large stock of long-lived capital raises the marginal product of short-lived capital. This attracts investment in obsolete short-lived capital, while prices of long-lived capital decline with vintage.

Unlike the fast case, when $\hat{q}$ declines, the ratio $A/B$ rises, because a decline in $\hat{q}$ lowers the interest rate $r$. This makes long-lived capital more attractive since long-lived capital will last relatively longer. The result does not occur in the fast case since the rates of obsolescence of capital types adjust such that the sum of the rates of depreciation and of obsolescence is the same across the different capital types.

Although the allocations of those inputs skew toward newer technology as $\hat{q}$ rises, unlike in the fast case, the motion of vintage short-lived capital is affected by investment in vintage short-lived capital as well as by physical depreciation. The ratio of investment in vintage short-lived capital to the existing short-lived capital, $I^B_{\underline{v}}/B_{\underline{v}}$, rises as $\hat{q}$ falls, because a smaller $\hat{q}$ makes investment in vintage short-lived capital more attractive.

The investment in obsolete technology geometrically discontinuously lowers the speed of diffusion of technology. The ratios of the aggregate production across vintages with technology from time $T$ to the whole production–which I call degrees of diffusion– are,

$$1 - e^{-(\frac{\hat{q}+\alpha\delta^A+\beta\delta^B}{\alpha+\beta}+\hat{N})(t-T)},$$

for the fast case; and,

$$1 - e^{-(\delta^A+\frac{\hat{q}}{\alpha}+\hat{N})(t-T)},$$

for the slow case. Note that the initial slope of the curve increases as $\hat{q}$ increases at the constant rate of $1/\alpha$ in the slow case, but at the constant rate of $1/(\alpha+\beta)$ in the fast case. The faster rate of increase in the slow case is because of the reallocation of investment from obsolete short-lived capital towards the frontier capital types.

# 3 Discussion

In the last section, the steady state analysis of the model reveals two distinct investment patterns: (i) if the rate of progress, $\hat{q}$, is above a threshold–which is the product of long-lived capital's share and the difference in the rates of depreciation–then all new investment concentrates on the frontier technology; (ii) otherwise, some investment is allocated to obsolete, short-lived capital to exploit the existing excessive stock of long-lived capital with obsolete technologies.

An implication of this result is that if the rate of progress is below the threshold, then, surprisingly, the prices of obsolete short-lived capital remain stable over time even when the rate of progress is positive. In this case, estimates of the rate of progress derived from changes in equipment prices would be systematically biased downward.

There are two important interpretations of the combinations of vintage compatible capital: (i) components of a system and the system itself that is aggregated by a certain type of intangible capital; and (ii) physical and another certain type of

Table 2: Two types of intangible capital.

| Type | Intangible capital |
| --- | --- |
| Integrator of system | Organization, Network, Process, Scheme, Product design |
| Direct complement | Software, Human capital, R&D, Brand, Manual, Product design |

intangible capital. This section discusses roles of the two types of intangible capital for/as long-lived capital, and why consideration of the combinations are important, and assesses empirical relevance of the model in relation to the combinations.

## 3.1 Intangible Capital and Two Combinations of Complementary Capital

Although intangible capital has long received little attention in the official statistics of economic analysis, recent literature has raised the importance of intangible capital in production (Hall (2001), Atkeson and Kehoe (2005), and McGrattan and Prescott (2005)). While various types of intangible capital are suggested in the literature,[10] the Bureau Economic Analysis has recently started including software (1999) in investment in the official statistics and releasing R&D satellite accounts (2006) with the National Science Foundation "to better account for intangible assets." Corrado et al. (2006) shows that intangible capital's income accounts for 15% of total income in the nonfarm business sector during the period 2000–2003, while that of physical capital accounts for 25%. Their growth accounting analysis shows that the growth rates of both output and labor productivity with intangible capital are higher than those without intangible capital. The importance of intangible capital rivals that of physical capital in the modern economy.

Intangible capital considered in literature can be classified to two types depending on its roles in view of the complementary capital. Intangible capital may be an

[10]Although a complete list is still under discussion, an incomplete list includes: software (Corrado et al. (2006)), R&D (Prucha and Nadiri (1996) and Corrado et al. (2006)), brand (McGrattan and Prescott (2005), Corrado et al. (2006)), organization (Atkeson and Kehoe (2005), McGrattan and Prescott (2005), Corrado et al. (2006)), monopoly franchise (Hall (2001), McGrattan and Prescott (2005)), firm-specific human capital (Laitner and Stolyarov (2003)), and product designs (Laitner and Stolyarov (2003)).

integrator of a system, which is the complement of components of the system; or it may be the direct complement of physical capital. Table 2 summarizes a possible classification of these types of intangible capital. The first type of intangible capital works as an integrator of components into a system. Organization, network, process, scheme, and product design integrates components into a system of production. The second type is intangible capital that directly works with physical capital. Software, human capital, R&D, brand, manual, and product design consist important part of inputs of production together with physical capital.[11]

