Generalized Comparative Statics for Political Economy Models

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17 December 2014

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Abstract

The Median Voter Theorem is an extremely popular result in Political Economy that holds only if the policy space is unidimensional. This assumption restricts its use to a class of very simple problems. In most applications in the literature this implied an oversimplification of the problem studied, which is one of the possible explanations for the lack of empirical support for several predictions derived with this tool.

In this paper I show that under suitable restrictions on individual preferences a Median Voter Theorem can be derived even if the policy space is multidimensional and I derive the comparative statics of the resulting model induced by a change in the pivotal voter.

I show that this tool can invalidate the predictions of the Meltzer-Richard model of size of goverment and that it can be useful to study other Political Economy problems that cannot be analyzed using the traditional framework, including games in which players have a richer strategy set than the policy vector to be chosen.

Keywords: median voter, multidimensionality, monotone comparative statics
PACS: 71.35.-y, 71.35.Lk, 71.36.+c
2014 MSC: 23-557

1. Introduction

The model of electoral competition proposed by Downs (1957) is a simple and useful tool that has proved to be extremely successful in the Political Economy literature. The model delivers very strong predictions: under suitable assumptions the Median Voter Theorem states that the unique equilibrium is the policy that is most preferred by the median voter, which implies that the levels and the comparative statics of the political equilibrium reduces to the

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ones of a single pivotal individual. The ease of calculation and interpretation of this prediction had to face a general lack of empirical support. A famous example is given by an influential paper by Meltzer and Richard (1981), who have shown that in a simple general equilibrium economy the size of the governmental expenditures is monotonic increasing in the mean to median income ratio. The large body of empirical studies[2] that has followed their paper has provided very little support to this hypotesis, with a majority of these analyses showing no statistically significant relationship between the two variables of interest and a number of papers that found a significant relationship but with opposite sign in comparison with the one implied by the Metlzer and Richard result.

A possible explanation for this poor performance of the model is a direct consequence of the restrictive assumptions that one has to impose in order to achieve the existence of a Majority Voting equilibrium in Downsian model, and one in particular is relevant for this and for several other examples: the unidimensionality of the policy space. In Melzer and Richard's paper a crucial consequence of this restriction is that the policy space is made of two variables: a linear tax rate and a lump-sum grant, that are connected by a balanced governmental budget constraint such that the effective choice of voters is reduced to a unidimensional policy space. The oversimplification of this setting is evident in several aspects. For instance in most countries direct redistribution is just one component of governmental spending and does not usually represent the largest share, given that usually expenditures in direct provision of Public Goods and services and other welfare policies accounts for a larger share of the public budget. Moreover the choice of a very simple tax system is likely to influence the result.

Unfortunately the attempt to model the political interaction in a Downsian model when the policy space is multidimensional has to face extremely burdersome assumptions in order to achieve existence of a pure strategy equilibrium (see section 2). Following a successful stream of literature (Levy 2004, 2005, Roemer, etc.) my approach is to make the political interaction slightly more complex (and realistic) in comparison with the one implied in Downsian models by introducing coalitions (or factions) as intermediate bodies between the voters and the policies that have to be implemented.

The crucial idea of this approach is that individual citizens have limited ability to commit to specific policies, but they are allowed to form coalitions whose role is to increase the space of policies that a faction can credibily commit to implement after elections, namely a coalition can propose any policy in the Pareto set of its members. This assumption is common in the recent Political Economy literature (see Levy 2004, 2005, Roemer 1999).

In this paper I do not explicitly model the process of coalition formation, but I require coalitions to be stable in a peculiar and relatively weak way: a coalition $\mathcal{A}$ is stable if there is at least a vector of policies x in its Pareto set such that there is no other vector of policies $x'$with the following features: (i)

[2]For a review of the literature about this topic see de Mello, Tiongson 2006.

x' makes each member of a subcoalition $\mathcal{A}' \subset \mathcal{A}$ strictly better off with respect to x; (ii) x' is in the Pareto set of the subcoalition $\mathcal{A}'$; (iii) x' is preferred to x by the society as a whole according to some social preference relation (that is going to be Majority Voting); (iv) there is no policy x'' in the Pareto set of the complementary subcoalition $\mathcal{A} \backslash \mathcal{A}'$ that is preferred to x' by the society. The details of this concept of stability will be given in section 2, but this description is sufficient to understand how flexible this concept of stability is: it is very unlikely that a coalition can be stable if for any policy that this coalition can put forward a subcoalition can deviate and propose a policy that makes all its member strictly better off (i) (ii), that is supported by the overall society against the policy initially proposed (iii) and such that the remaining members of the original coalition do not have access to any alternative that represents a "credible threat" and that can discourage this deviation (iv).

In this paper I show that under this notion of stability and some specific assumptions on individual preferences a Median Voter Theorem and a monotone comparative statics of the equilibrium outcomes can be derived in a multidimensional policy space. This result may be used to shed light on the effect of the restriction of unidimensionality of the policy space in some common applications in the literature.

The paper is structured as follows. Section 2 summarizes the existing literature about models of Political Economy in a multidimensional policy space and highlights why none of the existing models is suitable to analyze sufficiently complex problems of comparative statics. In section 3 I describe the model of political interaction and the notion of stability that I will use in the rest of the paper and the restriction on individual preferences that I need to impose. In section 4 I present the two main results of this paper: the Generalized Median Voter Theorem and the Monotone Comparative statics of the equilibrium outcome; moreover I show how my findings can be interpreted as a generalization of some results in the literature, I also describe some features of the coalition structures that can emerge in equilibrium and of the Social Choice function that is implied by this model. Section 5 introduces a generalization of the model in which a more complex game is played such that agents do not only vote over different policy vectors but they also have access to a richer strategy space. I show that when the resulting game has some specific characteristics (similar to the ones of a game with strategic complementarities) a monotone comparative statics result similar to the one in section 4 can be derived. Section 6 describes two possible applications: the first one shows how the result in Meltzer and Richard's paper is not robust to a small change in the environment and that in a rather simple but more realistic setting their monotonicity result cannot be achieved or if it is it has the opposite direction in comparison with the one they derive in their simpler setting. This result may help to shed light on the reasons that underpin the lack of empirical support to the main prediction of their model. The second one is an application of the extension in section 5 and it shows that interesting prediction can be derived if a famous game, namely the *Arms Race*, is played in a Political Economy framework. Sections 6 concludes providing some comments about the importance and the limits of the results in the paper.

2. Literature

The seminal contributions of Hotelling (1929) , Black (1948) and Downs (1957) gave rise to the success in the Political Economy literature of the so-called Downsian models of political competition, which proved to be extremely successful and it is still popular in recent applications. The reason of this success relies in the simple and powerful result that this model delivers under suitable restrictions on individual preferences: the Median Voter Theorem. The crucial consequence of this result is that the equilibrium choice is going to be the policy that is most preferred by a single individual (Pivotal voter or Condorcet winner), which is the median individual. This implies in turn that all predictions about levels and comparative statics of the equilibrium outcomes are very easy to derive and to interpret in relations to the changes of the identity and preferences of the pivotal individual.

Unfortunately it is well known that the conditions for the existence of a Condorcet winner in a multidimensional policy space are extremely burdensome (see Plott 1967; Davis, DeGroot, Hinich 1972; and Grandmont, 1978). This implies that in order to study problems that are characterized by a sufficiently rich policy space one has to rely to an alternative model of political interaction.

In the Political Economy literature there are several examples of models that meet this requirement; in this section I will mention the most popular ones and explain why none of them is suitable to answer questions about the comparative statics of the equilibrium outcomes if the number of available policies is sufficiently large.

The first and popular example is given by Citizen-Candidate models first proposed by Osborne and Slivinski (1996) and Besley and Coate (1997); this class of models is based on the assumption each voter can run for elections but she cannot commit to any policy that is not in her set of ideal points. Under this (rather restrictive) assumption the existence of a political equilibrium is ensured, but multiplicity of equilibria[3] is a typical outcome. For instance with the same set of voters there may be equilibria with only one candidate running unopposed, equilibria with two candidates or more, and each of these cases is characterized by a different set of policies that are implemented in equilibrium. This implies that the model is not suitable to answer questions about policy outcomes and their comparative statics because the set of policy vectors that can be equilibria is usually too large to deliver any useful prediction. The problem of multiplicity is shared with the Party Unanimity Nash Equilibrium (PUNE) proposed by Roemer (1999).

[3]There is a particular case, highlighted in Besley and Coate (1997) in which the Citizen Candidate model delivers a unique equilibium. In section 4.3 I describe this case and show that it can be interpreted as an extreme case of the model I present in this paper.

The model developed by Levy (2004, 2005) is based on the idea that citizens can expand their ability to commit to policies different from their own ideal point by forming coalitions such that each coalition can propose any policy that is in the Pareto set of its members. A peculiar notion of coalition stability ensures existence of an equilibrium in a multimensional policy space even if the individual preferences are relatively complex (i.e. individuals differ in two paramenters that enter their utility function). On the other hand the predictions about levels and comparative statics of the equilibrium outcomes can be derived analytically only for problems in which the policy space or the individual preferences are restricted in such a way that a very small number of policy vectors can be chosen in an equilibrium. for instance in the application described in Levy (2005) there are only three policies that can be be chosen in any equilibrium). This make Levy's model unsuitable to analyze more complex problems.

Finally we have a relative large literature about Probabilistic Voting Models (Lindbeck and Weibull, 1987; Enelow and Hinich, 1989) that under not very restrictive assumptions ensure the existence and uniqueness of a voting equilibrium. These models do not deliver any result that makes the political equilibrium equivalent to the ideal point of a single "pivotal individual", therefore the kind of comparative statics that can emerge is generally more complex to derive and its interpretation is not always straightforward. Given that the equilibium outcomes of Probabilistic Models depend in principle on the preferences of all voters, then all comparative statics exercises have to be related to some charasteristic of the whole population, for instance a feature of its distribution that can be summarized by a unidimensional parameter. An example of this approach is in a paper by Dotti (2014) in which in a model of public provision of a private good the comparative statics of the equilibrium outcome induced by a marginal mean preserving spread in the income distribution of voters is derived.

