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Tenryu, Yohei and Kamei, Keita

Graduate School of Economics, Osaka University, Graduate School of Economics, Kyoto University, Japan Society for the Promotion of Science

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Yohei Tenryu *
Graduate School of Economics, Osaka University
Japan Society for the Promotion of Science
and
Keita Kamei †
Graduate School of Economics, Kyoto University
Japan Society for the Promotion of Science

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Abstract

In this study, we analyze a dynamic duopoly game in which firms can use advertising and price as competitive tools. The market is assumed to be completely covered in the sense that all consumers purchase a product from one of the two firms. We assume that advertising creates a positive externality. Thus, each firm voluntarily advertises to persuade consumers to buy its products over those of the other firm, even though the firms compete with one another in price. Two cases are considered: an interior case and a corner case. In this situation, we investigate how changes in consumer preference and firm technology level affect advertising, profits, and economic welfare and highlight the differences between the two cases.

Keywords Advertising, vertical product differentiation, differential games, duopoly.
JEL Classification Numbers C72, C73, L13, M37.

*Machikaneyama 1-7, Toyonaka, Osaka 560-0043, Japan. Tel: +81-6-6850-5272. Email: mail@yoheitenryu.jp
†Yoshida Honmachi, Sakyo-Ku, Kyoto 606-8501, Japan. Tel: +81-75-753-7197. Email: keita.kamei@gmail.com

1 Introduction

Advertising is an important behavior of firms. Firms that produce nearly physically identical goods are thought to advertise to increase their market share. A considerable amount of literature has analyzed this type of advertising (Jørgensen (1982), Feichtinger et. al. (1994), Dockner et. al. (2000), and Huang, Leng, and Liang (2012)). However, advertising can be also interpreted as a public good; within an industry where firms compete with one another, each firm voluntarily advertises to persuade customers to buy its products over those of other firms (Friedman (1983), Roberts and Samuelson (1988), Martin (1993), and Piga (1998)). This situation occurs because voluntary advertising is known to create a positive externality. Thus, advertising by one firm benefits all other firms within an industry that produce the same industrial products. As the number of customers increases, all firms within the industry increase their profits. Voluntary advertising is frequently used by emerging industries, in which format competitions such as that which occurred between Blu-ray and HD DVD manufacturers, are common. Firms that produce products with a unique proprietary format use advertising to increase their market size.

Piga (1998) and Tenryu and Kamei (2013, 2014) investigated the relationship between advertising and production quality using the product differentiation model. Piga (1998) used the Hotelling (1929) location model and obtained results showing that market and advertising shares are positively correlated. Furthermore, he showed that industry size increases with the difference in firm production efficiency[1]: when the efficiency difference increases, the larger-share firm increases its advertising as the lower-share firm decreases it. Tenryu and Kamei (2013, 2014) adopted a vertical product differentiation model in their work. These authors first considered (2013) a covered market in which all consumers buy a product. In this scenario, the firm with the largest market share has the largest advertising share, and a positive relationship exists between the difference in product quality and the number of customers in the industry. An increase in quality difference also leads to an expansion of advertising by both types of firm. In subsequent work, Tenryu and Kamei (2014) considered a situation in which the market was not covered by consumers. In this case, a technological improvement by a low-quality firm led to an increase in the advertising and profit of the firm if technological gap between firms is relatively large.

In this study, we focus on a covered market and extend the previous advertising model of Tenryu and Kamei (2013) in two directions. First, following the work of Wauthy (1996), we consider two cases of a covered market: an interior case and a corner case. Tenryu and Kamei (2013) previously focused on an interior case alone. Second, we analyze consumer behavior using the utility-based approach. Consumer behavior has received little investigation in the existing literature, including the work of Piga (1998) and Tenryu and Kamei (2013). Several previous studies (Colombo and Lambertini (2003); Lambertini (2005); Lambertini and Palestini (2009); Bertuzzi and Lambertini (2010)) have used the utility-based approach to investigate the role of advertising investment. However, these works used the Hotelling (1929) location model and did not analyze voluntary advertising. Under these extensions, we investigate how changes in consumer preference and firm technology level

[1]In the work of Piga (1998), the production efficiency depends on the marginal costs.

affect advertising, profit, and economic welfare and highlight the differences between the two cases.

The main results of the paper may be summarized as follows. First, a high-quality firm possesses the largest market share, advertising share, and profit in both the interior and corner cases. Second, an increase in the minimum willingness to pay has different effects on the profits of the firms and economic welfare in both cases. Third, assuming a relatively small preference dispersion and relatively large technological gap, a rise in the technology level of a low-quality firm increases the advertising and profit of the firm in both cases. Finally, changes in preference dispersion and the technological gap differently affect individual utility and consumer surplus.

The remainder of the paper is structured as follows. Section 2 provides the basic setup of the model. Section 3 derives the steady states and characterizes the domains of the interior and corner cases. Section 4 analyzes producer surplus. We investigate how the main parameters of the model affect the total advertising volume, total number of customers in the market, and firm profits. In Section 5, we also investigate the effects of the parameters on individual consumers and consumer surplus. Section 6 concludes the work.

2 The Model

The basic structure of the firms and consumers in the present study is the same as that presented in Tenryu and Kamei (2013, 2014). In this economy, there exists a high-quality firm, H, and a low-quality firm, L. The high-quality firm produces high-quality goods, and the low-quality firm produces low-quality ones. The technology level of each firm is exogenously given by s_i for $i \in \{L, H\}$ and is assumed to satisfy the relation, $s_H > s_L$.

The consumers are uniformly distributed along a line with density N and have several preferences for goods, as defined by $\theta \in [\underline{\theta}, \bar{\theta}]$. This parameter represents each consumer's marginal willingness to pay, and $\bar{\theta}$ ($\underline{\theta}$) exogenously indicates the maximum (and minimum) value.[2] Given these preferences and the covered market, each consumer is assumed to buy a good from either the high-quality or low-quality firm. The indirect utility function at time t is $u_i(\theta, t) = \theta s_i - p_i(t)$, $i \in \{L, H\}$, where $u_i(\theta, t)$ represents the instantaneous utility generated from consumption of a good i at time t and $p_i(t)$ is good i's price at time t. Utility is assumed to be nonnegative. The lifetime discounted present value of utility for a consumer with θ is defined as follows:

$$\bar{u}_i(\theta) = \int_0^\infty (\theta s_i - p_i(t)) e^{-\rho t} dt, \quad i \in \{L, H\},$$

where ρ is the discount rate.

Because all consumers in the market purchase one good unit from either firm H or L, a threshold divides the market. This threshold characterizes a consumer who has no prefer-

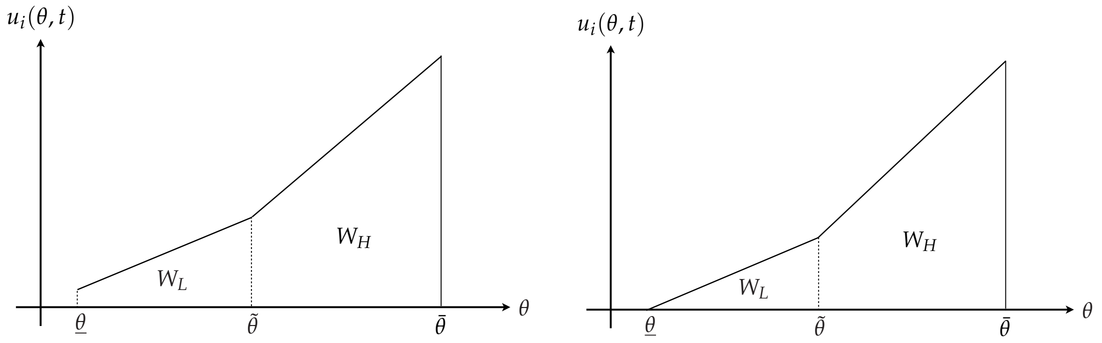
[2] Because Tenryu and Kamei (2013) concentrated on firm analysis, they assumed $\underline{\theta} = 0$. In this situation, however, negative consumer utility is possible. To avoid this scenario, we assume $\underline{\theta} > 1$, as shown in Assumption 1 below.

ence for buying the high-quality or the low-quality good, and may be defined as follows:

$$\tilde{\theta}(t) = \frac{p_H(t) - p_L(t)}{s_H - s_L}. \tag{1}$$

Therefore, at any given time, consumers with the preference, $\theta \in [\underline{\theta}, \tilde{\theta}]$ buy the low-quality good and consumers with the preference, $\theta \in [\tilde{\theta}, \bar{\theta}]$ buy the high-quality good. Accordingly, $N(t)(\bar{\theta} - \underline{\theta})$ represents the number of consumers in the industry at time t.

In addition, following the work of Wauthy (1996) we distinguish between an interior case and a corner case. We consider both cases because the utility of a consumer with the minimum willingness to pay, $\underline{\theta}$, may be either positive or negative. These situations are illustrated in Figure 1.



(a) Interior case (b) Corner case

Figure 1: Utility function

We designate the case in which utility is positive at $\underline{\theta}$ as the interior case and the case in which utility is zero as the corner case. In the interior case, no consumer is assumed to have a preference θ under $\underline{\theta}$.[3] Conversely, in the corner case, a consumer with preference $\underline{\theta}$ has no preference between buying product L and refraining from buying the product. Such a customer is defined as the solution to $\underline{\theta} s_L - p_L(t) = 0$, which is represented as p_L^C. Variables with the superscript C indicate equilibrium values in the corner case. Because s_L and $\underline{\theta}$ are given, firm L commits its price to p_L^C over time. Given the price and the opponent's strategy, firm L chooses its advertising strategy and firm H chooses its price and advertising strategies to maximise their lifetime profits.

We impose the following assumption on the maximum and minimum willingness to pay:

Assumption 1. *Assume* $\underline{\theta} > 1$ *and* $\bar{\theta} > 4\underline{\theta} - 3$.

This guarantees that both firms earn positive profits, utility is non-negative, and both interior and corner cases exist.[4] We can summarize the domain of the model in Figure 2.

[3] Tenryu and Kamei (2013) considered this case.

[4] The domains in which interior and corner cases exist are determined by the equilibrium prices, as is discussed in Section 3.4 below.

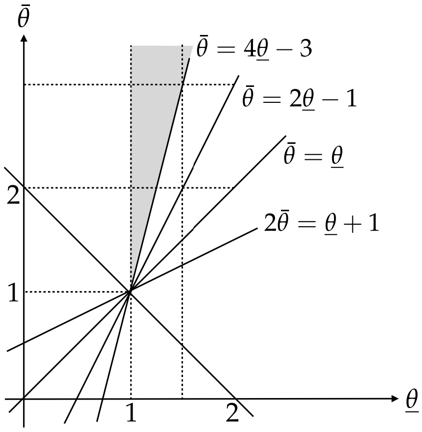


Figure 2: The restriction for $\bar{\theta}$ and $\underline{\theta}$

3 Characterization of the equilibrium path

3.1 Interior case

First, we consider the interior case in which both firms decide their strategies given $\bar{\theta}$, $\underline{\theta}$, s_i, the initial state, and the opponent's strategies. From the indirect utility functions, the demand functions may be represented as follow:

$$N(t)y_L(t) = N(t)(\tilde{\theta}(t) - \underline{\theta}) = N(t)\left(\frac{p_H(t) - p_L(t)}{s_H - s_L} - \underline{\theta}\right), \tag{2}$$

$$N(t)y_H(t) = N(t)(\bar{\theta} - \tilde{\theta}(t)) = N(t)\left(\bar{\theta} - \frac{p_H(t) - p_L(t)}{s_H - s_L}\right), \tag{3}$$

where $y_L(t)$ and $y_H(t)$ are defined as the market shares of firm L and H, respectively, because both firms face a common density of consumers, as explained below.

The sum of the discounted present value of the profit for firm i, V_i, is represented by the following equation:

$$V_i = \int_0^\infty \pi_i(t)e^{-\rho t}dt = \int_0^\infty \left[N(t)y_i(t)(p_i(t) - s_i) - \mu A_i(t)^2\right] e^{-\rho t}dt, \tag{4}$$

where $\pi_i(t)$ is firm i's profit at time t, $A_i(t)$ is the investment in advertisement at time t, $\mu A_i(t)^2$ is the investment cost at time t, and μ is the exogenous positive parameter. Following the work of Tenryu and Kamei (2013, 2014), we assume that each firm's unit cost is a linear function of its technology level and that the firms' discount rates are equal to the consumer's discount rate, ρ.

