Subsistence, saturation and irrelevance in preferences

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26 October 2014

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December 14, 2014

Abstract

This paper provides a microeconomic analysis of subsistence consumption in the consumer theory framework. Our key concepts are 'irrelevance' of a good in certain consumption bundles (increasing its amount does not make the consumer better off) and an 'unhappy set' (any bundle outside such a set is preferred to all bundles inside). Using these concepts we axiomatize subsistence and saturation induced irrelevance (SSI) preferences. We also axiomatize a generalized version of Leontief (GL) preferences, for which irrelevance is solely driven by complementarity. Irrelevance in SSI preferences results from the presence of unhappy sets; for GL preferences irrelevance is driven by their absence.

JEL Classifications: D11, O12, O15

Keywords: Subsistence, saturation, irrelevance, unhappy sets

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"Of the nonpossession of the matter of subsistence in such quantity as is necessary to the support of life, death is the consequence: and such natural death is preceded by a course of suffering much greater than what is attendant on the most afflictive violent deaths employed for the purpose of punishment."
—Jeremy Bentham, *Pannomial Fragments* (1843).

1 Introduction

Subsistence is the minimum amount of basic necessities essential for a person's survival. Depending on the context, it can be expressed alternatively in terms of income (e.g., \$1.25 per day) or in terms of nutrition such as a certain daily calorie requirement. It forms the basis of poverty measurements: "...absolute poverty lines are often based on estimates of the cost of basic food needs (i.e., the cost a nutritional basket considered minimal for the healthy survival of a typical family), to which a provision is added for non-food needs." (World Bank, 2013). As extreme poverty and hunger continue to pose a major global challenge, subsistence remains a useful concept for policymakers. For instance, effective policies to end hunger require knowledge of not only the number of hungry people, but also their *food deficit* or the depth of hunger, which is measured by "comparing the average amount of dietary energy that undernourished people get from the foods they eat with the minimum amount of dietary energy they need to maintain body weight and undertake light activity." (Food and Agricultural Organization of the United Nations, 2014).

Jeremy Bentham [2], one of the founding fathers of utility theory, considered "securing the existence of, and sufficiency of, the matter of subsistence for all the members of the community" an important milestone towards achieving "the all embracing end—the greatest happiness of the greatest number of the individuals belonging to the community in question." Yet an adequate treatment of subsistence consumption is lacking in a standard utility maximization setting.[1] The Stone-Geary utility function is widely used to model subsistence.[2] However, under this utility function a consumer is compelled to consume above the subsistence level regardless of the price of the basic good, thus assuming away the problem of poor people. In the Stone-Geary framework every

[1]Stigler [15] comments: "Occasionally it was stated that the marginal utility of a necessity falls rapidly as its quantity increases and the like; and there were some mystical references to the infinite utility of subsistence. These were *ad hoc* remarks, however, and were not explicitly developed parts of the formal theory."

[2]Rebelo [9] and Steger [13] use the Stone-Geary function to study the role of subsistence in economic growth. See Sharif [12] for a survey of measurement issues of subsistence. Subsistence consumption has also been associated with Giffen behavior, i.e., upward sloping demand curve (Jensen and Miller [4]).

consumer is prosperous by default, ignoring the possibility that a rise in food price may take an individual from prosperity to poverty.

We present a microeconomic analysis of subsistence consumption in the consumer theory framework.[3] We axiomatize subsistence consumption in a setting where an individual makes choices over two goods: a basic good which is a necessity such as food and a non-basic good which can represent a composite of other commodities.[4] In developing our theory, we appeal to two distinct aspects of a basic necessity. First, the individual requires a minimum critical level of this good. This is the subsistence requirement. If this requirement is not met, then the non-basic good is not useful. The non-basic good is useful only if the consumption of the basic good exceeds the subsistence requirement. This calls for non-homotheticity in preferences, an issue which has not received adequate attention (see Ray [8]). The second aspect is saturation, which is in line with the concept of 'abundance' proposed by Bentham [2]. Saturation implies that once the individual has consumed sufficiently large amounts of the basic good, consuming more of it may not be beneficial.

Subsistence and saturation generate *irrelevance* of one of the goods. A good is irrelevant at a consumption bundle if increasing its amount without changing the amount of the other good keeps the consumer indifferent. The non-basic good is irrelevant when the subsistence requirement is not met, while the basic good becomes irrelevant when its saturation is reached. Incorporating these features, we define *subsistence and saturation induced irrelevance* (SSI) preferences. For such preferences there are potentially three zones in the commodity space. Apart from the two zones where one of the goods is irrelevant, there can be an intermediate region (where the consumption of the basic good has exceeded the subsistence level but not yet reached saturation) in which none of the goods is irrelevant. In this region the individual has a standard consumer preference where two goods can be imperfect substitutes. SSI preferences thus enrich consumer theory by allowing for the existence of poverty and prosperity in different regions of the commodity space. This formalizes Bentham's concepts of subsistence and abundance in terms of individual preference. Theorem 1 axiomatizes SSI preferences.

Irrelevance can also be induced by complementarity between the two goods. If an individual prefers two spoons of sugar with every cup of tea and has one cup of tea,

[3]The Sanskrit word for bare subsistence *grāsācchādana* makes the components of subsistence particularly clear. It is a composite consisting of two words: *grāsa* (food) and *ācchādana* (clothing).

[4]Jensen and Miller [5] also consider a two-good setting to study subsistence behavior. However, both goods in their model are basic goods (food items that contribute calories) and there is substitutability between them. Substitutability across different basic goods as an optimization problem was first analyzed by Stigler [14].

then sugar becomes irrelevant after two spoons. For such preferences (called Leontief preferences), complementarity between the goods implies that at any consumption bundle one of the two goods is saturated and therefore rendered irrelevant. Theorem 2 axiomatizes a generalized version of the Leontief (GL) preferences.

Apart from the notion of irrelevance, the other key concept that is central for our axiomatizations is an *unhappy set*. Given a preference relation, a set of consumption bundles is said to be an unhappy set if every bundle outside this set is preferred to all bundles inside the set. This captures the state of a poor person who has extreme urge to come out of poverty. To see how the notions of irrelevance and unhappy sets are connected in our axioms, call a set of consumption bundles irrelevant in a certain good if that good is irrelevant at all bundles of the set. For SSI preferences, the zone where the subsistence requirement is not met is the largest unhappy set that is irrelevant in the non-basic good. But for GL preferences, if a set is irrelevant in any good, it can never be an unhappy set. Thus roughly speaking, SSI and GL preferences are characterized by the presence or the absence of unhappiness in irrelevance. It is a case of too little versus too much. Irrelevance of the non-basic good in SSI preferences stems from the fact that there is too little of the basic good. For GL preferences, irrelevance of a good is driven by too much of that good in relation to the other good.

von-Neumann and Morgenstern [16] introduced the notion of external stability as part of a solution concept in cooperative game theory. Unhappy sets can be interpreted as sets that have 'strong external instability'. For a preference relation a set of consumption bundles is 'externally stable' if for any bundle outside this set we can find a bundle in this set which is preferred to it. So a set is 'not externally stable' if there exists a bundle outside this set which is at least as good as all bundles inside. Our definition of unhappy set strengthens this notion of not external stability since we require that each bundle in an unhappy set is dominated (not just weakly but strictly) by all bundles (and not just by one bundle) outside the set.

To the best of our knowledge subsistence requirement has never been incorporated in the preference based approach of consumer behavior. Fishburn [3] axiomatized lexicographic preferences.[5] A lexicographic preference imposes a linear order on the two goods making it discontinuous. In SSI, if the consumer is in the subsistence zone, then the preference ordering over all bundles having different amounts of the basic good follows lexicographic order. However, this order breaks down when we compare any two bundles in the subsistence zone with the same amount of the basic good.

[5]Axiomatizations of other different consumer preferences include Milnor [7], Maskin [6], Segal and Sobel [11]. The main difference of our approach from this literature is that our axioms are on the regions of irrelevance embedded in SSI and GL preferences.

In SSI these two bundles lie on the the same indifferent curve while in lexicographic this is not the case. This is why unlike a lexicographic preference, SSI is continuous. We also show that unhappy sets can be used to axiomatize lexicographic preferences (Proposition 1).

Basu and Van [1] use subsistence and lexicographic ordering to define the preference of a household over two goods: a consumption good and a binary choice on whether or not to send the child to work. The preference is specified using the luxury axiom which says that a household will send its child to work only if its consumption without child labor income drops below the subsistence level. In contrast to SSI, the luxury axiom induces lexicographic order in the non-subsistence zone since in this zone the household does not send the child to work even if child labor yields higher consumption.

The paper is organized as follows. After providing the main framework in Section 2, we axiomatize SSI preferences in Section 3. In Section 4 we present axiomatization of GL preferences. Finally, implications of irrelevance and proofs of the results are provided in an appendix.

2 The main framework

Consider the problem of an individual in a two-good setting where the set of goods is $\{1, 2\}$. The individual has a consumption set $X = X_1 \times X_2$ where $X_i = \mathbb{R}_+$ for $i \in \{1, 2\}$, and $X = \mathbb{R}^2_+$. A consumption bundle is $x = (x_1, x_2) \in X$ where x_i stands for the amount of good i. Generic points in X will be denoted by x, y, z. If for all $i \in \{1, 2\}$: (a) $x_i > y_i$, then we say $x > y$, (b) $x_i \geq y_i$, then $x \geq y$ and (c) $x_i = y_i$, then $x = y$.

The individual's preference on X is defined using the binary relation "at least as good as". We write $x \succsim y$ for "x is at least as good as y". The preference relation $\succsim$ on X is rational (that is, complete and transitive).[6] The strict preference is defined as $x \succ y \Leftrightarrow [x \succsim y]$ and not $[y \succsim x]$. The indifference relation is defined as $x \sim y \Leftrightarrow [x \succsim y]$ and $[y \succsim x]$. The preference relation $\succsim$ on X is *continuous* if for any sequence of pairs of bundles $\{(x^n, y^n)\}$ with $x^n \succsim y^n$ for all n, $\lim_{n\to\infty} x^n = x$, and $\lim_{n\to\infty} y^n = y$, we have $x \succsim y$. It is *monotone* if for any $x, y \in X$ with $x > y$, we have $x \succ y$.

[6] The preference relation $\succsim$ on X is *complete* if for any $x, y \in X$, either $x \succsim y$ or $y \succsim x$ or both. It is *transitive* if for any $x, y, z \in X$, $[x \succsim y$ and $y \succsim z] \Rightarrow x \succsim z$.