In order to deal with more general comparative statics questions it is necessary to apply a tool that allows to summarizes the preferences of the society into the choice of a single individual, such that the comparative statics induced by changes in the environment can be derived and interpreted easily and condition for its monotonicity can be imposed in a simple and intuitive way. These features, that are ensured by the Median Voter Theorem in the unidimensional case, can be achieved in a multidimensional policy space thanks to the model presented in this paper.

3. The Model

3.1. Setting

Consider a voting game with n voters (n odd) such that each voter $i \in \mathcal{N}$ is denoted by a vector of parameters $\theta_i \in \Theta$. Assume $(\Theta, \preccurlyeq)$ is a totally ordered set for some transitive, reflexive, antisymmetric order relation $\preccurlyeq$. This allows me to establish a total order in the set of players $\mathcal{N}$, such that for all $i, j \in \mathcal{N}$

we have $i \leq j$ if and only if $\theta_i \preccurlyeq \theta_j$. For instance suppose θ is individual income, then $\theta \in [\underline{\theta}, \overline{\theta}]$ and Θ is a totally ordered set under the order relation $\leq$.

Each individual $i \in \mathcal{N}$ is endowed with a reflexive, complete and transitive preference ordering $\succeq^i$ that can be represented by a continuous and $\theta - concave$ utility function[4] $F : X \times \Theta \to R$.

The policy space X is a subset of the the d-dimensional real space R^d. In order to characterize X it is useful to recall some definitions.

Let $(L, \leqslant)$ be a partially ordered set, with the transitive, reflexive, antisymmetric order relation $\leqslant$. For x and y elements of X, let $x \vee y$ denote the least upper bound, or join, of x and y in X, if it exists, and let $x \wedge y$ denote the greatest lower bound, or meet of x and y in X, if it exists. The set L is a lattice if for every pair of elements x and y in L, the join $x \vee y$ and meet $x \wedge y$ do exist as elements of L. Similarly, a subset X of L is a sublattice of L if X is closed under the operations meet and join. A sublattice X of a lattice L is a convex sublattice of L, if $x \leqslant z \leqslant y$ and x, y in X implies that z belongs to X, for all elements x, y, z in L. Finally, a sublattice X of S is complete if for every nonempty subset X' of X, $\inf(X')$ and $\sup(X')$ both exist and are elements of X.

Recall the d-dimensional real space R^d is a partially ordered set under the transitive, reflexive, antisymmetric order relation $\leq$[5]. Moreover R^d is a lattice given the definition above. Now we have all the elements to characterize the policy space X. Let $X \subseteq R^d$ be a convex sublattice of R^d, then $(X, \leq)$ is a partially orderet set with order relation $\leq$. An example of a policy space that satisfies my assumption is given by the family of sets $Y = \{y | y \in [a, b]^d\}$ where $a, b \in R^d$.

Subset of voters can form coalitions $\mathcal{A} \subseteq \mathcal{N}$. The role of coalitions in this model is to increase the effective policy space available to the voters. Define $p_{X,\mathcal{A}}(a) \equiv \{b \in X : \ b \succeq^i a \, \forall i \in \mathcal{A}, b \not\preceq^i a \, \forall i \in \mathcal{A}\}$ to be the set of allocation in X that are Pareto superior to a for coalition $\mathcal{A}$. We assume that a coalition can propose any policy in the Pareto set of its members, i.e. $x^{\mathcal{A}} \in \mathcal{P}(\mathcal{A})$ where $\mathcal{P}(\mathcal{A}) \equiv \{a \in X : p_{X,\mathcal{A}}(a) = \oslash\}$. If a coalition is a singleton then the Pareto set reduces to the set of ideal points of its unique member (as in a citizen-candidate model).

3.2. Stability

In order to define a stability in this model we need to characterize a coalition structure and the preferences of each coalition. A coalition stucture is defined as a partition $\mathbb{P}$ of $\mathcal{N}$, i.e. a set of subsets of $\mathcal{N}$ such that $\oslash \notin \mathbb{P}$, $\cup_{\mathcal{A} \in \mathbb{P}} \mathcal{A} = \mathcal{N}$ and if $\mathcal{A}, \mathcal{B} \in \mathbb{P}$ with $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{A} \cap \mathcal{B} = \oslash$.

[4]For any function f defined on the convex subset X of R^d , we say that f is concave in direction $v \neq 0$ if, for all x, the map from the scalar s to $f(x + sv)$ is concave. (The domain of this map is taken to be the largest interval such that $x + sv$ lies in X.) We say that f is $i - concave$ if it is concave in direction v for any $v > 0$ with $v_i = 0$. See Quah (2007).

[5]For $x, y \in R^d$ $x \leq y$ if and only if $x_i \leq y_i$ for all $i = 1, 2, ..., d$.

We define a complete social preference relation $\succ$ and $\succeq$ such that $\succ$ is irreflexive i.e. $x \not\succ x$ and $\succeq$ is reflexive i.e. $x \succeq x$ and the weak and strong relations are dual, i.e. $a \succeq b \Leftrightarrow \neg b \succ a$ ($\succeq$ is not necessarily transitive). Given this preference relation we can define $P_A(a) \equiv \{b \in A : b \succ a\}$ where $A \subseteq X$ to be the strictly preferred set of a in A and $K(A) \equiv \{a \in A : P_A(a) = \emptyset\}$ to be the set of $P-maximal$ alternatives in A, or the Core.

The crucial aspect of my concept of stability relies on the idea of "credible threat". I define $S_{\mathcal{A}}(a) \equiv \{b \in \mathcal{P}(\mathcal{A}'), \mathcal{A}' \subseteq \mathcal{A} : b \succ^i a \,\forall i \in \mathcal{A}', b \succ c \,\forall c \in \mathcal{P}(\mathcal{A}\backslash\mathcal{A}')\}$ to be the set of "credible threats" to a. $S_{\mathcal{A}}(a)$ corresponds to the set of policies that are strictly preferred to a by each member of any subcoalition $\mathcal{A}' \subseteq \mathcal{A}$ and that are preferred by the society to any policy that can be proposed by the residual coalition $\mathcal{A}\backslash\mathcal{A}'$. Using this concept we can define the S-Core (SK) to be the set of $S-maximal$ alternatives in A. i.e. $SK(\mathcal{A}) = \{a \in \mathcal{P}(\mathcal{A}) : S_{\mathcal{A}}(a) = \emptyset\}$ is the set of policies that do not face any "credible threat" from any subcoalition of $\mathcal{A}$.

Using this structure we can now define a concept of stability for a coalition structure in this game:

Definition 1. A coalition $\mathcal{A}$ is stable if and only if $SK(\mathcal{A})$ is nonempty.

It is useful to give an example of why a coalition that does not satisfy the definition above is unlikely to survive. Suppose $SK(\mathcal{A}) = \emptyset$. Then for any $a \in \mathcal{P}(\mathcal{A})$, $\exists b \in \mathcal{P}(\mathcal{A}')$ and $\mathcal{A}' \subseteq \mathcal{A}$ such that $b \succ^i a \;\forall i \in \mathcal{A}'$ and $b \succeq c \,\forall c \in \mathcal{P}(\mathcal{A}\backslash\mathcal{A}')$, i.e. there exists a subset of the coalition $\mathcal{A}$ and a policy $b \in \mathcal{P}(\mathcal{A}')$ such that b is strictly preferred to a by all members of the subcoalition $\mathcal{A}'$ and b is also preferred by the society as a whole to any policy c that the remaining part of the original coalition $\mathcal{A}\backslash\mathcal{A}'$ can propose.

It is natural to consider this coalition structure unstable because for any policy chosen by this coalition in its Pareto Set (e.g. through some form of bargaining), the choice of this policy would not be self-enforcing because a subcoalition $\mathcal{A}'$ can deviate and propose a different policy that makes each member of the subcoalition strictly better off, that is preferred by the society as a whole, and such that the remaining part of the original coalition $\mathcal{A}\backslash\mathcal{A}'$ cannot prevent this deviation because there is no feasible "punishment" policy that can represent a credible threat for the deviators.

Definition 2. A stable coalition structure is a partition $\mathbb{P}$ of $\mathcal{N}$ such that all the coalitions $\mathcal{A}^j \in \mathbb{P}$ are stable.

3.3. Preferences

In order to establish our result I need to restrict individual and social preferences. The kind of restrictions I am going to use are very common in the many fields of Economic Theory.

About individual preferences the assumptions are *Supermodularity (SM)* and *Strict Single Crossing Property (SSCP).*

Recall that individual preferences can be represented by a function $F : X \times \Theta \to R$. A function F satisfies:

(i) *SM* if and only if $F(a\vee b,\theta)-F(a,\theta)\geq F(b,\theta)-F(a\wedge b,\theta)$ for all $\theta\in\Theta$ and for all $a,b\in X$.

(ii) *SSCP* in (a,θ) if and only if $F(a,\overline{\theta})-F(b,\overline{\theta})>F(a,\underline{\theta})-F(b,\underline{\theta})$ for all $a\geq b$ and $a\neq b$, and for all $\overline{\theta},\underline{\theta}\in\Theta$ such that $\overline{\theta}>\underline{\theta}$.

Given the individual preferences described above, I define $M(i)$ to be set of ideal points of an individual i, i.e. $M(i)\equiv\{y|y\in\arg\max_{x\in X}F(x,\theta_i)\}$[6].

About social preferences I am assuming *Majority Voting,* which is the most common and widely used criterion in order to establish a social preference relation. Formally $a\succeq b$ if and only if $\sum_{i=1}^{n}1[F(a,\theta_i)\geq F(b,\theta_i)]>n/2$. Notice that this preference relation does not necessarily imply a *tournament*, i.e. it is possible that $a\neq b$ and $a\sim b$.

3.4. Equilibrium

Now that the setting is complete we need to define an equilibrium for this voting game.

Definition 3. A policy vector $a\in A$ is a winning policy if and only if it is in the Core of A, i.e. $a\in K(A)$.

Given that our Social Choice function is the Majority Voting this is equivalent to say that a is a Majority Voting Equilibrium of the set of alternatives A.

Suppose a coalition structure is made of h coalitions $\mathcal{A}^j$ for $j=1,2,...,h$. then we can define an equilibrium for the voting game as follows.