As example illustrated by the combustion engines in Introduction, larger system typically is longer-lived than its components. On the other hand, the rates of depreciation of intangible capital proposed in the literature greatly vary depending on its types. Corrado et al. (2006) employs 20% and 60% as the rates of depreciation of R&D capital and brand equity respectively. Since the rates of depreciation proposed in literature include both obsolescence and physical depreciation, the physical depreciation of intangible capital–which is the focus of my model–will be substantially smaller than these figures. For instance, it is hard to believe that computer software–with an assumed depreciation rate of 33% in Corrado et al. (2006)–on a hard disk and on an installation CD, or product designs on paper physically depreciates more quickly than does physical capital. Therefore, there will be both shorter-lived intangible capital (e.g., brands) and longer-lived intangible capital (e.g., computer software) compared with associated types of physical capital.

The roles of system capital/intangible capital as complement to component/physical capital are illustrated by the following example. Consider production technology that uses a ship. A ship consists of various kinds of components. In this case, purchase of components corresponds to the investment in components/physical capital, and assembly of the ship from the components corresponds to the investment in system capital. In addition, manuals of a specific type of ship may be complementary intangible capital. You can produce transportation/recreational service as output using labor, the ship that is a set of components and a system, and the manuals. You then reinvest a part of the revenue from output in the components or system of the

[11]Note that product design can be either integrator of system or direct complement capital depending on the roles of it in production. On the one hand, when producing equipment, its production requires design of this equipment, which can be considered as direct complementary input. On the other hand, when producing services using the equipment, this production consists of components of equipment and equipment as a system integrated by the product design.

ship. Components depreciates more quickly than the ship's system or manual.[12] Now suppose one of the four engines of a ship breaks down. If the change in ship model develops quickly enough, you would refrain from replacing the engine and keep using the ship as it allows your savings to be used for purchasing a new ship instead, because it has much better features than the obsolete one. Otherwise, you would replace the malfunctioned engine with a new one but designed for the obsolete model in order to keep using the existing system/manuals of the obsolete ship.[13]

## 3.2 Empirical Relevance

I discuss five types of empirical relevance of the model: (i) investment in obsolete technologies; (ii) maintenance and repair; (iii) investment boom and recession; (iv) measurement of technological progress; and (v) the size of the threshold.

The first type of relevance is observed for investment in obsolete technologies. Figure 5 in Felli and Ortalo-Magne (1998) shows that there was continued investment in obsolete steam locomotives after the introduction of newer-type, diesel locomotives. In this case, short-lived capital is locomotives; and long-lived capital may be a specific fuel supply network (system) and existing mechanics' know-how about specific types of locomotives (intangible). The model interprets that in order to utilize the existing coal fuel supply network or human capital, there has been investment in steam locomotives that are less productive than diesel locomotives.[14]

Second, maintenance and repair as a substitute for investment in obsolete short-

[12] Table 3 in Fraumeni (1997) shows that the rates of depreciation of ships/boats and internal combustion engines are 6.1% and 21%, respectively, which assures the validity of this thought experiment. Manuals do not deteriorate as engines.

[13] The roles of intangible and physical capital may reverse depending on the production technology. For example, consider the *Coca-Cola Company* which produces and sells Coca-Cola using its factories (physical capital) and brand name (intangible capital). Suppose the depreciation rate of its brand name is 60% as suggested above, and far exceeds that of their factories, and the rate of development of beverages is slow. Then, advertisements for Coca-Cola can be interpreted as an investment in obsolete short-lived intangible capital to keep using the obsolete existing stock of long-lived factories.

[14] Other example are found in production with cotton spinning (Saxonhouse and Wright (2000)), and with steel furnaces (Nakamura and Ohashi (2007)). Data in Nakamura and Ohashi (2007) show that the declining rate of the capacity size of open-hearth furnaces (OHFs) in Japan for 10 years after the introduction of more productive basic oxygen furnaces (BOFs) and for 5 years after the peak usage of OHFs were about 5% and 9% respectively, which are both much smaller than the rates of depreciation of metalworking machines in the U.S. official statistics, of 12%. This implies there had been investment in obsolete OHF technology after the new BOF technology became available. Potential list of the complementary intangible capital would include human capital and system of production process that involves furnaces.

Table 3: Service life, warranty period, and change in prices of two types of equipment.

| Equipment type | PC | Appliance |
|---|---|---|
| Service life[a] | 7 years | 10 years |
| Warranty period[b] | 1 year | up to lifetime |
| Change in price[c] | 0.0467 | 3.014 |

[a] From Table 3 in Fraumeni (1997).

[b] Figure on PC is from Toshiba's notebooks, and that of appliance is from Kitchen Aid's refrigerators, dishwashers, and washers.

[c] Ratios of prices of 1983 to 1947 in Gordon (1990).

lived capital–which is suggested by McGrattan and Schmitz (1999) and Mullen and Williams (2004)–can be explained by the model. Table 3 shows service life, warranty period, and change in prices of PCs and appliances. Appliances have a longer warranty period and service life than PCs. These figures are counter-intuitive if physical depreciation from wear and tear is the dominant factor of the service life, since appliances have more moving parts than a PC. A potential explanation is that there is replacement of parts for appliances, but not for PCs because vintage-specific technological progress of PCs is faster than that of appliances as the model suggests. This is supported by the price-changes data in Table 3.