2.1 SSI preferences

A basic necessity such as food has two key features. The first feature is the subsistence requirement: the individual requires a minimum critical level of the necessity. If this requirement is not met, other goods are not useful. The second feature is saturation. Beyond a point, consuming more of it may not be beneficial. For a preference relation in a two-good setting, the common aspect of these two features is 'irrelevance' in one of the two goods:

Definition 1 Good 2 is *irrelevant at a bundle* x if $x \sim (x_1, y_2)$ for all $y_2 > x_2$. Similarly Good 1 is *irrelevant at a bundle* x if $x \sim (y_1, x_2)$ for all $y_1 > x_1$. For $i \in \{1, 2\}$, Good i is *relevant at a bundle* x if it is not irrelevant there.

Let $i, j \in \{1, 2\}$ and $i \neq j$. We say that *a bundle y involves x_i* if $y_i = x_i$. Thus, the set of all bundles involving x_i is $\{y \in X | y_i = x_i\}$.

Definition 2 Consider a preference relation $\succsim$ on X which is rational, continuous and monotone. It is *subsistence and saturation induced irrelevance preference* (or an SSI preference) with respect to Good 1 if it satisfies the following properties.

(I) *Subsistence*: $\exists\ \underline{Q} \in (0, \infty)$ such that

- (a) *Subsistence zone* $[0, \underline{Q}]$: for every $x_1 \in [0, \underline{Q}]$, Good 2 is irrelevant at all bundles involving x_1;
- (b) *Weak non-subsistence zone* $(\underline{Q}, \infty)$: for every $x_1 > \underline{Q}$, $\exists\ y_1 \in (\underline{Q}, x_1)$ such that Good 2 is relevant at some bundle involving y_1.

(II) *Weak saturation*: $\exists\ x_2 \in X_2$ and $\overline{Q}(x_2) \in \mathbb{R}_+$ such that Good 1 is irrelevant at x if $x_1 \geq \overline{Q}(x_2)$ and it is relevant at x if $x_1 < \overline{Q}(x_2)$.

Definition 2 has zones of subsistence, weak non-subsistence and weak saturation in preferences. Here Good 1 is the basic good and $\underline{Q}$ stands for the subsistence threshold. Good 2 is the non-basic good. For instance, if Good 1 represents food, then $\underline{Q}$ stands for the critical amount of food that corresponds to the minimum calorie requirements of the individual. The subsistence zone specifies that if the consumption of Good 1 is below this critical level, then Good 2 does not have any benefit (property I(a)).

Property I(b) says that once the amount of Good 1 exceeds $\underline{Q}$ we can always find a bundle with lower amount of Good 1 at which Good 2 is relevant. In other words, for any $x_1 = \underline{Q} + \varepsilon$ (where $\varepsilon > 0$, no matter how small), there is $y_1 \in (\overline{Q}, \overline{Q} + \varepsilon)$ such that Good 2 is relevant at some bundle involving y_1. Once $x_1 > \underline{Q}$, we are in the weak non-subsistence zone in that the total irrelevance of Good 2 disappears there. As we shall see, the properties of SSI preference ensure the existence of a subset of the weak non-subsistence zone that is a non-subsistence zone in a stronger sense.

Property (II) of the definition says that there is at least one $x_2 \in X_2$ and a corresponding threshold $\overline{Q}(x_2)$ such that for consumption bundles involving x_2, any unit of Good 1 beyond $\overline{Q}(x_2)$ has no benefit. This captures the saturation aspect of a basic good in a weak sense.[7]

The SSI preference has two implications that are stated in Observation 1. First, there is a natural order between the threshold of subsistence and any threshold of weak saturation: for any $x_2 \in X_2$ where weak saturation holds, we have $\underline{Q} \leq \overline{Q}(x_2)$. Second, for any consumption bundle where the amount of the basic good exceeds the weak saturation level (that is, $x_1 > \overline{Q}(x_2)$), the non-basic good is necessarily beneficial. Formally, call an interval $(a, \infty) \subseteq X_1$ a *strong non-subsistence zone with respect to Good* 1 if for every $x_1 \in (a, \infty)$ there is a bundle involving x_1 at which Good 2 is relevant. We show that if weak saturation holds for $x_2 \in X_2$, then the interval $(\overline{Q}(x_2), \infty)$ is a strong non-subsistence zone. That is, for every $x_1 > \overline{Q}(x_2)$, there is a bundle involving x_1 at which Good 2 is relevant.

Observation 1 *For an SSI preference consider any $x_2 \in X_2$ at which weak saturation (property* (II)*) holds. Then $\underline{Q} \leq \overline{Q}(x_2)$, with strict inequality if $x_2 > 0$. Moreover, the interval $(\overline{Q}(x_2), \infty)$ is a strong non-subsistence zone with respect to Good* 1.

Example 1 For $x = (x_1, x_2) \in \mathbb{R}^2_+$ consider the following continuous utility function

$$u(x) = \begin{cases} x_1 \text{ if } x_1 \leq 1, \\ 1 + \min\{\sqrt{(x_1 - 1)x_2}, x_2\} \text{ if } (1 < x_1 \leq 2) \text{ or } (x_1 > 2 \text{ and } x_2 \leq 1), \\ 1 + \min\left\{\sqrt{(x_1 - 1)x_2}, \frac{1+\sqrt{1+4(x_1-1)(x_2-1)}}{2}\right\} \text{ if } x_1 > 2 \text{ and } x_2 > 1. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 1. This utility function represents an SSI preference with respect to Good 1 with subsistence zone $[0, 1]$, that is, $\underline{Q} = 1$. Note that $(1, \infty) \subset X_1$ is a strong (and hence weak) non-subsistence zone. The weak saturation property holds for $x_2 \in [0, 2]$. For any such x_2, there is $\overline{Q}(x_2) = x_2 + 1$ (see Figure 1) such that Good 1 is irrelevant at x if $x_1 \geq \overline{Q}(x_2)$ and relevant if $x_1 < \overline{Q}(x_2)$. Note that $\overline{Q}(0) = \underline{Q}$ and $\overline{Q}(x_2) > \underline{Q}$ for $x_2 \in (0, 2]$. As shown in

[7] A stronger notion of saturation requires that such a property holds *for every* $x_2 \in X_2$. Formally, there is *strong saturation* with respect to Good 1 if for every $x_2 \in X_2$, $\exists\, \overline{Q}(x_2) \in \mathbb{R}_+$ such that Good 1 is irrelevant at x if $x_1 \geq \overline{Q}(x_2)$ and it is relevant at x if $x_1 < \overline{Q}(x_2)$.

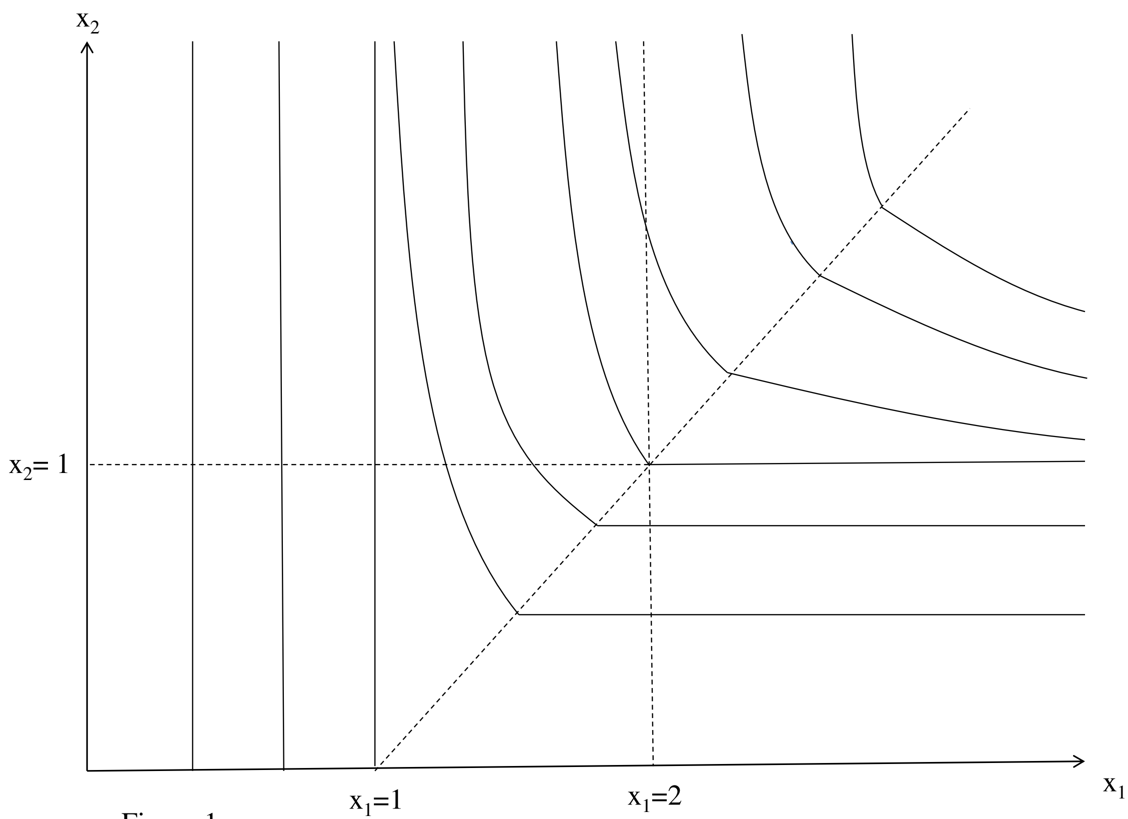

Figure 1

Observation 1, $(x_2+1,\infty) \subset X_1$ is a strong non-subsistence zone for every $x_2 \in [0,2]$. Finally observe that at any bundle with $x_1 > 1$ and $x_2 > 2$, both goods are relevant.

Let us see a particular context in which the properties of Example 1 are meaningful. Suppose Good 1 is food and Good 2 is physical activity, broadly construed. For instance, it can stand for labor which can bring monetary benefits or it can be sports which keeps the individual healthy. In the subsistence zone the individual does not have sufficient nutrition, so he is too weak to have any benefit from physical activity. Above this zone, for any $x_2 \leq 2$, there is a threshold (given by x_2+1) beyond which Good 1 is not useful. This is because to sustain low levels of physical activity the individual does not require an ever increasing amount of food and saturation is reached after a while. In particular, when $x_2 = 0$ (no physical activity), saturation is reached at the subsistence level (see Figure 1). There is no point of saturation once $x_2 > 2$. More nutrition is needed to carry out more demanding physical activities, which explains the absence of saturation for higher values of x_2.

Examples 2,3 that follow present SII preferences with strong saturation, in which for any $x_2 \in X_2$, the saturation threshold $\overline{Q}(x_2)$ is a constant independent of x_2. Whether saturation in an SII preference is weak or strong depends on how the non-basic good relates with the basic good. In the following examples Good 2 is considered to be a non-basic consumption good, which does not share the same relation that physical activity has with food. For such cases it is more natural that regardless of the amount of Good 2, saturation of Good 1 is reached once the individual has consumed sufficiently large amounts of it.

2.2 SSI preferences and Stone-Geary utility functions

Stony-Geary utility functions are often used to model subsistence. We provide examples of an SII preference to point out some drawbacks of Stone-Geary utility functions.