Definition 4. A pure strategy equilibrium is a coalition structure $\mathbb{P}^*=\{\mathcal{A}^j\}_{j=1}^{h}$, a policy profile $A^*=\{a^j\}_{j=1}^{h}$ and a set of winning policies $W(A^*)\subseteq A^*$ such that: (i) $\mathbb{P}^*$ is a stable coalition structure; (ii)$a^j\in SK(\mathcal{A}^j)$ for all $j=1,2,..,h$; (iii) $W(A^*)$ is nonempty.

In other words in an equilibrium each coalition is stable and is represented by one of the policy vectors that makes it stable, and the winning policy is a Majority Voting equilibrium of the reduced games in which the policy space ir reduced to $A^*\subseteq X$.

4. Results

The main result of this paper is stated in the following theorem:

Theorem 5. *(Median Voter Theorem). (i) An equilibrium of the voting game exists. (ii) In any equilibrium the set of winning policies W is a subset of the set of ideal points of the median voter m, i.e. $W\subseteq M(m)$. (iii) If the median voter has a unique ideal point, this policy is going to be the one chosen in any equilibrium.*

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[6]Notice that the completeness of X implies compactness in the order-interval topology. On bounded sets in R^d, the order-interval topology coincides with the Eucilidean topology (Birkhoff 1967). Hence $M(i)\neq\oslash$ for all $i\in\mathcal{N}$.

In order to prove this result we need to introduce some additional notation. Suppose the coalition $\mathcal{A}$ has k members. Consider a set of $k\times 1$ weighting vectors $\Lambda^{\mathcal{A}} \equiv \{\lambda : \sum_{i\in\mathcal{A}} \lambda_i = 1\}$ for each coalition $\mathcal{A}$ and a function $G : X\times\Lambda^{\mathcal{A}}\times\Theta \to R$ defined as follows: $G(x, \lambda, \Theta) = \sum_{i\in\mathcal{A}} \lambda_i F(x, \theta_i)$.

Lemma 6. *If F is a continuous function and X is a convex set then any point a in the Pareto set of $\mathcal{A}$ is a solution to* $\max_{x\in X}\ G(x, \lambda^a, \Theta)$ *for some vector* $\lambda^a \in \Lambda^{\mathcal{A}}$.

Proof. M.W.G., Proposition 16.E.2. □

We need to define four additional objects:

(i) a vector$\underline{\lambda}^{A,j}$ such that $\underline{\lambda}_i^{A,j} = \lambda_i \forall i \in \mathcal{A} : \theta_i < \theta_j$, $\underline{\lambda}_i^{A,j} = 0\ \forall i \in \mathcal{A} : \theta_i > \theta_j$, $\underline{\lambda}_j^{A,j} = \sum_{\substack{i\in\mathcal{A}\\ i\geq j}} \lambda_i$;

(ii) a vector $\bar{\lambda}^{A,j}$ such that $\bar{\lambda}_i^{A,j} = \lambda_i \forall i \in \mathcal{A} : \theta_i > \theta_m$, $\bar{\lambda}_i^{A,j} = 0\ \forall i \in \mathcal{A} : \theta_i < \theta_j$, $\bar{\lambda}_j^{A,j} = \sum_{\substack{i\in\mathcal{A}\\ i\geq j}} \lambda_i$;

(iii) the set $\Lambda^{A,j} = \{\underline{\lambda}^{A,j}, \lambda^A, \bar{\lambda}^{A,j}\}$;

(iv) an order relation $\leq^\lambda$given by: $\lambda^1 \leq^\lambda \lambda^2$ iff $\lambda_i^1 \geq \lambda_i^2\ \forall i \leq m$ and $\lambda_i^1 \leq \lambda_i^2$ $\forall i \geq m$. It follows that $(\Lambda^{A,j}, \leq^\lambda)$ is a totally ordered set.

Lemma 7. *If F satisfies SM and SSCP then the Pareto Set $\mathcal{P}(\mathcal{A})$ of a coalition of players $\mathcal{A} \subseteq \mathcal{N}$ is such that $x \in \mathcal{P}(\mathcal{A})$ only if $x \geq \sup\{M(l)\}$ and $x \leq \inf\{M(h)\}$ where $l = \inf(\mathcal{A})$ and $h = \sup(\mathcal{A})$.*

Proof. See appendix. □

Lemma 8. *The function $G(x, \lambda, \Theta)$ satisfies (i) SM in x and (ii) SCP in (x, λ) $\forall\lambda \in \Lambda^{A,j}$.*

Proof. (i) *SM*. G is the sum of *SM* functions so it is supermodular (proof in Milgrom, Shannon, 1994). (ii) *SCP*. Using the definition of supermodularity, G is supermodular if and only if: $G(\bar{x}, \lambda^A, \Theta) - G(\underline{x}, \lambda^A, \Theta) \geq G(\bar{x}, , \underline{\lambda}^{a,j}, \Theta) - G(\underline{x}, \underline{\lambda}^{a,j}, \Theta)\ \forall \bar{x} \geq \underline{x}, \lambda \in \vec{\Lambda}^{a,j}$. Use the definitions of G and $\underline{\lambda}^{a,j}$:

$$[G(\bar{x}, \lambda^A, \Theta) - G(\underline{x}, \lambda^A, \Theta)] - [G(\bar{x}, \underline{\lambda}^{A,j}, \Theta) - G(\underline{x}, \underline{\lambda}^{A,j}, \Theta)] =$$

$$= (\sum_{\substack{i\in\mathcal{A}\\ i\geq j}} \lambda_i[F(\bar{x}, \theta_i) - F(\underline{x}, \theta_i)]) - (\sum_{\substack{i\in\mathcal{A}\\ i\geq j}} \lambda_i)[F(\bar{x}, \theta_j) - F(\underline{x}, \theta_j)] =$$

$$= \sum_{\substack{i\in\mathcal{A}\\ i\geq j}} \lambda_i\left([F(\bar{x}, \theta_i) - F(\underline{x}, \theta_i)] - [F(\bar{x}, \theta_j) - F(\underline{x}, \theta_j)]\right)$$

Notice that $[F(\bar{x}, \theta_i) - F(\underline{x}, \theta_i)] - [F(\bar{x}, \theta_j) - F(\underline{x}, \theta_j)] \geq 0\ \forall i \geq j$ and $\lambda_i \geq 0$ $\forall i$ hence the sum above is also weakly positive, which implies $[G(\bar{x}, \lambda^A, \Theta) -$

$G(\underline{x}, \lambda^A, \Theta)] - [G(\bar{x}, \underline{\lambda}^{A,j}, \Theta) - G(\underline{x}, \underline{\lambda}^{A,j}, \Theta)] \geq 0$. Similarly one can show that this is also true for $(\lambda^A, \bar{\lambda}^{A,j})$and $(\underline{\lambda}^{A,j}, \bar{\lambda}^{A,j})$. Finally notice that given that X is a convex sublattice of R^d and $\underline{x} \leq \tilde{x} \leq \bar{x}$, then $\tilde{x} \in X$. □

Lemma 9. *If $x \in \mathcal{P}(\mathcal{A})$ then $\exists x' \in \mathcal{P}(\mathcal{A}^{\leq j})$ with $\mathcal{A}^{\leq j} = \{i \in \mathcal{A} : i \leq j\}$ such that $x' \leq x$.*

Proof. Milgrom-Shannon's monotone comparative statics implies: $\widetilde{M}(\mathcal{A}, \lambda) = \arg\max_{x \in X, \lambda \in \Lambda} G(x, \lambda, \Theta)$ is monotone nondecreasing in λ. Hence $\exists x' \in \widetilde{M}(\mathcal{A}', \underline{\lambda}^{A,j})$ such that $x' \leq x$. Given that, together with the $\theta - concavity$ of F and the convexity of X (which together imply a convex utility possibility set), using the result in Mas Colell, Proposition 16.E.2, it follows that $x \in \mathcal{P}(\mathcal{A}^j)$, i.e. x is in the Pareto set of coalition $\mathcal{A}^j = \{i \in \mathcal{A} : i \leq j\}$. Q.E.D. □

Notice that we cannot exclude that $x' = x$.

Lemma 10. *If $x' \in \widetilde{M}(\mathcal{A}, \lambda^{A,j})$ and $x \in \widetilde{M}(\mathcal{A}', \underline{\lambda}^{A,j})$ and $x' \leq x$, $x' \neq x$, then $F(x', \theta_j) \geq F(x, \theta_j)$ and $F(x', \theta_i) > F(x, \theta_i)$ $\forall i < j$.*

Proof. We know $x' \leq x$ and $G(x', \underline{\lambda}^{A,j}, \Theta) \geq G(x, \underline{\lambda}^{A,j}, \Theta)$ from Monotone comparative statics. Suppose $F(x', \theta_j) < F(x, \theta_j)$. Then it must be true that $\sum_{i \in \mathcal{A}} \underline{\lambda}_i^{A,j} [F(x', \theta_i) - F(x', \theta_i)] > F(x', \theta_j) - F(x, \theta_j)$. Using $\sum_{i \in \mathcal{A}} \underline{\lambda}_i^{A,j} = 1$ the above can be rearranged as follows:

$$\sum_{i \in \mathcal{A}} \underline{\lambda}_i^{A,j} \left([F(x', \theta_i) - F(x', \theta_i)] - [F(x', \theta_j) - F(x, \theta_j)] \right) > 0$$

. Notice that $x' \leq x$ and $i \leq j$ $\forall i \in \mathcal{A}$, hence $SSCP$ implies $[F(x', \theta_i) - F(x', \theta_i)] - [F(x', \theta_j) - F(x, \theta_j)] \leq 0$ $\forall i \in \mathcal{A}$ and hence

$$\sum_{i \in \mathcal{A}} \underline{\lambda}_i^{A,j} \left([F(x', \theta_i) - F(x', \theta_i)] - [F(x', \theta_j) - F(x, \theta_j)] \right) \leq 0$$

which leads to a contradiction. Hence it must be true that $F(x', \theta_j) \geq F(x, \theta_j)$. Given that $x' \leq x$, $x' \neq x$, $SSCP$ implies $F(x', \theta_i) > F(x, \theta_i)$ $\forall i < j$. □

Lemma 11. *The coalition $\mathcal{A}^m$ (could be a singleton) that includes the median voter m is stable only if $a^m \in M(m)$.*

Proof. Suppose $a^m \notin M(m)$ (A1).