The state variable evolves according to the following state equation:

$$\dot{N}(t) = \alpha(A_H(t) + A_L(t)) - \lambda N(t), \quad N(0) \geq 0, \tag{5}$$

where $\alpha(>0)$ is the advertising efficiency parameter, $\lambda(>0)$ is the depreciation rate, and $N(0)$ is the initial stock. This law of motion implies that advertising is cooperative behavior, in the sense that advertising by one firm benefits both firms. Therefore, the advertising may be interpreted as a public good.[5]

To solve the duopolistic game defined above, we use the Hamiltonian function. The Hamiltonian equation for firm i is represented as follows:

$$\mathcal{H}_i^I = N(t)y_i(t)(p_i(t) - s_i) - \mu A_i(t)^2 + \phi_i(t)[\alpha(A_H(t) + A_L(t)) - \lambda N(t)] \tag{6}$$

We may thereby obtain the following optimality conditions of both firms. Firm i's optimal conditions are represented as follow:

$$\frac{\partial \mathcal{H}_i^I}{\partial p_i} = 0, \tag{7}$$

$$\frac{\partial \mathcal{H}_i^I}{\partial A_i} = 0 \quad \Longleftrightarrow \quad \phi_i(t) = \frac{2\mu}{\alpha} A_i(t), \tag{8}$$

$$\dot{\phi}_i(t) = -\frac{\partial \mathcal{H}_i^I}{\partial N} + \rho\phi_i(t) = (\lambda + \rho)\phi_i(t) - y_i(t)(p_i(t) - s_i), \tag{9}$$

$$0 = \lim_{t\to\infty} \phi_i(t)N(t)e^{-\rho t} \tag{10}$$

From (7), we obtain the following optimal prices:

$$p_H^I = \frac{(2\bar{\theta} - \underline{\theta})(s_H - s_L) + 2s_H + s_L}{3}, \quad \text{and} \quad p_L^I = \frac{(\bar{\theta} - 2\underline{\theta})(s_H - s_L) + s_H + 2s_L}{3} \tag{11}$$

Variables with the superscript I indicate equilibrium values in the interior case. These prices are guaranteed to be positive under the assumption $s_H > s_L$ and are constant over time.[6] We can easily confirm that p_H^I is always higher than p_L^I.

These prices lead to an equilibrium threshold, calculated as $\tilde{\theta}^I = \frac{\bar{\theta}+\underline{\theta}+1}{3}$. This threshold represents an increasing function of $\bar{\theta}$ and $\underline{\theta}$ but is independent of firm technology levels. Each firm's market share, therefore, may be calculated as follows:

$$y_H^I = \bar{\theta} - \tilde{\theta}^I = \frac{2\bar{\theta} - \underline{\theta} - 1}{3}, \quad \text{and} \quad y_L^I = \tilde{\theta}^I - \underline{\theta} = \frac{\bar{\theta} - 2\underline{\theta} + 1}{3}.$$

Under Assumption 1, firm i's unit profit $(p_i^I - s_i)$ the threshold, and both firms' market shares are always positive. The threshold value is independent of both firms' technology levels. In the corner case below and the uncovered market case (Tenryu and Kamei (2014)), this situation is not observed. In addition, we can confirm that the market share of firm H is

[5]In our model, the state equation and firms' profit functions are linear with respect to the state variable. This situation is called the linear state game: in this case the control variables are independent from the state variable, and the open-loop equilibrium is Markov perfect. (See Dockner, et al. (2000), section 7.3.)

[6]For p_L^I to be positive, $\frac{s_H}{s_L}$ must be larger than $\frac{\bar{\theta}-2\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}$. This condition is always satisfied because $\frac{\bar{\theta}-2\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}$ is smaller than 1.

always higher than that of firm L and that both shares are increasing (decreasing) functions of $\bar{\theta}$ ($\underline{\theta}$) but independent of s_i.

Next, we may use (9) and (11) to obtain steady state co-state variables and the following equation:

$$\phi_i^I = \frac{y_i^I(p_i^I - s_i)}{\lambda + \rho}. \tag{12}$$

This equation and (8) immediately lead to equilibrium advertising strategies:

$$A_H^I = \frac{\alpha(2\bar{\theta} - \underline{\theta} - 1)^2(s_H - s_L)}{18\mu(\lambda + \rho)}, \quad \text{and} \quad A_L^I = \frac{\alpha(\bar{\theta} - 2\underline{\theta} + 1)^2(s_H - s_L)}{18\mu(\lambda + \rho)}. \tag{13}$$

From (5), $\dot{N} = 0$ leads to the following equation:

$$N^I = \frac{\alpha}{\lambda}(A_H^I + A_L^I) = \frac{\alpha^2(s_H - s_L)[4(\bar{\theta} - \underline{\theta})^2 + (\bar{\theta} - 1)^2 + (\underline{\theta} - 1)^2]}{18\lambda\mu(\lambda + \rho)}. \tag{14}$$

This equation represents the steady state consumer density. Additionally, by solving (14), we derive the trajectory of the consumer density as $N^I(t) = N^I + [N(0) - N^I]e^{-\lambda t}$. This result implies that, if the initial density is higher (lower) than the steady state value, the function monotonically decreases (increases) and converges to the steady state.

3.2 Transversality conditions and stability

In this subsection, we confirm that the equilibrium path satisfies the transversality conditions and that the dynamic system is stable. We use equations (2), (3), and (11) to solve the differential equations for the costate variables, (9). The following equations are obtained:

$$\phi_H^I(t) = \frac{(2\bar{\theta} - \underline{\theta} - 1)^2(s_H - s_L)}{9(\lambda + \rho)} + \left[\phi_H^I(0) - \frac{(2\bar{\theta} - \underline{\theta} - 1)^2(s_H - s_L)}{9(\lambda + \rho)}\right] e^{(\lambda+\rho)t},$$
$$\phi_L^I(t) = \frac{(\bar{\theta} - 2\underline{\theta} + 1)^2(s_H - s_L)}{9(\lambda + \rho)} + \left[\phi_L^I(0) - \frac{(\bar{\theta} - 2\underline{\theta} + 1)^2(s_H - s_L)}{9(\lambda + \rho)}\right] e^{(\lambda+\rho)t}.$$

By substituting these equations and the consumer density function, $N(t)$, we may prove that the transversality conditions are satisfied. The result is summarized in the following lemma:

Lemma 1. *The transversality conditions of firm L and firm H are satisfied if and only if the initial costate variables are as follow:*

$$\phi_H^I(0) = \frac{(2\bar{\theta} - \underline{\theta} - 1)^2(s_H - s_L)}{9(\lambda + \rho)} \quad \text{and} \quad \phi_L^I(0) = \frac{(\bar{\theta} - 2\underline{\theta} + 1)^2(s_H - s_L)}{9(\lambda + \rho)}. \tag{15}$$

Proof. The proof follows that presented in Appendix A of Tenryu and Kamei (2014). □

The condition (15) requires that both costate variables are constant at the steady state value. This situation means that the advertising strategies and revenue per density for each firm[7] are also constant at the steady state value. From (8) and (12), we obtain the following equation:

$$\phi_i^I = \frac{2\mu}{\alpha} A_i^I = \frac{y_i^I(p_i^I - s_i)}{\lambda + \rho}. \tag{16}$$

Therefore, all variables except $N^I(t)$ are constant, and the complete dynamical system of the model may be represented with only the law of motion for consumer density (5). As discussed above, the consumer density function converges to the steady state value, and the dynamical system is thus globally stable.

3.3 Corner case

Let us consider the corner case. As discussed in section 2, the optimal price for firm L is $p_L^C = \underline{\theta} s_L$. Because $\underline{\theta}$ and s_L are exogenously given, the price p_L^C is constant for any time t. As for the interior case, we may derive the following demand functions:

$$N(t)y_L(t) = N(t)(\tilde{\theta}(t) - \underline{\theta}) = N(t)\left(\frac{p_H(t) - p_L^C}{s_H - s_L} - \frac{p_L^C}{s_L}\right), \tag{17}$$

$$N(t)y_H(t) = N(t)(\bar{\theta} - \tilde{\theta}(t)) = N(t)\left(\bar{\theta} - \frac{p_H(t) - p_L^C}{s_H - s_L}\right). \tag{18}$$

The sum of the discounted present value of the profit for firm i and the state equation are the same as (4) and (5), respectively. Unless the minimum willingness to pay is greater than 1, firm L's profit cannot be positive.[8] Firm L's Hamiltonian equation, therefore, is as follows:

$$\mathcal{H}_L^C = N(t)y_L(t)(p_L^C - s_L) - \mu A_L(t)^2 + \phi_L(t)[\alpha(A_H(t) + A_L(t)) - \lambda N(t)].$$

Because firm L commits its price to p_L^C, the price is not a control variable; firm L chooses its optimal advertising strategy as the only control variable. Therefore, first-order conditions may be given as (8), (9), and (10). In contrast, firm H's problem is the same as that in the interior case, and first-order conditions are thus given as (7) $-$ (10). Substituting firm L's price into the first-order condition with price H, we obtain firm H's optimal price as follows:

$$p_H^C = \frac{\bar{\theta}(s_H - s_L) + s_H + \underline{\theta} s_L}{2}. \tag{19}$$

[7] Firm i's revenue at time t is $N^I(t)y_i^I(p_i^I - s_i)$. When $N(t) = 1$, the firm's revenue is represented as $y_i^I(p_i^I - s_i)$. We call this value firm i's revenue per density.

[8] The instantaneous profit of firm L is $\pi_L(t) = N(t)y_L(t)(\underline{\theta} - 1)s_L - \mu A_L(t)^2$, such that if $\underline{\theta}$ is smaller than 1, the firm's profit cannot be positive. Therefore, the first condition in Assumption 1 is required.

This price is also constant over time because it depends on only exogenous parameters, and it is higher than p_L^C.[9] The threshold value of the corner case is, therefore, $\tilde{\theta}^C = \frac{\bar{\theta}(s_H - s_L) + s_H - \underline{\theta} s_L}{2(s_H - s_L)}$. In contrast to that of the interior case, this threshold depends on each firm's technology and is an increasing function of $\bar{\theta}$ and s_H but an decreasing function of $\underline{\theta}$ and s_L. Therefore, each firm's market share may be obtained as follows:

$$y_H^C = \bar{\theta} - \tilde{\theta}^C = \frac{(\bar{\theta} - 1)s_H - (\bar{\theta} - \underline{\theta})s_L}{2(s_H - s_L)} \quad \text{and} \quad y_L^C = \tilde{\theta}^C - \underline{\theta} = \frac{(\bar{\theta} - 2\underline{\theta} + 1)s_H - (\bar{\theta} - \underline{\theta})s_L}{2(s_H - s_L)}.$$

Within the domain of the corner case, both shares are positive, and y_H^C is larger than y_L^C.[10] These parameters differ from those of the interior case in that y_i^C is affected by s_i and that $\underline{\theta}$ has an opposite effect on $\tilde{\theta}^C$ and y_H^C.

The equilibrium advertising strategies and steady state consumer density are represented as follow:

$$A_H^C = \frac{\alpha[(\bar{\theta} - 1)s_H - (\bar{\theta} - \underline{\theta})s_L]^2}{8\mu(\lambda + \rho)(s_H - s_L)} \quad \text{and} \quad A_L^C = \frac{\alpha[(\bar{\theta} - 2\underline{\theta} + 1)s_H - (\bar{\theta} - \underline{\theta})s_L](\underline{\theta} - 1)s_L}{4\mu(\lambda + \rho)(s_H - s_L)}, \tag{20}$$

$$N^C = \frac{\alpha^2[(\bar{\theta} - 1)^2 s_H^2 - 2\{(\bar{\theta} - \underline{\theta})^2 + (\underline{\theta} - 1)^2\}s_H s_L + (\bar{\theta} - \underline{\theta})(\bar{\theta} - 3\underline{\theta} + 2)s_L^2]}{8\lambda\mu(\lambda + \rho)(s_H - s_L)^2}. \tag{21}$$

As with (14), (21) is represented as the sum of both firms' advertising investments, and positive advertising investments therefore lead to positive consumer density.

Finally, in the same way as that presented in the previous section, we can confirm that the transversality condition holds and that the dynamic system of the model is globally stable.

Corollary 1. *The transversality conditions of firm L and firm H are satisfied if and only if the initial costate variables are as follow:*

$$\phi_H^C(0) = \frac{[(\bar{\theta} - 1)s_H - (\bar{\theta} - \underline{\theta})s_L]^2}{4(\lambda + \rho)(s_H - s_L)} \quad \textit{and} \quad \phi_L^C(0) = \frac{[(\bar{\theta} - 2\underline{\theta} + 1)s_H - (\bar{\theta} - \underline{\theta})s_L](\underline{\theta} - 1)s_L}{2(\lambda + \rho)(s_H - s_L)}.$$

3.4 Domain of the interior and corner cases.

In the previous subsection, we computed equilibrium candidates corresponding to the interior and corner cases. In this subsection, following the work of Wauthy (1996), we identify the parameter constellations for which the candidates effectively yield the corresponding market outcomes.

