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Abstract

In this paper we investigate the minimization process of the exact cost function for a continuous review (Q,R) inventory model with non-negative reorder point and fixed lead-time. Backorders are allowed and the unit shortage cost is used to determine the expected annual shortage cost. Provided that the lead-time demand has J-shaped or unimodal distribution satisfying specific assumptions we derive the general condition when the minimum cost is attained at a positive reorder point or at a reorder point equal to zero. Based on this condition a general algorithm is developed. Some numerical experimentation based on this algorithm using parameter values from the relevant literature indicates that with large demand uncertainty measured by the coefficient of variation the optimal inventory policies lead to excessively large orders and zero reorder points.

Keywords: Inventory; Continuous review model; Exact cost function; Convexity; Cost parameter values; General algorithm.

JEL Codes: C61; C63; M11; M21.

1. Introduction

The continuous review (Q,R) inventory model with stochastic demand, fixed lead-time and backorders has been studied extensively in the area of inventory control. For this type of review policy, when the inventory position (on hand plus on order minus backorders) drops to the reorder point R then an order of size Q is placed and is delivered after a fixed period of time (lead-time) has elapsed. Provided that there is never more than one order outstanding at any point in time, which this in turn means that the lead-time demand never exceeds the order quantity, the values of the decision variables (Q,R) are determined by minimizing the annual total cost function resulting from the sum of the annual expected ordering, holding and shortage costs. Given, however, that different ways to compute the holding and shortage costs have been suggested in the relevant literature, this type of continuous review policy can be differentiated according to the form of the annual total cost objective function.

Regarding the calculation of the annual expected holding cost, Hadley & Whitin (1963) were the first who derived an exact expression for the expected on-hand inventory at any point in time. Their analysis was based on the assumptions that the lead-time demand has the Poisson distribution and each order delivery brings the on hand inventory level above the non-negative reorder point. Extending the original Poisson results of Hadley & Whitin, among others Zheng (1992), Platt et al. (1997), and Lau & Lau (2002) gave the following exact expression

$$I_{ex} = \frac{1}{Q} \int_{R}^{Q+R} \left\{ \int_{0}^{Y} (Y - x) f(x) dx \right\} dY \tag{1}$$

for the expected on-hand inventory at any point in time when the lead-time demand is a random variable, X, with probability density function $f(x)$ and cumulative distribution function $F(x)$. The derivation of (1) was made under the conditions that the lead-time demand never exceeds Q and the lead-time is constant. These conditions ensure that the inventory position Y is uniformly distributed between R and $Q+R$. In the meantime, however, due to the complexity of (1), approximations for I_{ex} have been suggested.

Hadley & Whitin (1963) were again the first who gave the widely known approximate expression $(Q/2+\text{safety stock})$ for I_{ex}. However, according to the authors this expression is valid only when the probability of stock-outs is sufficiently small or equivalently R is sufficiently larger than the expected lead-time demand. Approximate expressions for I_{ex} have been also developed for the case of small R (e.g., Holt et al., 1960; Wagner, 1985; Love, 1979; Yano, 1985), and a detailed review of some of them can be found in Lau & Lau (2002) who also developed their own approximation. Surprisingly enough, in this review paper, the authors concluded that the Hadley-Whitin approximation is more accurate than some other expressions approximating (1) which handle situations in which the stock-out probability is not sufficiently low.

The second factor which differentiates the form of the annual cost function is the way of computing the expected shortage cost. Three shortage cost models are available in the literature (e.g., Silver et al., 1998; Lau et al., 2002a; Lau & Lau, 2008), where each one of them has its own way to evaluate the expected size of backorders incurred per year. In specific terms, the first model assumes that only a fixed cost per stock-out occasion is known, in which case the annual expected size of backorders is given by the product of the expected number of cycles per year and the stock-out probability. On the contrary, the second model considers a shortage cost per unit backordered and the resulting expected size of backorders is

$$S(R)=\int_{R}^{\infty}(x-R)f(x)dx\,.$$

Finally, the third model, taking into account the time factor in evaluating the shortage cost, uses the shortage cost per unit backordered per year. In this case the expression which gives the annual expected size of backorders becomes

$$S_{tm} = \frac{1}{Q} \int_{R}^{Q+R} \left\{ \int_{Y}^{\infty} (x - Y) f(x) dx \right\} dY .$$

Combining the exact expression (1) with each one of the above models evaluating the annual expected shortage cost, the following three alternative *exact annual cost functions* are produced:

$$C(Q,R) = \frac{A \cdot D}{Q} + h \cdot I_{ex} + B \cdot \frac{D}{Q} \left[1 - F(R) \right], \tag{2a}$$

$$C(Q,R) = \frac{A \cdot D}{Q} + h \cdot I_{ex} + s \cdot \frac{D}{Q} \cdot S(R), \tag{2b}$$

where, A is the fixed ordering cost, B the cost per stock-out occasion, h the holding cost per unit per year, s the shortage cost per unit backordered, s' the shortage cost per unit per year, and D the annual expected demand.