(i) If $a^m \geq \underline{x}_m(\leq)$ for any $\underline{x}_m = \inf\{M(m)\}$ and $a^m \wedge x_m \in \mathcal{P}(\mathcal{A}^m)$. A1 implies $F(x_m, \theta_m) > F(a^m, \theta_m)$. $SSCP$ implies $F(x_m, \theta_i) > F(a^m, \theta_i)$ and $x_m \succ^i a^m$ $\forall i \in \mathcal{N} : \theta_i \leq \theta_m(\geq)$. Recall that any $c \in \mathcal{P}(\mathcal{A}^m \backslash \mathcal{A}^{\leq m})$ is $c \geq \underline{x}_m$(because of Lemma 7). Hence either $c \in M(m)$ or $\sum_{i=1}^n 1[F(\underline{x}_m, \theta_i) > F(c, \theta_i)] > n/2$ $\forall c \in \mathcal{P}(\mathcal{A} \backslash \mathcal{A}')$ which implies $x_m \succ c \rightarrow a^m \notin SK(\mathcal{A}^m)$.

(ii) If $a^m \not\geq x_m, a^m \not\leq x_m$ and $a^m \wedge x_m \in \mathcal{P}(\mathcal{A})$., Consider $a^m \vee x_m(a^m \wedge x_m)$. Revealed preferences imply $F(x_m, \theta_m) \geq F(a^m \vee x_m, \theta_m)$. QSM implies $F(a^m \wedge x_m, \theta_m) \geq F(a^m, \theta_m)$. $SSCP$ implies $F(a^m \wedge x_m, \theta_i) > F(a^m, \theta_i)$ $\forall i \in \mathcal{N} : \theta_i \leq \theta_m$. Recall that any $c \in \mathcal{P}(\mathcal{A}^m \backslash \mathcal{A}^{\leq m})$ is $c \geq \underline{x}_m \geq a^m \wedge x_m$. Hence either $c \in M(m)$ or $\sum_{i=1}^{n} 1[F(a^m \wedge x_m, \theta_i) \geq F(c, \theta_i)] > n/2$ which implies $a^m \wedge x_m \succ c$.

(iii) If and $a^m \wedge x_m \notin \mathcal{P}(\mathcal{A}^m)$. $a^m \wedge x_m$. Recall that X is a convex set and $F(x, \theta)$ is $\theta - concave$, hence as $a^m \in A^m$ it has to be the solution to a problem in the form $a^m \in \arg\max_{x \in X} G(x, \lambda^m, \Theta)$. Now if $a^m \wedge x_m$ is not part of the Pareto set of $\mathcal{A}^m$, consider the following alternative: $\tilde{x} \in \widetilde{M}(\mathcal{A}, \lambda^{a^m, m})$ (see Lemma 3). We know from Lemma 3 that $\tilde{x} \leq a^m$. First of all notice that $\widetilde{M}(\mathcal{A}, \lambda^{a^m, m}) = \widetilde{M}(\mathcal{A}', \lambda')$ for some λ', which imples that $\tilde{x} \in \mathcal{P}(\mathcal{A}')$, i.e. it is in the Pareto set of $\mathcal{A}'$. We need to show that $\tilde{x} \neq a^m$ and that $\tilde{x} \succeq^i a^m$ $\forall i \in \mathcal{A}'$. Suppose $\tilde{x} = a^m \rightarrow a^m \in \mathcal{P}(\mathcal{A}')$. But from point (b) we know that $F(a^m \wedge x_m, \theta_i) > F(a^m, \theta_i) \forall i \in \mathcal{N} : \theta_i \leq \theta_m \rightarrow a^m \notin \mathcal{P}(\mathcal{A}') \rightarrow$Contraddiction. Hence $\tilde{x} \neq a^m$ and $\tilde{x} \leq a^m$. Moreover, Lemma 4 implies $\tilde{x} \succeq^m a^m$. This means that $F(\tilde{x}, \theta_m) \geq F(a^m, \theta_m)$ and because $\tilde{x} \neq a^m$ SSCP implies $F(\tilde{x}, \theta_i) > F(a^m, \theta_i)$ $\forall i \in \mathcal{N} : \theta_i \leq \theta_m$. Recall that any $c \in \mathcal{P}(\mathcal{A}^m \backslash \mathcal{A}^{\leq m})$ is $c \geq \underline{x}_m \geq \tilde{x}$. Hence either $c \in M(m)$ or $\sum_{i=1}^{n} 1[F(\tilde{x}, \theta_i) \geq F(c, \theta_i)] > n/2$ which implies $\tilde{x} \succ c \rightarrow a^m \notin SK(\mathcal{A}^m)$.

Now Suppose $a^m \in M(m)$, and in particular say $a^m = \bar{x}_m = \sup\{M(m)\}$ ($= \underline{x}_m \inf\{M(m)\}$). Consider any coalition $\mathcal{A}^{\leq m}$ such that $\theta_i < \theta_m (\geq)$ $\forall i \in \mathcal{A}^{\leq m}$. From Lemma 7 we know that any $b \in \mathcal{P}(\mathcal{A}^{\leq m})$ it must be true that $b \leq x_m (\geq)$. Optimality implies $F(x_m, \theta_m) > F(b, \theta_m)$. *SSCP* implies $F(x_m, \theta_i) \geq F(b, \theta_i)$ and $x_m \succeq^i b$ $\forall i \in \mathcal{N} : \theta_i \geq \theta_m (\leq)$. Hence $\sum_{i=1}^{n} 1[F(x_m, \theta_i) \geq F(b, \theta_i)] > n/2$ $\forall b \in \mathcal{P}(\mathcal{A}^{\leq m})$ which implies $x_m \succ b$ $\forall b \in \mathcal{P}(\mathcal{A}^m) \rightarrow a^m \in SK(\mathcal{A}^m)$.

Finally Consider any coalition $\mathcal{A}^m$ such that $\theta_i \leq \theta_m (\geq)$ $\forall i \in \mathcal{A}^m$ and $a^m = \overline{x}_m (\underline{x}_m)$. From Lemma 7 we know that any $b \in \mathcal{P}(\mathcal{A}^m)$ it must be true that $b \leq \overline{x}_m (\geq \underline{x}_m)$. This implies $F(\overline{x}_m, \theta_m) > F(b, \theta_m)$. *SSCP* implies $F(\overline{x}_m, \theta_i) > F(b, \theta_i)$ and $\overline{x}_m \succ^i b$ $\forall i \in \mathcal{N} : \theta_i \geq \theta_m (\leq)$. Hence $\sum_{i=1}^{n} 1[F(\overline{x}_m, \theta_i) \geq F(b, \theta_i)] > n/2$ $\forall b \in \mathcal{P}(\mathcal{A}^m)$ which implies $\overline{x}_m \succ a$ $\forall a \in \mathcal{P}(\mathcal{A}^m) \rightarrow P_{\mathcal{P}(\mathcal{A}^m)}(a^m) = \oslash \leftrightarrow a^m \in K(A^m)$. □

Lemma 12. *Any coalition $\mathcal{A}^j$ that does not contain the median voter m is stable only if $\exists a^j$ such that either of the following is true: (i) $a^j \in M(m)$; (ii) $a^j \geq x_m$ for all $x_m \in M(m)$; (iii) $a^j \leq x_m$ for all $x_m \in M(m)$.*

Proof. Suppose $a^j \notin M(m)$ and $a^j \ngeq x_m, a^j \nleq x_m$. There are three possible cases.

(i) say $x_k \in M(k)$ and $\forall k \in \mathcal{A}^j$ it is true either $x_k \geq a^j$ or $x_k \leq a^j$. Consider x_j such that $F(x_j, \theta_m) \geq F(x_k, \theta_m)$. In particular consider $\underline{x}_j = \inf\{\min_{\substack{k \in \mathcal{A}^j \\ k > m}} x_k\}$ or $\bar{x}_j = \sup\{\max_{\substack{k \in \mathcal{A}^j \\ k < m}} x_k\}$. Suppose $x_j = \underline{x}_j$ $(\bar{x}_j)$. Optimality implies $F(x_j, \theta_j) > F(a^j, \theta_j)$. Notice that because $x_j \neq a^j$ *SSCP* implies $F(x_j, \theta_i) > F(a^j, \theta_i)$ $\forall i \in \mathcal{N} : \theta_i > \theta_m (<)$. Notice that both $\underline{x}_j$ and $\bar{x}_j$ are in the Pareto set $\mathcal{P}(\mathcal{A}^{<j}) = \mathcal{P}(\{i \in \mathcal{A} : i < m\})$ $(\mathcal{P}(\mathcal{A}^{>j}) = \mathcal{P}(\{i \in \mathcal{A} : i > m\}))$ because they are the highest (lower) ideal points of some member of the subcoalition $\mathcal{A}^{<j}$ (see Lemma 7). Finally notice that any policy $b \in \mathcal{P}(\mathcal{A}^j \backslash \mathcal{A}^{<j})$ must be $b \in M(m)$ or $b \leq x_j$ $(\geq)$ (because of Lemma 7). Hence if $x_j \neq a^j$ then

$\sum_{i=1}^{n} 1[F(x_j, \theta_i) \geq F(b, \theta_i)] > n/2 \ \forall b \in \mathcal{P}(\mathcal{A}^j \backslash \mathcal{A}^{<j})$ which implies $x_j \succ b \forall b \in \mathcal{P}(\mathcal{A}^j \backslash \mathcal{A}^{<j}) \rightarrow a^j \notin SK(\mathcal{A}^j)$.

(ii) $\exists x_k \in M(k)$, $k \in \mathcal{A}^j$, $\theta_k > \theta_m(<)$ such that $x_k \not\geq a^j$, $x_k \not\leq a^j$ (A2). Consider $x_k \wedge a^j$. Notice that (A2) implies $x_k \wedge a^j \neq a^j$. Optimality implies $F(x_k, \theta_k) \geq F(x_k \vee a^j, \theta_k)$. *SM* implies $F(x_k \wedge a^j, \theta_k) \geq F(a^j, \theta_k)$. *SSCP* implies $F(x_k \wedge a^j, \theta_i) \geq F(a^j, \theta_i) \ \forall i \in \mathcal{N} : \theta_i \leq \theta_m$. Hence $\sum_{i=1}^{n} 1[F(x_k \wedge a^j, \theta_i) \geq F(a^j, \theta_i)] > n/2$. which implies $x_k \wedge a^j \succ a \ \forall a \in \mathcal{P}(\mathcal{A}^j)$.