The corner case exits when there exists a range of parameter values where neither condition $p_L^P > \underline{\theta} s_L$ nor $p_L^I < \underline{\theta} s_L$. The price p_L^P represents firm L's equilibrium price in an

[9] The condition required for $p_H^C > p_L^C$ to be satisfied is $\frac{s_H}{s_L} > \frac{\bar{\theta} + \underline{\theta}}{\bar{\theta} + 1}$. As discussed below, this condition is satisfied within the existing region of the corner case, $\frac{s_H}{s_L} \in \left(\frac{\bar{\theta} + \underline{\theta} - 2}{\bar{\theta} - \underline{\theta} + 1}, \frac{\bar{\theta} - \underline{\theta}}{\bar{\theta} - 4\underline{\theta} + 3}\right)$.

[10] The domain is derived in the next subsection.

uncovered market, which is obtained as $p_L^P = \frac{s_L\bar{\theta}(s_H - s_L) + 3s_H s_L}{4s_H - s_L}$.[11] The former condition implies that, in this situation, an uncovered market exists. Under the latter condition, the price in the interior case is lower than that in the corner case, and consumers thus purchase good L at the price p_L^I. Therefore, the market is not covered at equilibrium when $\frac{s_H}{s_L} \in \left[\frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}, \infty\right)$, the market is covered by firm L quoting a price just sufficient to cover the market when $\frac{s_H}{s_L} \in \left[\frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}, \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}\right)$, and the market is covered in the usual sense when $\frac{s_H}{s_L} \in \left(1, \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}\right)$.[12][13] In the following section, we analyze the interior and corner cases in accordance with these domains.

4 Producer surplus

In this section, we investigate how the control variables, state variable, and firms' lifetime profits respond to changes in preference dispersion and technological gap. In the next subsection, we calculate the firms' lifetime profits on the equilibrium path and investigate which firm earns more profit. We then calculate comparative statics for prices, advertising, consumer density, and lifetime profits. For notational simplicity, we define a firm's instantaneous revenue per density, $y_i^k(p_i^k - s_i)$, as z_i^k, $k \in \{I, C\}$.

4.1 Firms' lifetime profits

Firm i's lifetime profit is defined by (4). By substituting equilibrium values obtained above into (4), we obtain the following equation:

$$V_i^k = \int_0^\infty \left[N(t)z_i^k - \mu(A_i^k)^2\right] e^{-\rho t} dt = \frac{N(0)z_i^k}{\lambda+\rho} + \frac{\lambda N^k z_i^k - (\lambda+\rho)\mu(A_i^k)^2}{\rho(\lambda+\rho)}, \;\; k \in \{I, C\}. \tag{22}$$

Furthermore, we may use (16) to rewrite this equation. The equation obtained below enables us to understand how advertising investment affects profit:

$$V_i^k = \frac{2\mu}{\rho}\left[\frac{\rho N(0)}{\alpha}A_i^k + \frac{(A_i^k)^2}{2} + A_i^k A_j^k\right], \;\; j \neq i, \;\; k \in \{I, C\}. \tag{23}$$

We can confirm that a positive externality exists, as represented by the third term in the square brackets. Each firm, therefore, benefits not only from its own advertising investment but also from tits opponent's advertising investment. Using this equation, we may obtain the following proposition:

[11] See Tenryu and Kamei (2014) for the derivation of this equation.

[12] If the second condition of Assumption 1 is not satisfied, then the partial market cannot exit. Because we follow the work of Wauthy (1996) and do not exclude the existence of an uncovered market, we assume $\bar{\theta} > 4\underline{\theta} - 3$.

[13] Our model differs from that of Wauthy (1996) in terms of production cost. He assumes zero production cost whereas we assume a linear production cost. In this case, p_L^I is always positive and thus there is no market preempted by firm H.

Proposition 1. *In a covered market, the investment and lifetime profit of firm H are always larger than those of firm L.*

Proof. See Appendix A. □

4.2 Comparative statics for prices, shares, and advertising

In this subsection, we investigate how each firm responds to changes in the maximum and minimum willingness to pay and firm technology levels. Table 1 summarizes the effects of changes in maximum preference, minimum preference, and firm technology levels on prices, threshold, market shares, and advertising investment.

Table 1: Effects of increases in maximum and minimum willingness to pay and firm technology on price, threshold, market share, and advertising investment.

(a) Interior case

	p_L^I	p_H^I	$\tilde{\theta}^I$	y_L^I	y_H^I	A_L^I	A_H^I
$\bar{\theta}$	+	+	+	+	+	+	+
$\underline{\theta}$	−	−	+	−	−	−	−
s_H	+	+	0	0	0	+	+
s_L	$*$	−	0	0	0	−	−

(b) Corner case

	p_L^C	p_H^C	$\tilde{\theta}^C$	y_L^C	y_H^C	A_L^C	A_H^C
$\bar{\theta}$	0	+	+	+	+	+	+
$\underline{\theta}$	+	+	−	−	+	$*$	+
s_H	0	+	+	+	−	+	+
s_L	+	−	−	−	+	$*$	−

0 indicates no effect, and $*$ indicates an ambiguous effect.

Table 1 states that the effect of $\underline{\theta}$ on the parameters varies between the two cases. In the interior case, how p_L^I responds to an increase in s_L is ambiguous.[14] In the corner case, it should be noted that the effects of $\underline{\theta}$ or s_L on firm L's advertising ambiguous. We investigate these situations in the following lemma[15]:

Lemma 2. *If $\bar{\theta} > 8\underline{\theta} - 7$, $\frac{\partial A_L^C}{\partial \underline{\theta}}$ is always positive, and if $4\underline{\theta} - 3 < \bar{\theta} \leq 8\underline{\theta} - 7$,*

$$\frac{\partial A_L^C}{\partial \underline{\theta}} \begin{cases} < 0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \leq \frac{s_H}{s_L} < \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3} \\ = 0 & \text{if } \frac{s_H}{s_L} = \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3} \\ > 0 & \text{if } \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3} < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}. \end{cases}$$

Furthermore, if $\bar{\theta} > 10\underline{\theta} - 9$, $\frac{\partial A_L^C}{\partial s_L}$ is negative; if $5\underline{\theta} - 4 \leq \bar{\theta} \leq 10\underline{\theta} - 9$,

$$\frac{\partial A_L^C}{\partial s_L} \begin{cases} < 0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \leq \frac{s_H}{s_L} < \frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1} \\ = 0 & \text{if } \frac{s_H}{s_L} = \frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1} \\ > 0 & \text{if } \frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1} < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}; \end{cases}$$

[14] The derivative of p_L^I with respect to s_L is $\frac{\partial p_L^I}{\partial s_L} = \frac{-\bar{\theta}+2\underline{\theta}+2}{3}$. Thus $\frac{\partial p_L^I}{\partial s_L} \gtreqless 0$ if and only if $\bar{\theta} \lesseqgtr 2\underline{\theta} + 2$.

[15] The remaining results for advertising are given in Appendix B.

and if $4\underline{\theta} - 3 < \bar{\theta} \leq 5\underline{\theta} - 4$, $\frac{\partial A_L^C}{\partial s_L}$ *is positive.*

Proof. See Appendix C. □

In contrast to the interior case, if the preference dispersion is relatively large in the corner case, firm L increases (decreases) its advertising investment in response to an increase in $\underline{\theta}$ (s_L).

4.3 Comparative Statics for Consumer Density and Profits

Using the results obtained above, we can investigate how the parameters affect consumer density. These effects are summarized as follow:

Proposition 2. *The instantaneous consumer density,* $N^k(t)$, $k \in \{I, C\}$, *and the steady state consumer density,* N^k, *are increasing functions of the maximum willingness to pay and the technology level of firm H and decreasing functions of the technology level of firm L.* $N^I(t)$ *and* N^I *are decreasing functions, and* $N^C(t)$ *and* N^C *are increasing functions, of the minimum willingness to pay,* $\underline{\theta}$.

Proof. The consumer density at time t is represented as $N^k(t) = N^k + [N(0) - N^k]e^{-\lambda t}$. We may take its derivative with respect to $\psi \in \{\bar{\theta}, \underline{\theta}, s_H, s_L\}$, yielding the following equation:

$$\frac{\partial N^k(t)}{\partial \psi} = \left(1 - \frac{1}{e^{\lambda t}}\right) \frac{\partial N^k}{\partial \psi},$$

Because , λt is positive for any time $t > 0$, the value in parentheses is positive, whereas thus the sign of $\frac{\partial N^k(t)}{\partial \psi}$ is determined by the change in the steady state value. Therefore, the steady state value should be investigated.

The consumer density at the steady state is represented as $N^k = \frac{\alpha}{\rho}(A_H^k + A_L^k)$. The signs of the derivatives of N^k are determined by that of the sum of the derivatives of A_H^k and A_L^k. For the interior case, the results are clear from Appendix B.

In the corner case, because advertising is an increasing function of $\bar{\theta}$ and s_H, consumer density is also an increasing function of these parameters. Because the resulting situation is not obtained immediately, we investigate the signs of the derivative of N^C. Differentiating (21) with respect to $\underline{\theta}$ yields the following equation:

$$\frac{\partial N^C}{\partial \underline{\theta}} = \frac{\alpha^2[2(\bar{\theta} - 2\underline{\theta} + 1)s_H - (2\bar{\theta} - 3\underline{\theta} + 1)s_L]}{4\lambda\mu(\lambda + \rho)(s_H - s_L)}.$$

As shown in Appendix B, we may confirm that this equation is positive. Similarly, differentiating (21) with respect to s_L and rearranging, the result yields the following equation:

$$\frac{\partial N^C}{\partial s_L} = -\frac{\alpha^2[(\bar{\theta} - \underline{\theta})(\bar{\theta} - 3\underline{\theta} + 2)(s_H - s_L)^2 + (\underline{\theta} - 1)^2 s_H^2]}{8\lambda\mu(\lambda + \rho)(s_H - s_L)^2}.$$

Under Assumption 1, $\bar{\theta} - 3\underline{\theta} + 2$ is positive and thus the derivative of N^C with respect to s_L is negative. □

This proposition implies that effect of the parameters on consumer density is analogous to their effect on firm H's advertising, even in a situation in which firm L's advertising behavior differs from that of firm H. In other words, the advertising investment by firm H has a dominant effect on the consumer density. The same result is obtained for an uncovered market. (See Tenryu and Kamei (2014).)

In the corner case, the reason why firm L invests more in advertising even if consumer density decreases is because the firm has an opportunity to increase its profits. In the interior case, however, firm L does not have this opportunity. To analyze this result, we investigate how the parameters $\bar{\theta}, \underline{\theta}, s_H$, and s_L affect the firms' lifetime profits. For analytical simplicity, we assume that the initial consumer density is zero. This model thus represents an emerging industry.

Assumption 2. *We assume that the initial consumer density is zero, $N(0) = 0$.*

By differentiating (23) with respect to the parameters, we obtain the following proposition:

Proposition 3. *The effects of the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L on firms' lifetime profits are summarized below:*

Interior case

	V_L^I	V_H^I
$\bar{\theta}$	$+$	$+$
$\underline{\theta}$	$-$	$-$
s_H	$+$	$+$
s_L	$-$	$-$

Corner case

	V_L^C	V_H^C
$\bar{\theta}$	$+$	$+$
$\underline{\theta}$	$*$	$+$
s_H	$+$	$+$
s_L	$*$	$-$

$*$ indicates an ambiguous effect.