In the current paper we study the convexity of (2b) when the lead-time demand is a non-negative continuous random variable which has a unimodal distribution satisfying specific assumptions or J-shaped distribution with decreasing probability density function. Under these two types of distributions, for the first time we state when the unique minimum cost is determined through mathematical optimization. Particularly, taking the first order conditions from the minimization of (2b) with respect to Q and R and following an analogous approach to that of Das (1988) and Chung et al. (2009) we rewrite the cost function in terms only of R. Transferring in that way the analysis from the three dimensional to two dimensional space, we derive a general condition which identifies the following three cases: (a) the cost function is convex and has a unique minimum determined through mathematical optimization, (b) the cost function is not convex but it has a unique minimum determined through mathematical optimization, and (c) the cost function is increasing at an increasing rate in the entire domain of R in which case the minimum cost occurs at the smallest value of R, that is, at $R = 0$. The relevance of the general condition in determining whether or not the minimum cost will be obtained through mathematical optimization lies in the fact that it does not depend on the form of the lead-time demand distribution as this condition is expressed in terms of the annual expected demand, the variance of the lead-time demand and the three cost parameters, A, h and s.

To the extent of our knowledge, there has been little research on studying the convexity of the cost functions (2a) and (2b), focused mainly on problems encountered during the minimization process. Specifically, for Normally distributed lead-time demands and through the use of numerical examples, Lau et al. (2002b) found that, solving iteratively the first-order conditions from the minimization of (2b) with respect to Q and R and using relatively low values for s, at some stages the iterative procedure led to negative service levels. This had as a result the procedure to break down and nonsensical solutions to be obtained. To resolve the so called “degeneracy problem”, the authors derived an altered form of (2b) to include also the case of negative reordered points and presented the minimization of the new cost function through the Excel’s Solver under specific parameter values. Unfortunately, the degeneracy problem is also encountered even when we replace I_{ex} with the Hadley-Whitin approximation $(Q/2 + \text{safety stock})$ in (2a) and (2b). Although some explanations have been given by several authors (e.g., Lau & Lau, 2002; Lau et al., 2002a) to handle the “degeneracy problem”, in the current work we overcome the problem considering that R takes on only non-negative values.

On the other side, the convexity of the exact cost function (2c) has been studied extensively in the inventory literature. Zheng (1992) proved the convexity of (2c) based on the results of Zipkin (1986) who showed that the expected size of backorders is a jointly convex function of Q and R. Under discrete demand, Federgruen & Zheng (1992) developed a surprisingly simple and efficient algorithm to reach the minimum cost. But according not only to these authors but also to Platt et al. (1997), the algorithm is valid provided that $-\left(h \cdot I_{ex} + s' \cdot S_{tm}\right)$ is a unimodal function. The same algorithm was used by Zhao et al. (2012) to find the minimum cost in a single-item system with limited resource for goods in on-hand inventory and outstanding orders. For Poisson distributed lead-time demand, Guan & Zhao (2011) proved the convexity of (2c) for any given Q and R, noting, however, the computational difficulties in determining the minimum cost. Finally, significant research was also made for determining the minimum cost when the Hadley & Whitin (1963) approximation $\left(Q/2 + \text{safety stock}\right)$ is used in (2) instead of I_{ex}. Results on studying the convexity problem and/or cost minimization procedures can be found in Das (1983a,1983b, 1988), Silver et al. (1998), Lau et al. (2002a), Chung et al. (2009), Cobb et al. (2013) and Halkos et al. (2014).

