Secular stagnation and decline: a simplified model

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ABSTRACT

Presented is a mathematical model of single-product economy describing a nominal economic growth and a nominal economic decline. Based on the model of economic dynamics, policies handling the gravity of the secular stagnation are furnished. First, transition of the secular stagnation into the secular decline is to be prevented. Second, a two-stage economic policy against the secular stagnation should be entertained. The first stage is to promote a policy of advancing the additional demand for products to counterbalance the additional supply of products by external suppliers. The second stage is to sustain a policy of savings and investments to stipulate an economic growth where the savings and investments are to be committed with a modest acceleration. Two stages of the alleviating economic policy can be executed concurrently.

JEL Classification Numbers: E32, O11, C61

Keywords: economic growth, business fluctuations, secular stagnation

1 Introduction

The *secular stagnation* hypothesis with respect to today's economical conditions had been put forward by outstanding modern economist Lawrence Henry Summers.[1] His ideas were consonant with my findings of conjectural economic situations, which I had been analyzing working with the models of an economic growth (Krouglov, 2006, 2009). However, his pragmatic research allowed me to transfer my theoretical constructs from an imaginary realm of virtual reality into the practical realm of modern-day economic life.

Previously, I presented a mathematical model of economic growth based on the differential equations describing demand and supply forces in economy. The model showed that both constant-rate and constant-acceleration savings and investments were correspondingly causing a restricted (long-term impact) and an unrestricted (long-term impact) economic growth. Processes were examined by using a hypothetical single-product economy.

Here, I complement the model with a contrasting situation where external supply of the product is provided on the market. I will show that the external supply provided with either a constant rate or a constant acceleration is correspondingly causing a restricted (*secular stagnation*) and an unrestricted (*secular decline*) economic decline.

Below I describe a mathematical model of the economic market. The economic forces acting on the market represent both inherent market forces of demand and supply complemented with an impact of savings or external supply. The market forces will be expressed through the system of ordinary differential equations.

For completeness, I will include the situation of an economic growth since the contrast of economic growth with economic stagnation and decline makes the obtained theoretical results more practical and thoughtful.

[1] Reflections of Larry Summers on the secular stagnation hypothesis are available in a Vox eBook.

2 Single-Product Economy at Undisturbed State

Concepts and methodology presented in this section are based on the framework of mathematical dynamics of economic systems developed in Krouglov, 2006; 2009.

When there are no disturbing economic forces, the market is in equilibrium position, i.e., the supply of and demand for product are equal, they are developing with a constant rate and a price of the product is fixed.

I assume the market had been in an equilibrium until time $t = t_0$, volumes of the product supply $V_S(t)$ and demand $V_D(t)$ on market were equal, and they both were developing with a constant rate r_D^0. The product price $P(t)$ at that time was fixed,

$$V_D(t) = r_D^0 (t - t_0) + V_D^0 \tag{1}$$

$$V_S(t) = V_D(t) \tag{2}$$

$$P(t) = P^0 \tag{3}$$

where $V_D(t_0) = V_D^0$.

When the balance between the product supply and demand is broken, the market is experiencing economic forces, which act to bring the market to a new equilibrium position.

3 Nominal Economic Growth in a Single-Product Economy

An economic growth is fundamentally driven by the process of saving and investment.

I will present a model where a constant-rate and a constant-acceleration saving and investment are causing restricted economic growth and unrestricted economic growth correspondingly.

3.1. Model of a Continuous Constant-Rate Saving and Investment

According to this scenario, I assume the amount of product saving and investment $S_I(t)$ on the market increases since time $t = t_0$ according to the following formula,

$$S_I(t) = \begin{cases} 0, & t < t_0 \\ \delta_S (t - t_0), & t \ge t_0 \end{cases} \tag{4}$$

where $S_I(t) = 0$ for $t < t_0$ and $\delta_S > 0$.