(iii) Is $x_k \wedge a^j$ part of the Pareto set of $\mathcal{A}^{<j}$? Recall that X is a convex set and $F(x, \theta)$ is $\theta - concave$, hence as $a^j \in \mathcal{P}(\mathcal{A}^j)$ it has to be the solution to a problem in the form $a^j \in \arg\max_{x \in X} G(x, \lambda^j, \Theta)$. Now if $x_k \wedge a^j$ is not part of the Pareto set of $\mathcal{A}^j$, consider the following alternative: $\tilde{x} \in \widetilde{M}(\mathcal{A}, \lambda^{a^j,k})$ (see Lemma 6). We know from Lemma 9 that $\tilde{x} \leq a^j$. First of all notice that $\widetilde{M}(\mathcal{A}, \lambda^{a^j,k}) = \widetilde{M}(\mathcal{A}', \lambda')$ for some λ', which imples that $\tilde{x} \in \mathcal{P}(\mathcal{A}')$, i.e. it is in the Pareto set of $\mathcal{A}^{<j}$.We need to show that $\tilde{x} \neq a^j$ and that $\tilde{x} \succeq^i a^j$ $\forall i \in \mathcal{A}^{<j}$. Suppose $\tilde{x} = a^j \rightarrow a^j \in \mathcal{P}(\mathcal{A}^{<j})$. From point (ii) we know that $F(x_k \wedge a^j, \theta_i) > F(a^j, \theta_i) \ \forall i \in \mathcal{N} : \theta_i \leq \theta_m \rightarrow a^j \notin \mathcal{P}(\mathcal{A}^{<j}) \rightarrow$ Contraddiction. Hence $\tilde{x} \neq a^j$ and $\tilde{x} \leq a^j$. Moreover, Lemma 10 implies $\tilde{x} \succeq^j a^j$. This means that $F(\tilde{x}, \theta_j) \geq F(a^j, \theta_j)$ and because $\tilde{x} \neq a^j$ *SSCP* implies $F(\tilde{x}, \theta_i) > F(a^j, \theta_i)$ $\forall i \in \mathcal{N} : \theta_i \leq \theta_j$. Recall that any $c \in \mathcal{P}(\mathcal{A} \backslash \mathcal{A}^{<j})$ is $c \geq \underline{x}_j \geq \tilde{x}$. Hence either $c \in M(m)$ or $\sum_{i=1}^{n} 1[F(\tilde{x}, \theta_i) \geq F(c, \theta_i)] > n/2$ which implies $\tilde{x} \succ c \rightarrow a^j \notin SK(\mathcal{A}^j)$. □

4.1. Proof of main result (Theorem 5)

The main result of this paper is stated in the following theorem:

> *(i) An equilibrium exists. (ii) In any equilibrium the set of winning policies W is a subset of the set of ideal points of the median voter m, i.e. $W \subseteq M(m)$. (iii) If the median voter has a unique ideal point, this policy is going to be the one chosen in any equilibrium.*

Proof. The results in Lemma 11 and Lemma 12 imply that the only policies that can be proposed by stable coalitions in equilibrium are either $a^m \in M(m)$ or $a^l \leq a^m$ or $a^h \geq a^m$. Recall optimality implies $F(a^m, \theta_m) > F(a^l, \theta_m)$ and *SSCP* implies $F(a^m, \theta_i) > F(a^l, \theta_i) \ \forall i \in \mathcal{N} : \theta_i \geq \theta_m$. Similarly $F(a^m, \theta_m) > F(a^h, \theta_m)$ and *SSCP* implies $F(a^m, \theta_i) > F(a^h, \theta_i) \ \forall i \in \mathcal{N} : \theta_i \leq \theta_m$. Hence a Majority Voting equilibrium among the proposed policies exists, which is also the policy chosen in an equilibrium of the coalitional game (i). The total order in the policy available in all reduced games generated by a stable coalition structure implies the Majority Voting equilibrium must be always some $a^m \in M(m)$ (ii). The proof of (iii) is straighforward from (i) and (ii). □

Corollary 13. *(i) The equilibrium policy is in the Core of a winning coalition, i.e. $x \in W \rightarrow x \in K(\mathcal{A}^m)$ for some winning coalition $\mathcal{A}^m$. Moreover, (ii) the equilibrium policy is in the Core of the reduced game, i.e. $x \in W \rightarrow x \in K(\mathcal{A}^s)$ for any equilibrium policy profile $\mathcal{A}^s$.*

Proof. Straightforward from Theorem 5. □

4.2. Comparative statics

Define E to be the space of possible equilibrium policies, i.e. $x \in E$ if and only if x is a winning policy in the voting game.

Lemma 14. *The space of possible equilibrium policies E is a sublattice of X.*

Proof. Recall that a subset X of L is a sublattice of L if X is closed under the operations meet and join. It is easy to show that (i) if $\mathcal{A}^m = \{m\}$ and $M(m) \cap M(i) = \oslash$ for all $i \neq m$ then $E = M(m)$; (ii) if $\mathcal{A}^m = \{m\}$ and $M(m) \cap M(i) \neq \oslash$ for some $i \neq m$ then $E = \{\overline{x}_m, \underline{x}_m\}$; (iii) if $\mathcal{A}^m \neq \{m\}$ then $E = \{\overline{x}_m, \underline{x}_m\}$. Recall that $M(m)$ is a convex sublattice of X (see Milgrom, Shannon 1994). Moreover $E = \{\overline{x}_m, \underline{x}_m\}$ is a sublattice of X because $E \subseteq X$ and $\overline{x}_m \vee \underline{x}_m = \overline{x}_m$, $\overline{x}_m \wedge \underline{x}_m = \underline{x}_m$ hence $\{\overline{x}_m \vee \underline{x}_m, \overline{x}_m \wedge \underline{x}_m\} \in E$ therefore it satisfies the definition of sublattice. □

The notion of monotonicity is the same as in Milgrom, Shannon (1994) and it is related to the Strong Set order, namely given two sets Y, Z we say that Y is grater than or equal to Z in the Strong Set order ($Y \geq_s Z$) if for any $y \in Y$ and $z \in Z$ we have $y \vee z \in Y$ and $y \wedge z \in Z$. This leads to the second important statement in this paper.

Theorem 15. *(Monotone Comparative Statics). The set of policies E that can be supported in an equilibrium of the voting game is monotonic nondecreasing in θ_m.*

Proof. From the proof of Lemma 11 consider case (i). Given that $E = M(m)$ and $M(m)$is monotonic nondecreasing in θ_m (Milgrom, Shannon 1994, Theorem 4), then E is as well. Consider cases (ii), (iii). Suppose $\theta_{m'} > \theta_m$, then given that $M(m') \geq_s M(m)$ (Milgrom, Shannon 1994), then it must be true that $\overline{x}_{m'} \geq \overline{x}_m$ and $\underline{x}_{m'} \geq \underline{x}_m$, which implies $\{\overline{x}_{m'}, \underline{x}_{m'}\} \geq_s \{\overline{x}_m, \underline{x}_m\}$. Q.E.D. □

This result is potentially very important in order to establish the direction of the change in policy induced by a change in the distribution of θ even if the individual objective function F is not C^2 and therefore the First Order Conditions of the maximization problem cannot be used in order to calculate the comparative statics of interest. The reason of this is that the result is based on Milgrom-Shannon's monotone comparative statics which is very general. One caveat is that in order to establish the existence of a Political Equilibrium in my model the additional assumption of continuity of F makes my comparative statics result slightly less general than the one in their paper.

4.3. Citizen Candidate Model

A class of models that allow for the existence of a political equilibrium even if the policy space is multidimensional is the one of Citizen-Candidate models (Besley and Coate 1997; Osborne and Slivinski, 1996). The crucial assumption of this class of models is that each voter can run for elections as a candidate, and that each candidate i can credibly commit only to a policy that is in the set of her ideal points $x_i \in M(i)$. The main shortcoming of this class of models if one aims to get predictions about the policy choice of a certain group of individuals is the multiplicity of equilibria. There is neverthless a case in which this model delivers a unique equilibrium, and this case is described in Corollary 2 (ii) of Besley and Coates 1997. In their model a citizen faces a cost δ to run for elections, and:

> (ii) if x_i is a strict Condorcet winner in the set of alternatives $\{x_j : j \in \mathcal{N}\}$ and if $x_i \neq x_0$, then a political equilibrium exists in which citizen i runs unopposed for sufficiently small δ.

Consider one particular stable coalition structure in the model presented in this paper, namely the one in which each coalition is a singleton. In this case the Pareto set of each coalition coincide with the set of ideal points of its single member. In this setting, my assumptions about individual preferences (*SM* and *SSCP*) are sufficient to ensure that there is at least one $x_m \in M(m)$ who is a Condorcet winner, as a consequence of Corollary 17 (ii).

Therefore we can conclude that:

(i) the result in Besley and Coate for $\delta \to 0$ can be interpreted as a particular case of equilibrium of the coalitional game model presented in this paper;

(ii) the restrictions on individual preferences I assumed in this paper (*SM* and *SSCP*) are also sufficient conditions for a unique equilibrium in which the median voter run unopposed and implements a policy that is in the set of her ideal points in the Citizen-Candidate model whenever $\delta \to 0$.

4.4. Coalition Structure

In this section I provide some example of stable and unstable coalition structures in this framework. A key aspect of a stable coalition structure in this model is given by the following statement.