Based on the aforementioned discussion and remarks the rest of the paper is organized as follows. In Section 2 we write the cost function (2b) as function only of R and derive analytic forms for the linear and quadratic loss functions when lead-time demand distribution is Gamma, Log-Normal and Weibull. These analytic forms are obtained from the *N*th truncated moment expressions given by Jawitz (2004). In Section 3, under unimodal and J-shaped lead-time demand distributions satisfying certain assumptions we obtain the general condition to have a unique minimum after solving the first order conditions from the cost function minimization. In the same section, from that condition we obtain the range of the cost parameters values in order the optimal reorder point to be equal to zero. In Section 4, we present a general algorithm for the minimization process of the cost function and applying this algorithm to a set of parameter values used in Lau & Lau (2002) we investigate the managerial implications of increasing demand uncertainty on the optimal target inventory measures. Finally, the last section concludes the paper summarizing the most important findings.

2. The Cost Function in the two dimensional space

Let X be a continuous non-negative random variable representing the demand in the lead-time with mean μ and variance σ^2. Providing that $R \geq 0$, Lau et al. (2002b) offered a simplified expression for (1) which is

$$I_{ex} = \frac{Q}{2} + R - \mu + \frac{\Theta(R)}{2Q}, \tag{3}$$

where $\Theta(R) = \int_R^\infty (x - R)^2 f(x)dx = \int_R^\infty x^2 f(x)dx - 2R\int_R^\infty x f(x)dx + R^2[1 - F(R)]$, (4)

and $F(R)$ is the cumulative distribution function of X evaluated at R. Replacing (3) in (2b) and taking $\partial C(Q,R)/\partial Q = 0$ and $\partial C(Q,R)/\partial R = 0$, the solution of first-order conditions for the minimization of the exact annual cost function gives

$$Q(R) = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S(R) + \Theta(R)}, \tag{5a}$$

and

$$F(R) = 1 - \frac{h[Q(R) - S(R)]}{s \cdot D}, \tag{5b}$$

where $S(R) = \int_R^\infty (x - R) f(x)dx = \int_R^\infty x f(x)dx - R[1 - F(R)]$. (6)

Solving iteratively (5a) and (5b) until convergence is achieved (e.g., Hadley & Whitin, 1963; Silver et al., 1998; and Lau et al., 2002b), the optimal pair of values (Q^*, R^*) is obtained. Regarding the order quantity, we distinguish between the notation Q (or Q^*) meaning a given number and the notation $Q(R)$ which illustrates a function of R derived after solving the first-order conditions for a minimum of the cost function (2b).

Substituting $Q(R)$ for Q first in (3) and then in (2b), and performing some algebraic manipulation in the resulting expression of the cost function, (2b) is transformed to a function only of R, and is written as

$$C(R) = h[Q(R) + R - \mu]. \tag{7}$$

The stated assumption in the introductory section that *"there is never more than one order outstanding at any point in time"* is true only if at each delivery the lead-time demand never exceeds the order quantity (Lau et al., 2002b). This also means that $Q(R) > \mu$, which in turn leads to a positive $C(R)$ for any $R \geq 0$.

To compute $Q(R)$ and $C(R)$ we need analytic expressions for the general functions $\Theta(R)$ and $S(R)$ which are given in (4) and (6) respectively. Assuming that X has certain probability distribution (e.g., Gamma, Log-Normal etc), such analytic expressions can be derived using the solutions of integrals $m_1 = \int_R^{\infty} x f(x) dx$ and $m_2 = \int_R^{\infty} x^2 f(x) dx$ which are obtained directly from the formulae reported in table 1 of Jawitz (2004).

From (4) and (6), it is deduced that when $R \to 0$ we have $S(R) \to \mu$ and $\Theta(R) \to \mu^2 + \sigma^2$, while if $R \to \infty$ then $S(R)$ and $\Theta(R)$ tend to zero. These limits are justified as follows. Since $S(R)$ expresses the expected shortage (or backorders size) per inventory cycle, $\Theta(R)$ equals to the sum of the squared expected shortage plus the variance of the shortage. So, when $R \to \infty$ the lead-time becomes infinity, the shortage goes to zero and its mean and variance tend also to zero. On the contrary, the fact that $R \to 0$ implies that shortage tends to be identical with the lead-time demand verifying in that way the aforementioned limits of $S(R)$ and $\Theta(R)$.