Economic forces trying to bring the market into a new equilibrium position are described by the following ordinary differential equations with regard to the product supply $V_S(t)$, demand $V_D(t)$, and price $P(t)$ on the market (see Krouglov, 2006; 2009),

$$\frac{dP(t)}{dt} = -\lambda_P \left(V_S(t) - V_D(t) - S_I(t) \right) \tag{5}$$

$$\frac{d^2 V_S(t)}{dt^2} = \lambda_S \frac{dP(t)}{dt} \tag{6}$$

$$\frac{d^2 V_D(t)}{dt^2} = -\lambda_D \frac{d^2 P(t)}{dt^2} \tag{7}$$

In Eqs. (5) – (7) above the values $\lambda_P, \lambda_S, \lambda_D \ge 0$ are constants.[2]

Let me introduce a new variable $D(t) \equiv \left(V_S(t) - V_D(t) - S_I(t) \right)$ representing the volume of product surplus (or shortage) on the market. Therefore, behavior of $D(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 D(t)}{dt^2} + \lambda_P \lambda_D \frac{dD(t)}{dt} + \lambda_P \lambda_S D(t) = 0 \tag{8}$$

[2] The coefficients characterize price inertness, supply inducement, and demand amortization correspondingly.

with the initial conditions, $D(t_0) = 0$, $\frac{dD(t_0)}{dt} = -\delta_S$.

Similar to Eq. (8), the product price $P(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2P(t)}{dt^2} + \lambda_P \lambda_D \frac{dP(t)}{dt} + \lambda_P \lambda_S \left(P(t) - P^0 - \frac{\delta_S}{\lambda_S} \right) = 0 \tag{9}$$

with the initial conditions, $P(t_0) = P^0$, $\frac{dP(t_0)}{dt} = 0$.

Let me introduce another variable $P_1(t) \equiv P(t) - P^0 - \frac{\delta_S}{\lambda_S}$ to simplify an analysis of the product price behavior. Then behavior of the variable $P_1(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2P_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dP_1(t)}{dt} + \lambda_P \lambda_S P_1(t) = 0 \tag{10}$$

with the initial conditions, $P_1(t_0) = -\frac{\delta_S}{\lambda_S}$, $\frac{dP_1(t_0)}{dt} = 0$.

The behavior of solutions for both $D(t)$ and $P_1(t)$ described by Eqs. (8) and (10) depends on the roots of the corresponding characteristic equations (Piskunov, 1965; Petrovski, 1966). Note that Eqs. (8) and (10) have the same characteristic equation.

When the roots of characteristic equation are complex-valued (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$) both the variable $D(t)$ and variable $P_1(t)$ experience damped oscillations for time $t \ge t_0$. When the roots of characteristic equation are real and different (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$) both the variable $D(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$. When the roots of characteristic equation are real and equal (i.e., $\frac{\lambda_P^2\, \lambda_D^2}{4} = \lambda_P\, \lambda_S$) both the variable $D(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$ as well.

It takes place $D(t) \to 0$ and $P_1(t) \to 0$ for $t \to +\infty$ if the roots of characteristic equation are complex-valued ($\frac{\lambda_P^2\, \lambda_D^2}{4} < \lambda_P\, \lambda_S$), real and different ($\frac{\lambda_P^2\, \lambda_D^2}{4} > \lambda_P\, \lambda_S$), or real and equal ($\frac{\lambda_P^2\, \lambda_D^2}{4} = \lambda_P\, \lambda_S$).

Therefore, it takes place for the product price $P(t)$, for the product demand $V_D(t)$, for the product supply $V_S(t)$, and for the product saving and investment $S_I(t)$ if $t \to +\infty$,

$$P(t) \to P^0 + \frac{\delta_S}{\lambda_S} \tag{11}$$

$$V_D(t) \to r_D^0 (t - t_0) + V_D^0 - \frac{\lambda_D}{\lambda_S} \delta_S \tag{12}$$

$$V_S(t) \to \left(r_D^0 + \delta_S\right)(t - t_0) + V_D^0 - \frac{\lambda_D}{\lambda_S} \delta_S \tag{13}$$

$$S_I(t) = \delta_S (t - t_0) \tag{14}$$

To perform analyze of the economic growth let me introduce a new variable $E_D(t) \equiv P(t) \times r_D(t)$ where $r_D(t) \equiv \frac{dV_D(t)}{dt}$, i.e., a rate of nominal demand for the product, which roughly represents the product earning on the market.

I make the comparison of variable $E_D(t)$, a rate of nominal demand changed by the saving and investment $S_I(t)$ on the market, with the variable $\tilde{E}_D(t)$, a rate of original nominal demand unchanged by the saving and investment, for $t \to +\infty$, $E_D(t) \to \left(P^0 + \frac{\delta_S}{\lambda_S}\right) r_D^0$ and $\tilde{E}_D(t) \to P^0\, r_D^0$.

Thus, we can clearly see that if saving and investment $S_I(t)$ on the market is done with a constant rate $\delta_S > 0$ then the quantitative value of that rate δ_S has a restricted impact on the rate of nominal demand $E_D(t)$.