Example 2 Let $0 < \underline{Q} < \overline{Q} < \infty$. For $x = (x_1, x_2) \in \mathbb{R}^2_+$ define the *net-usefulness function* $g : X_1 \to \mathbb{R}_+$ as

$$g(x_1) := \begin{cases} 0 & \text{if } 0 \leq x_1 \leq \underline{Q}, \\ x_1 - \underline{Q} & \text{if } \underline{Q} < x_1 < \overline{Q}, \\ \overline{Q} - \underline{Q} & \text{if } x_1 \geq \overline{Q}. \end{cases} \tag{1}$$

That is, $g(x_1) = \max\{x_1 - \underline{Q}, 0\} + \min\{\overline{Q} - x_1, 0\}$. The net-usefulness function $g(.)$ is continuous, non-decreasing and piecewise linear. This function captures the usefulness of the basic good beyond subsistence requirement. Using the net-usefulness function, consider the following utility function, where $0 < \alpha < 1$.

$$u(x) = \min\{x_1, \underline{Q}\} + [g(x_1)]^\alpha x_2^{1-\alpha}. \tag{2}$$

Equivalently,

$$u(x) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq \underline{Q}, \\ \underline{Q} + \left(x_1 - \underline{Q}\right)^\alpha x_2^{1-\alpha} & \text{if } \underline{Q} < x_1 < \overline{Q}, \\ \underline{Q} + \left(\overline{Q} - \underline{Q}\right)^\alpha x_2^{1-\alpha} & \text{if } x_1 \geq \overline{Q}. \end{cases} \tag{3}$$

The preference represented by (3) is an SSI preference (with strong saturation). We normalize the price of Good 2 to be 1. The price of Good 1 is $p > 0$ and the income is $w > 0$. Let $\underline{w}(p) = p\underline{Q}$, $\overline{w}(p) = p\overline{Q}$ and $\widehat{w}(p) = \overline{w}(p) + (1-\alpha)p(\overline{Q} - \underline{Q})/\alpha$. The (unique) solution $x^* = (x_1^*, x_2^*)$ to this utility maximization problem is

$$x^* = \begin{cases} \left(\frac{w}{p}, 0\right) & \text{if } w \in (0, \underline{w}(p)], \\ \left(\underline{Q} + \frac{\alpha(w - \underline{w}(p))}{p}, (1-\alpha)(w - \underline{w}(p))\right) & \text{if } w \in \left(\underline{w}(p), \widehat{w}(p)\right), \\ \left(\overline{Q}, (w - p\overline{Q})\right) & \text{if } w \geq \widehat{w}(p). \end{cases} \tag{4}$$

Observe that the utility function (3) is different from the Stone-Geary utility function. For $\underline{Q} < x_1 < \overline{Q}$ in (3), $u(x)$ resembles the Stone-Geary utility function, but there is a qualitative difference. The minimum income required to achieve the subsistence consumption $\underline{Q}$ is $\underline{w}(p) = p\underline{Q}$ which is a function of p. This important aspect, intrinsic to subsistence, is missing from the Stone-Geary utility function as it implicitly assumed that any consumer *always* has enough wealth to stay outside the subsistence zone without any reference to the price of the basic (subsistence) good. However, an increase in the price of certain basic good such as foodgrains may very well push a consumer from non-subsistence to subsistence zone.

Also note that if $w \geq \overline{w}(p) = p\overline{Q}$, the consumer can afford the saturation level $\overline{Q}$ of the basic good. However, for the interval $[\overline{w}(p), \widehat{w}(p))$, it is optimal to buy less than saturation level of the basic good and more of the non-basic good.

The next example highlights the possibility of *subsistence inertia*: even if a consumer has adequate income to buy more than subsistence level of the basic good ($w > p\underline{Q}$) and avail both goods, it might still be optimal to not to buy the non-basic good at all. That is, even outside the subsistence zone a consumer may continue to buy only the basic good.

Example 3 Let $0 < \underline{Q} < \overline{Q} < \infty$. Using the net-usefulness function $g(.)$ defined in Example 2, for $x = (x_1, x_2) \in \mathbb{R}^2_+$ consider the utility function

$$u(x) = \min\{x_1, \overline{Q}\} + g(x_1)x_2 \tag{5}$$

which can be equivalently written as

$$u(x) = \begin{cases} x_1 & \text{if } 0 \leq x_1 \leq \underline{Q}, \\ x_1 + (x_1 - \underline{Q})x_2 & \text{if } \underline{Q} < x_1 < \overline{Q}, \\ \overline{Q} + (\overline{Q} - \underline{Q})x_2 & \text{if } x_1 \geq \overline{Q}. \end{cases} \tag{6}$$

Note that this utility function also represents an SSI preference (with strong saturation). As before, the price of Good 2 is normalized at 1, the price of Good 1 is $p > 0$ and the income is $w > 0$. Denote $\underline{w}(p) = p\underline{Q}$ and $\overline{w}(p) = p\overline{Q}$.

For $0 < p \leq 1/(\overline{Q} - \underline{Q})$, the (unique) solution $x^* = (x_1^*, x_2^*)$ to this utility maximization problem is

$$x^* = \begin{cases} \left(\frac{w}{p}, 0\right) & \text{if } w \in (0, \overline{w}(p)], \\ \left(\overline{Q}, w - \overline{w}(p)\right) & \text{if } w \geq \overline{w}(p). \end{cases} \tag{7}$$

For $p > 1/(\overline{Q} - \underline{Q})$, let $\delta = p(\overline{Q} - \underline{Q}) - 1 > 0$. The (unique) solution $x^* = (x_1^*, x_2^*)$ to this utility maximization problem is

$$x^* = \begin{cases} \left(\frac{w}{p}, 0\right) & \text{if } w \in (0, \underline{w}(p) + 1], \\ \left(\frac{\underline{w}(p)+w+1}{2p}, \frac{w-\underline{w}(p)-1}{2}\right) & \text{if } w \in (\underline{w}(p) + 1, \overline{w}(p) + \delta), \\ \left(\overline{Q}, w - \overline{w}(p)\right) & \text{if } w \geq \overline{w}(p) + \delta. \end{cases} \tag{8}$$

Note that when $w > \underline{w}(p)$, the consumer can afford more than the subsistence level $\underline{Q}$ of Good 1 and positive amount of Good 2. Yet there exist intervals $(\underline{w}(p), \overline{w}(p)]$ for (7) and $(\underline{w}(p), \underline{w}(p) + 1]$ for (8) where it is optimal not to buy Good 2 at all. For (7) buying Good 2 (keeping Good 1 fixed at $\overline{Q}$) is optimal only when income is above $\overline{w}(p)$. However, for (8), although the consumer can afford the saturation level $\overline{Q}$ of Good 1 if $w > \overline{w}(p)$, there is an interval $[\overline{w}(p), \overline{w}(p) + \delta)$ where it is optimal to buy less than the saturation level of Good 1 and more of Good 2.

These examples demonstrate a glimpse of the intricacies of consumer behavior that can be associated with SSI preferences.

2.3 SII and lexicographic preferences

The preference relation $\succsim$ on X is a *lexicographic preference with linear order* $1 <_0 2$ on the two goods if the following hold: $x \succsim y$ if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Consider two bundles that are in the subsistence zone of an SSI preference. If they have different amounts of Good 1 (like points $x = (x_1, x_2)$ and $z = (z_1, z_2)$ in Figure 1), then their preference ordering in the SSI is same as lexicographic. However, if the bundles have the same amount of Good 1 (like points $x = (x_1, x_2)$ and $y = (x_1, y_2)$ in Figure 1), the orderings of SSI and lexicographic are very different. Such bundles lie in the same indifference curve for SSI, while for lexicographic, they are strictly ordered in terms of the amount of Good 2. Indeed, SSI preference is continuous while lexicographic is not. On the other hand, lexicographic is strong monotone, while SSI is not.[8]

Lexicographic order with subsistence was used by Basu and Van [1] to define a household's preference for the consumption good relative to child labor (which is binary choice on whether or not to send the child to work). By contrast, our objective is to analyze preferences with subsistence and saturation for the basic good relative to the non-basic good in a standard utility theory framework.

[8] The preference relation $\succsim$ on X is *strong monotone* if for any $x, y \in X$ such that $x \geq y$ and $x \neq y$, $x \succ y$.

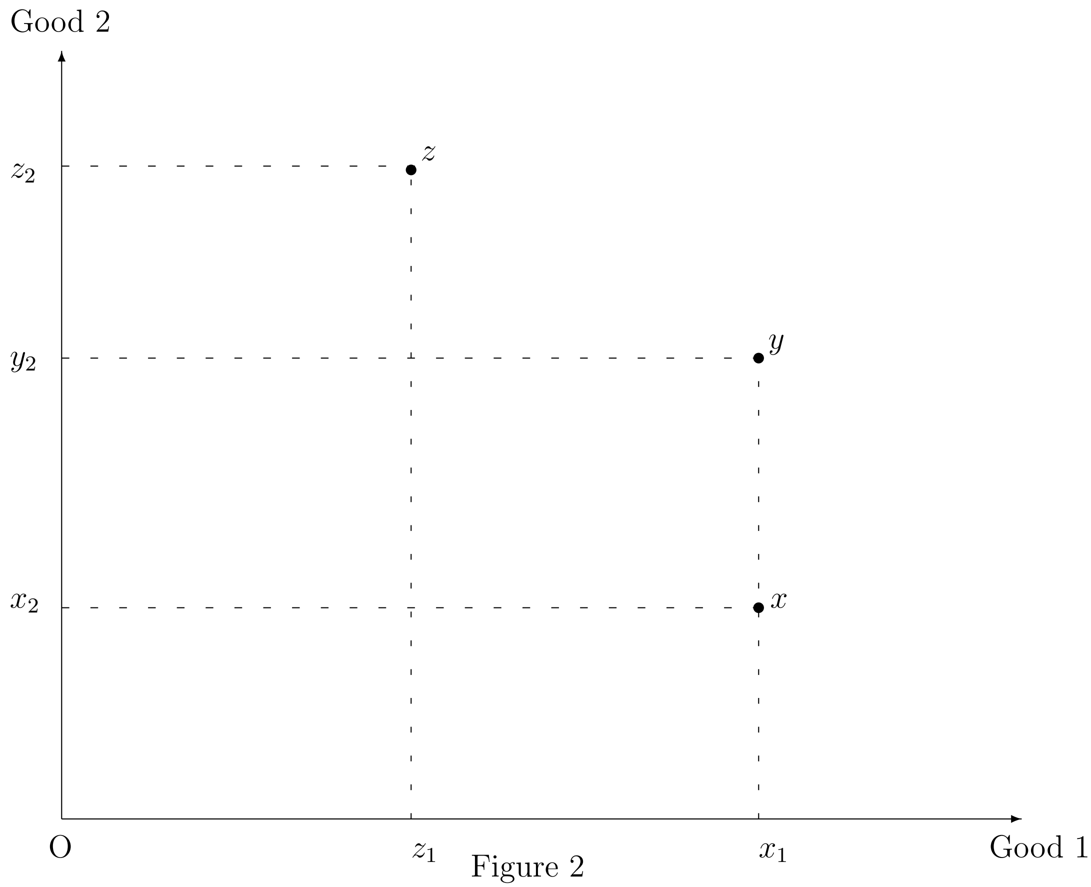

Figure 2

2.4 Unhappy sets

We introduce the notion of unhappy sets which will be used in our axiomatizations.