3. Minimization process of the Cost Function

From (7) and using the derivatives $dS(R)/dR = -[1 - F(R)]$ and $d\Theta(R)/dR = -2S(R)$, we obtain $C'(R) = -h \cdot V(R)$ and $C''(R) = h \cdot g(R)/[Q(R)]^3$, where

$$V(R) = -\frac{dQ(R)}{dR} - 1, \tag{8}$$

$$g(R) = \left\{ \frac{s}{h} Df(R) + [1 - F(R)] - \left[\frac{dQ(R)}{dR} \right]^2 \right\} [Q(R)]^2, \tag{9}$$

and

$$\frac{dQ(R)}{dR} = -\frac{s \cdot D[1 - F(R)]/h + S(R)}{Q(R)}. \tag{10}$$

From the forms of $C'(R)$ and $C''(R)$ we deduce that (a) the range of function $g(R)$ determines whether $C(R)$ is convex or not, and (b) provided that $C''(R) > 0$, namely $C(R)$ is convex, the range of function $V(R)$ determines whether or not there is a unique value $R^* > 0$ for which $C'(R^*) = 0$.

The range of $g(R)$ and $V(R)$ is determined by investigating how the first derivatives $V'(R) = -g(R)/[Q(R)]^3$ and $g'(R) = u(R)[Q(R)]^2$ respond to changes of R, that is, when R increases from zero to infinity, given that

$$u(R) = \left\{ \frac{s}{h} D f'(R) - f(R) \right\}. \quad (11)$$

This investigation is carried out in the remaining of this section when the lead-time demand distribution belongs to one of the following two types of skewed distributions: (a) J-shaped with $f'(R) < 0$, and (b) unimodal satisfying the following two assumptions:

Assumption 1: $\lim_{R \to 0} f(R) = \lim_{R \to \infty} f(R) = 0$.

Assumption 2: Given that the mode of distribution occurs at R_m, there is only one value $R_o < R_m$ for which $u(R_o) = 0$, with $u(R) > 0$ for $R < R_o$, and $u(R) < 0$ for $R > R_o$.

It is proved that assumption 2 is true for the unimodal Gamma(α, β) and Weibull(α, β), with $\alpha > 1$, and for the Log-Normal (λ, θ) as the latter one is unimodal for any λ and θ.

At this point it is important to mention that we have chosen Gamma, Weibull and Log-Normal because, under these distributions we can handle large demand variability when R is always positive. According to Gallego et al. (2007) when the demand coefficient of variation (CV) is large, it is preferable to describe the demand by non-negative skewed distributions instead of the Normal. This is one reason why the Normal distribution has not been included in our analysis as Normal offers tractable results and good approximations for target inventory measures only when the demand has relatively low coefficient of variation, preferably below 0.3 (e.g., Lau, 1997; Syntetos & Boylan, 2008; Janssen et al., 2009; Kevork, 2010). The second reason is that, according to Lau & Lau (2002) under Normally distributed lead-time demand with low CV the Hadley & Whitin (1963) approximation behaves well even when service levels are not large.

Given the above analysis when X has a J-shaped distribution with $f'(\mathrm{R})<0$ it can be proved that $\mathrm{g}(\mathrm{R})$ is positive for any $\mathrm{R}\geq 0$, and when it holds

$$(\mathrm{s/h})^2\mathrm{D}^2-(2\mathrm{A/h})\mathrm{D}-\sigma^2>0 \tag{12}$$

there exists a single $\mathrm{R}^*>0$ for which $\mathrm{V}(\mathrm{R}^*)=0$. Hence for this type of distribution the cost function is always convex since $\mathrm{C}''(\mathrm{R})=\mathrm{h}\cdot\mathrm{g}(\mathrm{R})/[\mathrm{Q}(\mathrm{R})]^3>0$. Further, condition (12) ensures that there is a positive R^* for which $\mathrm{C}'(\mathrm{R}^*)=-\mathrm{h}\cdot\mathrm{V}(\mathrm{R}^*)=0$. The value of R^* is obtained after solving either the equation $(\mathrm{dQ}(\mathrm{R})/\mathrm{dR})+1=0$ or the system of the first order conditions (5a) and (5b). Both ways lead to the same equation which is

$$\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}[1-F(\mathrm{R}^*)]+\mathrm{S}(\mathrm{R}^*)=\mathrm{Q}(\mathrm{R}^*). \tag{13}$$