Lemma 16. *Any coalitions that include either (a) individuals with index* $(i \leq m)$ *or (b) individuals with index* $(j \geq m)$ *is always stable. Therefore a coalition structure* $\mathbb{P}$ *is stable if each coalition* $\mathcal{A} \in \mathbb{P}$ *satisfies either (a) or (b).*

Proof. Straightforward from Lemma 11 and Lemma 12 and the definition of a stable coalition structure. □

For illustrative purposes it may be useful to analyse a case in which the policy space is bi-dimensional, i.e. $X \subseteq R^2_+$, 5 players i.e. $\mathcal{N} = 5$, $m = 3$ and in which individuals have unique ideal points (black dots). Fig. 1A, 1B, 1C all represent stable coalition structures because condition (a) is satisfied. Notice that Fig. 1C corresponds to the case of the Citizen-Candidate model described in the previous section. Finally Fig. 1D represents a case in which condition (a) is violated, hence it may not represent a stable coalition structure.

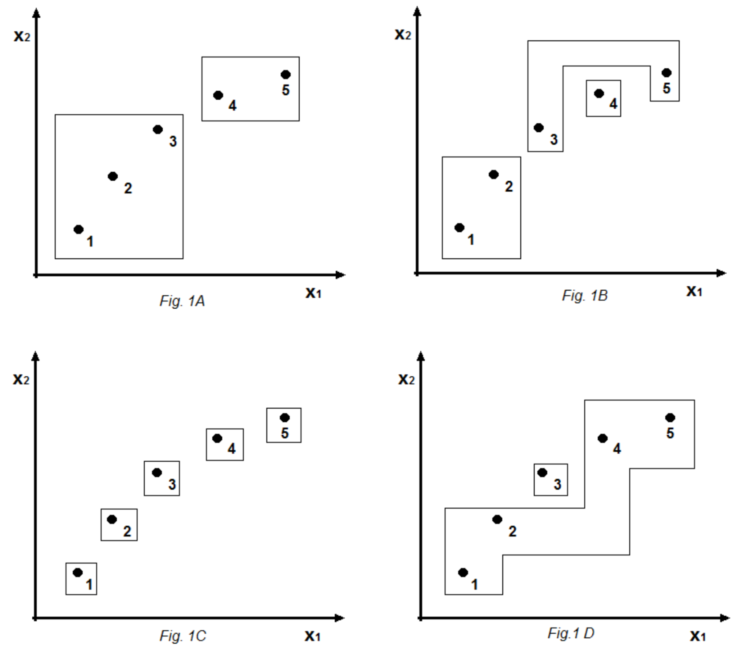

Fig. 1A Fig. 1B Fig. 1C Fig.1 D

These examples show that under my assumptions about individual preferences (*SM* and *SSCP*) the Median Voter Theorem result that emerge in the Citizen-Candidate model is robust to settings in which a much richer policy space is actually available to the voters, and hence it does not crucially depend on the strong restrictions that the model proposed by Besley and Coates implies on this aspect of the political interaction.

4.4.1. Conjecture: Ray-Vohra Stability

Under the assumptions of *SM* and *SSCP* a coalition structure is stable if and only if it is stable in the sense of Ray and Vohra (1997).

This concept of strability is similar to the one I propose in this paper, except for the fact that the profitability from a deviation for a subcoalition is evaluated keeping into account the possibility of future additional deviations. hence it can be considered a recursive version of the S-Core.

If this conjecture can be proved a strong link will arise between the political equilibrium concept presented in this paper and the one in Levy (2004, 2005), which is based on the stability concept proposed by Ray and Vohra.

4.5. Social Choice function

It is useful to analyse the characteristics of the Social Choice function generated by this Political Economy model. We know from Arrow's Impossibility Theorem (Arrow 1950; Gibbard 1973; Satterthwaite 1975) that there is no Social Welfare Function that satisfies at the same time *unrestricted domain (UD), non-dictatorship (ND), Pareto efficiency (PE), and independence of irrelevant* *alternatives* (*IIA*). Bergson (1976) has shown that Citizen-Candidate class of models imply a violation of *IIA*.

In my model *IIA* is not generally satisfied because the restrictions on the policy space X (which has to be a convex sublattice of R^d) are crucial in order to ensure a stable outcome that satisfies the Median Voter theorem.

On the other hand *Unrestricted Domain (UD)* is obviously violated given the restrictions on individual preferences, in the same way in which the Spence-Mirrlees conditions imply a violation of *UD* in the traditional Median Voter analysis of Downs (1957).

It is easy to verify that *ND* and *PE* are satisfied.

5. Games with Strategic Complementarities

Modify the setting in 2.1 in the following way. Consider a game with k players $j \in \mathcal{J}$ with a subset $\mathcal{N} \subset \mathcal{J}$ of n players who are also voters (n odd); Each player $j \in \mathcal{J}$ is endowed with a reflexive, complete and transitive preference ordering $\succeq^i$ that can be represented by a continuous and $\theta-$ *concave* utility function $F : X \times \Theta \times \Delta \to R$.

Define Δ as the set of vectors of strategies that each player can take outside of the voting game, with typical element $\delta = \{\delta_1, \delta_2, ..., \delta_k\}$, $\delta \in \Delta$.

The game can be either (i) simultaneous or (ii) sequential, i.e. at $t = 1$ voters play the voting game and the policy is chosen; at $t = 2$ each player j chooses $y \in Y_j$.

The voting game requires minimum changes in the definitions, in particular individual preference relations over policies have to be defined conditional on beliefs about other players' strategies, e.g. $\succeq^i (\tilde{\delta}_{-i})$ describes the preference relation of individual i given beliefs $\tilde{\delta}_{-i}$. Consequently the set of policies that are Pareto superior to a for coalition $\mathcal{A}$ becomes $p_{X,\mathcal{A}}(a, \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}}) \equiv \{b \in X : b \succeq^i (\tilde{\delta}_{-i})a \; \forall i \in \mathcal{A}, b \not\preceq^i (\tilde{\delta}_{-i})a \; \forall i \in \mathcal{A}\}$ and the Pareto set of coalition $\mathcal{A}$ becomes $\mathcal{P}(\mathcal{A}, \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}}) \equiv \{a \in X : \; p_{X,\mathcal{A}}(a, \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}}) = \oslash\}$. Similarly social preferences are now given by $\succeq (\{\tilde{\delta}_{-i}\}_{i\in\mathcal{N}})$ such that

$a \succeq (\{\tilde{\delta}_{-i}\}_{i\in\mathcal{N}})b$ if and only if $\sum_{i=1}^{n} 1[\max_{\delta_i} F(a, \theta_i, \delta_i, \tilde{\delta}_{-i}) \geq \max_{\delta_i} F(b, \theta_i, \delta_i, \tilde{\delta}_{-i})] > n/2$.

Similarly one can modify the definitions in section 3 as follows:

$P_A(a, (\{\tilde{\delta}_{-i}\}_{i\in\mathcal{N}})) \equiv \{b \in A : \; b \succ (\{\tilde{\delta}_{-i}\}_{i\in\mathcal{N}})a\}$ is the strictly preferred set;

$K(\mathcal{A}, (\{\tilde{\delta}_{-i}\}_{i\in\mathcal{N}})) \equiv \{a \in A : \; P_A(a, (\{\tilde{\delta}_{-i}\}_{i\in\mathcal{N}})) = \oslash\}$ is the Core;

$S_{\mathcal{A}}(a, \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}}) \equiv \{b \in \mathcal{P}(\mathcal{A}', \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}}), \mathcal{A}' \subseteq \mathcal{A} : b \succ^i (\tilde{\delta}_{-i})a \; \forall i \in \mathcal{A}', b \succ c \; \forall c \in \mathcal{P}(\mathcal{A}\backslash\mathcal{A}', \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}})\}$ is the set of "credible threats"

$SK(\mathcal{A}) = \{a \in \mathcal{P}(\mathcal{A}) : \; S_{\mathcal{A}}(a, \{\tilde{\delta}_{-i}\}_{i\in\mathcal{A}}) = \oslash\}$ is the S-Core.

Definition 17. A pure strategy equilibrium of this game is:

(a) a coalition structure $\mathbb{P}^* = \{\mathcal{A}^j\}_{j=1}^{h}$, a policy profile $A^* = \{a^j\}_{j=1}^{h}$ and a set of winning policies $W(A^*) \subseteq A^*$ such that such that *given beliefs about the strategies* $\tilde{y}$: (i) $\mathbb{P}^*$ is a stable coalition structure; (ii)$a^j \in SK(\mathcal{A}^j)$ for all $j = 1, 2, .., h$; (iii) $W(A^*)$ is nonempty;

(b) a strategy profile $\delta \in \Delta$ such that $\delta_i \in \arg\max_{\delta \in \Delta_i} F(\tilde{x}, \theta_i, \delta, \tilde{\delta}_{-i})$ for each $j \neq m$ and $(x, \delta_m) \in \arg\max_{x \in X, \delta \in \Delta_m} F(x, \theta_m, \delta, \tilde{\delta}_{-m})$ for $i \in m$.

With this new definition of an equilibrium for the game we can state the following result:

Theorem 18. *(i) An equilibrium exists. (ii) In any equilibrium the set of winning policies W is a subset of the set of ideal points of the median voter m. (iii) The largest and smallest pure strategy equilibria (and serially undominated strategy profiles) (a^*_m, δ^*) and (a_{*m}, δ_*), are monotone nondecreasing functions of the parameter that identifies the median voter θ_m.*

Proof. See Appendix B. □

6. Applications

6.1. Meltzer-Richard revisited

In one influential paper Meltzer and Richard (1981) analyse in a unidimensional political economy model the relationship between income distribution of a society and the extent of redistributive policies. One famous result in in their paper is that

> [..] An increase in mean income relative to the income of the decisive voter increases the size of the government.

In this section I will show how this result does not have to emerge in a political equilibrium if a richer policy space is available to voters. Moreover I will show that is the progressivity of the tax sistem that should monotone nondecreasing in the income of the median voter.