When $\underline{\theta}$ increases, there exists a $\bar{\theta}_a \in (5\underline{\theta} - 4, 6\underline{\theta} - 5)$ such that, if $\bar{\theta} > \bar{\theta}_a$, V_L^C increases, and if $4\underline{\theta} - 3 < \bar{\theta} \le \bar{\theta}_a$,

$$\frac{\partial V_L^C}{\partial \underline{\theta}} \begin{cases} > 0 & \text{if } s_a < \frac{s_H}{s_L} < \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}, \\ = 0 & \text{if } \frac{s_H}{s_L} = s_a, \\ < 0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \le \frac{s_H}{s_L} < s_a, \end{cases}$$

where

$$s_a = \frac{s_H}{s_L} = \frac{(\bar{\theta} - 2\underline{\theta} + 1)^3 + \sqrt{D_a}}{(\bar{\theta} - \underline{\theta})^2(\theta - 4\underline{\theta} + 3)},$$

$$D_a = (\bar{\theta} - 2\underline{\theta} + 1)^6 - (\bar{\theta} - \underline{\theta})^2(\bar{\theta} - \underline{\theta})(\theta - 4\underline{\theta} + 3)[(\bar{\theta} - \underline{\theta})(\bar{\theta} - 4\underline{\theta} + 3) - 2(\bar{\theta} - 2\underline{\theta} + 1)(\underline{\theta} - 1)].$$

When s_L increases, there exists an s_b such that, if $\bar{\theta} > 7.181\bar{\theta} - 6.181$, V_L^C decreases, and if $4\underline{\theta} - 3 < \bar{\theta} \le 7.181\bar{\theta} - 6.181$,

$$\frac{\partial V_L^C}{\partial s_L} \begin{cases} < 0 & \text{if } s_c \le \frac{s_H}{s_L} < s_b \\ = 0 & \text{if } \frac{s_H}{s_L} = s_b \\ > 0 & \text{if } s_b < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}, \end{cases}$$

where
$$s_c = \begin{cases} \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} & \text{if } 4\underline{\theta}-3 < \bar{\theta} < 5\underline{\theta}-4 \\ \frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1} & \text{if } 5\underline{\theta}-4 \leq \bar{\theta} \leq 7.181\bar{\theta}-6.181. \end{cases}$$

Proof. See Appendix D. □

Firm L cannot increase its profit in the interior case but can do so in the corner case. This result is associated with the firm's advertising behavior. Let us consider the interior case. As $\underline{\theta}$ increases, the heterogeneity of preference becomes small; that is, $\bar{\theta}-\underline{\theta}$ declines. In this situation, firms should reduce their prices to remain their market shares (see Table 1). Even if the firms lower their prices, however, y_i^I declines so that profit per density, z_i^I, also declines. As shown by (16), the reduction of revenue per density causes a decrease in advertising investment, which leads to a loss of consumer density. As a result, the profits of both firms decrease. In the case of a rise in s_L, the unit profits of both firms, $p_i^I - s_i$, decline while their market shares remain unchanged. Thus, profits per density and advertising investments of both firms also decrease. These changes reduce consumer density and the firms' lifetime profits.

Next, let us consider the corner case. As $\underline{\theta}$ increases, both firms should reduce their prices. Firm L, however, commits its price to $p_L^C = \underline{\theta} s_L$ so that an increase in $\underline{\theta}$ causes p_L^C to increase. Because firm H knows this commitment, the firm has an incentive to increase its price and is thus able to widen its share. This situation leads to rincreases in z_H and advertising. As a consequence, firm H enjoys an additional profit gain. In contrast, firm L loses market share but can increase unit profit through its own price increase. If firm L increases its revenue per density, its lifetime profit also increases. Even in the case where z_L^C declines, an increase in advertising by firm H benefits firm L's lifetime profit. This situation occurs if the preference dispersion is large or if the dispersion is small but the technological gap is relatively large.

When s_L is improved, firm H extends its share by decreasing its price but cannot increase revenue per density. This situation leads to reductions in advertising by firm H and consumer density. Therefore, the lifetime profit of firm H decreases. In contrast, if the technological gap is relatively large, the lifetime profit of firm L increases. This result occurs because firm L can raise z_L^C by increasing its price at the expense of its market share (see Lemma 2). In this situation, the firm can mitigate the loss of consumer density by an advertising increase.

In an uncovered market, the effect of s_L on the profit of firm L is similar to its effect in the corner case.[16] When the technology gap is relatively large, firm L enjoys an additional profit because it can increase both its price and market share.

5 Welfare analysis

As in the previous section, we investigate how increases in the maximum willingness to pay, minimum willingness to pay, firm H's technology, and firm L's technology affect the

[16] In a partially covered market, $\underline{\theta}$ has no direct effect on the parameters.

economic welfare. In our model, consumers are heterogeneous with respect to product preference, such that an increase in the parameters may have different effects on individual consumer and consumer surplus (calculated as the sum of individual utility). Hence, in the next subsection we first consider individual consumers and then analyze consumer surplus.

5.1 Individual utility

The instantaneous utility of the consumer of θ is defined as $u_i(\theta, t) = \theta s_i - p_i(t)$. As discussed above, the prices are constant over time. This situation means that the instantaneous utility of each consumer is constant and separated from the discount rate. Hence, we may represent $u_i(\theta, t)$ as $u_i(\theta)$ and obtain the individual discounted sum of utility as follows:

$$\bar{u}_i^k(\theta) = \int_0^\infty (\theta s_i - p_i^k)e^{-\rho t}dt = \frac{1}{\rho}u_i^k(\theta), \quad i \in \{L, H\}, \quad k \in \{I, C\},$$

The function implies that effect of the parameters on lifetime utility is that of instantaneous utility multiplied by the reciprocal of the discount rate. We may derive the lifetime utility of the consumers who buy good i as follows:

$$\bar{u}_H^I(\theta) = \frac{1}{\rho}u_H^I(\theta) = \frac{(3\theta - 2\bar{\theta} + \underline{\theta} - 2)s_H + (2\bar{\theta} - \underline{\theta} - 1)s_L}{3\rho} \tag{24}$$

$$\bar{u}_L^I(\theta) = \frac{1}{\rho}u_L^I(\theta) = \frac{(3\theta + \bar{\theta} - 2\underline{\theta} - 2)s_L - (\bar{\theta} - 2\underline{\theta} + 1)s_H}{3\rho}. \tag{25}$$

Because these equations are increasing functions of θ, if the utility of consumers with preference $\underline{\theta}$ is positive, the utility of other consumers is also positive. Within the domain of the interior case discussed in section 3.4, we can confirm that consumers of $\underline{\theta}$ obtain positive utility. Using (24) and (25), we may obtain the following results:

Proposition 4. *In the interior case, the effects of increases in the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L on individual utility are summarized below.*

	$\bar{u}_L^I(\theta)$	$\bar{u}_H^I(\theta)$
$\bar{\theta}$	$-$	$-$
$\underline{\theta}$	$+$	$+$
s_H	$-$	$*$
s_L	$+$	$+$

$*$ indicates an ambiguous effect.

When firm H's technology is improved, the change in the utility of consumers who purchase good H may be represented as follows:

$$\frac{\partial \bar{u}_H^I(\theta)}{\partial s_H} \begin{cases} > 0 & if\ \frac{2\bar{\theta} - \underline{\theta} + 2}{3} < \theta \leq \bar{\theta} \\ = 0 & if\ \theta = \frac{2\bar{\theta} - \underline{\theta} + 2}{3} \\ < 0 & if\ \tilde{\theta}^I \leq \theta < \frac{2\bar{\theta} - \underline{\theta} + 2}{3}. \end{cases}$$

Proof. By differentiating (24) and (25) with respect to the different parameters, we obtain the above results. □

Similarly, we may derive the lifetime utility of consumers who purchase good i in the corner case as follows:

$$\bar{u}_H^C(\theta) = \frac{1}{\rho} u_H^C(\theta) = \frac{(2\theta - \bar{\theta} - 1)s_H + (\bar{\theta} - \underline{\theta})s_L}{2\rho} \tag{26}$$

$$\bar{u}_L^C(\theta) = \frac{1}{\rho} u_L^C(\theta) = \frac{(\theta - \underline{\theta})s_L}{\rho}. \tag{27}$$

These equations are also increasing functions of θ. In the corner case, consumers with $\underline{\theta}$ obtain zero utility, but other consumers enjoy positive utility. From (26) and (27), we may derive the following proposition:

Proposition 5. *In the corner case, the effects of increases in the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L on individual utility are summarized below:*

	$\bar{u}_L^I(\theta)$	$\bar{u}_H^I(\theta)$
$\bar{\theta}$	0	$-$
$\underline{\theta}$	$-$	$-$
s_H	$-$	$*$
s_L	$*$	$+$

$*$ indicates an ambiguous effect.

When firm H's technology is improved, the change in utility of consumers who purchase good H may be represented as follows:

$$\frac{\partial \bar{u}_H^C(\theta)}{\partial s_H} \begin{cases} > 0 & \text{if } \frac{\bar{\theta}+1}{2} < \theta \leq \bar{\theta} \\ = 0 & \text{if } \theta = \frac{\bar{\theta}+1}{2} \\ < 0 & \text{if } \tilde{\theta}^C \leq \theta < \frac{\bar{\theta}+1}{2}. \end{cases}$$

When firm L's technology is improved, the utility of consumers with $\underline{\theta}$ does not change, but the utility of all other consumers with increases.

These propositions are illustrated in Figures 3 and 4. The parameters have similar effects on utility, except for $\underline{\theta}$. When $\bar{\theta}$ increases, firm H increases its price because new consumers with stronger preferences buy good H (see Table 1). However, consumers with $\theta \in [\underline{\theta}, \bar{\theta}]$ maintain their appreciation for goods; that is, the first term in the utility function does not change. Therefore, these consumers are worse off. When s_H increases, consumers who buy good H appreciate the good to a greater degree; that is, the first term in the utility function increases. At the same time, the price also increases. These effects on utility are in opposition, leading to an ambiguous total effect. As illustrated in Figures 3 and 4, the utility of consumers with higher $\theta \in [\hat{\theta}^k, \bar{\theta}]$ increases but that of consumers with lower $\theta \in [\underline{\theta}, \hat{\theta}^k)$ decreases. However, in the corner case, p_L^C is independent of $\bar{\theta}$ and s_H such that utility of consumers who buy good L remains unchanged. When s_L rises, the reduction of p_H^k benefits consumers who purchase good H. Regarding the utility of consumers who buy good L, the rise in their appreciation for the good dominates the price increase, even if p_L^k increases. In the case of an increase in $\underline{\theta}$, the heterogeneity of preference becomes small and it is advantageous for firms to reduce their prices, which benefits all consumers (Figure 3(b)). In the corner case, however, firm L commits its price to $p_L^C = \underline{\theta} s_L$ such that it cannot reduce its price. Firm H knows this situation and can thus increase its price. Because the appreciation of consumers buying good H is unchanged, their utility declines (Figure 4(b)).