Definition 3 For a preference relation $\succsim$ on X, a set $S \subseteq X$ is an *unhappy set* if for any $y \notin S$, $y \succ x$ for every $x \in S$.

For any preference relation $\succsim$ on X, the empty set and the set X are both unhappy sets. Both lower contour and strict lower contour sets of any $x \in X$ are unhappy sets.[9] Specifically, S is an unhappy set if and only if $S = \cup_{x \in S} L(x)$.[10]

Remark 1 For discontinuous preferences, lower contour sets may not be closed but one can find closed unhappy sets. Consider a lexicographic preference with linear order $1 <_0 2$. For any $x \in X$, $L(x) = \{y \in X \mid y_1 < x_1\} \cup \{y \in X \mid y_1 = x_1, y_2 \leq x_2\}$ which is not closed. Let $T(x_1) = \{y \in X \mid y_1 \leq x_1\} = L(x) \cup \{y \in X \mid y_1 = x_1, y_2 > x_2\}$. Clearly, $L(x) \subset T(x_1)$ and $T(x_1)$ is a *closed* set. Note that $T(x_1)$ is an unhappy set. To see this, consider any $y \in T(x_1)$ and $z \notin T(x_1)$. Since $z_1 > x_1 \geq y_1$, we have $z \succ y$, which shows that $T(x_1)$ is an unhappy set.

[9] For any $\succsim$ on X and $x \in X$, the lower contour set of x is $L(x) = \{y \in X \mid x \succsim y\}$ and the strict lower contour set of x is $\bar{L}(x) = \{y \in X \mid x \succ y\}$.

[10] Take any $x \in S$. Since $x \in L(x)$, it is immediate that $S \subseteq \cup_{x \in S} L(x)$. For the converse, if $y \in L(x)$ and $x \in S$, then by definition of an unhappy set $y \in S$. Therefore, $L(x) \subseteq S$ for all $x \in S$. Hence $\cup_{x \in S} L(x) \subseteq S$.

3 Axiomatization of SSI preferences

Let $B_1 := \{x \in X|$ Good 2 is irrelevant at $x\}$, $B_2 := \{x \in X|$ Good 1 is irrelevant at $x\}$. Thus the set B_1 (the set B_2) is the set of all *bundles* at which Good 2 (Good 1) is irrelevant.

Also define $A_1 := \{x_1 \in X_1 | \exists x_2 \in X_2$ such that $x = (x_1, x_2) \in B_1\}$ and $A_2 := \{x_2 \in X_2 | \exists x_1 \in X_1$ such that $x = (x_1, x_2) \in B_2\}$. Therefore, for $i \neq j$, $A_i \subseteq X_i$ is the set of all *elements* x_i for which there exists a bundle involving x_i at which Good j is irrelevant.

We characterize SSI preferences using Axiom 1 and Axiom 2. Axiom 1 requires that irrelevance of the non-basic good is at least partially driven by inadequacy of the basic good. Axiom 2 requires that there exists at least one bundle where the basic good is irrelevant. Thus for each of the two goods there is a structural transition in preference. Theorem 1 shows that this requirement uniquely characterizes SSI preferences.

Axiom 1 *Unhappiness driven irrelevance:* B_1 has an unhappy subset of positive area.

Axiom 2 B_2 is non-empty.

Theorem 1 *Consider a preference relation* $\succsim$ *on* $X = \mathbb{R}^2_+$ *which is rational, continuous and monotone. The following statements are equivalent.*

(SSI1) *The preference relation* $\succsim$ *on* X *satisfies Axiom* 1 *and Axiom* 2*.*

(SSI2) *The preference relation* $\succsim$ *on* X *is a SSI preference with respect to* X_1*.*

Idea of the proof: To see that SSI satisfies Axiom 1 and Axiom 2, first observe that the set $\mathbb{S} := \{x \in X \mid x_1 \in [0, \underline{Q})\} \subseteq B_1$ since Good 2 is irrelevant for any consumption bundle in $\mathbb{S}$. Moreover, by monotonicity of preference any bundle in $X \setminus \mathbb{S}$ is strictly preferred to any bundle in $\mathbb{S}$. Hence $\mathbb{S}$ is an unhappy set of positive area implying Axiom 1. Property (II) of SSI implies Axiom 2.

To prove that Axiom 1 and Axiom 2 imply SSI preference we use the following chain of arguments. First, by Axiom 1 there is a subset $S \subseteq B_1$ which is an unhappy set of positive area. Therefore, there exists $x \in S$ such that $x_1 > 0$ (see Figure 2). By monotonicity of preference x is at least as good as any point in the rectangle $OExA$. Since S is an unhappy set, any point in this rectangle must be in S and hence in B_1. In particular for all points lying on the line OE, Good 2 is irrelevant and hence all indifference curves are parallel vertical lines in this region. This indicates the existence of a subsistence zone and in particular the interval OE is its subset. To fully characterize the subsistence zone we have to use Axiom 2.

Axiom 2 requires non-emptiness of B_2 and hence there exists $y = (y_1, y_2) \in B_2$. The point $(y_1, 0)$ must be to the right of OE, otherwise two indifference curves will intersect. This implies that B_2 cannot have an unhappy set of positive area. To see this note that if B_2 has an unhappy subset of positive area, then, by using the arguments of the previous paragraph, there will be a region where indifference curves are parallel horizontal lines each meeting the horizontal axis which is not possible (see IC(y)in Figure 2).

Finally, non-emptiness of B_2 gives rise to weak saturation. More importantly the subsistence zone must be bounded since $(y_1, 0)$ and anything to the right of it cannot belong to the subsistence zone. Existence of non-subsistence follows.

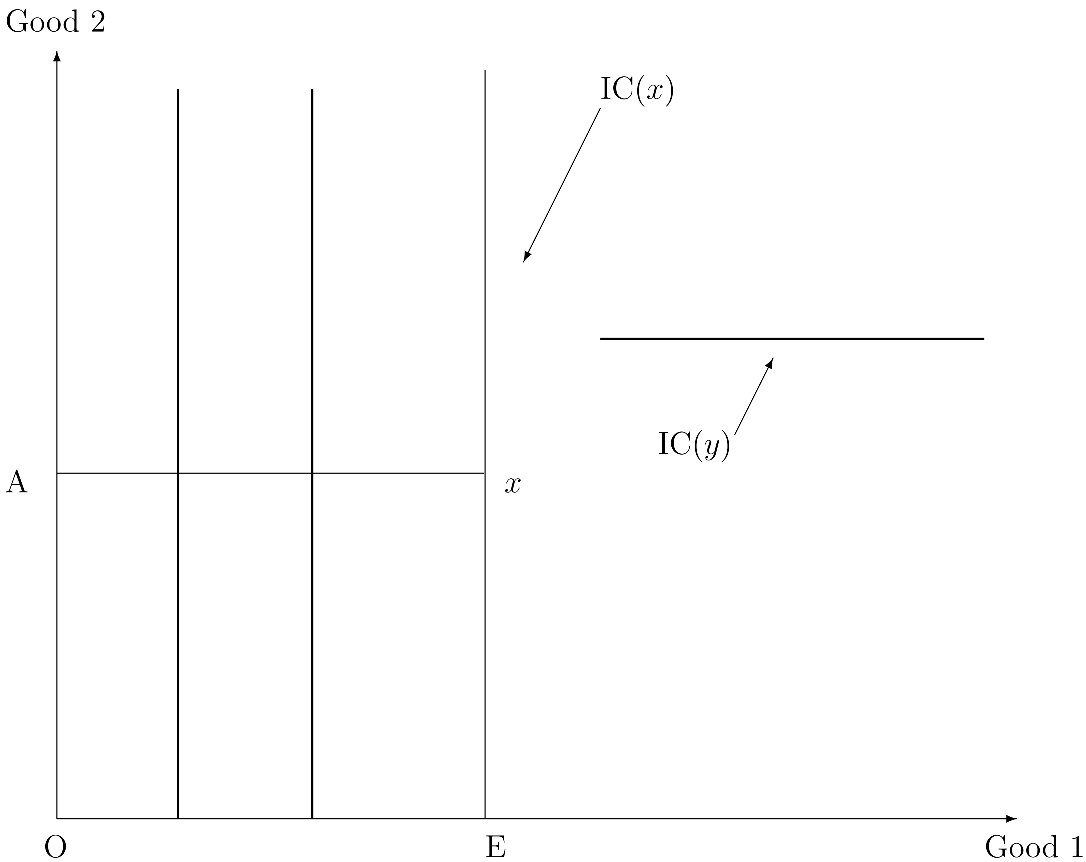

Figure 3

3.1 Robustness of Axiom 2

The generalized Leontief preference (defined in the next section) satisfies Axiom 2 but not Axiom 1. Hence we only need to check the robustness of Axiom 2 which requires that the set B_2 must be non-empty. This is useful not only to generate weak saturation, but it is also necessary for the existence of a non-subsistence zone. Without it, a non-subsistence zone might not exist. Without a reference to a situation of non-subsistence, the notion of subsistence may not be meaningful.

Corollary 1 *Consider a preference relation $\succsim$ on $X = \mathbb{R}^2_+$ which is rational, continuous and monotone. The following statements are equivalent.*

(S1) *The preference relation $\succsim$ on X satisfies Axiom* 1.

(S2) *For the preference relation $\succsim$ on X, either property* (I) *of Definition* 2 *holds, or Good* 2 *is irrelevant at all bundles.*

Recall that in property (I) of Definition 2, the subsistence zone is $[0, \underline{Q}]$ for $0 < \underline{Q} < \infty$, which results in a non-subsistence zone $(\underline{Q}, \infty)$. The preference in (S2) of Corollary 1 includes the case where $\underline{Q} = \infty$, in which case there is no non-subsistence zone with respect to Good 1, rendering the other Good 2 to be irrelevant at all bundles. In that case $T(x_1) = \{y \in X \mid y_1 \in [0, x_1]\}$ is an unhappy set for all $x_1 \in X$. Lexicographic preference shares the same property but is strong monotone (so both goods are relevant at all bundles). In fact, strong monotonicity together with this property characterizes the lexicographic preference.