My setup is similar to the one in Meltzer-Richard (*MR*), with the difference that the budget of the government is spent not only in in-cash redistribution, but also in Public Goods. Suppose an individual i has utility function:

$$V_i = U(c_i, 1 - h_i) + v(Y)$$

where c_i is i's consumption of private goods, Y is the the quantity of Public Goods that is provided by the government and h_i is i's hours of work. The tax system is the same as in *MR*, namely individual post-tax income is determined by a linear tax rate $t = 1 - x$ and by a lump-sum grant g. Then the after tax income will be given by:

$$c_i = x\omega_i h_i + g$$

The government has to break even, such that the governmental budget constraint is given by:

$$(1 - x)\sum_{j=1}^{n}[\omega_j h_j] - Y - ng \geq 0$$

Assuming that the constraint above is binding in equilibrium we can substitute in the previous one:

$$c_i = x\omega_i h_i + (1-x)\overline{y} - Y/n$$

And then into the individual utility function:

$$V_i = U(x\omega_i h_i + (1-x)\overline{y} - Y/n, 1-h_i) + v(Y)$$

Lemma 19. *In the augmented MR model, if income inequality is sufficiently high, (i) the necessary conditions for the size of the government being monotonic increasing in the mean/median income ratio cannot be met; Moreover, (ii) the size of the government is monotonic nondecreasing in the median income (at constant mean) and (iii) the expenditure in Public goods is nondecreasing in the median income (at constant mean).*

Proof. See Appendix C. □

This result suggests that, differently from what Meltzer and Richard claim in their paper, a society with lower wage inequality may still have a larger government size in comparison with a more unequal one, but that the former should have a less progressive tax system.

Lemma 20. *In the augmented MR model the progressivity of the tax system is monotone nonincreasing in the in the median income (at constant mean) if the tax elasticity of labour supply is not strongly positive.*

Proof. Define the tax rate of an individual with income y_i as $T_i = \frac{(1-x)y_i - g}{y_i}$. The progressivity $PR_i = PR(x, g, y_i)$ of the tax system is decreasing in y_m:$PR_i = \left.\frac{\partial T_i}{\partial y_i}\right|_{x,g} = \frac{g}{y_i^2}$. Recall $y_i = xh_i\omega_i + g$. Differentiate w.r.t. $d\omega_m$ to get $dy_m = \left(\omega_m \frac{\partial h_m}{\partial \omega_m} + h_m + \omega_m \frac{\partial h_m}{\partial x}\frac{\partial x}{\partial \omega_m} + \omega_m \frac{\partial h_m}{\partial g}\frac{\partial g}{\partial \omega_m}\right) d\omega_m$. Hence $\frac{d\omega_m}{dy_m} = 1 \Big/ \left(\frac{\partial y_m}{\partial \omega_m} + \omega_m \frac{\partial h_m}{\partial x}\frac{\partial x}{\partial \omega_m} + \omega_m \frac{\partial h_m}{\partial g}\frac{\partial g}{\partial \omega_m}\right) > 0$ as long as $\eta_{h_m,t}$ is not strongly positive. Now the derivative of interest is:

$$\left.\frac{\partial PR_i}{\partial y_m}\right|_{\overline{y}} = \left(\frac{\partial PR_i}{\partial x}\left.\frac{\partial x}{\partial \omega_m}\right|_{\overline{y}} + \frac{\partial PR_i}{\partial g}\left.\frac{\partial g}{\partial \omega_m}\right|_{\overline{y}}\right)\frac{d\omega_m}{dy_m} =$$

$$= \left[-\frac{2g}{y_i^4}\left(\frac{\partial y_i}{\partial x}\left.\frac{\partial x}{\partial \omega_m}\right|_{\overline{y}} - \frac{\partial y_i}{\partial g}\overline{y}\left[\frac{\partial x}{\partial \omega_m} + \frac{1}{n}\frac{\partial Y}{\partial \omega_m}\right]\right) + \frac{1}{y_i^2}\left.\frac{\partial g}{\partial \omega_m}\right|_{\overline{y}}\right]\frac{d\omega_m}{dy_m} < 0 \forall i$$

Recall that $\frac{\partial y_i}{\partial g} \leq 0$ because the grant generates a pure income effect, $\frac{\partial y_i}{\partial x}$ is small negative or positive if $\eta_{h_m,t}$ is not strongly positive, $\left.\frac{\partial g}{\partial \omega_m}\right|_{\overline{y}} < 0$ and $\left.\frac{\partial x}{\partial \omega_m}\right|_{\overline{y}} > 0$ from Lemma (...). Hence we get: $\left.\frac{\partial PR_i}{\partial y_m}\right|_{\overline{y}} < 0$. Q.E.D. □

6.2. Arms Race in a Democratic country

Consider a democratic country whose citizens vote about a linear tax rate, a lump-sum grant and the level of spending in national defence in presence of an external threating nation. Notice that all citizens have same preferences about national defence. The utility function of a citizen with income is:

$$U_i(x, y, I_i, Y) = f(x, y) + u(c_i, l_i)$$

where x is the national spending in defence, y is the level of spending of the rival country, c_i is the expenditure in consumption and l_i is the hours of leisure enjoyed by citizen i. Following the literature about arms race I assume $f_{xy} \geq 0$, i.e. an increase in expenditures in national defence in a rival country increases the marginal utility of governmental spending in national defence., and $u_{cl} \geq 0$. Individual i's consumption is given by her disposable income such that $c_i \leq (1-t)h_i\theta_i + g$ and the governmental budget constraint is balanced: $t\sum_{i=1}^{n} h_i\theta_i \geq x + ng$.

Lemma 21. *If the difference between median and mean pre-tax income is sufficiently small and a change in the tax system that benefit the riches and harm the poors is implemented by the government, then the same government will also increase the expenditure in national defence. Moreover in equilibrium all rival countries will increase their expenditures in national defence.*

Proof. See Appendix D. □

7. Conclusions

This paper proposes a model of political interaction in which voters can form coalitions in order to increase the space of policies that can be proposed and in which this coalitions are required to be stable in a peculiar sense. I show that the assumptions of Supermodularity and Strict Single Crossing Property of voters' objective functions are sufficient for the existence of a political equilibrium in a multidimensional policy space. Moreover I show that under the same assumptions a version of the Median Voter Theorem holds and as a consequence a monotone comparative statics result of the equilibrium outcomes is derived.

The paper describes a tool that can be useful to correct the predictions delivered by traditional Downsian models for some common Political Economy questions in the literature and potentially to explain the poor empirical performance of these predictions.

A feature that emerges is that the model is sufficiently flexible to deal with games that are more complex than simple voting games and can deliver interesting answer to questions in the field that cannot be easily analysed in the traditional framework.

I claim that this results can be applied to addess a number of different questions in the field and to light shed on some controversial results in the literature and that it represents an elegant an parsimonious way of dealing with one of the most commons problems that emerge if one aims to model Political Choices in an economic model. Moreover, the model is sufficiently general to be suitable todescribe many different Political Economy problems and to incorporate some results that are well established in the literature as special cases of my framework.

Despite of these achivements the assumptions that must be satisfied in order to prove the two main results of the paper are relatively restrictive so their applicability should be evaluated in relationship to the credibility of those assumptions in each specific application.

AppendixA. Lemma 5

If F satisfies SM and SSCP then the Pareto Set $\mathcal{P}(\mathcal{A})$ of a coalition of players $\mathcal{A} \subseteq \mathcal{N}$ is such that $x \in \mathcal{P}(\mathcal{A})$ only if $x \geq \sup\{M(l)\}$ and $x \leq \inf\{M(h)\}$ where $l = \min(\mathcal{A})$ and $h = \max(\mathcal{A})$.

Proof. Suppose $y \ngeq \overline{x}_l$ but $y \in \mathcal{P}(\mathcal{A})$. Because of the optimality of $\overline{x}_l$ and because X is a lattice, it must be true that $F(\overline{x}_l, \theta_l) \geq F(y \wedge \overline{x}_l, \theta_l)$. Supermodularity implies $F(y \vee \overline{x}_l, \theta_l) \geq F(y, \theta_l)$. Notice that $y \ngeq \overline{x}_l$ implies $y \vee \overline{x}_l \neq y$. Hence the Strict Single Crossing Property implies $F(y \vee \overline{x}_l, \theta_i) \geq F(y, \theta_i)$ $\forall \theta_i > \theta_l$. Given that $\theta_i > \theta_l$ is true for all $\theta \in \mathcal{A}, \theta \neq \theta_l$ we have that $\exists x \in X$ such that $F(x, \theta) \geq F(y, \theta)$ $\forall \theta \in \mathcal{A}$ and $F(x, \theta) > F(y, \theta)$ for at least one $\theta \in \mathcal{A}$, i.e $p_{X,\mathcal{A}}(y) \neq \oslash$. Hence $y \notin \mathcal{P}(\mathcal{A})$. Similarly one can show that $x \in A$ only if $x \leq \underline{x}_h$. Q.E.D. □

AppendixB. Theorem 18

Define the following game. A nonempty set N indexes the players, and each player's strategy set is S_i, partially ordered by $\geq$. The space of strategy profiles is then S, and player i has payoff function $\pi_i(z_i, z_{-i})$. Following Milgrom, Shannon 1994 such a game has (ordinal) strategic complementarities if for every i: (1) S_i is a compact lattice; (2) π_i is upper semi-continuous in z_i for z_{-i} fixed, and continuous in z_{-i} for fixed z_i; (3) π_n is quasisupermodular in z_i and satisfies the single crossing property in $(z_i; z_{-i})$. Say $z_i = \delta_i$, $z_{-i} = (x, \delta_{-i})$ for all $i \neq m$ and $z_m = (x, \delta_i)$, $z_{-m} = \delta_{-i}$.

INCOMPLETE.

AppendixC. Lemma 19

In the augmented MR model, if income inequality is sufficiently high, (i) the necessary conditions for the size of the government being monotonic increasing in the mean/median income ratio cannot be met; Moreover, (ii) the size of the government is monotonic nondecreasing in the median income (at constant mean) and (ii) the expenditure in Public goods is nondecreasing in the median income (at constant mean).