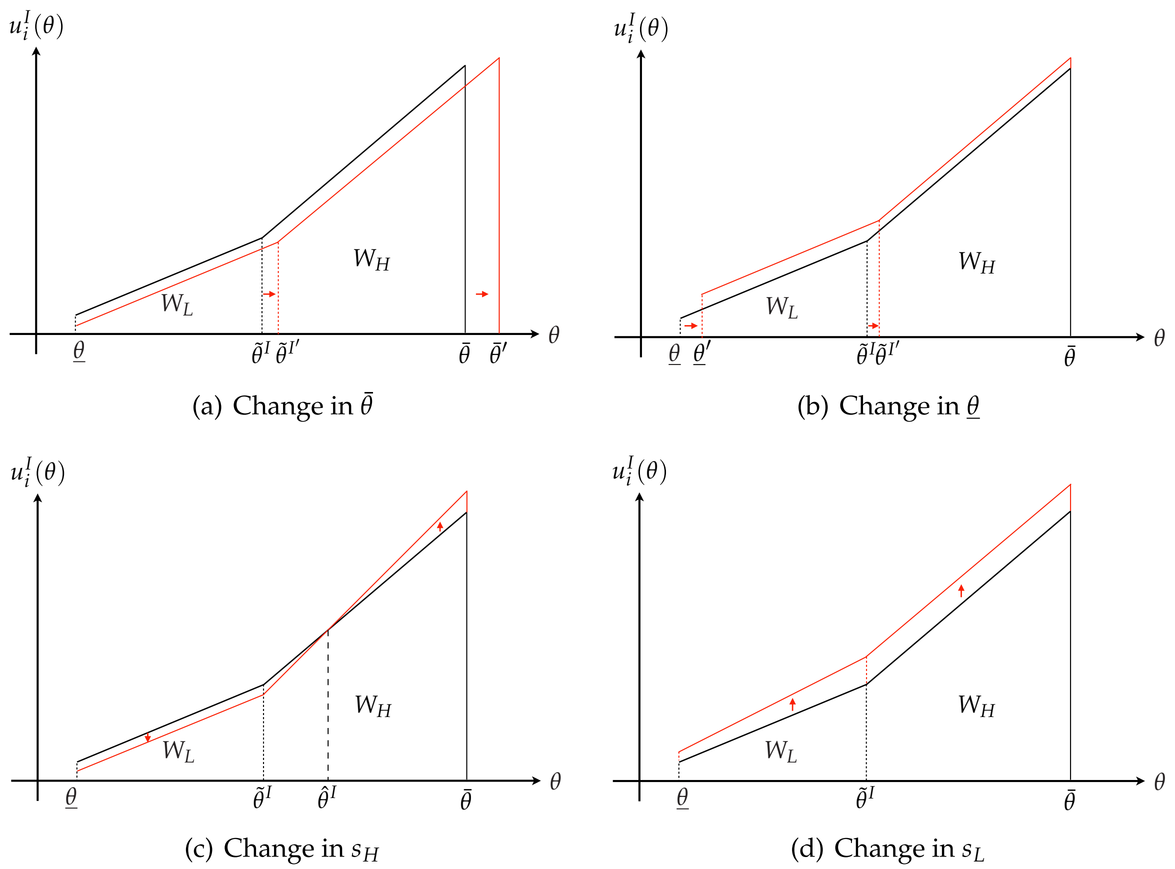
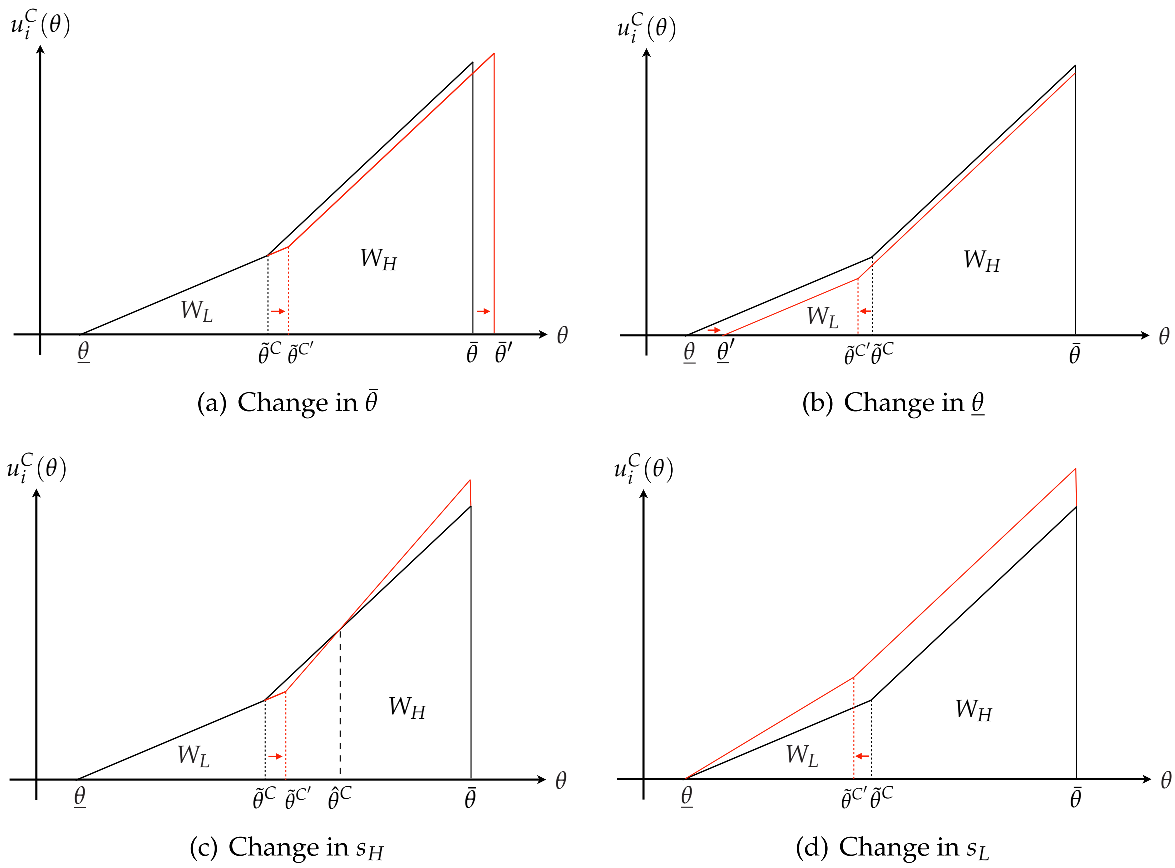


Figure 3: Comparative statics for individual utility (1): the interior case



(a) Change in $\bar{\theta}$

(b) Change in $\underline{\theta}$

(c) Change in s_H

(d) Change in s_L

Figure 4: Comparative statics for individual utility (2): the corner case

5.2 Consumer surplus

In this subsection, we analyze economic welfare. To do so, we must consider the evolution of consumer density, $N^k(t)$. We define the lifetime consumer surplus as follows ;

$$W = \int_{\underline{\theta}}^{\tilde{\theta}^k} \int_0^{\infty} N(t)(\theta s_L - p_L^k)e^{-\rho t} dt d\theta + \int_{\tilde{\theta}^k}^{\bar{\theta}} \int_0^{\infty} N(t)(\theta s_H - p_H^k)e^{-\rho t} dt d\theta, \quad k \in \{I, C\}.$$

For notational simplicity, we represent the sum of utility that consumers derive from purchasing good i as W_i^k:

$$W_L^k \equiv \int_{\underline{\theta}}^{\tilde{\theta}^k} (s_L\theta - p_L^k)d\theta, \quad \text{and} \quad W_H^k \equiv \int_{\tilde{\theta}^k}^{\bar{\theta}} (s_H\theta - p_H^k)d\theta. \tag{28}$$

Because instantaneous utility is constant and the state variable is independent of consumer preference, no interaction exists between θ and t. Therefore, the lifetime surplus function can be rearranged as follows:

$$W^k = \int_0^{\infty} N(t)e^{-\rho t}(W_L^k + W_H^k)dt = \frac{\lambda N^k(W_L^k + W_H^k)}{\rho(\lambda + \rho)}, k \in \{I, C\}. \tag{29}$$

As discussed below, the effects of the parameters can be divided into advertising and utility effects. The advertising effect is represented as the change in consumer density, and the utility effect is represented by the change in the sum of utility, $W_L^k + W_H^k$. We first investigate how the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L affect the sum of utility. The results are summarized in the following lemma:

Lemma 3. *The effects of the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L on instantaneous consumer surplus are summarized below.*

	$W_L^I + W_H^I$	$W_L^C + W_H^C$
$\bar{\theta}$	$+$	$+$
$\underline{\theta}$	$*$	$-$
s_H	$*$	$+$
s_L	$+$	$+$

$*$ indicates an ambiguous effect.

When $\underline{\theta}$ increases, the change in $W_L^I + W_H^I$ may be represented as follows:

$$\frac{\partial(W_L^I + W_L^I)}{\partial \underline{\theta}} \begin{cases} > 0 & \text{if } \frac{7\bar{\theta}-2\underline{\theta}-5}{7\bar{\theta}-11\underline{\theta}+4} < \frac{s_H}{s_L} < \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \\ = 0 & \text{if } \frac{s_H}{s_L} = \frac{7\bar{\theta}-2\underline{\theta}-5}{7\bar{\theta}-11\underline{\theta}+4} \\ < 0 & \text{if } 1 < \frac{s_H}{s_L} < \frac{7\bar{\theta}-2\underline{\theta}-5}{7\bar{\theta}-11\underline{\theta}+4}. \end{cases}$$

When s_L increases, the change in $W_L^I + W_H^I$ may be represented as follows:

$$\frac{\partial(W_L^I + W_L^I)}{\partial s_H} \begin{cases} > 0 & \text{if } 4\underline{\theta} - 3 < \bar{\theta} < 6.098\underline{\theta} - 5.098 \\ = 0 & \text{if } \bar{\theta} = 6.098\underline{\theta} - 5.098 \\ < 0 & \text{if } 6.098\underline{\theta} - 5.098 < \bar{\theta}. \end{cases}$$

Proof. See Appendix E. □

The effects of $\underline{\theta}$ and s_H in the interior case are ambiguous. An increase in $\underline{\theta}$ causes both firms to reduce their prices, while consumers with lower θ refrain from buying and exit the market. For a relatively large (small) technological gap, the former (latter) trend dominates, causing instantaneous consumer surplus to increase (decrease). Regarding s_H in the interior case, a rise in s_H causes p_L^I to increase, which leads to the loss of consumer surplus for relatively large preference dispersion.

Finally, we analyze lifetime consumer surplus using these results.

Proposition 6. *The effects of the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L on lifetime consumer surplus are summarized below:*

	W^I	W^C
$\bar{\theta}$	$+$	$+$
$\underline{\theta}$	$-$	$*$
s_H	$+$	$+$
s_L	$*$	$*$

$*$ indicates an ambiguous effect.

In the interior case, when $\underline{\theta}$ increases, there exists a $\bar{\theta}_b \in (4\underline{\theta}-3, 5\underline{\theta}-4)$ such that, for the region where $\bar{\theta} > \bar{\theta}_b$, $\frac{\partial W^I}{\partial s_L}$ is negative. Otherwise,

$$\frac{\partial W^I}{\partial s_L} \begin{cases} > 0 & \text{if } \Theta < \frac{s_H}{s_L} < \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \\ = 0 & \text{if } \frac{s_H}{s_L} = \Theta \\ < 0 & \text{if } 1 < \frac{s_H}{s_L} < \Theta, \end{cases}$$

where

$$\Theta \equiv \frac{4(2\bar{\theta}-\underline{\theta}-1)^2 + 6(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)}{(2\bar{\theta}-\underline{\theta}-1)(4\bar{\theta}-5\underline{\theta}+1) + (\bar{\theta}-2\underline{\theta}+1)(5\bar{\theta}-4\underline{\theta}-1)}.$$

In the corner case, when $\underline{\theta}$ increases, there exist a $\bar{\theta}_c \in (4\underline{\theta}-3, 5\underline{\theta}-4)$ and an s_b such that,

$$\frac{\partial W^C}{\partial \underline{\theta}} \begin{cases} > 0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \leq \frac{s_H}{s_L} < s_b \\ = 0 & \text{if } \frac{s_H}{s_L} = s_b \\ < 0 & \text{if } s_b < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}. \end{cases}$$

For $\bar{\theta} > \bar{\theta}_c$, $\frac{\partial W^C}{\partial \underline{\theta}}$ is positive. When s_L increases, there exist a $\bar{\theta} \in (5\underline{\theta}-4, 6\underline{\theta}-5)$ and an s_c such that, for $4\underline{\theta}-3 < \bar{\theta} \leq \bar{\theta}_d$,

$$\frac{\partial W^C}{\partial s_L} \begin{cases} < 0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \leq \frac{s_H}{s_L} < s_c \\ = 0 & \text{if } \frac{s_H}{s_L} = s_c \\ > 0 & \text{if } s_c < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}. \end{cases}$$

For $\bar{\theta} > \bar{\theta}_d$, $\frac{\partial W^C}{\partial s_L}$ is negative.

Proof. See Appendix F. □

An increase in $\bar{\theta}$ has a positive effect on lifetime consumer surplus. This result occurs because, although individual consumers are worse off, new consumers with higher θ enter the market and buy a product. Therefore, consumer density increases. As for $\underline{\theta}$ in the interior case, even if the instantaneous consumer surplus rises, the advertising effects dominates and the lifetime consumer surplus therefore declines. In the corner case, instantaneous consumer surplus declines, while the advertising effect dominates, when the preference dispersion is relatively large. Therefore, lifetime consumer surplus is improved. If the diversion is relatively small, however, the change of lifetime consumer surplus is determined by the technological gap. When s_H increases, the advertising effect is very large, and lifetime consumer surplus increases in both the interior and corner cases. In the case of s_L, the signs are ambiguous in both cases. If the diversity of preference and technological gap are relatively small, the advertising effect is so large that lifetime consumer surplus decreases.

6 Conclusion

In this paper, we have described a differential duopoly game in which each firm voluntarily advertises to persuade consumers to buy its products over that of the other firm. We have assumed that the technology levels of firms are exogenously given. Under this situation, we have analyzed both interior and corner cases and investigated how the diversity of preference and technological gap affect advertising, profit, and economic welfare using the vertical product differentiation model.

We have shown that effects of the minimum willingness to pay and the technology of firm L on firm L's lifetime profit may differ in both cases. In the corner case, firm L has the opportunity to increase its profits. This result was similar to that previously obtained for an uncovered market (Tenryu and Kamei (2014)). We have also shown that changes in preference dispersion and technological gap differently affect individual utility and consumer surplus.

The model described here represents only a beginning to the investigation of consumer behavior using the utility-based approach. This investigation could be extended in several directions. One avenue of future research is to consider a nonlinear state game and the resulting interaction between prices and advertising investments. The advertising investment strategies in this case would be represented by Cobb-Douglas or quadratic function instead of a linear function. Another direction for research is to introduce another advertising model into the present model. For example, we could use the Lanchester model to extend the present model to two industries model.

Appendix A. Proof of proposition 1

In the interior case, as seen from (13), we may easily confirm that firm H's advertising investment is always larger than firm L's.

$$A_H^I - A_L^I = \frac{\alpha(\bar{\theta} - \underline{\theta})(\bar{\theta} + \underline{\theta} - 2)(s_H - s_L)}{6\mu(\lambda + \rho)} > 0.$$

Similarly, in the corner case from (20), we obtain the following equation:

$$A_H^C - A_L^C = \frac{\alpha[(\bar{\theta} - 1)^2(s_H - s_L)^2 + (\underline{\theta} - 1)^2(4s_H - s_L)s_L]}{8\mu(\lambda + \rho)(s_H - s_L)} > 0.$$

According to (23), we may compare the firms' profits.

$$V_H^I - V_L^I = \mu(A_H^I - A_L^I)\left[\frac{2N(0)}{\alpha} + \frac{A_H^I + A_L^I}{\rho}\right] > 0.$$

Similarly, we can confirm that $V_H^C > V_L^C$.