Proposition 1 *A complete and strong monotone preference relation $\succsim$ on X is a lexicographic preference with linear order $1 <_0 2$ if and only if $T(x_1) = \{y \in X \mid y_1 \in [0, x_1]\}$ is an unhappy set for all $x_1 \in X_1$.*

4 Generalized Leontief preferences

In the previous section we characterized the SSI preference that had the feature that each good had stretches of irrelevance. The only well-known preference where irrelevance in both goods exists is the Leontief preference. For this preference at least one good is irrelevant at any bundle $x \in X$. Thus irrelevance spans the entire domain of preference. Formally, a preference relation $\succsim$ on X is the *Leontief preferences* if there exists $a > 0$ such that for any $x, y \in X$, $x \succsim y$ if and only if $\min\{ax_1, x_2\} \geq \min\{ay_1, y_2\}$. Observe that there exists a linear function $F(x_1) = ax_1$ such that given any $x_1 \in X_1$, both goods are irrelevant at $(x_1, F(x_1))$, Good 1 is irrelevant at $(y_1, F(x_1))$ for any $y_1 > x_1$ and Good 2 is irrelevant at (x_1, y_2) for any $y_2 > F(x_1)$. The ratio $1/a$ is the fixed coefficient of substitutability between the two goods. For Leontief preference there is no apparent pressing need to keep the substitution fixed across the two goods. For example, with one cup of tea a day, a consumer may want two spoons of sugar, but if the same consumer drinks ten cups of tea a day, he may take less than twenty spoons of sugar if he is diabetic. Thus for the Leontief preference the proportion of substitutability may well vary as we vary the amount of any one good (say Good 1). Incorporating this generality of variable substitutability, ceteris paribus, we define the 'generalized Leontief' preference as follows.

Definition 4 The preference relation $\succsim$ on X is a *generalized Leontief preference* (or a GL preference) if there exists an onto (surjective)[11] and increasing function $F : X_1 \to X_2$ with $F(0) = 0$ such that for any $x_1 \in X_1$:

(i) at any bundle $(x_1, F(x_1))$, both goods X_1 and X_2 are irrelevant,

(ii) Good 1 is irrelevant at any bundle $(y_1, F(x_1))$ for $y_1 > x_1$, and

(iii) Good 2 is irrelevant at any bundle (x_1, y_2) for $y_2 > F(x_1)$.

Observe that since F is onto and increasing, it is also one-to-one and continuous. The domain of the inverse function of F is X_2.

Axiom 3 *Irrelevance without unhappiness:* Neither B_1 nor B_2 has an unhappy subset of positive area.

Axiom 4 *Spanning axiom:* $A_1 = X_1$, $A_2 = X_2$ and $B_1 \cup B_2 = X$.

We axiomatize GL preferences using these two axioms. Monotonicity of preference ensures that B_1 and B_2 both cannot have an unhappy set of positive area (Lemma 1). Hence Axiom 3 is the compliment of Axiom 1 (allowing for relabeling of the goods).

Theorem 2 *Consider a preference relation $\succsim$ on $X = \mathbb{R}^2_+$ which is rational, continuous and monotone. The following statements are equivalent.*

(GL1) *The preference relation $\succsim$ on X satisfies Axiom 3 and Axiom 4.*

(GL2) *The preference relation $\succsim$ on X is a generalized Leontief preference.*

Idea of the proof: For GL preferences, $A_i = X_i$ for $i = 1, 2$. Moreover, $B_1 = \{x \in X \mid x_2 \geq F(x_1)\}$, $B_2 = \{x \in X \mid x_2 \leq F(x_1)\}$ and so $B_1 \cup B_2 = X$. Therefore, we have Axiom 4. For any $x \in B_i$, there exists $y \notin B_i$ such that x is indifferent to y. Hence there does not exist an unhappy subset of B_i. Thus Axiom 3 holds.

[11] A function $F : X_1 \to X_2$ is an *onto* or a *surjective* function if for any $x_2 \in X_2$, $\exists\, x_1 \in X_1$ such that $F(x_1) = x_2$.

To see the converse consider any $x = (x_1, x_2) > (0, 0)$ and, given Axiom 4, assume without loss of generality that $x \in B_1$. If $(x_1, 0) \in B_1$, then $S = \{y \in X \mid y_1 \in [0, x_1)\}$ is an unhappy set of positive area. Since $B_1 \cup B_2 = X$ (Axiom 4) and indifference curves cannot intersect it follows that $S \subset B_1$, contradicting Axiom 3. So we must have $(x_1, 0) \in B_2$. Then the indifference curve containing x cannot meet the horizontal axis. Hence there exists $y_2 \in (0, x_2]$ such that $(x_1, y_2) \in B_2$ and $(x_1, z_2) \in B_1$ for all $z_2 \geq y_2$. Let $y_2 = F(x_1)$. Using $A_i = X_i$ (for $i = 1, 2$) it can be shown that $F(.)$ is an onto and increasing function with $F(0) = 0$.

4.1 Robustness of axioms

Axiom 3 and Axiom 4 have three requirements: (i) $A_i = X_i$ for $i = 1, 2$, (ii) $B_1 \cup B_2 = X$, and (iii) none of B_1 and B_2 has an unhappy subset of positive area. In each of the following examples, only one of requirements (i)-(iii) is violated, and we see that we do not get the generalized Leontief preference.

Example 4 Consider the preference represented by utility function u where $k > 0$.

$$u(x_1, x_2) = \begin{cases} \min\{x_1/(k - x_1), x_2\} & \text{if } x_1 < k, \\ x_2 & \text{if } x_1 \geq k. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 4. For this example, $B_1 \cup B_2 = X$ and none of B_1 and B_2 has an unhappy set of positive area. However, $A_1 = [0, k)$ although $A_2 = X_2$. This preference is not GL but is "locally Leontief" (for $x_1 < k$) with saturation of Good 1 at $x_1 = k$.

Example 5 Consider the preference represented by utility function u where $k > 0$.

$$u(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 \leq k, \\ k + \min\{x_1 - k, x_2\} & \text{if } x_1 > k. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 5. For this example, Axiom 4 holds. But Axiom 3 does not hold since B_1 has an unhappy subset of positive area. The set $\{(x_1, x_2) | x_1 \in [0, k], x_2 \in X_2\} \subset B_1$ is an unhappy set. We get "locally Leontief" (for $x_1 > k$) and subsistence with respect to Good 1 for $x_1 \leq k$.

Example 6 Consider the preference represented by the utility function

$$u(x_1, x_2) = \begin{cases} x_2 & \text{if } x_2 \leq x_1/2, \\ (x_1 + x_2)/3 & \text{if } x_1/2 < x_2 < 2x_1, \\ x_1 & \text{if } x_2 \geq 2x_1. \end{cases}$$

Some indifference curves of this preference are drawn in Figure 6. For this example, Axiom 3 holds and $A_i = X_i$ for $i = 1, 2$. However, $B_1 \cup B_2 \neq X$.