Therefore, the change in the rate of nominal demand $\Delta E_D(t) \equiv E_D(t) - \tilde{E}_D(t)$ for the product on the market, which roughly represents the difference of the product earning or an economic growth, is equal for time $t \to +\infty$, $\Delta E_D(t) \to \frac{\delta_S}{\lambda_S} r_D^0$.

Another observation that can be made is that eventual change of the rate of nominal demand $\Delta E_D(t)$ is constrained by the value $\frac{\delta_S}{\lambda_S} r_D^0 > 0$. Therefore, saving and investment on the market with a constant rate $\delta_S > 0$ doesn't produce a profound terminal effect on the rate of nominal demand $E_D(t)$.

Nevertheless, the model shows that amount of eventual rate of nominal demand adjustment $\frac{\delta_S}{\lambda_S} r_D^0 > 0$ is directly proportional to the rate of saving and investment $\delta_S > 0$ in the market (the limitary value of $\Delta E_D(t)$ is also directly proportional to the initial rate of demand $r_D^0 > 0$ and, correspondingly, inversely proportional to a supply inducement constant $\lambda_S \geq 0$ inherent to the economic system).

For completeness, let me introduce another variable $E_S(t) \equiv P(t) \times r_S(t)$ where $r_S(t) \equiv \frac{dV_S(t)}{dt}$, i.e., a rate of nominal supply for the product, which roughly represents the product (internal) production.

I make the comparison of variable $E_S(t)$, a rate of nominal supply changed by the saving and investment $S_I(t)$ on the market, with the variable $\tilde{E}_S(t)$, a rate of original nominal supply unchanged by the saving and investment, for $t \to +\infty$, $E_S(t) \to \left(P^0 + \frac{\delta_S}{\lambda_S} \right) \left(r_D^0 + \delta_S \right)$ and $\tilde{E}_S(t) \to P^0 \, r_D^0$.

Therefore, the difference in the rate of nominal supply $\Delta E_S(t) \equiv E_S(t) - \tilde{E}_S(t)$ of the product on the market, which roughly represents a difference of the product production or a production growth, is equal, for time $t \to +\infty$, $\Delta E_S(t) \to \delta_S \left(P^0 + \frac{r_D^0 + \delta_S}{\lambda_S} \right) > 0$ when $\delta_S > 0$.

Note that limitary value of variable $\Delta E_S(t)$ doesn't have extremal points in the region $\delta_S > 0$. In fact, the variable $\Delta E_S(t)$ has minimal limitary value when $\delta_S = -\frac{1}{2} \left(P^0 \lambda_S + r_D^0 \right) < 0$. Then, respectively

$$\min\{\Delta E_S(t)\} \to -\frac{\lambda_S}{4} \left(P^0 + \frac{r_D^0}{\lambda_S} \right)^2 < 0 \text{ for } t \to +\infty.$$

I will analyze economic implications of the obtained results in a following section.

3.2. *Model of a Continuous Constant-Acceleration Saving and Investment*

According to this scenario, I assume the amount of product saving and investment $S_I(t)$ on the market increases since time $t = t_0$ according to the following formula,

$$S_I(t) = \begin{cases} 0, & t < t_0 \\ \delta_S (t - t_0) + \dfrac{\varepsilon_S}{2}(t - t_0)^2, & t \geq t_0 \end{cases} \tag{15}$$

where $S_I(t) = 0$ for $t < t_0$, $\delta_S \geq 0$, and $\varepsilon_S > 0$.

Economic forces trying to bring the product supply $V_S(t)$, demand $V_D(t)$, and price $P(t)$ on the market into a new equilibrium position are described by Eqs. (5) – (7).

Let me use again the variable $D(t) \equiv (V_S(t) - V_D(t) - S_I(t))$ representing the volume of product surplus (or shortage) on the market. The behavior of $D(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 D(t)}{dt^2} + \lambda_P \lambda_D \frac{dD(t)}{dt} + \lambda_P \lambda_S D(t) + \varepsilon_S = 0 \tag{16}$$

with the initial conditions, $D(t_0) = 0$, $\dfrac{dD(t_0)}{dt} = -\delta_S$.