Appendix B. The remaining derivatives of A_H^C and A_L^C

Note that the following relations hold.

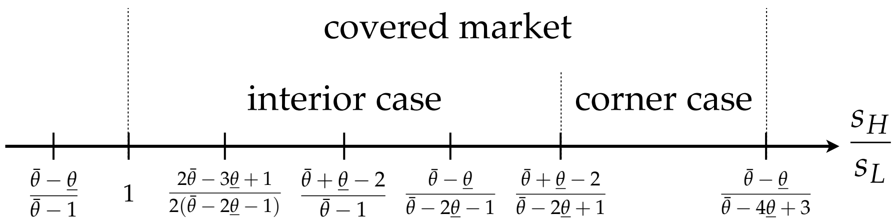


Figure 5: The Relationships among Parameters (1)

Because the interior case is clear, we need prove only the corner case. Differentiating (20) with respect to the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L yields the following equations:

$$\frac{\partial A_H^C}{\partial \bar{\theta}} = \frac{\alpha[(\bar{\theta} - 1)s_H - (\bar{\theta} - \underline{\theta})s_L]}{4\mu(\lambda + \rho)} > 0,$$

$$\frac{\partial A_H^C}{\partial \underline{\theta}} = \frac{\alpha[(\bar{\theta} - 1)s_H - (\bar{\theta} - \underline{\theta})s_L]s_L}{4\mu(\lambda + \rho)(s_H - s_L)} > 0,$$

$$\frac{\partial A_H^C}{\partial s_H} = \frac{\alpha[(\bar{\theta} - 1)s_H - (\bar{\theta} - \underline{\theta})s_L][(\bar{\theta} - 1)s_H - (\bar{\theta} + \underline{\theta} - 2)s_L]}{8\mu(\lambda + \rho)(s_H - s_L)^2} > 0,$$

$$\frac{\partial A_H^C}{\partial s_L} = -\frac{\alpha[(\bar{\theta}-1)s_H - (\bar{\theta}-\underline{\theta})s_L][(\bar{\theta}-2\underline{\theta}+1)s_H - (\bar{\theta}-\underline{\theta})s_L]}{8\mu(\lambda+\rho)(s_H-s_L)^2} < 0,$$
$$\frac{\partial A_L^C}{\partial \bar{\theta}} = \frac{\alpha(\underline{\theta}-1)s_L}{4\mu(\lambda+\rho)} > 0,$$
$$\frac{\partial A_L^C}{\partial s_H} = \frac{\alpha(\underline{\theta}-1)^2 s_L^2}{4\mu(\lambda+\rho)(s_H-s_L)^2} > 0.$$

Appendix C. Proof of Lemma 2

First, we differentiate (20) with respect to $\underline{\theta}$.

$$\frac{\partial A_L^C}{\partial \underline{\theta}} = \frac{\alpha[(\bar{\theta}-4\underline{\theta}+3)s_H - (\bar{\theta}-2\underline{\theta}+1)s_L]s_L}{4\mu(\lambda+\rho)(s_H-s_L)}$$

The sign is determined by the relationship between $\frac{s_H}{s_L}$ and $\frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}$. To confirm this, we compare the upper and lower bounds in the corner case with $\frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}$ as follows.

$$\frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3} - \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3} = -\frac{\underline{\theta}-1}{\bar{\theta}-4\underline{\theta}+3} < 0$$
$$\frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3} - \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} = -\frac{(\bar{\theta}-8\underline{\theta}+7)(\underline{\theta}-1)}{(\bar{\theta}-2\underline{\theta}+1)(\bar{\theta}-4\underline{\theta}+3)}$$

These calculations imply that, if $\bar{\theta} > 8\underline{\theta}-7$, the lower bound is larger than $\frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}$ and that, if $4\underline{\theta}-3 < \bar{\theta} \le 8\underline{\theta}-7$, $4\underline{\theta}-3 < \bar{\theta} \le 8\underline{\theta}-7$ is between the upper and lower bounds. Therefore, we can summarize the result as presented above.

Similarly, we differentiate (20) with respect to s_L.

$$\frac{\partial A_L^C}{\partial s_L} = \frac{\alpha(\underline{\theta}-1)[(\bar{\theta}-2\underline{\theta}+1)s_H^2 - 2(\bar{\theta}-\underline{\theta})s_H s_L + (\bar{\theta}-\underline{\theta})s_L^2]}{4\mu(\lambda+\rho)(s_H-s_L)^2}. \tag{30}$$

The sign is determined by the equation $(\bar{\theta}-2\underline{\theta}+1)s_H^2 - 2(\bar{\theta}-\underline{\theta})s_H s_L + (\bar{\theta}-\underline{\theta})s_L^2$. Let us check the value of this equation at the lower bound. We substitute $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$ into the equation as follows:

$$s_L^2\left[\frac{(\bar{\theta}+\underline{\theta}-2)^2}{\bar{\theta}-2\underline{\theta}+1} - \frac{2(\bar{\theta}-\underline{\theta})(\bar{\theta}+\underline{\theta}-2)}{\bar{\theta}-2\underline{\theta}+1} + (\bar{\theta}-\underline{\theta})\right] = -\frac{(\bar{\theta}-5\underline{\theta}+4)(\underline{\theta}-1)}{\bar{\theta}-2\underline{\theta}+1}s_L^2.$$

This calculation implies that, at the lower bound, the equation is nonpositive if $\bar{\theta} \ge 5\underline{\theta}-4$ and positive if $4\underline{\theta}-3 < \bar{\theta} < 5\underline{\theta}-4$. The abscissa of the vertex is smaller than the lower bound value[17] such that, in the region where $4\underline{\theta}-3 < \bar{\theta} < 5\underline{\theta}-4$, the equation and (30)

[17] See Appendix B.

are positive. Similarly, let us check the equation value at the upper bound by substituting $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}s_L$ into the equation as follows:

$$s_L^2\left[\frac{(\bar{\theta}-2\underline{\theta}+1)(\bar{\theta}-\underline{\theta})}{(\bar{\theta}-4\underline{\theta}+3)^2}-\frac{2(\bar{\theta}-\underline{\theta})}{\bar{\theta}-4\underline{\theta}+3}+(\bar{\theta}-\underline{\theta})\right]=-\frac{(\bar{\theta}-\underline{\theta})(\bar{\theta}-10\underline{\theta}+9)(\underline{\theta}-1)}{(\bar{\theta}-4\underline{\theta}+3)^2}. \tag{31}$$

This calculation implies that, at the upper bound, the equation is negative if $\bar{\theta} > 10\underline{\theta}-9$ and nonnegative if $4\underline{\theta}-3 < \bar{\theta} \leq 10\underline{\theta}-9$. Thus, in the region where $\bar{\theta} > 10\underline{\theta}-9$, the equation and (30) are negative.

Within $5\underline{\theta}-4 \leq \bar{\theta} \leq 10\underline{\theta}-9$, the sign of the equation changes from negative to positive at the solution, $s_H = \frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1}$. Therefore we can summarize these results as the table given in the proposition.

Appendix D. Proof of Proposition 3

We differentiate (23) with respect to the parameters as follows:

$$\frac{\partial V_i^k}{\partial \psi}=\frac{2\mu}{\rho}\left[(A_i^k+A_j^k)\frac{\partial A_i^k}{\partial \psi}+A_i^k\frac{\partial A_j^k}{\partial \psi}\right], \quad i,j(\neq i)\in\{H,L\}, k\in\{I,C\}, \tag{32}$$

where $\psi \in \{\bar{\theta}, \underline{\theta}, s_H, s_L\}$. In the interior case, the effects of the parameters $\bar{\theta}$, $\underline{\theta}$, s_H, and s_L on both firms' advertising behavior are the same. Thus, (32) has the same sign for each parameter.

In the corner case, if ψ is $\bar{\theta}$ or s_H, $\frac{\partial A_i^I}{\partial \psi}$ is positive such that $\frac{\partial V_i^I}{\partial \psi}$ is positive. In contrast, if ψ is $\underline{\theta}$ or s_L, the effects of these parameters on both firms' advertising investment may differ because of the indeterminacy of firm L's advertising behavior. The change in consumer density is represented by the sum of the marginal advertising effects as follows:

$$\frac{\partial N^C}{\partial \psi}=\frac{\alpha}{\lambda}\left(\frac{\partial A_H^C}{\partial \psi}+\frac{\partial A_L^C}{\partial \psi}\right).$$

According to Proposition 2, this result is uniquely determined: the effect of $\underline{\theta}$ is positive, and that of s_L is negative. Using this result, we can rearrange (32) and investigate the effects of the parameters on firm H's lifetime profit.

$$\frac{\partial V_H^C}{\partial \underline{\theta}}=\frac{2\mu}{\rho}\left[A_H^C\underbrace{\left(\frac{\partial A_H^C}{\partial \underline{\theta}}+\frac{\partial A_L^C}{\partial \underline{\theta}}\right)}_{(+)}+A_H^C\underbrace{\frac{\partial A_H^C}{\partial \underline{\theta}}}_{(+)}\right]>0,$$

$$\frac{\partial V_H^C}{\partial s_L}=\frac{2\mu}{\rho}\left[A_H^C\underbrace{\left(\frac{\partial A_H^C}{\partial s_L}+\frac{\partial A_L^C}{\partial s_L}\right)}_{(-)}+A_H^C\underbrace{\frac{\partial A_H^C}{\partial s_L}}_{(-)}\right]<0.$$

Finally, we investigate the effects of $\underline{\theta}$ and s_L on firm L's lifetime profit. We only prove the cases in which $\underline{\theta}$ and s_L differently affect advertising and the consumer density. First, for a change in $\underline{\theta}$, we consider the case in which $4\underline{\theta}-3 \leq \theta < 8\underline{\theta}-7$ and $\frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} < \frac{s_H}{s_L} < \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}$.

$$\frac{\partial V_L^C}{\partial \underline{\theta}} = \frac{2\mu}{\rho}\left[A_L^C \underbrace{\left(\frac{\partial A_H^C}{\partial \underline{\theta}} + \frac{\partial A_L^C}{\partial \underline{\theta}}\right)}_{(+)} + A_H^C \frac{\partial A_L^C}{\partial \underline{\theta}}\right] = \frac{\alpha^2 s_L \Sigma_a(\bar{\theta},\underline{\theta},s_H,s_L)}{32\mu^2(\lambda+\rho)^2(s_H-s_L)}, \tag{33}$$

where

$$\begin{aligned}\Sigma_a(\bar{\theta},\underline{\theta},s_H,s_L) = {} & (\bar{\theta}-1)^2(\bar{\theta}-4\underline{\theta}+3)s_H^2 - 2(\bar{\theta}-2\underline{\theta}+1)^3 s_H s_L \\ & + (\bar{\theta}-\underline{\theta})[(\bar{\theta}-\underline{\theta})(\bar{\theta}-4\underline{\theta}+3) - 2(\bar{\theta}-2\underline{\theta}+1)(\underline{\theta}-1)]s_L.\end{aligned}$$

Because the sign of (33) depends on that of Σ_a, we focus on the sign of this value. We check the function value at the upper bound of the technology gap between the firms. The inflection point of Σ_a is $s_H = \frac{(\bar{\theta}-2\underline{\theta}+1)^3}{(\bar{\theta}-1)^2(\bar{\theta}-4\underline{\theta}+3)}s_L$, which is smaller than $s_H = \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}s_L$, and Σ_a is positive at $s_H = \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}s_L$. Next, we check the sign at the lower bound of the technology gap, $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$. After tedious calculations, we obtain the following equation:

$$\Sigma_a = \left(\frac{2(\underline{\theta}-1)s_L}{\bar{\theta}-2\underline{\theta}+1}\right)^2 [3\bar{\theta}^3 - (22\underline{\theta}-13)\bar{\theta}^2 + (31\underline{\theta}^2 - 18\underline{\theta}-4)\bar{\theta} - 16\bar{\theta}^3 + 17\underline{\theta}^2 - 8\underline{\theta} + 4]$$

The sign is determined by that of the cubic function in the square brackets. The cubic function reaches a local minimum at $\bar{\theta} \approx 4.0353\underline{\theta} - 3.0353$ and becomes negative at $\bar{\theta} = 4\underline{\theta} - 3$. This result means that, for $\bar{\theta} = 4\underline{\theta}-3$, the function stays positive after it changes from negative to positive. The function has a solution between $(5\underline{\theta}-4, 6\underline{\theta}-5)$. We represent the solution as $\bar{\theta}_a$.[18] The function is positive at $\bar{\theta} \in (\bar{\theta}_a, 8\underline{\theta}-7)$, and thus Σ_a is also positive. Conversely, for $\bar{\theta} \in (4\underline{\theta}-3, \bar{\theta}_a]$, we can confirm that

$$\Sigma_a(\bar{\theta},\underline{\theta},s_H,s_L)\begin{cases} >0 & \text{if } \frac{s_H}{s_L} \in \left(s_a, \frac{\bar{\theta}-2\underline{\theta}+1}{\bar{\theta}-4\underline{\theta}+3}\right), \\ =0 & \text{if } \frac{s_H}{s_L} = s_a, \\ <0 & \text{if } \frac{s_H}{s_L} \in \left(\frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}, s_a\right), \end{cases}$$

where s_a is a solution of Σ_a.

$$s_a = \frac{s_H}{s_L} = \frac{(\bar{\theta}-2\underline{\theta}+1)^3 + \sqrt{D_a}}{(\bar{\theta}-\underline{\theta})^2(\theta-4\underline{\theta}+3)},$$

where

$$D_a = (\bar{\theta}-2\underline{\theta}+1)^6 - (\bar{\theta}-\underline{\theta})^2(\bar{\theta}-\underline{\theta})(\theta-4\underline{\theta}+3)[(\bar{\theta}-\underline{\theta})(\bar{\theta}-4\underline{\theta}+3) - 2(\bar{\theta}-2\underline{\theta}+1)(\underline{\theta}-1)]$$

[18] When $\underline{\theta}=2$, $\bar{\theta}^a$ is 6.67918.