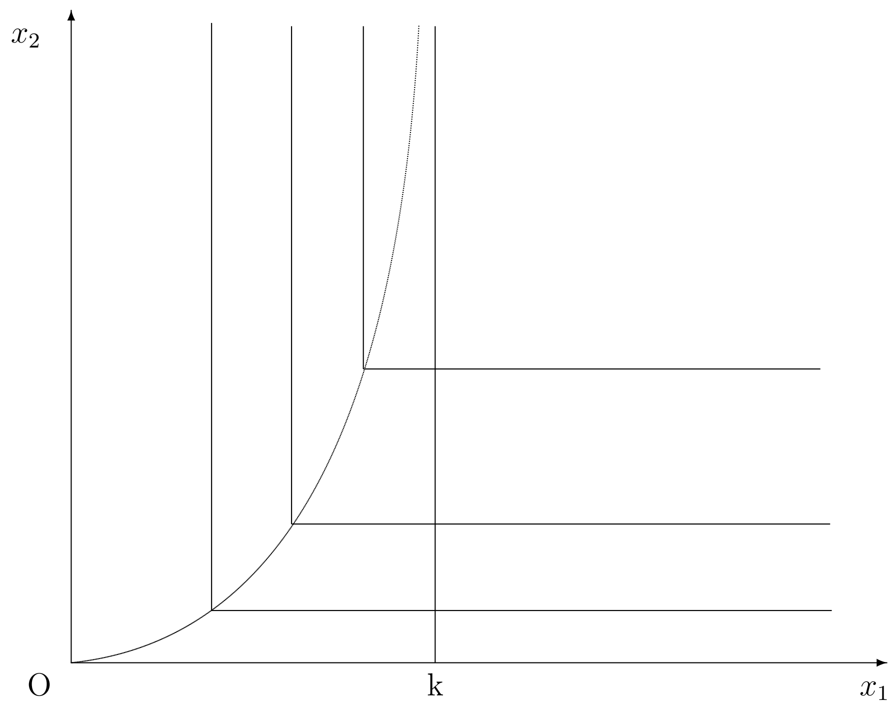

Figure 4

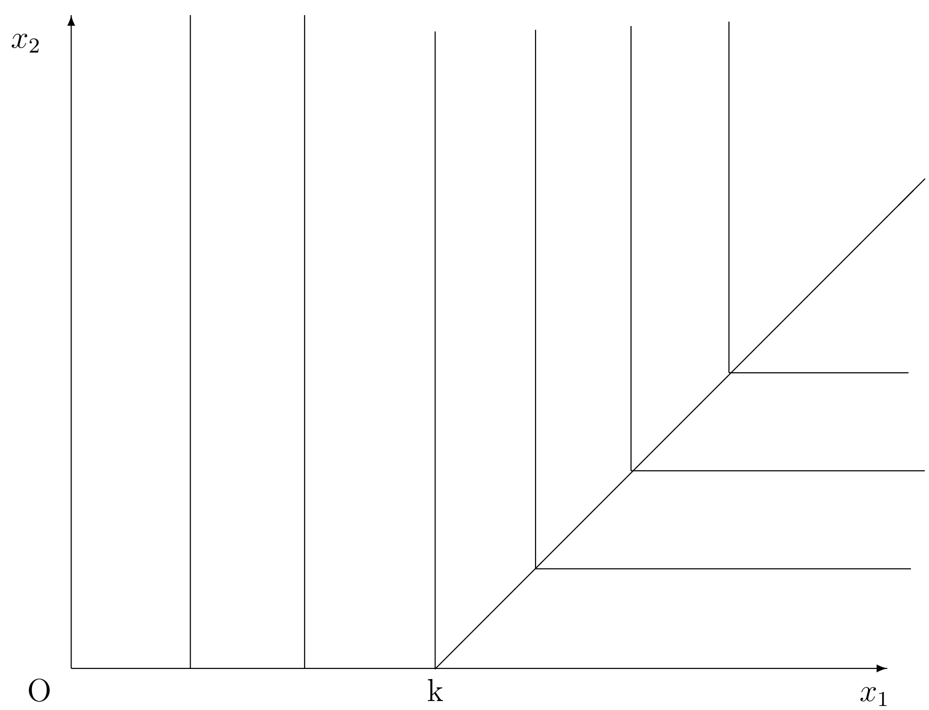

Figure 5

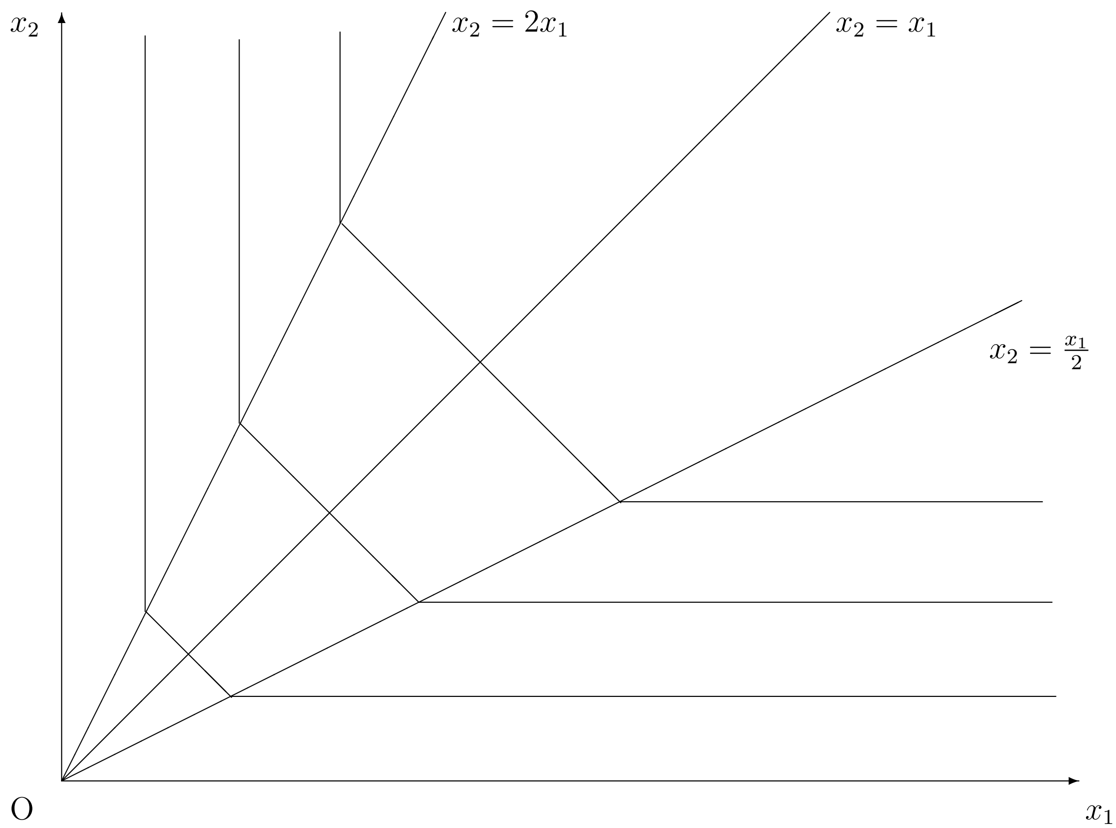

Figure 6

5 Appendix

Proof of Observation 1: Let $x_2 \in X_2$ be such that property (II) holds there. For the first part, suppose on the contrary that $\underline{Q} > \overline{Q}(x_2)$. Then $(\overline{Q}(x_2), x_2) \sim (\underline{Q}, x_2)$ (by property (II)) and $(\underline{Q}, x_2) \sim (\underline{Q}, y_2)$ for any $y_2 > x_2$ (by property (I)(a)). By transitivity, $(\overline{Q}(x_2), x_2) \sim (\underline{Q}, y_2)$ for any $y_2 > x_2$ which violates monotonicity. So we must have $\underline{Q} \leq \overline{Q}(x_2)$.

Let $x_2 > 0$. If $\underline{Q} = \overline{Q}(x_2) = Q$, then $(Q, 0) \sim (Q, x_2)$ (by (I)(a)) and $(Q, x_2) \sim x$ for any $x_1 > Q$ (by (II)), implying $(Q, 0) \sim x$ for any $x_1 > Q$ which violates monotonicity. So we must have $\underline{Q} < \overline{Q}(x_2)$ if $x_2 > 0$.

To prove that $(\overline{Q}(x_2), \infty)$ is a strong non-subsistence zone with respect to Good 1, we have to show that for any $x_1 > \overline{Q}(x_2)$, Good 2 is relevant at some bundle involving x_1. Suppose, on the contrary, $\exists\, x_1 > \overline{Q}(x_2)$ such that Good 2 is irrelevant at all bundles involving x_1. Then $x \sim (x_1, y_2)$ for any $y_2 > x_2$. But since $x_1 > \overline{Q}(x_2)$, by property (II) we have $x \sim (\overline{Q}(x_2), x_2)$. By transitivity, $(x_1, y_2) \sim (\overline{Q}(x_2), x_2)$, which violates monotonicity, a contradiction. ■

5.1 Irrelevance: some implications

We define two functions $f_1, f_2 : X \to \{0, 1\}$ that captures the notion of irrelevance.

$$f_1(x) \equiv \begin{cases} 0 \text{ if } x \sim (y_1, x_2) \text{ for all } y_1 \geq x_1, \\ 1 \text{ otherwise.} \end{cases}$$

$$f_2(x) \equiv \begin{cases} 0 \text{ if } x \sim (x_1, y_2) \text{ for all } y_2 \geq x_2, \\ 1 \text{ otherwise.} \end{cases}$$

The function $f_1(x)$ captures irrelevance of Good 1 at bundle x. Similarly, the function $f_2(x)$ captures irrelevance of Good 2 at bundle x. Observation 2 shows that if a good is irrelevant at a bundle, then it continues to remain so for all bundles where its quantity is increased keeping the quantity of the other good unchanged. Observation 2 also shows that the converse is true, which is proved using continuity of the preference relation.

Observation 2 (i) $f_2(x) = 0 \Leftrightarrow f_2(x_1, y_2) = 0$ *for all* $y_2 > x_2$ *and* (ii) $f_1(x) = 0 \Leftrightarrow f_1(y_1, x_2) = 0$ *for all* $y_1 > x_1$.

Proof: We prove (i), proof of (ii) is similar. Let $f_2(x) = 0$. Then $x \sim (x_1, y_2)$ for any $y_2 > x_2$. Hence $(x_1, y_2) \sim (x_1, z_2)$ for any $z_2 > y_2 > x_2$, implying that $f_2(x_1, y_2) = 0$.

Conversely, let $f_2(x_1, y_2) = 0$ for all $y_2 > x_2$. Then $(x_1, y_2) \sim (x_1, z_2)$ for all $z_2 > y_2 > x_2$. Let $x^n = (x_1, x_2 + 1/n)$ and $y^n = (x_1, y_2 + 1/n)$ for $n = 1, 2, \ldots$. Then $x^n \sim y^n$, and hence $x^n \succsim y^n$ for $n = 1, 2, \ldots$. Since $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = (x_1, y_2)$, by continuity we have $x \succsim (x_1, y_2)$. Since $y_2 > x_2$, by monotonicity we have $(x_1, y_2) \succsim x$. We then conclude that $x \sim (x_1, y_2)$ for any $y_2 > x_2$, proving that $f_2(x) = 0$. ■

We conclude from Observation 2 that for every $x_i \in A_i$, $\exists\ \alpha_i(x_i) \in X_j = \mathbb{R}_+$ such that

$$f_j(x) = \begin{cases} 0 \text{ if } x_j \geq \alpha_i(x_i), \\ 1 \text{ otherwise.} \end{cases} \tag{9}$$

It follows from (9) that $B_i = \{x \in X | x_i \in A_i, x_j \geq \alpha_i(x_i)\}$. For $x_i \in A_i$, let $B_i(x_i)$ be the set of all bundles involving x_i at which good j is irrelevant, that is, $B_i(x_i) := \{y \in X | y_i = x_i, y_j \geq \alpha_i(x_i)\}$. It is immediate that $B_i = \cup_{x_i \in A_i} B_i(x_i)$. For any $x_i \in X_i$, define the set of all bundles involving x_i as $M_i(x_i) := \{y \in X | y_i = x_i\}$. Observe that for any $x_i \in A_i$, $B_i(x_i) \subseteq M_i(x_i)$. Moreover $B_i(x_i) = M_i(x_i)$ if and only if $\alpha_i(x_i) = 0$. The last equality implies that good j is irrelevant at *all* bundles involving x_i.

Observation 3

(i) *Let $x_i, y_i \in A_i$ and $y_i < x_i$. Then $x \succ y$ for any $x \in B_i(x_i)$ and $y \in B_i(y_i)$.*

(ii) *Let $x_i > 0$. If $B_i(y_i) = M_i(y_i)$ for all $y_i \in [0, x_i)$, then $x_i \in A_i$ and $B_i(x_i) = M_i(x_i)$.*

Proof: Without loss of generality (w.l.o.g.), let $i = 1$.

(i) Let $y \in B_1(y_1)$. Consider any $z_2 > \max\{y_2, \alpha_1(x_1)\}$. Then $(x_1, z_2) \in B_1(x_1)$. Since $x_1 > y_1$ and $z_2 > y_2$, by monotonicity $(x_1, z_2) \succ y$. Since $(x_1, z_2) \sim x$ for any $x \in B_1(x_1)$ the result follows from transitivity.

(ii) Consider two sequences $x^n = (x_1 - 1/n, x_2)$, $y^n = (x_1 - 1/n, 0)$ where $x_2 > 0$ and $n > 1/x_1$. Since $y_1 \in A_1$ and $\alpha_1(y_1) = 0$ for $y_1 \in [0, x_1)$, we have $x^n, y^n \in M_1(x_1 - 1/n) = B_1(x_1 - 1/n)$. Hence $x^n \sim y^n$ and in particular, $y^n \succsim x^n$. Since $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = (x_1, 0)$, by continuity we have $(x_1, 0) \succsim x$. Since $x_2 > 0$, by monotonicity we have $x \succsim (x_1, 0)$, implying that $x \sim (x_1, 0)$ for any $x_2 > 0$. This proves the result. ■

Consider any two arbitrary bundles at both of which Good j is irrelevant. The first part of Observation 3 shows that the preference ordering of these two bundles is completely determined by amounts of Good i. The second part shows that if for any $y_i < x_i$, Good j is irrelevant at all bundles involving y_i, then Good j is also irrelevant at all bundles involving x_i.