Third, the model implies that acceleration in the rate of vintage-specific technological progress can cause an abrupt reallocation of investment towards modern capital–consistent with investment booms that are concentrated in certain "high-tech" equipment. In this case, organizational capital of firms can be interpreted as complementary long-lived capital. There is a widely accepted observation that the economic boom in the late 1990s coincided with the diffusion of IT.[15] While typical growth models consider investment in IT equipment as a source of improvement in productivity, the current model provides a different viewpoint: the concentration of investment in IT equipment is a result of a higher rate of vintage-specific technological change.

Forth, the model implies that estimated vintage-specific technological progress using changes in equipment prices over time may be underestimated. Greenwood et al.

[15] For example, see Oliner and Sichel (2003) and Jorgenson et al. (2007).

(1997) use change in Gordon (1990)'s equipment prices to measure the rate of progress. If the economy involves short-lived equipment and long-lived system/intangible capital, and has experienced the slow case, estimates of the progress result in downward bias, because during the slow case there is no change in the price of equipment even when the progress is positive.

Finally, is it possible that an economy experiences the slow case? Suppose that the share of intangible capital is 15% as suggested by Corrado et al. (2006); and the difference in the rates of depreciation of physical and intangible capital is 10%. Then, an ad hoc threshold will be $\alpha(\delta^B - \delta^A) = 0.015$, which is about the same order of the growth rate of labor productivity in the postwar U.S. economy. Although the rate of vintage-specific technological progress is typically smaller than that of labor productivity, it is possible that the economy fluctuates around the threshold and the cases would differ at times. This implies that: an abrupt reallocation of investment is possible; and values in Greenwood et al. (1997) may be biased downward.

# 4 Conclusion

The existence of heterogeneous complimentary capital yields two distinctive investment patterns: (i) if the rate of technological progress is above a threshold– the product of long-lived capital's share and the difference in the rates of depreciation–then all new investment concentrates on the capital types that embody frontier technology; (ii) otherwise, a part of the investment is allocated to obsolete short-lived capital to exploit existing obsolete long-lived capital.

The result provides a new explanation for the observed investment in physical capital with obsolete technologies. An important implication is that change in prices of physical capital does not necessarily reflect the rate of technological progress. Another implication is that an acceleration in the rate of technological progress can cause an abrupt reallocation of investment towards modern capital, consistent with investment booms that are concentrated in certain "high-tech" equipment.

As a consequence of the conventional assumptions of the model, the results can be applied to broader types of economic activities that involves vintage compatible heterogeneous capital. The result of the model fit well with the evidence from the real economy. Introduction of simple capital heterogeneity in a conventional vintage growth model provides promising implications for the investment patterns, suggesting that economists pay closer attention to capital heterogeneity.

Avenues for future research include both theoretical and empirical. Theoretically, important applications include characterizing transition and generalizing production function. Transition of the model expands the applicability of the model to broader exercise of the real economy. Generalization of the production function improves the promises of the model.

Empirically testable implications include: investment/repair patterns across different types of production process/machines that have different rates of technological progress; and measuring the true rate of vintage-specific technological change. The analysis of investment/repair patterns would confirm the degree of the importance of the capital heterogeneity in production. The measurement of the true rate of technological progress is important to correctly understand its accurate contribution to the economic growth. Quantifying the relative importance of system capital and intangible capital will be another interest.

These applications would reveal the significance of the result of this study–the optimal allocation of investment across vintages depends on the trade-off between the remaining stocks of the complementary obsolete capital and the advancement of the frontier technology.

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# A Appendix

## A.1 Proofs

### A.1.1 Lemma 1 (Vintage Aggregation)

(i) Agent's profit maximization conditions are:

$$MPA_{\underline{v}} = \alpha \frac{Y_v}{A_{\underline{v}}} = P^A_{\underline{v}} R^A_{\underline{v}} \quad = \quad P^A_{\underline{v}} \left[ r + \delta^A - \hat{P}^A_{\underline{v}} \right], \tag{16}$$

$$MPB_{\underline{v}} = \beta \frac{Y_v}{B_{\underline{v}}} = P^B_{\underline{v}} R^B_{\underline{v}} \quad = \quad P^B_{\underline{v}} \left[ r + \delta^B - \hat{P^B_{\underline{v}}} \right], \tag{17}$$

$$MPL = (1 - \alpha - \beta) \frac{Y_v}{L_v} \quad = \quad W, \tag{18}$$

where the $R^X_v$ is the rate of return of $X_v$, $r$ is the real interest rate, hat ($\hat{}$) denotes the time derivative of the natural log of argument, and prices are normalized at each unit of output. Note that the net of the rate of return, depreciation, and obsolescence is identical to real interest rate, and that marginal product of labor and wage ($MPL$ and $W$) do not have vintage subscript because labor is not vintage-specific.