Additionally, if X has a unimodal distribution satisfying assumptions 1, 2 and condition (12) is true, then it can be shown that: (a) $\mathrm{g}(\mathrm{R})$ intersects the horizontal axis at a unique positive value R_1 so that $\mathrm{g}(\mathrm{R})<0$ for $0\leq\mathrm{R}<\mathrm{R}_1$ and $\mathrm{g}(\mathrm{R})>0$ for $\mathrm{R}>\mathrm{R}_1$ and (b) $\mathrm{V}(\mathrm{R})$ intersects the horizontal axis at $\mathrm{R}^*>\mathrm{R}_1$ for which $\mathrm{V}(\mathrm{R}^*)=0$, $\mathrm{V}(\mathrm{R})>0$ for $\mathrm{R}<\mathrm{R}^*$ and $\mathrm{V}(\mathrm{R})<0$ for $\mathrm{R}>\mathrm{R}^*$. Therefore, it is deduced that the cost function is not convex as $\mathrm{C}''(\mathrm{R})=\mathrm{h}\cdot\mathrm{g}(\mathrm{R})/[\mathrm{Q}(\mathrm{R})]^3<0$ for $0\leq\mathrm{R}<\mathrm{R}_1$, and $\mathrm{C}(\mathrm{R})$ will have a unique minimum at $\mathrm{R}^*>0$ since $\mathrm{C}'(\mathrm{R}^*)=0$ and $\mathrm{C}''(\mathrm{R}^*)>0$. This value of $\mathrm{R}^*>0$ is obtained by solving again equation (13).

From the above analysis it is also realized that, for both types of distributions, when $(\mathrm{s/h})^2\mathrm{D}^2-(2\mathrm{A/h})\mathrm{D}-\sigma^2<0$ then for $\mathrm{R}\geq 0$ the function $\mathrm{g}(\mathrm{R})$ is always positive while $\mathrm{V}(\mathrm{R})$ is always negative. This means that when R increases on the interval $(0,\infty)$ then $\mathrm{C}(\mathrm{R})$ increases at an increasing rate, as $\mathrm{C}'(\mathrm{R})=-\mathrm{h}\cdot\mathrm{V}(\mathrm{R})>0$ and $\mathrm{C}''(\mathrm{R})=\mathrm{h}\cdot\mathrm{g}(\mathrm{R})/[\mathrm{Q}(\mathrm{R})]^3>0$.

Hence, $C(R)$ is convex but an extreme value does not exist under a strict mathematical framework. In this case, however, we shall consider as minimum the lowest point of the $C(R)$ curve which is located at $R^* = 0$. Then this minimum cost is given by

$$C(0) = h\left(\lim_{R^* \to 0} Q(R^*) - \mu\right) = h\left(\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu + \mu^2 + \sigma^2} - \mu\right). \tag{14}$$

Finally, when $(s/h)^2 D^2 - (2A/h)D - \sigma^2 = 0$ it holds $g(R) > 0$ and $V(R) < 0$ meaning that for J-shaped distributions $\lim_{R \to 0} g(R) = +\infty$, $\lim_{R \to 0} V(R) = 0$ and for unimodal distributions $\lim_{R \to 0} g(R) = \lim_{R \to 0} V(R) = 0$. Therefore for both types of distributions $C(R)$ is flat at $R = 0$ and starts to increase at an increasing rate for $R > 0$. In this case the lowest point of the $C(R)$ curve occurs at $R^* = 0$ and the minimum cost is given again by (14).

Closing this section, we note that the usefulness of condition (12) is twofold. First, solving the inequality with respect to one of the cost parameters keeping the other two fixed we obtain threshold values which determine the range values of the cost parameters in order the unique minimum to be attained for $R^* > 0$ or $R^* = 0$. Second, these threshold values are independent of the form of the lead-time demand distribution and to compute them we need to know only the mean and the variance of the lead-time demand. In Table 2 we give the range values of s, A and h in order the minimum cost to occur at a positive R value. When this happens the threshold value is the minimum for s and the maximum for A and h.

Table 2: Interval values of the cost parameters for a minimum cost at a positive reorder point.

Cost		
shortage	**ordering**	**holding**
$\sqrt{2\frac{A}{D}h + \frac{h^2}{D^2}\sigma^2} \le s < +\infty$	$0 \le A \le \frac{\frac{s^2D^2}{h} - h\sigma^2}{2D}$	$0 \le h \le \frac{-AD + \sqrt{A^2D^2 + \sigma^2 s^2 D^2}}{\sigma^2}$

4. An algorithm for the solution approach

The solution steps for finding the minimum of the cost function $C(R)$ defined in (7) when the lead-time demand has a J-shaped distribution with $f'(R) < 0$ or a unimodal distribution satisfying assumptions 1 and 2 described in section 3, are summarized into the following general algorithm:

Step 1: Give values to the parameters: s, A, h, D, μ and σ^2.