Proof. Suppose there is an internal maximum. Then FOCs of the median voter imply:

$$\frac{\partial V_m}{\partial x} = U_c(x\omega_m h_m + (1-x)\overline{y} - Y/n, 1-h_i)\left(\omega_m h_m - \overline{y} + (1-x)\frac{\partial \overline{y}}{\partial x}\right) = 0$$

$$\frac{\partial V}{\partial Y} = -U_c(x\omega_i h_i + (1-x)\overline{y} - Y/n, 1-h_i)/n + v_Y$$

We want to study the comparative statics induced by changes of the median income y_m at constant mean income $\overline{y}$. Recall that for C^2 functions a sufficient condition for the *SSCP* to apply is $\left.\frac{\partial^2 V}{\partial x_j \partial \omega_i}\right|_{\overline{y}} > 0 \forall j, i$. First of all I will study the derivative w.r.t. ω_m. In this example we have:

$$V_{x\omega_m} \equiv \left.\frac{\partial^2 V}{\partial x \partial \omega_m}\right|_{\overline{y}} = \left[U_{cc} \cdot \left(xh_m + x\omega_i \frac{\partial h_m}{\partial \omega_m}\right) - U_{cl} \cdot \frac{\partial h_i}{\partial \omega_i}\right]\left(\omega_m h_m - \overline{y} + (1-x)\frac{\partial \overline{y}}{\partial x}\right) +$$

$$+U_c(x\omega_m h_m + (1-x)\overline{y} - Y/n, Y)\left(h_m + \omega_m \frac{\partial h_m}{\partial \omega_m}\right) =$$

Notice that in an internal equilibrium FOCs imply either $\omega_m h_m - \overline{y} + (1-x)\frac{\partial \overline{y}}{\partial x} = 0$ or $U_c(x\omega_m h_m + (1-x)\overline{y} - Y/n, 1-h_m)\left(\omega_m h_m - \overline{y} + (1-x)\frac{\partial \overline{y}}{\partial x}\right) = 0$. Hence using FOCs the condition reduces to:

$$V_{x\omega_m} > 0 \leftrightarrow \left[U_{cc} \cdot \left(xh_m + x\omega_m \frac{\partial h_m}{\partial \omega_m}\right) - U_{cl} \cdot \frac{\partial h_m}{\partial \omega_m}\right]\left(\omega_m h_m - \overline{y} + (1-x)\frac{\partial \overline{y}}{\partial x}\right) > 0$$

$$V_{xY} \equiv \left.\frac{\partial^2 V}{\partial x \partial Y}\right|_{\overline{y}} = 0$$

$$V_{Y\omega} \equiv \left.\frac{\partial^2 V}{\partial Y \partial \omega_m}\right|_{\overline{y}} = -U_{cc} \cdot \left(x\omega_m \frac{\partial h_m}{\partial \omega_m} + xh_m\right)/n + U_{cl} \cdot \frac{\partial h_m}{\partial \omega_m} > 0$$

$$V_{Y\omega} \equiv \left.\frac{\partial^2 V}{\partial Y \partial \omega_m}\right|_{\overline{y}} = -U_{cc} \cdot x\frac{\partial y_m}{\partial \omega_m}/n + U_{cl} \cdot \frac{\partial h_m}{\partial \omega_m}/n > 0$$

if h_m has low wage elasticity and /or $U_{cl} \approx 0$. This implies that V is supermodular in (x, Y) and satisfies the *SSCP* in (x, Y, ω) and hence using Milgrom-Shannon we know that the set of ideal points is monotone nondecreasing in ω. Hence in our model of political interaction if the median voter changes to an individual with higher ω(and hence higher income y) and the average is constant, the linear tax rate $t = (1-x)$ will be weakly in equilibrium, as in Meltzer-Richard. Alternatively if the solution is internal one can check that:

$$\left.\frac{\partial x}{\partial \omega_m}\right|_{\overline{y}} = -\frac{V_{x\omega}}{V_{xx}} > 0$$

$$\left.\frac{\partial Y}{\partial \omega_m}\right|_{\overline{y}} = -\frac{V_{x\omega}}{V_{YY}} > 0$$

$$V_{xx} \equiv \left.\frac{\partial^2 V}{\partial x^2}\right|_{\overline{y}} = \left(U_{cc} \cdot \left[\omega_m h_m + x\omega_m \frac{\partial h_m}{\partial x} - \overline{y}\right] - U_{cl}\frac{\partial h_m}{\partial x}\right)\left(\omega_m h_m - \overline{y} + (1-x)\frac{\partial \overline{y}}{\partial x}\right)$$

$$V_{YY} \equiv \left.\frac{\partial^2 V}{\partial Y^2}\right|_{\overline{y}} = U_{cc}(1 - \frac{\partial h_m}{\partial Y}x\omega_m)/n^2 + v_{YY}$$

Finally, we can recover the sign of the change in g induced by a marginal increase in ω_m using the budget constraint:

$$g = (1-x)\overline{y} - Y/n$$

hence (assuming that g is differentiable, explain why):

$$\left.\frac{\partial g}{\partial \omega_m}\right|_{\overline{y}} = -\overline{y}\frac{\partial x}{\partial \omega_m} - \frac{1}{n}\frac{\partial Y}{\partial \omega_m}$$

Define the size of the government as follows:

$$S(\overline{y}, y_m, x, g) \equiv \sum_{i=1}^{n} \mathbb{I}\left[(1-x)h_i\omega_i - g \geq 0\right]\left((1-x)h_i\omega_i - g\right)$$

i.e. the size of the government is the total revenue from taxes.

$$\left.\frac{\partial S}{\partial \omega_m}\right|_{\overline{y}} = \sum_{i=1}^{n} 1\left[(1-x)h_i\omega_i - g \geq 0\right]\left(-\frac{\partial x}{\partial \omega_m}y_i + (1-x)\frac{\partial y_i}{\partial x}\frac{\partial x}{\partial \omega_m} - \frac{\partial g}{\partial \omega_m}\right) =$$

$$= -\frac{\partial x}{\partial \omega_m}\sum_{i=k}^{n} y_i + (1-x)\sum_{i=k}^{n}\left(\frac{\partial y_i}{\partial x}\frac{\partial x}{\partial \omega_m}\right) - n\frac{\partial g}{\partial \omega_m} =$$

$$= -\frac{\partial x}{\partial \omega_m}\sum_{i=k}^{n} y_i + (1-x)\sum_{i=k}^{n}\left(\frac{\partial y_i}{\partial x}\frac{\partial x}{\partial \omega_m}\right) - n\left(-\overline{y}\frac{\partial x}{\partial \omega_m} + (1-x)\frac{\partial \overline{y}}{\partial x}\frac{\partial x}{\partial \omega_m} - \frac{1}{n}\frac{\partial Y}{\partial \omega_m}\right) =$$

$$-\frac{\partial x}{\partial \omega_m}\left[\sum_{i=1}^{k-1} y_i - (1-x)\sum_{i=1}^{k-1}\left(\frac{\partial y_i}{\partial x}\frac{\partial x}{\partial \omega_m}\right)\right] + \frac{\partial Y}{\partial \omega_m} =$$

Rearranging:

$$= -\frac{\partial x}{\partial \omega_m}\left[\sum_{i=1}^{k-1} y_i\left(1 + \eta_{y_i,t}\right)\right] + \frac{\partial Y}{\partial \omega_m} =$$

Now:

$$dy_m = \left(\omega_m \frac{\partial h_m}{\partial \omega_m} + h_m + \omega_m \frac{\partial h_m}{\partial x}\frac{\partial x}{\partial \omega_m}\right)d\omega_m$$

Hence

$$\frac{d\omega_m}{dy_m} = 1 \Bigg/ \left(\frac{\partial y_m}{\partial \omega_m} + \omega_m \frac{\partial h_m}{\partial x}\frac{\partial x}{\partial \omega_m}\right) > 0$$

as long as $\eta_{h_m,t}$ is not strongly positive.

$$\left.\frac{\partial S}{\partial y_m}\right|_{\overline{y}} = \left[-\frac{\partial x}{\partial \omega_m}\frac{\sum_{i=1}^{k-1} y_i\left(1+\eta_{y_i,t}\right)}{V_{xx}} + \frac{\partial Y}{\partial \omega_m}\right]\frac{d\omega_m}{dy_m}$$

Consider to extreme case: $y_i = 0$ for all $i \leq j$ with $m \leq j$, $y_i > 0$ for all $i > j$. If an internal equilibrium exists this implies $\left.V_{xx}\right|_{\overline{y}} = U_{cc}\overline{y}\left(\overline{y} - (1-x)\frac{\partial \overline{y}}{\partial x}\right)$ and $\sum_{i=1}^{k-1} y_i\left(1+\eta_{y_i,t}\right) = 0$ because $\frac{\partial h_m}{\partial x} = 0$, therefore: $\left.\frac{\partial S}{\partial y_m}\right|_{\overline{y}} > 0$. □

AppendixD. Lemma 19

Proof. Substitute the individual and the governmental budget constraint into the utility function. The resulting objective function of citizen i is $V(x, y, T; \theta_i) = f(x, y) + u\left((1-t)h_i\theta_i + t\overline{y} - x/n, 1-h_i\right)$. First derivatives of V are: $V_x = f_x - u_c/n$; $V_t = -u_c\left(h_i\theta_i - \overline{y}\right)$. Second derivatives at constant $\overline{y}$ (see definition of V_{ij} in Appendix B) are: $V_{t\theta_i} = \left[-u_{cc}\left(h_i + \theta_i\frac{\partial h_i}{\partial \theta_i}\right)(1-t) + u_{cl}\frac{\partial h_i}{\partial \theta_i}\right](y_i - \overline{y}) - u_c\left(h_i + \theta_i\frac{\partial h_i}{\partial \theta_i}\right) < 0$; $V_{x\theta_i} = -u_{cc}\left(h_i + \theta_i\frac{\partial h_i}{\partial \theta_i}\right)(1-t) + u_{cl}\frac{\partial h_i}{\partial \theta_i} > 0$; $V_{xt} = u_{cc}\left(y_i - \overline{y}\right) > 0$. Notice that $\lim_{y_i \to \overline{y}} V_{t\theta_i} = 0$; $\lim_{y_i \to \overline{y}} V_{xt} = 0$; $\lim_{y_i \to \overline{y}} V_{tx\theta} > 0$. Hence there must be a threshold $\hat{y}(\overline{y})$ such that if $\hat{y}(\overline{y}) \leq y_m < \overline{y}$ then the comparative statics is: (i) x is increasing in y_m; (ii) g is decreasing in y_m; (iii) the comparative statics of t is ambiguous. This implies that a less progressive tax system will be implemented and the amount of transfers to poor individuals will fall and at the same time the expenditure in national defence will rise. Moreover, using (SECTION 5) we know that the expenditure in defence of all other countries must be weakly higher in equilibrium. □

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