Using (1) and (16)-(18), relationship of marginal products of capital types of vintages $v$ and $v'$ is characterized by,

$$\left[ \frac{MPA_{\underline{v}}}{MPA_{\underline{v'}}} \right]^{\alpha} \left[ \frac{MPB_{\underline{v}}}{MPB_{\underline{v'}}} \right]^{\beta} = \frac{q_{\underline{v}}}{q_{\underline{v'}}}. \tag{19}$$

Now, using (1), (3) - (6), and (16) - (19), the aggregate output will be,

$$\begin{aligned} Y &= \int_{T_0}^{t} Y_v \, dv \\ &= \int_{T_0}^{t} q_{\underline{v}} A^{\alpha}_{\underline{v}} B^{\beta}_{\underline{v}} L^{1-\alpha-\beta}_v \, dv \\ &= \int_{T_0}^{t} q_{\underline{v}} \left[ \frac{A_{\underline{t}}}{L_t} \frac{MPA_{\underline{t}}}{MPA_{\underline{v}}} L_v \right]^{\alpha} \cdot \\ &\quad \left[ \frac{B_{\underline{t}}}{L_t} \frac{MPB_{\underline{t}}}{MPB_{\underline{v}}} L_v \right]^{\beta} L^{1-\alpha-\beta}_v \, dv \\ &= \left[ \frac{A_{\underline{t}}}{L_t} \right]^{\alpha} \left[ \frac{B_{\underline{t}}}{L_t} \right]^{\beta} \int_{T_0}^{t} q_{\underline{v}} \frac{q_{\underline{t}}}{q_{\underline{v}}} L_v \, dv \\ &= q_{\underline{t}} A^{\alpha} B^{\beta} L^{1-\alpha-\beta}, \end{aligned}$$

which is equation (7).

(ii) Use (4), (8), (9), and (16) -(18) to show

$$A \quad = \quad \int_{T_0}^{t} \left[ \frac{q_{\underline{v}}}{q_{\underline{t}}} \right]^{\frac{1}{\alpha+\beta}} \left[ \frac{B_{\underline{v}}/A_{\underline{v}}}{B_{\underline{t}}/A_{\underline{t}}} \right]^{\frac{\beta}{\alpha+\beta}} A_{\underline{v}} \, dv$$

$$
\begin{aligned}
&= q_{\underline{t}}^{-\frac{1}{\alpha+\beta}} \left[\frac{B_{\underline{t}}}{A_{\underline{t}}}\right]^{-\frac{\beta}{\alpha+\beta}} \int_{T_0}^{t} \left[q_{\underline{v}} A_{\underline{v}}^{\alpha} B_{\underline{v}}^{\beta}\right]^{\frac{1}{\alpha+\beta}} \, dv \\
&= q_{\underline{t}}^{-\frac{1}{\alpha+\beta}} \left[\frac{B}{A}\right]^{-\frac{\beta}{\alpha+\beta}} \int_{T_0}^{t} J_{\underline{v}} \, dv \\
&= q_{\underline{t}}^{-\frac{1}{\alpha+\beta}} \left[\frac{B}{A}\right]^{-\frac{\beta}{\alpha+\beta}} J,
\end{aligned}
$$

which is the equitation (10). From (1), (3), (7) - (9), and (18), we have

$$
\left[\frac{J_{\underline{v}}}{L_v}\right]^{\alpha+\beta} = \frac{Y_v}{L_v} = \frac{Y}{L} = \left[\frac{J}{L}\right]^{\alpha+\beta},
$$

which provides (11) and (12).

■

### A.1.2 Proposition 1 (Investment Scheme)

The proof has four steps. It shows that: (1) constant price-changes of two types of capital; (2) uniqueness of price-changes in the same investment scheme; (3) distribution of schemes; and (4) investment scheme.

The scheme (d) is not allowed across vintages, because if different vintages $v$ and $v'$ are in scheme (d), both capital types' prices must be one and therefore $MPA_{\underline{v}} = MPA_{\underline{v'}}$ and $MPB_{\underline{v}} = MPB_{\underline{v'}}$, which breaks the condition (19).

**(1) Constant Price-Changes:** In a steady state, (16) and (17) imply that both $MPA_{\underline{v}}$ and $MPB_{\underline{v}}$ grow at constant rates since $Y_v$, $A_{\underline{v}}$, and $B_{\underline{v}}$ all grow at constant rates. Now, suppose $\hat{P}_{\underline{v}}^{A} > M\hat{P}A_{\underline{v}}$. Then, $MPA_{\underline{v}}/P_{\underline{v}}^{A} = r^* + \delta^A - \hat{P}_{\underline{v}}^{A}$ declines and therefore the growth rate of $P_{\underline{v}}^{A}$ accelerates over time. Then, $P_{\underline{v}}^{A}$ reaches one with positive growth rate in a finite time, which breaks the condition, $P_{\underline{v}}^{A} \in [0, 1]$.