Step 2: If $(s/h)^2 D^2 - 2(A/h)D - \sigma^2 > 0$ then go to Step 3, otherwise go to Step 6.

Step 3: Find analytic forms for the functions $F(R)$, $f(R)$, $S(R)$ and $\Theta(R)$ (for the distributions Gamma, Weibull, and Log-Normal such analytic forms are offered in Table 1) and go to step 4.

Step 4: Find the optimum reorder point, R^*, by solving the equation

$$\frac{s \cdot D\left[1 - F\left(R^*\right)\right] / h + S\left(R^*\right)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S\left(R^*\right) + \Theta\left(R^*\right)}} - 1 = 0, \tag{15a}$$

and go to Step 5.

Step 5: Compute the optimal order quantity and the minimum total cost respectively from

$$Q^* = \sqrt{2\frac{A}{h}D + 2\frac{s}{h} \cdot D \cdot S\left(R^*\right) + \Theta\left(R^*\right)}, \tag{15b}$$

$$C\left(Q^*, R^*\right) = h\left(Q^* + R^* - \mu\right), \tag{15c}$$

and go to Step 7.

Step 6: Set $R^* = 0$ and compute the optimal order quantity and the minimum total cost from

$$Q^* = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu + \mu^2 + \sigma^2}, \tag{15d}$$

$$C\left(Q^*, 0\right) = h\left(Q^* - \mu\right). \tag{15e}$$

Step 7: End of algorithm.

Assigning to A, h, D, μ the values which have been used by Lau & Lau (2002) in their experimental framework and applying them to the general algorithm, we give in Table 3 the optimal target inventory measures under different combinations of sizes for the coefficient of variation (CV) and the shortage cost per unit backordered, s. Especially for s, we selected both larger and smaller sizes than its threshold value which determines the range where the optimal reorder point is positive or zero. From Table 3 we observe that given the CV size as s gets smaller the order quantity increases while the minimum cost, the reorder point and the service level decline.

Different trends for some of the four target inventory measures are observed when A or h increases with the remaining parameter values to be kept fixed. Particularly, from Tables 4 and 5 it can be verified numerically that when A rises then the order quantity and the minimum cost increase while the reorder point and the service level decline. If on the other hand h is getting larger apart from the minimum cost which increases the remaining three target inventory measures decrease. Furthermore, increasing the size of CV, keeping all the other parameter values fixed, results in (a) larger order quantities and minimum costs, and (b) smaller reorder points and service levels. Therefore, optimal inventory policies with large demand uncertainty expressed by the size of CV lead to excessively large orders, zero reorder points and higher minimum costs.

Table 3: Optimal target inventory measures when A=70, h=0.6, D=10000 and μ=300.

CV	s	Lead-time Demand Distribution	Exact Cost Function		
			service level	(Q*,R*)	C(Q*,R*)
0.2	1.5	Gamma	0.938	(1560.64, 397.07)	994.63
		Log-Normal	0.937	(1565.02,398.61)	998.17
	0.1	Gamma	0.079	(1617.63,219.61)	922.34
		Log-Normal	0.079	(1614.49,222.35)	922.11
	0.05*	Gamma	0	(1710.83,0)	846.50
		Log-Normal	0	(1710.83,0)	846.50
0.52	1.5	Gamma	0.934	(1646.82,562.83)	1145.79
		Log-Normal	0.933	(1682.22,555.52)	1162.64
		Rayleigh	0.935	(1619.47,560.37)	1127.91
	0.1	Gamma	0.070	(1743.14,108.73)	931.13
		Log-Normal	0.071	(1721.10,129.13)	930.14
		Rayleigh	0.069	(1762.43,90.76)	931.92
	0.05*	Gamma	0	(1716.95,0)	850.17
		Log-Normal	0	(1716.95,0)	850.17
		Rayleigh	0	(1716.95,0)	850.17
1	1.5	Exponential	0.927	(1856.71,783.60)	1404.18
		Log-Normal	0.923	(1959.04,694.37)	1412.05
	0.1	Exponential	0.056	(1856.71,17.26)	944.38
		Log-Normal	0.057	(1815.32,56.97)	943.37
	0.05*	Exponential	0	(1735.90,0)	861.54
		Log-Normal	0	(1735.90,0)	861.54
2	1.5	Gamma	0.899	(2606.67,894.37)	1920.62
		Log-Normal	0.901	(2564.77,684.91)	1769.81
	0.1	Gamma	0.013	(1945.08,0.00)	987.05
		Log-Normal	0.013	(1937.05,7.97)	987.01
	0.05*	Gamma	0	(1812.00,0)	907.20
		Log-Normal	0	(1812.00,0)	907.20
4	1.5	Gamma	0.847	(4081.43,207.21)	2393.18
		Log-Normal	0.873	(3334.35,494.66)	2117.41
	0.1*	Gamma	0	(2205.30,0)	1143.18
		Log-Normal	0	(2205.30,0)	1143.18
	0.05*	Gamma	0	(2088.86,0)	1073.32
		Log-Normal	0	(2088.86,0)	1073.32
6	1.5	Gamma	0.830	(4537.01,7.73)	2546.85
		Log-Normal	0.853	(3855.93,363.05)	2351.39
	0.1*	Gamma	0	(2581.34,0)	1368.81
		Log-Normal	0	(2581.34,0)	1368.81
	0.05*	Gamma	0	(2482.61,0)	1309.56
		Log-Normal	0	(2482.61,0)	1309.56