Next, let us consider the effect of s_L on V_L^C. According to Lemma 2, if $\frac{\partial A_L^C}{\partial s_L}$ is nonpositive, we can easily confirm that $\frac{\partial V_L^C}{\partial s_L}$ is negative. Thus, we need only consider the cases where (i) $4\underline{\theta} - 3 < \bar{\theta} < 5\underline{\theta} - 4$ and (ii) $5\underline{\theta} - 4 \leq \bar{\theta} \leq 10\underline{\theta} - 9$ and $\frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1} < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}$.

$$\frac{\partial V_L^C}{\partial s_L} = \frac{2\mu}{\rho}\left[A_L^C \underbrace{\left(\frac{\partial A_H^C}{\partial s_L} + \frac{\partial A_L^C}{\partial s_L}\right)}_{(-)} + A_H^C \frac{\partial A_L^C}{\partial s_L}\right] = \frac{\alpha^2(\underline{\theta}-1)\Sigma_b(\bar{\theta},\underline{\theta},s_H,s_L)}{32\mu^2(\lambda+\rho)^2(s_H-s_L)^2}, \tag{34}$$

where

$$\begin{aligned}\Sigma_b(\bar{\theta},\underline{\theta},s_H,s_L) = {} & (\bar{\theta}-1)^2(\bar{\theta}-2\underline{\theta}+1)s_H^3 - 4(\bar{\theta}-\underline{\theta})[(\bar{\theta}-\underline{\theta})^2 + (\underline{\theta}-1)^2]s_H^2 s_L \\ & + (\bar{\theta}-\underline{\theta})[(\bar{\theta}-2\underline{\theta}+1)(\bar{\theta}-3\underline{\theta}+2) + 4(\bar{\theta}-\underline{\theta})^2]s_H s_L^2 - 2(\bar{\theta}-\underline{\theta})^2[(\underline{\theta}-2\underline{\theta}+1)]s_L^3.\end{aligned}$$

The function Σ_b is negative and decreasing at $s_H = s_L$, which means that it changes to increasing at the local minimum for the region where $s_H > s_L$. In addition, for $\bar{\theta} > 4\underline{\theta} - 3$, Σ_b is negative at both $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$ and $s_H = \frac{(\bar{\theta}-\underline{\theta})+\sqrt{(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)}}{\bar{\theta}-2\underline{\theta}+1}s_L$. However, the value of Σ_b at $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}s_L$ is ambiguous. Substituting $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}s_L$ into Σ_b yields the following equation:

$$\Sigma_b = -\frac{4(\bar{\theta}-\underline{\theta})^2(\underline{\theta}-1)^2 s_L^3}{(\bar{\theta}-4\underline{\theta}+3)^3}[3\bar{\theta}^2 - 2(13\underline{\theta}-10)\bar{\theta} + 32\underline{\theta}^2 - 38\underline{\theta} + 9].$$

This equation is nonnegative for $4\underline{\theta} < \bar{\theta} \leq 7.181\underline{\theta} - 6.181$ and negative for the other region. Therefore, for $4\underline{\theta} < \bar{\theta} \leq 7.181\underline{\theta} - 6.181$, there exists a θ-intercept for Σ_b. This result means that Σ_b is negative under the θ-intercept but positive over it. For the region where $\bar{\theta} > 7.181\underline{\theta} - 6.181$, Σ_b is negative.

Appendix E. Proof of Lemma 3

Note that the following relations hold:

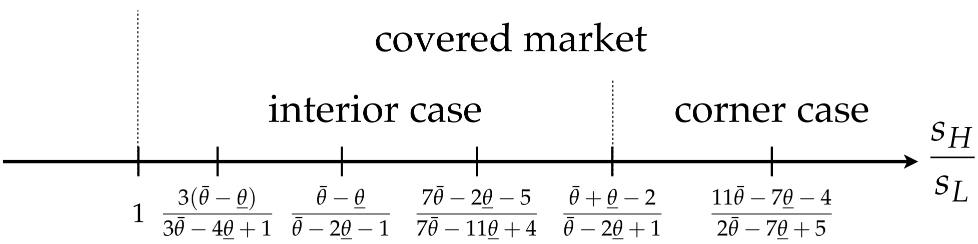


Figure 6: The Relationships among Parameters (2)

Let us consider the interior case. Using (24), (25), and (28), we may obtain the sum of the instantaneous utility as follows:

$$W_L^L + W_H^I = \frac{1}{18}\begin{bmatrix} [3(2\bar{\theta}-\underline{\theta}-1)(\underline{\theta}-1)-2(\bar{\theta}-2\underline{\theta}+1)^2]s_H \\ +[3(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)+2(2\bar{\theta}-\underline{\theta}-1)^2]s_L \end{bmatrix}.$$

We differentiate this equation with respect to the parameters.

$$\frac{\partial(W_L^I+W_H^I)}{\partial\bar{\theta}} = \frac{(-2\bar{\theta}+7\underline{\theta}-5)s_H+(11\bar{\theta}-7\underline{\theta}-4)s_L}{9} > 0$$

$$\frac{\partial(W_L^I+W_H^I)}{\partial\underline{\theta}} = \frac{(7\bar{\theta}-11\underline{\theta}+4)s_H-(7\bar{\theta}-2\underline{\theta}-5)s_L}{9} \tag{35}$$

$$\frac{\partial(W_L^I+W_H^I)}{\partial s_H} = \frac{3(2\bar{\theta}-\underline{\theta}-1)(\underline{\theta}-1)-2(\bar{\theta}-2\underline{\theta}+1)^2}{18} \tag{36}$$

$$\frac{\partial(W_L^I+W_H^I)}{\partial s_L} = \frac{3(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)+2(2\bar{\theta}-\underline{\theta}-1)^2]s_L}{18} > 0.$$

As for (35), we obtain the following equation:

$$\frac{\partial(W_L^I+W_L^I)}{\partial\underline{\theta}}\begin{cases} >0 & \text{if } \frac{7\bar{\theta}-2\underline{\theta}-5}{7\bar{\theta}-11\underline{\theta}+4} < \frac{s_H}{s_L} < \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \\ =0 & \text{if } \frac{s_H}{s_L} = \frac{7\bar{\theta}-2\underline{\theta}-5}{7\bar{\theta}-11\underline{\theta}+4} \\ <0 & \text{if } 1 < \frac{s_H}{s_L} < \frac{7\bar{\theta}-2\underline{\theta}-5}{7\bar{\theta}-11\underline{\theta}+4}. \end{cases}$$

To determine the sign of (36), we arrange the numerator as $-2\bar{\theta}^2+2(7\underline{\theta}-5)\bar{\theta}-11\underline{\theta}^2+8\underline{\theta}+1$. For the region where $\bar{\theta} > 4\underline{\theta}-3$, the equation is decreasing and has one solution, $\bar{\theta} \approx 6.098\underline{\theta}-5.098$. Therefore, we obtain the following equation:

$$\frac{\partial(W_L^I+W_H^I)}{\partial s_H}\begin{cases} >0 & \text{if } 4\underline{\theta}-3 < \bar{\theta} < 6.098\underline{\theta}-5.098 \\ =0 & \text{if } \bar{\theta} = 6.098\underline{\theta}-5.098 \\ <0 & \text{if } 6.098\underline{\theta}-5.098 < \bar{\theta}. \end{cases}$$

Similarly, let us consider the corner case. Using (26), (27), and (28), we may obtain the sum of the instantaneous utility as follows:

$$W_L^C + W_H^C = \frac{1}{18(s_H-s_L)^2}\begin{bmatrix} (\bar{\theta}-1)^2 s_H^3 + \{(\bar{\theta}-2\underline{\theta}+1)^2 - 2(\bar{\theta}-1)(\underline{\theta}-1)\}s_H^2 s_L \\ -(\bar{\theta}-\underline{\theta})(5\bar{\theta}-7\underline{\theta}+2)s_H s_L^2 + 3(\bar{\theta}-\underline{\theta})^2 s_L^3 \end{bmatrix}.$$

Using the relations in Figures 1 and 2 and differentiating this equation with respect to the parameters yields the following equations:

$$\frac{\partial(W_L^C+W_H^C)}{\partial\bar{\theta}} = \frac{(\bar{\theta}-1)s_H+3(\bar{\theta}-\underline{\theta})s_L}{4} > 0,$$

$$\frac{\partial(W_L^C+W_H^C)}{\partial\underline{\theta}} = -\frac{[(3\bar{\theta}-4\underline{\theta}+1)s_H-3(\bar{\theta}-\underline{\theta})s_L]s_L}{4(s_H-s_L)} < 0,$$
$$\frac{\partial(W_L^C+W_H^C)}{\partial s_H} = \frac{\{(\bar{\theta}-1)s_H-(\bar{\theta}+\underline{\theta}-2)s_L\}\{(\bar{\theta}-1)s_H-(\bar{\theta}-\underline{\theta})s_L\}}{8(s_H-s_L)^2} > 0,$$
$$\frac{\partial(W_L^C+W_H^C)}{\partial s_L} = \frac{3(\bar{\theta}-\underline{\theta})^2(s_H-s_L)^2+(\underline{\theta}-1)^2s_H^2}{8(s_H-s_L)^2} > 0.$$

Appendix F. Proof of Proposition 6

We differentiate (29) with respect to $\psi \in \{\bar{\theta}, \underline{\theta}, s_H, s_L\}$ and obtain the following equation:

$$\frac{\partial W^k}{\partial\psi} = \frac{\lambda}{\rho(\lambda+\rho)}\left[\underbrace{N^k\frac{\partial(W_L^k+W_H^k)}{\partial\psi}}_{\text{the utility effect}} + \underbrace{(W_L^k+W_H^k)\frac{\partial N^k}{\partial\psi}}_{\text{the advertising effect}}\right].$$

Because the effect of the parameters is divided into the utility and advertising effects, we only prove the case where these effects differently impact the lifetime consumer surplus, W^k. First, we consider the interior case. For $\underline{\theta}$, we obtain

$$\begin{aligned}\frac{\partial W^I}{\partial\underline{\theta}} =& \frac{\alpha^2\lambda(s_H-s_L)}{162\mu\lambda\rho(\lambda+\rho)^2}\\ &\times\left[\begin{array}{l}(\bar{\theta}-2\underline{\theta}+1)\{(2\bar{\theta}-\underline{\theta}-1)(14\bar{\theta}-13\underline{\theta}-1)\\ \quad+(\bar{\theta}-2\underline{\theta}+1)(15\bar{\theta}-21\underline{\theta}+6)\}s_H\\ -\{[(2\bar{\theta}-\underline{\theta}-1)^2+(\bar{\theta}-2\underline{\theta}+1)^2](7\bar{\theta}-2\underline{\theta}-5)\\ \quad+[3(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)+2(2\bar{\theta}-\underline{\theta}-1)^2](4\bar{\theta}-5\underline{\theta}+1)\}s_L\end{array}\right].\end{aligned}$$

This equation is negative because the following relation is always satisfied.

$$\frac{s_H}{s_L} < \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} < \frac{\begin{array}{c}[(2\bar{\theta}-\underline{\theta}-1)^2+(\bar{\theta}-2\underline{\theta}+1)^2](7\bar{\theta}-2\underline{\theta}-5)\\ +[3(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)+2(2\bar{\theta}-\underline{\theta}-1)^2](4\bar{\theta}-5\underline{\theta}+1)\end{array}}{\begin{array}{c}(\bar{\theta}-2\underline{\theta}+1)\{(2\bar{\theta}-\underline{\theta}-1)(14\bar{\theta}-13\underline{\theta}-1)\\ +(\bar{\theta}-2\underline{\theta}+1)(15\bar{\theta}-21\underline{\theta}+6)\}\end{array}}.$$