Next, suppose $\hat{P}_{\underline{v}}^{A} < M\hat{P}A_{\underline{v}}$. Then, $P_{\underline{v}}^{A}$ reaches zero in a finite time with negative growth rate and either it breaks the condition, $P_{\underline{v}}^{A} \in [0, 1]$, or firms get rid of the capital, which breaks the constant growth.

Therefore, $\hat{P}_{\underline{v}}^{A} = M\hat{P}A_{\underline{v}}$ and thus $P_{\underline{v}}^{A}$ must grow at a constant rate. Then, $\hat{P}_{\underline{v}}^{A} \leq 0$ since otherwise $P_{\underline{v}}^{A}$ exceeds one in a finite time, which breaks $P_{\underline{v}}^{A} \in [0, 1]$. Similar arguments apply to the prices of physical capital, $P_{\underline{v}}^{B}$.

Thus, both $P^A_{\underline{v}}$ and $P^B_{\underline{v}}$ must grow at constant rates, and

$$M\hat{P}A_{\underline{v}} = \hat{P^A_{\underline{v}}} \leq 0, \tag{20}$$
$$M\hat{P}B_{\underline{v}} = \hat{P^B_{\underline{v}}} \leq 0. \tag{21}$$

**(2) Uniqueness of Price-Changes in the Same Investment Scheme:** Consider the vintage $v$ and $v'$ that are in scheme (c). Since there is no continuous positive investment, $\hat{J}_{\underline{v}} = \hat{J}_{\underline{v'}}$, which implies $\hat{Y}_v = \hat{Y}_{v'}$ from (12). Therefore, (16), (17), (20), and (21) imply $\hat{P^A_{\underline{v}}} = M\hat{P}A_{\underline{v}} = M\hat{P}A_{\underline{v'}} = \hat{P^A_{\underline{v'}}}$. Argument for $P^B_{\underline{v}}$ and $P^B_{\underline{v'}}$ is the same.

For the vintage $v$ and $v'$ that are in scheme (a), $\hat{P^A_{\underline{v}}} = \hat{P^A_{\underline{v'}}} = 0$, and thus (19) and (20) implies $MPB_{\underline{v}}/MPB_{\underline{v'}} = P^B_{\underline{v}}/P^B_{\underline{v'}}$ is constant. The similar argument applies to the scheme (b).

Thus, if vintages $v$ and $v'$ are in a same investment scheme, then

$$\hat{P^A_{\underline{v}}} = \hat{P^A_{\underline{v'}}}, \tag{22}$$
$$\hat{P^B_{\underline{v}}} = \hat{P^B_{\underline{v'}}}. \tag{23}$$

**(3) Distribution of Scheme:** Using (9), (12), (16), (17), (22), and (23), observe that the relationships of prices across vintages of capital types in a same investment scheme are,

$$P^A_{\underline{v}} = \left[\frac{q_{\underline{v}}}{q_{\underline{v'}}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_{\underline{v}}/A_{\underline{v}}}{B_{\underline{v'}}/A_{\underline{v'}}}\right]^{\frac{\beta}{\alpha+\beta}} P^A_{\underline{v'}}; \tag{24}$$
$$P^B_{\underline{v}} = \left[\frac{q_{\underline{v}}}{q_{\underline{v'}}}\right]^{\frac{1}{\alpha+\beta}} \left[\frac{B_{\underline{v}}/A_{\underline{v}}}{B_{\underline{v'}}/A_{\underline{v'}}}\right]^{-\frac{\alpha}{\alpha+\beta}} P^B_{\underline{v'}}. \tag{25}$$

Now, suppose vintage $v$ is scheme (b) or (c) and vintage $v'$ is scheme (a). Then, since $[B_{\underline{v}}\hat{/}A_{\underline{v}}] > [B_{\underline{v'}}\hat{/}A_{\underline{v'}}]$, (24) implies,

$$\hat{P^A_{\underline{v}}} > \hat{P^A_{\underline{v'}}} = 0,$$

which cannot be held in a steady state because price of capital $A_{\underline{v}}$ exceeds one in a finite time. Similar argument applies to the remaining combinations of schemes (c) and (b) with (25).

Therefore, investment scheme is unique across vintages, and is either (a), (b), or (c).

**(4) Investment Scheme:**

(i) Suppose investment scheme is (a) $\forall\, v$. Then, $P^A_{\underline{v}} = 1$ and $\hat{P}^A_{\underline{v}} = 0$, while $\hat{P}^B_{\underline{v}} = -\hat{q}/\beta$ from (19). Then, $\widehat{\left[\frac{B_{\underline{v}}}{A_{\underline{v}}}\right]} = \widehat{\left[\frac{MPA_{\underline{v}}}{MPB_{\underline{v}}}\right]} = \hat{P}^A_{\underline{v}} - \hat{P}^B_{\underline{v}} = \hat{q}/\beta$. This requires disinvestment in $A_{\underline{v}}$ since $-[\delta^B - \delta^A] < \hat{q}/\beta$, which is not allowed by assumption.