5.2 SSI preferences

We start with the following definition. For $i = 1, 2$, a set $S \subseteq B_i$ is a *maximal unhappy subset* of B_i if (a) S is an unhappy set and (b) $\nexists\ T \subseteq B_i$ such that T is an unhappy set and $S \subset T$. Lemma 1 (that follows) will be used to prove Theorem 1. Part (I) of Lemma 1 shows that if for some $x_1 > 0$, Good 2 is irrelevant at all bundles involving any $y_1 \in [0, x_1]$ then Axiom 1 holds. Part (II) shows that the converse is also true. Moreover, if Axiom 1 holds, then B_1 has a unique maximal unhappy subset $\overline{S}$ which has the property that if $x = (x_1, x_2) \in \overline{S}$, then $(x_1, 0) \in \overline{S}$ and consequently Good 2 is irrelevant at all bundles involving x_1. Finally if Axiom 1 holds, then B_2 cannot have an unhappy subset of positive area.

Given Axiom 1, an immediate consequence of Lemma 1(I) is that the set $T(x_1) = \{y \in X \mid y_1 \in [0, x_1]\} \subseteq B_1$ is an unhappy set and the indifference curves in $T(x_1)$ are all parallel to the X_2 axis.

Lemma 1 (I) *If* $x_1 > 0$, $[0, x_1] \subseteq A_1$ *and* $B_1(y_1) = M_1(y_1)$ *for all* $y_1 \in [0, x_1]$, *then Axiom* 1 *holds.*

(II) *Suppose Axiom* 1 *holds.*

(i) *Let* $S \subseteq B_1$ *be an unhappy set of positive area. If* $x \in S$, *then* $\alpha_1(y_1) = 0$ *for all* $y_1 \in [0, x_1]$ *and* $\cup_{y_1 \in [0,x_1]} B_1(y_1) = \cup_{y_1 \in [0,x_1]} M_1(y_1) \subseteq S$.

(ii) B_1 *has a unique maximal unhappy subset* $\overline{S}$, *which has the following properties: Either* (a) $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ *or* (b) $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ *for some* $\overline{x}_1 \in (0, \infty)$, *or* (c) $\overline{S} = \cup_{y_1 \in \mathbb{R}_+} M_1(y_1) = \mathbb{R}^2_+$.

(iii) *Suppose* (a) *or* (b) *of* (ii) *holds. Then for every* $x_1 > \overline{x}_1$, $\exists\ y_1 \in (\overline{x}_1, x_1)$ *such that either* $y_1 \notin A_1$, *or* $y_1 \in A_1$ *and* $\alpha_1(y_1) > 0$.

(iv) B_2 *cannot have an unhappy subset of positive area.*

Proof of Lemma 1: (I) Let $y_1 \in [0, x_1]$. Let $T := \cup_{y_1 \in [0,x_1)} B_1(y_1) = \cup_{y_1 \in [0,x_1)} M_1(y_1) \subseteq B_1$. To prove that T is an unhappy set, first we show that $x \succ y$ for any $y \in T$. Observe that $x \in M_1(x_1) = B_1(x_1)$. Let $y \in T$. Then $y \in M_1(y_1) = B_1(y_1)$ for some $y_1 < x_1$.By Observation 3(i), we conclude that $x \succ y$.

To complete the proof we show that $z \succ y$ for any z such that $z_1 > x_1$. Monotonicity of preference implies that $z \succsim (x_1, 0)$ for any such z. From the preceding paragraph, we have $(x_1, 0) \succ y$ for any $y \in T$. By transitivity, $z \succ y$ for any $y \in T$. This proves that T is an unhappy set. As $x_1 > 0$, the area of T is positive. So Axiom 1 holds.

(II) (i) Let $S \subseteq B_1$ has positive area. Then $\exists\ x \in S$ where $x_1 > 0$. Consider such $x \in S$. Since $y \sim x$ for all $y \in B_1(x_1)$ and S is an unhappy set, we must have $B_1(x_1) \subseteq S$.

Next observe that if $\alpha_1(x_1) > 0$ for some $x \in S$, we can find y such that $y_1 = x_1$ and $y_2 \in [0, \alpha_1(x_1))$. Then $y \notin B_1$, so we have $y \notin S$. But $x \succsim y$ (by continuity and monotonicity of $\succsim$), which contradicts that S is an unhappy set. Hence for any $x \in S$, we must have $\alpha_1(x_1) = 0$, implying that $B_1(x_1) = M_1(x_1) \subseteq S$.

Now we show that if $x \in S$, then $y \in S$ for any y such that $y_1 < x_1$. To see this, consider z such that $z_1 = x_1$ and $z_2 > y_2$. Since $B_1(x_1) = M_1(x_1) \subseteq S$, we have $z \in S$. By monotonicity, $z \succ y$. As S is an unhappy set, we must have $y \in S$.

From the preceding paragraphs we conclude that if $x \in S$, then $\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$ and $\cup_{y_1 \in [0,x_1]} B_1(y_1) = \cup_{y_1 \in [0,x_1]} M_1(y_1) \subseteq S$. This proves (i).

(ii) First observe that if S, T are two subsets of B_1 that are both unhappy sets, then either $S \subseteq T$ or $T \subseteq S$. If neither holds, then $\exists\ x \in S, y \in T$ such that $x \notin T, y \notin S$. If $x_1 = y_1$, then $y \in M_1(x_1) \subseteq S$, a contradiction. So $x_1 \neq y_1$. W.l.o.g., let $y_1 < x_1$. But then from the last paragraph, we have $y \in M_1(y_1) \subseteq S$, again a contradiction.

Therefore, if Axiom 1 holds, then it has a unique maximal unhappy subset $\overline{S}$ and this set has positive area. From part (i) we conclude that either $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ or $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ for some $0 < \overline{x}_1 < \infty$, or $\overline{S} = \cup_{y_1 \in \mathbb{R}_+} M_1(y_1) = \mathbb{R}^2_+$.

(iii) If (a) or (b) of (ii) holds, then $y_1 \in A_1$ and $\alpha_1(y_1) = 0$ for all $y_1 \in [0, \overline{x}_1]$ (for (b), the result for $y_1 = \overline{x}_1$ follows from Observation 3(ii)). Suppose, on the contrary $\exists$ $x_1 > \overline{x}_1$ where the assertion (iii) does not hold. Then for every $y_1 \in (\overline{x}_1, x_1)$, we have $y_1 \in A_1$ and $\alpha_1(y_1) = 0$, so that $B_1(y_1) = M_1(y_1)$. Let $\widetilde{S}^* := \cup_{y_1 \in [0,x_1)} M_1(y_1)$. Then $\overline{S} \subset \widetilde{S}^* \subseteq B_1$. By part (I), $\widetilde{S}^*$ is an unhappy set, which contradicts (II)(ii).

(iv) Suppose on the contrary both B_1, B_2 have unhappy subsets of positive area. Then by part (II)(i), for $i = 1, 2$, $\exists\ x_i > 0$ such that $x_i \in A_i$ and $\alpha_i(x_i) = 0$. Then $(x_1, 0) \sim x$ (since $\alpha_1(x_1) = 0$) and $(0, x_2) \sim x \sim (y_1, x_2)$ for any $y_1 > x_1$ (since $\alpha_2(x_2) = 0$). This implies $(x_1, 0) \sim (y_1, x_2)$. But since $y_1 > x_1$ and $x_2 > 0$, by monotonicity we must have $(y_1, x_2) \succ (x_1, 0)$, a contradiction. This proves (iv). ■

Proof of Theorem 1: We first prove (SSI1)⇒ (SSI2).

Proof of subsistence property: Since Axiom 1 holds, by Lemma 1(II)(ii), B_1 has a unique maximal unhappy subset $\overline{S}$.

Now we show that $\overline{S} \neq \mathbb{R}^2_+$. To see this, first note that since Axiom 1 holds, by Lemma 1(II)(iv), B_2 cannot have an unhappy subset of positive area. Moreover, by Axiom 2, B_2 is non-empty and so is A_2. Let $x_2 \in A_2$, $y_1 > x_1 \geq \alpha_2(x_2)$ and

$y_2 = x_2$. Then $x, y \in B_2(x_2)$, so that $x \sim y$. If $\overline{S} = \mathbb{R}^2_+$, then $x, y \in \overline{S} \subseteq B_1$. As $x \in M_1(x_1) = B_1(x_1)$, $y \in M_1(y_1) = B_1(y_1)$ and $y_1 > x_1$, by Observation 3(i) we have $y \succ x$, a contradiction. So we must have $\overline{S} \neq \mathbb{R}^2_+$.

From the preceding paragraph and by Lemma 1(II)(ii) we conclude that either $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1)} M_1(y_1)$ or $\overline{S} = \cup_{y_1 \in [0,\overline{x}_1]} M_1(y_1)$ for some $\overline{x}_1 \in (0, \infty)$. In either case, by Observation 3(ii) we have $\alpha_1(y_1) = 0$ for all $y_1 \in [0, \overline{x}_1]$. Taking $\underline{Q} = \overline{x}_1$ proves part (a) of the *subsistence* property. Part (I)(b) of SSI preference with respect to Good 1 follows from Lemma 1(II)(iii).

Proof of weak saturation property: Since B_2 is non-empty, $\exists\ x_2 \in X_2$ and $\alpha_2(x_2) \geq 0$ such that Good 1 is relevant at x if $x_1 < \alpha_2(x_2)$ and it is irrelevant at x if $x_1 \geq \alpha_2(x_2)$. Taking $\overline{Q} = \alpha_2(x_2)$ proves the *weak saturation* property. From continuity and monotonicity of preference it also follows that $\underline{Q} = \overline{x}_1 \leq \overline{Q} = \alpha_2(x_2)$ and the inequality is strict if $x_2 > 0$.

We now prove (SSI2)$\Rightarrow$(SSI1). We consider the SSI preference with respect to Good 1 and show that it satisfies Axiom 1. Observe from the subsistence property that $[0, \underline{Q}] \subseteq A_1$ and $B_1(x_1) = M_1(x_1)$ for all $x_1 \in [0, \underline{Q}]$. Then by Lemma 1(I), it follows that Axiom 1 holds. To show that Axiom 2 holds, observe from the weak saturation property that $\{x \in X | x_1 \geq \overline{Q}\} \subseteq B_2$ so that B_2 is non-empty. ■

Proof of Corollary 1: We first prove (S1)$\Rightarrow$ (S2). Since Axiom 1 holds, by Lemma 1(II)(ii), B_1 has a unique maximal unhappy subset $\overline{S}$. If either (a) or (b) of Lemma 1(II)(ii) holds, then property (I) of Definition 2 holds. So suppose (c) of Lemma 1(II)(ii) holds, i.e., $\overline{S} = \mathbb{R}^2_+$. Then $A_1 = \mathbb{R}_+$ and $\alpha_1(x_1) = 0$ for all $x_1 \in \mathbb{R}_+$, implying that Good 2 is irrelevant at all bundles.