* these values are smaller than the threshold values of the shortage cost per unit backordered

Table 4: Optimal target inventory measures when s=1.5, h=0.6, D=10000 and μ=300.

CV	A	Lead-time Demand Distribution	Exact Cost Function		
			service level	(Q*,R*)	C(Q*,R*)
0.2	40	Gamma	0.953	(1186.89,406.91)	776.28
		Log-Normal	0.952	(1191.53,409.38)	780.55
	17000	Gamma	0.048	(23902.89,207.48)	14286.23
		Log-Normal	0.048	(23898.67,211.41)	14286.04
	19000*	Gamma	0	(25464.23,0)	15098.54
		Log-Normal	0	(25464.23,0)	15098.54
0.52	40	Gamma	0.949	(1273.51,593.87)	940.43
		Log-Normal	0.948	(1313.63,590.50)	962.48
		Rayleigh	0.950	(1244.04,586.71)	918.45
	17000	Gamma	0.047	(24022.42,94.26)	14290.01
		Log-Normal	0.047	(23998.40,116.98)	14289.23
		Rayleigh	0.047	(24043.17,74.55)	14290.63
	19000*	Gamma	0	(25464.64,0)	15098.78
		Log-Normal	0	(25464.64,0)	15098.78
		Rayleigh	0	(25464.64,0)	15098.78
1	40	Exponential	0.941	(1493.04,849.00)	1225.22
		Log-Normal	0.936	(1619.89,754.33)	1244.53
	17000	Exponential	0.047	(24106.65,14.49)	14292.69
		Log-Normal	0.047	(24066.87,52.72)	14291.76
	19000*	Exponential	0	(25465.92,0)	15099.55
		Log-Normal	0	(25465.92,0)	15099.55
2	40	Gamma	0.911	(2313.47,984.45)	1798.75
		Log-Normal	0.911	(2304.63,739.75)	1646.63
	17000	Gamma	0.047	(24127.09,0.00)	14296.26
		Log-Normal	0.047	(24110.57,16.02)	14295.95
	19000*	Gamma	0	(25471.23,0)	15102.74
		Log-Normal	0	(25471.23,0)	15102.74
4	40	Gamma	0.853	(3931.39,232.46)	2318.31
		Log-Normal	0.880	(3149.79,525.01)	2024.88
	17000	Gamma	0.046	(24149.47,0.00)	14309.68
		Log-Normal	0.046	(24145.08,4.27)	14309.61
	19000*	Gamma	0	(25492.42,0)	15115.45
		Log-Normal	0	(25492.42,0)	15115.45
6	40	Gamma	0.835	(4423.75,9.40)	2479.89
		Log-Normal	0.859	(3705.19,381.55)	2272.04
	17000	Gamma	0.045	(24186.70,0.00)	14332.02
		Log-Normal	0.045	(24184.71,1.95)	14331.99
	19000*	Gamma	0	(25527.70,0)	15136.62
		Log-Normal	0	(25527.70,0)	15136.62

* these values are smaller than the threshold values of the ordering cost

Table 5: Optimal target inventory measures when A=70, s=1.5, D=10000 and μ=300.