Next, suppose investment scheme is (b) $\forall\, v$. In this case, $\widehat{\left[\frac{B_{\underline{v}}}{A_{\underline{v}}}\right]} = -\hat{q}/\alpha$ from (19). Again this requires disinvestment in $B_{\underline{v}}$ since $-[\delta^B - \delta^A] > -\hat{q}/\alpha$ is the condition, which is not allowed by assumption.

Therefore, in a steady state with $\hat{q} > \alpha(\delta^B - \delta^A)$, investment scheme must be (c) $\forall\, v$.

(ii) Investment scheme (a) is impossible as in (i). Now, suppose investment scheme is (c) $\forall\, v$. There is no investment in vintage capital and thus $\hat{Y} = \hat{I^A} = \hat{I^B} = \hat{A}_{\underline{t}} = \hat{B}_{\underline{t}}$ from (2). But this is impossible because (25) implies that $P^B_{\underline{v}}$ exceeds one when $\hat{q} < \alpha(\delta^B - \delta^A)$. Therefore, investment scheme must be (b) when $\hat{q} < \alpha(\delta^B - \delta^A)$.

■

### A.1.3 Proposition 2 (Steady State)

**Allocation Across Capital Types:** By canceling $r$ from (16) and (17),

$$\left[\frac{\beta}{P^B_{\underline{v}} B_{\underline{v}}} - \frac{\alpha}{P^A_{\underline{v}} A_{\underline{v}}}\right] Y_v = [\delta^B - \hat{P}^B_{\underline{v}}] - [\delta^A - \hat{P}^A_{\underline{v}}]. \tag{26}$$

Since $Y/L = Y_t/L_t$, $A/L = A_{\underline{t}}/L_t$, and $B/L = B_{\underline{t}}/L_t$ from (1), (4), (5), and (7), by applying $v \to t$ and with per effective capita amounts, rewrite (26) as,

$$\frac{\beta a^\alpha b^{\beta-1}}{P^B_{\underline{t}}} - \frac{\alpha a^{\alpha-1} b^\beta}{P^A_{\underline{t}}} = [\delta^B - \hat{P}^B_{\underline{v}}] - [\delta^A - \hat{P}^A_{\underline{v}}].$$

(20) and (21) imply $P^A_{\underline{t}} = P^B_{\underline{t}} = 1$ since otherwise there is no investment. Then,

when investment scheme is in (b), from (19),

$$\hat{P^A_{\underline{v}}} = -\frac{\hat{q}}{\alpha}, \text{ and } \hat{P^B_{\underline{v}}} = 0, \tag{27}$$

and when investment scheme is in (c), from (24) and (25),

$$\hat{P^A_{\underline{v}}} = -\frac{\hat{q} + \beta(\delta^B - \delta^A)}{\alpha + \beta}, \text{ and } \hat{P^B_{\underline{v}}} = -\frac{\hat{q} - \alpha(\delta^B - \delta^A)}{\alpha + \beta}, \tag{28}$$

respectively. Then, use these result to observe the relationship (14).

**Aggregate Laws of Motion:** The laws of motion of the capital types of each vintage are,

$$\begin{aligned} \dot{A}_{\underline{v}} &= I^A_{\underline{v}} - \delta^A A_{\underline{v}}, \\ \dot{B}_{\underline{v}} &= I^B_{\underline{v}} - \delta^B B_{\underline{v}}. \end{aligned}$$

In a steady state, $P^A_{\underline{t}} = P^B_{\underline{t}} = 1$ from Proposition 1. Then, from (22), (23), rewrite (4) and (5) as,

$$A = \int_{T_0}^{t} P^A_{\underline{v}} A_{\underline{v}} \, dv, \tag{29}$$

$$B = \int_{T_0}^{t} P^B_{\underline{v}} B_{\underline{v}} \, dv. \tag{30}$$

By differentiating (29) and (30), obtain the laws of motion of aggregate capital:

$$\begin{aligned} \dot{A} &= \frac{\partial}{\partial t} \int_{T_0}^{t} P^A_{\underline{v}} A_{\underline{v}} \, dv \qquad (31) \\ &= \int_{T_0}^{t} [P^A_{\underline{v}} A_{\underline{v}}][\hat{P^A_{\underline{v}}} + \hat{A}_{\underline{v}}] \, dv + A_{\underline{t}} \\ &= \left[\hat{P^A_{\underline{v}}} - \delta^A\right] A + \int_{T_0}^{t} I^A_{\underline{v}} \, dv + I^A_{\underline{t}} \\ &= \left[\hat{P^A_{\underline{v}}} - \delta^A\right] A + I^A; \end{aligned}$$

and,

$$\dot{B} = \left[\hat{P^B_{\underline{v}}} - \delta^B\right] B + I^B. \tag{32}$$

Since $A$ grows at a constant rate, (31) implies $\hat{I^A} = \hat{A}$. Similarly, $\hat{I^B} = \hat{B}$. Then, from (2), $\hat{Y} = \hat{I^A} = \hat{I^B}$, and thus from (7),

$$\hat{A} = \hat{B} = \hat{Y} = \frac{\hat{q}}{1 - \alpha - \beta} + \hat{L} = \hat{N}.$$

Therefore, $a$ and $b$ are constant in a steady state.