Therefore, $\frac{\partial W^I}{\partial\underline{\theta}}$ is always negative. Similarly, for s_H, we obtain

$$\begin{aligned}\frac{\partial W^I}{\partial s_H} =& \frac{\alpha^2\lambda\{(2\bar{\theta}-\underline{\theta}-1)^2+(\bar{\theta}-2\underline{\theta}+1)^2\}}{324\mu\lambda\rho(\lambda+\rho)^2}\\ &\times\left[\begin{array}{l}3(2\bar{\theta}-\underline{\theta}-1)(\underline{\theta}-1)(s_H-s_L)\\ \quad+(2\bar{\theta}-\underline{\theta}-1)\{3(\underline{\theta}-1)s_H+2(\bar{\theta}+\underline{\theta}-2)s_L\}\\ \quad+(\bar{\theta}-2\underline{\theta}+1)\{-4(\bar{\theta}-2\underline{\theta}+1)s_H+(9\bar{\theta}-6\underline{\theta}-3)s_L\}\end{array}\right].\end{aligned}$$

We can easily confirm that, for the third term in the square brackets,

$$\frac{s_H}{s_L} < \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} < \frac{9\bar{\theta}-6\underline{\theta}-3}{4(\bar{\theta}-2\underline{\theta}+1)}.$$

Therefore, $\frac{\partial W^I}{\partial s_H}$ is always positive. Finally, for s_L, we obtain

$$\begin{aligned}\frac{\partial W^I}{\partial s_L} = &\frac{\alpha^2\lambda\{(2\bar{\theta}-\underline{\theta}-1)^2+(\bar{\theta}-2\underline{\theta}+1)^2\}}{324\mu\lambda\rho(\lambda+\rho)^2} \\ &\times\begin{bmatrix}\{(2\bar{\theta}-\underline{\theta}-1)(4\bar{\theta}-5\underline{\theta}+1)+(\bar{\theta}-2\underline{\theta}+1)(5\bar{\theta}-4\underline{\theta}-1)\}s_H \\ -2\{2(2\bar{\theta}-\underline{\theta}-1)^2+3(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)\}s_L\end{bmatrix}.\end{aligned}$$

The sign of this equation depends on the sign in the square brackets; that is,

$$\frac{\partial W^I}{\partial s_L} \gtreqless 0 \iff \frac{s_H}{s_L} \gtreqless \frac{4(2\bar{\theta}-\underline{\theta}-1)^2+6(\bar{\theta}-1)(\bar{\theta}-2\underline{\theta}+1)}{(2\bar{\theta}-\underline{\theta}-1)(4\bar{\theta}-\underline{\theta}+1)+(\bar{\theta}-2\underline{\theta}+1)(5\bar{\theta}-4\underline{\theta}-1)} \equiv \Theta.$$

We can easily confirm that the value of the right-hand side is greater than 1. We compare this equation with the upper bound value of the interior case, $\frac{s_H}{s_L} = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}$.

$$\Theta - \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} = \frac{3\{3\bar{\theta}^3-(19\underline{\theta}-10)\bar{\theta}^2+(25\underline{\theta}^2-12\underline{\theta}-4)\bar{\theta}-7\underline{\theta}^3-4\underline{\theta}^2+10\underline{\theta}-2\}}{(\bar{\theta}-2\underline{\theta}+1)\{(2\bar{\theta}-\underline{\theta}-1)(4\bar{\theta}-\underline{\theta}+1)+(\bar{\theta}-2\underline{\theta}+1)(5\bar{\theta}-4\underline{\theta}-1)\}}.$$

The sign is determined by the cubic function in the numerator. The cubic function reaches a local minimum at $\bar{\theta} \approx 3.407\underline{\theta}-2.407$ and is negative at $\bar{\theta} = 4\underline{\theta}-3$. This result means that, for $\bar{\theta} > 4\underline{\theta}-3$, the cubic function is increasing and positive after the largest $\bar{\theta}$-intercept. The cubic function, therefore, has one solution between $4\underline{\theta}-3$ and $5\underline{\theta}-4$, which is represented as $\bar{\theta}_b$.[19] For the region where $\bar{\theta} > \bar{\theta}_b$, the sign is positive, and thus $\frac{\partial W^I}{\partial s_L}$ is always negative. Otherwise, we obtain

$$\frac{\partial W^I}{\partial s_L}\begin{cases} > 0 & \text{if } \Theta < \frac{s_H}{s_L} < \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \\ = 0 & \text{if } \frac{s_H}{s_L} = \Theta \\ < 0 & \text{if } 1 < \frac{s_H}{s_L} < \Theta. \end{cases}$$

Next, we consider the corner case. As for $\underline{\theta}$, we obtain

$$\frac{\partial W^C}{\partial \underline{\theta}} = \frac{\alpha^2\lambda s_L \Sigma_c(\bar{\theta},\underline{\theta},s_H,s_L)}{32\mu\lambda\rho(\lambda+\rho)^2(s_H-s_L)^2},$$

where

$$\begin{aligned}\Sigma_c(\bar{\theta},\underline{\theta},s_H,s_L) = &-3(\bar{\theta}-1)^3 s_H^3 \\ &+2\{(\bar{\theta}-1)^3+2(\bar{\theta}-\underline{\theta})^2(5\bar{\theta}-8\underline{\theta}+3)+2(3\bar{\theta}-4\underline{\theta}+1)(\underline{\theta}-1)^2\}s_H^2 s_L\end{aligned}$$

[19] If $\underline{\theta} = 2$, $\bar{\theta}_b$ is 5.649.

$$
\begin{aligned}
&-(\bar{\theta}-\underline{\theta})\{12(\bar{\theta}-2\underline{\theta}+1)^2+(\bar{\theta}-1)(7\bar{\theta}-9\underline{\theta}+2)\}s_H s_L^2\\
&+9(\bar{\theta}-\underline{\theta})^2(\bar{\theta}-2\underline{\theta}+1)s_L^3.
\end{aligned}
$$

This function, Σ_c, is positive and increasing at $s_H = s_L$. This result means that, for $s_H > s_L$, the function changes from increasing to decreasing at the local maximum. In addition, the function is positive at $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$ and, at $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-2\underline{\theta}+1}s_L$,

$$
\begin{aligned}
\Sigma_c = &\frac{4(\bar{\theta}-\underline{\theta})^2(\underline{\theta}-1)s_L^3}{(\bar{\theta}-4\underline{\theta}+3)^3}\\
&\times[\bar{\theta}^3+(8\underline{\theta}-11)\bar{\theta}^2-(90\underline{\theta}^2-164\underline{\theta}+71)\bar{\theta}+120\underline{\theta}^3-270\underline{\theta}^2+188\underline{\theta}-39].
\end{aligned}
$$

The cubic function in the square brackets reaches a local minimum at $\bar{\theta} \approx 3.425\underline{\theta} - 2.425$ and is thus increasing for the region where $\bar{\theta} > 4\underline{\theta} - 3$. Furthermore, there exists a solution of the cubic function between $5\underline{\theta} - 4$ and $6\underline{\theta} - 5$, which is represented as $\bar{\theta}_c$.[20] Therefore, for $4\underline{\theta} - 3 < \bar{\theta} \le \bar{\theta}_c$, Σ_c is positive at $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$ and negative at $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-2\underline{\theta}+1}s_L$. This result means that there exists a solution of Σ_c between these boundaries, which is represented as $s_H = s_b s_L$. We summarize the result as follows:

$$
\frac{\partial W^C}{\partial \underline{\theta}} = \frac{\alpha^2 \lambda s_L \Sigma_c(\bar{\theta},\underline{\theta},s_H,s_L)}{32\mu\lambda\rho(\lambda+\rho)^2(s_H-s_L)^2}
\begin{cases}
>0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \le \frac{s_H}{s_L} < s_b\\
=0 & \text{if } \frac{s_H}{s_L} = s_b\\
<0 & \text{if } s_b < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}.
\end{cases}
$$

For $\bar{\theta} > \bar{\theta}_c$, $\Sigma_c$is positive such that $\frac{\partial W^C}{\partial \underline{\theta}}$ is positive.

As for s_L, we obtain

$$
\frac{\partial W^C}{\partial s_L} = \frac{\alpha^2 \lambda s_H \Sigma_d(\bar{\theta},\underline{\theta},s_H,s_L)}{32\mu\lambda\rho(\lambda+\rho)^2(s_H-s_L)},
$$

where

$$
\begin{aligned}
\Sigma_d(\bar{\theta},\underline{\theta},s_H,s_L) = &(\bar{\theta}-1)^3(\bar{\theta}-\underline{\theta})s_H^3 - \{2(\bar{\theta}-1)(\bar{\theta}-\underline{\theta})^2+(\bar{\theta}-2\underline{\theta}+1)^3(2\bar{\theta}-3\underline{\theta}+1)\\
&+2(\bar{\theta}-\underline{\theta})(\bar{\theta}-2\underline{\theta}+1)^2(\underline{\theta}-1)\}s_H^2 s_L + (\bar{\theta}-\underline{\theta})\{15(\bar{\theta}-\underline{\theta})^2(\underline{\theta}-1)\\
&+3(\bar{\theta}-\underline{\theta})(\bar{\theta}-2\underline{\theta}+1)(3\bar{\theta}-4\underline{\theta}+1)+(\bar{\theta}-3\underline{\theta}+2)(\underline{\theta}-1)^2\}s_H s_L^2\\
&-(\bar{\theta}-\underline{\theta})^2\{(\bar{\theta}-2\underline{\theta}+1)(2\bar{\theta}-3\underline{\theta}+1)+6(\bar{\theta}-\underline{\theta})(\underline{\theta}-1)\}s_L^3.
\end{aligned}
$$

The function, Σ_d, is negative and decreasing at $s_H = s_L$. This result implies that, for the region where $s_H > s_L$, the function changes from decreasing to increasing at the local minimum. In addition, the function is negative at $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$ and, at $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-2\underline{\theta}+1}s_L$,

$$
\Sigma_d = \frac{(\bar{\theta}-\underline{\theta})^2(\underline{\theta}-1)s_L^3}{(\bar{\theta}-4\underline{\theta}+3)^3}
$$

[20] If $\underline{\theta} = 2$, $\bar{\theta}_c$ is 6.075.

$$\times \Big[-5\bar{\theta}^4 + (37\underline{\theta} - 17)\bar{\theta}^3 - (70\underline{\theta}^2 - 29\underline{\theta} - 11)\bar{\theta}^2 \\ + (92\underline{\theta}^3 - 136\underline{\theta}^2 + 107\underline{\theta} - 43)\bar{\theta} - 48\underline{\theta}^4 + 100\underline{\theta}^3 - 82\underline{\theta}^2 + 19\underline{\theta} + 6 \Big]$$

Because the inflection point of the quartic function at the larger $\bar{\theta}$ is $\bar{\theta} \approx 2.893\underline{\theta} - 1.893$, the function is concave for $\bar{\theta} > 4\underline{\theta} - 3$. The function is positive at $\bar{\theta} = 4\underline{\theta} - 3$ and thus changes to negative at $\bar{\theta}_d$, which exists for $\bar{\theta} \in (5\underline{\theta} - 4, 6\underline{\theta} - 5)$. [21] Therefore, for $4\underline{\theta} - 3 < \bar{\theta} \leq \bar{\theta}_d$, Σ_d is negative at $s_H = \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1}s_L$ and non-negative at $s_H = \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-2\underline{\theta}+1}s_L$. This result means that there exists a solution of Σ_d between these boundaries, which is represented as $s_H = s_c s_L$. We summarize the result as follows:

$$\frac{\partial W^C}{\partial s_L} = \frac{\alpha^2 \lambda s_H \Sigma_d(\bar{\theta}, \underline{\theta}, s_H, s_L)}{32\mu\lambda\rho(\lambda+\rho)^2(s_H - s_L)} \begin{cases} < 0 & \text{if } \frac{\bar{\theta}+\underline{\theta}-2}{\bar{\theta}-2\underline{\theta}+1} \leq \frac{s_H}{s_L} < s_c \\ = 0 & \text{if } \frac{s_H}{s_L} = s_c \\ > 0 & \text{if } s_c < \frac{s_H}{s_L} < \frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-4\underline{\theta}+3}. \end{cases}$$

For $\bar{\theta} > \bar{\theta}_d$, Σ_d is negative such that $\frac{\partial W^C}{\partial s_L}$ is also negative.

[21] If $\underline{\theta} = 2$, $\underline{\theta}_d$ is 6.369.

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