CV	h	Lead-time Demand Distribution	Exact Cost Function		
			service level	(Q*,R*)	C(Q*,R*)
0.2	0.4	Gamma	0.949	(1903.09,404.53)	803.05
		Log-Normal	0.949	(1907.62.406.81)	805.78
	18	Gamma	0.627	(328.43,315.64)	6193.41
		Log-Normal	0.625	(330.93.313.28)	6195.68
	150*	Gamma	0	(403.65,0)	15547.50
		Log-Normal	0	(403.65,0)	15547.50
0.52	0.4	Gamma	0.947	(1987.80,588.93)	910.69
		Log-Normal	0.946	(2025.84,586.29)	924.85
		Rayleigh	0.948	(1959.56,581.80)	896.54
	18	Gamma	0.534	(455.45,286.14)	7948.53
		Log-Normal	0.529	(461.08,275.66)	7861.27
		Rayleigh	0.542	(444.93,299.20)	7994.37
	150*	Gamma	0	(428.86,0)	19329.70
		Log-Normal	0	(428.86,0)	19329.70
		Rayleigh	0	(428.86,0)	19329.70
1	0.4	Exponential	0.942	(2194.73,853.88)	1099.44
		Log-Normal	0.939	(2314.51,768.33)	1113.14
	18	Exponential	0.374	(709.61,140.46)	9901.24
		Log-Normal	0.388	(668.23,167.30)	9639.53
	150*	Exponential	0	(499.33,0)	29899.93
		Log-Normal	0	(499.33,0)	29899.93
2	0.4	Gamma	0.923	(2951.39,1101.39)	1501.11
		Log-Normal	0.922	(2978.64,814.27)	1397.17
	18	Gamma	0.144	(1013.44,0.34)	12848.14
		Log-Normal	0.149	(975.41,35.85)	12802.73
	150*	Gamma	0	(720.65,0)	63097.18
		Log-Normal	0	(720.65,0)	63097.18
4	0.4	Gamma	0.880	(4717.77,398.43)	1926.48
		Log-Normal	0.901	(3845.62,635.22)	1672.34
	18*	Gamma	0	(1451.82,0)	20732.74
		Log-Normal	0	(1451.82,0)	20732.74
	150*	Gamma	0	(1264.65,0)	144697.13
		Log-Normal	0	(1264.65,0)	144697.13
6	0.4	Gamma	0.864	(5377.64,32.85)	2044.20
		Log-Normal	0.887	(4407.69,492.34)	1840.01
	18*	Gamma*	0	(1976.81,0)	30182.58
		Log-Normal	0	(1976.81,0)	30182.58
	150*	Gamma	0	(1843.73,0)	231559.22
		Log-Normal	0	(1843.73,0)	231559.22

* these values are smaller than the threshold values of the holding cost

5. Conclusions

In this paper, for the continuous review (Q,R) inventory model with backorders and fixed lead-time we examined the minimization process of the exact annual cost function. This function is the sum of the annual expected ordering, holding and shortage costs. For the calculation of the expected annual holding cost we used the exact expression for the expected on-hand inventory at any point in time. Further, the shortage cost per unit backordered and the resulting size of backorders were used for the determination of the annual expected shortage cost. The investigation of the minimization process was carried out under J-shaped and unimodal distributions satisfying specific assumptions.

Expressing the cost function in terms only of the reorder point we derived a general condition to identify when the minimum of the cost function (a) is obtained through mathematical optimization and b) occurs when the reorder point takes on the value zero. The usefulness of this condition relies on the fact that interval values of the cost parameters are obtained in order the minimum cost to occur at zero reorder point. Further, the limits of these intervals are independent of the form of the lead-time demand distribution and to compute them we need, apart from the cost parameter values, the annual expected demand and the variance of the lead-time demand. Based on this condition we offer a general algorithm for finding the minimum of the cost function.

Finally, after some numerical experimentation applying parameter values taken from the inventory literature to this algorithm, we observed that as the cost per unit backordered declines we move from a situation where the unique minimum cost is attained at a positive reorder point to a situation where the minimum cost occurs at zero reorder point. The same trend holds when the ordering or the holding cost increases. Furthermore, as the coefficient of variation raises with fixed cost parameter values we result in larger optimal order quantities and larger minimum costs while the reorder points and service levels decline. From the managerial aspects of inventory this means that as demand uncertainty grows the optimal policies lead to excessively large orders, zero reorder points and higher minimum costs.

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