In per effective labor expressions, the sum of the laws of motion, (31) and (32), becomes

$$\dot{a} + \dot{b} = \sigma a^\alpha b^\beta - [\delta^A - \hat{P^A_{\underline{v}}} + \hat{N}]a - [\delta^B - \hat{P^B_{\underline{v}}} + \hat{N}]b. \tag{33}$$

Then, using (27), (28), and steady state property $\dot{a} = \dot{b} = 0$, obtain the result of (15).

**Uniqueness and Stability:** The relationship (14) can be written as

$$a = f(b). \tag{34}$$

Since (34) implies $\dot{a} = f'(b)\dot{b}$, (33) can be rewritten as

$$\dot{b} = \frac{\sigma f(b)^\alpha b^\beta - [\delta^A - \hat{P^A_{\underline{v}}} + \hat{N}]f(b) - [\delta^B - \hat{P^B_{\underline{v}}} + \hat{N}]b}{f'(b) + 1}. \tag{35}$$

Clearly, $\dot{b}(t) = 0$ when $b = 0$. Then, observe that the numerator of the right hand side of (35) can be rewritten as $\frac{b^2}{a\beta - b\alpha}[\{\sigma(\delta^B - \hat{P^B_{\underline{v}}} + \hat{P^A_{\underline{v}}} - \delta^A) + (\delta^A - \hat{P^A_{\underline{v}}} + \hat{N})\alpha - (\delta^B - \hat{P^B_{\underline{v}}} + \hat{N})\beta\}\frac{a}{b} - (\delta^A - \hat{P^A_{\underline{v}}} + \hat{N})(\frac{a}{b})^2\beta + (\delta^B - \hat{P^B_{\underline{v}}} + \hat{N})\alpha]$. The inside of the square brackets is positive when $\left[\frac{a}{b}\right]_{b \to +0} = \frac{\alpha}{\beta}$ and there is a unique $\frac{a}{b} > \frac{\alpha}{\beta}$ such that $b > 0$ and the inside of the brackets is zero. Thus, there exists a unique $b^* > 0$ where $\dot{b} = 0$.

Note that at $b^*$, (14) implies $a^* > 0$ and $\dot{a} = 0$. Observe that the first series of the Taylor approximation of the summarized law of motion of capital (35) is $\dot{b} \approx \frac{(\alpha+\beta-1)\{\beta(\delta^A - \hat{P^A_{\underline{v}}} + \hat{N})(a^*/b^*) + \alpha(\delta^B - \hat{P^B_{\underline{v}}} + \hat{N})(b^*/a^*)\}}{2\alpha\beta + \beta(1-\beta)(a^*/b^*) + \alpha(1-\alpha)(b^*/a^*)}(b - b^*)$, where the coefficient is negative. Therefore, at $a^*, b^*$, the economy is stable.

■

## A.2 Steady State If Economy Started at $T_0 = -\infty$

**Steady State Values:** Given the current $L$ and $q_{\underline{t}}$,

(i) [Fast Case] if $\alpha(\delta^B - \delta^A) < \hat{q}$, the economy is characterized as follows:

1. Aggregate Capital:

$$
\begin{aligned}
A &= a^* N; \\
B &= b^* N.
\end{aligned}
$$

2. Distribution of labor:

$$
L_v = \left[\frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} + \hat{N}\right] e^{-(\frac{\hat{q}+\alpha\delta^A+\beta\delta^B}{\alpha+\beta} + \hat{N})(t-v)} L.
$$

3. Distribution of capital:

$$
\begin{aligned}
A_{\underline{v}} &= \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} e^{-(\delta^A+\hat{N})(t-v)} A; \\
B_{\underline{v}} &= \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} e^{-(\delta^B+\hat{N})(t-v)} B.
\end{aligned}
$$

4. Allocation of investment:

$$
\begin{aligned}
I^A &= I^A_{\underline{t}} = \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} A; \\
I^B &= I^B_{\underline{t}} = \frac{\hat{q} + \alpha\delta^A + \beta\delta^B}{\alpha+\beta} B.
\end{aligned}
$$

where $\hat{N} = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L}$.

(ii) [Slow Case] otherwise, the economy is characterized as follows:

1. Aggregate Capital:

$$
\begin{aligned}
A &= a^* N; \\
B &= b^* N.
\end{aligned}
$$

2. Distribution of labor:

$$L_v = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A + \frac{\hat{q}}{\alpha} + \hat{N})(t-v)} L.$$

3. Distribution of capital:

$$\begin{aligned} A_{\underline{v}} &= \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A + \hat{N})(t-v)} A; \\ B_{\underline{v}} &= \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A + \frac{\hat{q}}{\alpha} + \hat{N})(t-v)} B. \end{aligned}$$

4. Allocation of investment:

$$\begin{aligned} I^A &= I^A_{\underline{t}} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] A; \\ I^B_{\underline{t}} &= \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] B; \\ I^B_{\underline{v}} &= \left[\delta^B - \delta^A - \frac{\hat{q}}{\alpha}\right] B_{\underline{v}}; \\ I^B &= I^B_{\underline{t}} + \int_{T_0}^{t} I^B_{\underline{v}}\, dv = \left[\delta^B + \hat{N}\right] B, \end{aligned}$$

where $\hat{N} = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L}$.