To prove (S2)$\Rightarrow$ (S1), if property (I) of Definition 2 holds, then from the proof of Theorem 1 it follows that Axiom 1 holds. Otherwise, $B_1 = \mathbb{R}^2_+$, which is itself an unhappy set of positive area. ■

Proof of Proposition 1: Suppose the preference $\succsim$ on X is lexicographic with linear order $1 <_0 2$. Consider any $x = (x_1, x_2) \in X$. Take any $y \in T(x_1)$ and any $z \notin T(x_1)$. Then $z_1 > x_1 \geq y_1$ and hence $z \succ y$.[12] This shows that $T(x_1)$ is an unhappy set for any $x_1 \in X_1$.

To prove the converse, take any $x, y \in X$. If $y_1 > x_1$, then $y \notin T(x_1)$ and since $T(x_1)$ is an unhappy set, $y \succ x$. Using completeness of preference,

$$\text{(a)} \qquad \text{if } x \succsim y, \text{ then } x_1 \geq y_1.$$

[12]For a lexicographic preference with linear order $1 <_0 2$, $z \succ y$ if and only if either $z_1 > y_1$ or $z_1 = y_1$ and $z_2 > y_2$.

If $y_1 \geq x_1$ and $y_2 > x_2$, then by strong monotonicity $y \succ x$. Therefore, by completeness and condition (a) if $x \succsim y$, then either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Hence we have lexicographic preference with linear order $1 <_0 2$. ∎

5.3 GL preferences

To prove Theorem 2 we will use the following lemmas. Given Axiom 4, Lemma 2 shows that if a good is irrelevant (relevant) at a bundle and its amount is decreased (increased), then it continues to be irrelevant (relevant) at the new bundle.

Lemma 2 *Suppose $\succsim$ satisfies Axiom 4.*

(I) *Let $i, j \in \{1,2\}$ and $i \neq j$. For any $x_i \in X_i$, $f_i(x)$ is non-decreasing in x_j.*

(II) *If $x_i \in A_i$, then $y_i \in A_i$ and $\alpha_i(y_i) \leq \alpha_i(x_i)$ for all $y_i \in [0, x_i)$.*

Proof: W.l.o.g. take $i = 1$ and $j = 2$.

(I) We have to show that $f_2(y_1, x_2) \leq f_2(x)$ for all $y_1 < x_1$ and $f_2(y_1, x_2) \geq f_2(x)$ for all $y_1 > x_1$. Since $f_2(.)$ equals 0 or 1, it is sufficient to show: (a) if $f_2(x) = 0$, then $f_2(y_1, x_2) = 0$ for all $y_1 < x_1$ and (b) if $f_2(x) = 1$, then $f_2(y_1, x_2) = 1$ for all $y_1 > x_1$. If (a) does not hold, then $\exists\ x$ and $y_1 < x_1$ such that $f_2(x) = 0$ and $f_2(y_1, x_2) = 1$, i.e., $(y_1, x_2) \notin B_1$. By Axiom 4, we must have $(y_1, x_2) \in B_2$, so that $\alpha_2(x_2) \leq y_1 < x_1$. Hence $(y_1, x_2), x \in B_2(x_2)$, implying $(y_1, x_2) \sim x$. Since $f_2(x) = 0$, we have $x \sim (x_1, z_2)$ for any $z_2 > x_2$. By transitivity, $(y_1, x_2) \sim (x_1, z_2)$ which violates monotonicity, so (a) must hold. If (b) does not hold, then $\exists\ z$ and $\tilde{z}_1 > z_1$ such that $f_2(z) = 1$ and $f_2(\tilde{z}_1, z_2) = 0$. Taking $x_1 = \tilde{z}_1$, $x_2 = z_2$ and $y_1 = z_1$ contradicts (a). Hence (b) must hold.

(II) If $x_1 \in A_1$, then $\exists\ \alpha_1(x_1) = x_2$ such that $f_2(x_1, y_2) = 0\ \forall\ y_2 \geq x_2$. By Lemma 2(I), for any $y_1 \in [0, x_1)$, we have $f_2(y_1, x_2) = 0$. By definition of $\alpha_1(.)$, we have $\alpha_1(y_1) \leq x_2 = \alpha_1(x_1)$ for all $y_1 \in [0, x_1)$. ∎

Since $A_i = X_i$ (by Axiom 4), $\alpha_i(.)$ is defined for any $x_i \in X_i$. Lemma 3 derives properties of this function and as a consequence we get the function $F(.)$ specified in the definition of GL preference.

Lemma 3 *Suppose the preference relation $\succsim$ on X satisfies Axiom 3 and Axiom 4. The following hold for $i, j \in \{1,2\}, i \neq j$.*

(I) *$\alpha_i(x_i) > 0$ for any $x_i > 0$.*

(II) *$\alpha_i(0) = 0$.*

(III) $\alpha_j(\alpha_i(x_i)) = x_i$.

(IV) *$\alpha_i(x_i)$ is increasing for all $x_i \geq 0$.*

(V) *$\alpha_i(x_i)$ is an onto function from X_i to X_j, i.e., for every $x_j \in X_j$, $\exists\ x_i \in X_i$ such that $\alpha_i(x_i) = x_j$.*

Proof: W.l.o.g., take $i = 1, j = 2$.

(I) Suppose on the contrary $\alpha_1(x_1) = 0$ for some $x_1 > 0$. Then by Lemma 2(II), $\alpha_1(y_1) = 0$ for all $y_1 \in [0, x_1]$. Then by Lemma 1(I), Axiom 1 holds, contradicting Axiom 3.

(II) Suppose on the contrary $\alpha_1(0) = x_2 > 0$. Let $y_2 \in (0, x_2)$. Then $(0, y_2) \notin B_1$ (since $y_2 < \alpha_1(0)$) and $(0, y_2) \notin B_2$ (since $0 < \alpha_2(y_2)$, part (I)), i.e., $y_2 \notin B_1 \cup B_2$, which contradicts Axiom 4.

(III) By (II), the result clearly hold for $x_1 = 0$, so let $x_1 > 0$. Then $\alpha_1(x_1) > 0$ (by (I)). Let $x_2 \in [0, \alpha_1(x_1))$. Then $x \notin B_1$, so by Axiom 4 we must have $x \in B_2$, implying that $\alpha_2(x_2) \leq x_1$ for all $x_2 \in [0, \alpha_1(x_1))$. By continuity,[13] we have $\alpha_2(\alpha_1(x_1)) \leq x_1$.

Denote $\alpha_1(x_1) = y_2$ and $\alpha_2(y_2) = y_1$. If $y_1 < x_1$, then $y, (x_1, y_2) \in B_2(y_2)$, so that $y \sim (x_1, y_2)$. Let $z_2 > y_2 = \alpha_1(x_1)$. Then $(x_1, z_2), (x_1, y_2) \in B_1(x_1)$, implying $(x_1, z_2) \sim (x_1, y_2)$. By transitivity, $y \sim (x_1, z_2)$, a contradiction (since $x_1 > y_1$ and $z_2 > y_2$). Hence we must have $y_1 \geq x_1$, i.e., $\alpha_2(\alpha_1(x_1)) \geq x_1$. From the conclusion of the previous paragraph, we conclude that $\alpha_2(\alpha_1(x_1)) = x_1$.

(IV) Since $\alpha_1(0) = 0$ and $\alpha_1(x_1) > 0$ for any $x_1 > 0$, $\alpha_1(x_1)$ is increasing at $x_1 = 0$. By Lemma 2(II), $\alpha_1(x_1)$ is non-decreasing. If it is not increasing for all $x_1 > 0$, $\exists\ x_1 > y_1 > 0$ such that $\alpha_1(x_1) = \alpha_1(y_1) = x_2 > 0$. By part (III), we then have $\alpha_2(x_2) = \alpha_2(\alpha_1(x_1)) = x_1$ and $\alpha_2(x_2) = \alpha_2(\alpha_1(y_1)) = y_1 < x_1$, a contradiction.

(V) By (II), the result holds for $x_2 = 0$. Suppose $\exists\ x_2 > 0$ such that $\alpha_1(x_1) \neq x_2$ $\forall\ x_1 \in X_1$. Since $\alpha_1(.)$ is continuous and $\alpha_1(0) = 0$, we must have $\alpha_1(x_1) < x_2$ for all $x_1 \in X_1$. By Axiom 4, $A_2 = X_2$. Hence $x_2 \in A_2$ and $\alpha_2(x_2)$ is well defined. Taking $x_1 = \alpha_2(x_2)$ above, we have $\alpha_1(\alpha_2(x_2)) < x_2$, which contradicts (III). ■

Proof of Theorem 2: $(L1) \Rightarrow (L2)$ By Axiom 4, for $i = 1, 2$, $A_i = X_i$ and $\alpha_i(x_i)$ is well defined for all $x_i \in X_i$. Note from Lemma 3 that $\alpha_1(.) : X_1 \to X_2$ is an increasing and onto function with $\alpha_1(0) = 0$ (the same property holds for $\alpha_2(.) : X_2 \to X_1$ and $\alpha_2(.)$ is the inverse function of $\alpha_1(.)$). Taking $F(x_1) = \alpha_1(x_1)$, by Lemma 3(III) it follows that (i)-(iii) of Definition 4 hold.

[13]Let $x_2 = \alpha_1(x_1)$. Suppose $\alpha_2(x_2) = y_1 > x_1$ and let $y_2 = x_2$. Then $y \succ x$. For any neighborhoods N_y, N_x around y, x we can find $z \in N_y, \widetilde{z} \in N_x$ such that $z_2 = \widetilde{z}_2 < x_2 = \alpha_1(x_1)$ and $z_1 > \widetilde{z}_1 \geq x_1$. Since $x_1 \geq \alpha_2(z_2)$, we have $z, \widetilde{z} \in B_2(z_2)$, so that $z \sim \widetilde{z}$. This contradicts continuity of $\succsim$ (see, e.g., Rubenstein [10]), proving that $\alpha_2(\alpha_1(x_1)) \leq x_1$.

$(L2) \Rightarrow (L1)$ Suppose the preference is generalized Leontief. Then for $i = 1, 2$, $A_i = X_i = \mathbb{R}_+$. For any $x_1 \in X_1$, we have $\alpha_1(x_1) = F(x_1)$ and for any $x_2 \in A_2$, we have $\alpha_2(x_2) = F^{-1}(x_2)$, and $F(0) = 0$. Hence $B_1(x_1) = \{(x_1, x_2)|x_2 \geq F(x_1)\}$ and $B_2(x_2) = \{(x_1, x_2)|x_1 \geq F^{-1}(x_2)\}$. So we have $B_i = \cup_{x_i \in X_i} B_i(x_i)$ for $i = 1, 2$, and $B_1 \cup B_2 = X$. Therefore, Axiom 4 holds.

It remains to show that Axiom 3 holds. If for some $i = 1, 2$, $\exists\ S \subseteq B_i$ such that S is an unhappy set of positive area, then $\exists\ x \in S$ such that $x_i > 0$. By Lemma 1(II)(i), this will imply that $\alpha_i(x_i) = 0$ for all $y_i \in [0, x_i]$, a contradiction since $\alpha_i(y_i) > 0$ for all $y_i > 0$. ■

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