**Derivation of Full Characterization:** I focus on the proof of (ii) [Slow Case] because the proof of (i) [Fast Case] is an easier case of that of the former one.

First, aggregate capital can be specified by

$$\begin{aligned} \frac{A}{N} &= a^*, \\ \frac{B}{N} &= b^*, \end{aligned}$$

which are from the per effective labor definition.

Then, consider the investment allocation between aggregate long-lived and short-lived capital. Since at the steady state $\dot{a} = \dot{b} = 0$, from (31), (32) and the proof of

Lemma 1, I have

$$\frac{I^A}{N} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] a^*, \tag{36}$$

$$\frac{I^B}{N} = \left[\delta^B + \hat{N}\right] b^*, \tag{37}$$

where $\hat{N} = \frac{\hat{q}}{1-\alpha-\beta} + n$.

(36) and (37) also imply the investment allocation between them is constant,

$$\frac{I^A}{I^A + I^B} = \frac{[\delta^A + g/\alpha + \hat{N}](a^*/b^*)}{[\delta^A + g/\alpha + \hat{N}](a^*/b^*) + [\delta^B + \hat{N}]} = \theta.$$

Since the investment in long-lived capital is only for the frontier vintage,

$$A_{\underline{v}} = A_{\underline{v}}(v)e^{-\delta^A(t-v)} = \theta\sigma Y(v)e^{-\delta^A(t-v)}. \tag{38}$$

Since $\hat{Y} = \hat{N} = \hat{N}$,

$$Y(v) = Ye^{-\hat{N}(t-v)}.$$

Thus, (38) can be written as

$$A_{\underline{v}} = \theta\sigma Y e^{-(\delta^A+\hat{N})(t-v)}. \tag{39}$$

Now, find the optimal allocation of labor, $L_v$. From (16) - (18), the proof of Lemma 1, and (39) with per effective labor notation,

$$L_v = P^A_{\underline{v}} \frac{A_{\underline{v}}}{A_{\underline{t}}} L_t = e^{-(\delta^A + \frac{\hat{q}}{\alpha} + \hat{N})(t-v)} L_t. \tag{40}$$

So, since the total amount of labor is $L = \int_{T_0}^t L_v \, dv$,

$$L_t = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] L. \tag{41}$$

Therefore, (40) and (41) can determine the distribution of the labor, $L_v$.

(40) combined with (36) provides distribution of long-lived capital,

$$A_{\underline{v}} = \frac{L_v}{L_t}\frac{A_{\underline{t}}}{P^A_{\underline{v}}} = \left[\delta^A + \frac{\hat{q}}{\alpha} + \hat{N}\right] e^{-(\delta^A+\hat{N})(t-v)} A. \tag{42}$$

Now consider vintage physical capital. Since $MPB_{\underline{v}} = MPB_{\underline{t}}$ and thus from (16) and (18)

$$\frac{B_{\underline{v}}}{L_v} = \frac{B_{\underline{t}}}{L_t}. \tag{43}$$

Using (42),

$$B_{\underline{t}} = A_{\underline{t}} \frac{B}{A} = \left[ \delta^A + \frac{\hat{q}}{\alpha} + \hat{N} \right] B.$$

Therefore,

$$B_{\underline{v}} = B_{\underline{t}} \frac{L_v}{L_t} = \left[ \delta^A + \frac{\hat{q}}{\alpha} + \hat{N} \right] e^{-(\delta^A + \frac{\hat{q}}{\alpha} + \hat{N})(t-v)} B.$$

On the other hand, in the per effective labor notation, (43) is,

$$B_{\underline{v}} = b_t \left[ \frac{q_{\underline{t}}}{q_{\underline{v}}} \right]^{\frac{1}{1-\alpha-\beta}} = b^* e^{\frac{\hat{q}}{1-\alpha-\beta}(t-v)}.$$

So,

$$\hat{B}_{\underline{v}} = \frac{\hat{q}}{1-\alpha-\beta} + \hat{L_v} = - \left[ \delta^A + \frac{\hat{q}}{\alpha} \right].$$

Therefore,

$$\frac{I^B_{\underline{v}}}{B_{\underline{v}}} = \hat{B}_{\underline{v}} + \delta^B = \delta^B - \delta^A - \frac{\hat{q}}{\alpha},$$

and thus

$$I^B_{\underline{v}} = \left[ \delta^B - \delta^A - \frac{\hat{q}}{\alpha} \right] B_{\underline{v}}.$$