If one uses a new variable $D_1(t) \equiv D(t) + \dfrac{\varepsilon_S}{\lambda_P \lambda_S}$, then Eq. (16) becomes,

$$\frac{d^2 D_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dD_1(t)}{dt} + \lambda_P \lambda_S D_1(t) = 0 \tag{17}$$

with the initial conditions, $D_1(t_0) = \dfrac{\varepsilon_S}{\lambda_P \lambda_S}$, $\dfrac{dD_1(t_0)}{dt} = -\delta_S$.

Similar to Eq. (16), the product price $P(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P(t)}{dt^2} + \lambda_P \lambda_D \frac{dP(t)}{dt} + \lambda_P \lambda_S \left(P(t) - P^0 - \frac{\delta_S}{\lambda_S} - \frac{\varepsilon_S}{\lambda_S}(t - t_0) \right) = 0 \tag{18}$$

with the initial conditions, $P(t_0) = P^0$, $\frac{dP(t_0)}{dt} = 0$.

Let me introduce a variable $P_1(t) \equiv P(t) - P^0 - \frac{\delta_S}{\lambda_S} - \frac{\varepsilon_S}{\lambda_S}(t - t_0) + \frac{\lambda_D}{\lambda_S^2}\varepsilon_S$ to simplify an analysis of the product price behavior. The behavior of variable $P_1(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dP_1(t)}{dt} + \lambda_P \lambda_S P_1(t) = 0 \qquad (19)$$

with the initial conditions, $P_1(t_0) = -\frac{\delta_S}{\lambda_S} + \frac{\lambda_D}{\lambda_S^2}\varepsilon_S$, $\frac{dP_1(t_0)}{dt} = -\frac{\varepsilon_S}{\lambda_S}$.

The behavior of solutions for $D_1(t)$ and $P_1(t)$ described by Eqs. (17) and (19) depends on the roots of the corresponding characteristic equations. Again Eqs. (17) and (19) have the same characteristic equation.

As before, when the roots of characteristic equation are complex-valued (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$) the variable $D_1(t)$ and variable $P_1(t)$ experience damped oscillations for time $t \geq t_0$. When the roots of characteristic equation are real and different (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$) the variable $D_1(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$. When the roots are real and equal (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$) the variable $D_1(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$.

It occurs $D_1(t) \to 0$ and $P_1(t) \to 0$ for $t \to +\infty$ if the roots of characteristic equation are complex-valued ($\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$), real and different ($\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$), or real and equal ($\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$).

It takes place for the product surplus (shortage) $D(t)$, for the product price $P(t)$, for the product demand $V_D(t)$, for the product supply $V_S(t)$, and for the product saving and investment $S_I(t)$ if $t \to +\infty$,

$$D(t) \to -\frac{\varepsilon_S}{\lambda_P \lambda_S} \tag{20}$$

$$P(t) \to \frac{\varepsilon_S}{\lambda_S}(t - t_0) + P^0 + \frac{\delta_S}{\lambda_S} - \frac{\lambda_D}{\lambda_S^2}\varepsilon_S \tag{21}$$

$$V_D(t) \to \left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_S \right)(t - t_0) + V_D^0 - \frac{\lambda_D}{\lambda_S}\delta_S + \frac{\lambda_D^2}{\lambda_S^2}\varepsilon_S \tag{22}$$

$$V_S(t) \to \left(r_D^0 + \delta_S - \frac{\lambda_D}{\lambda_S}\varepsilon_S \right)(t - t_0) + \frac{\varepsilon_S}{2}(t - t_0)^2 + V_D^0 - \frac{\lambda_D}{\lambda_S}\delta_S - \frac{\varepsilon_S}{\lambda_P \lambda_S} + \frac{\lambda_D^2}{\lambda_S^2}\varepsilon_S \tag{23}$$

$$S_I(t) = \delta_S (t - t_0) + \frac{\varepsilon_S}{2}(t - t_0)^2 \tag{24}$$

For analyze of the economic growth I use the variable $E_D(t) \equiv P(t) \times r_D(t)$ where $r_D(t) \equiv \frac{dV_D(t)}{dt}$, i.e., a rate of nominal demand for the product, which roughly represents the product earning on the market.

I make the comparison of variable $E_D(t)$, a rate of nominal demand changed by the saving and investment $S_I(t)$ on the market, with the variable $\tilde{E}_D(t)$, a rate of original nominal demand unchanged by the saving and investment, for $t \to +\infty$, $E_D(t) \to \left(\frac{\varepsilon_S}{\lambda_S}(t - t_0) + P^0 + \frac{\delta_S}{\lambda_S} - \frac{\lambda_D}{\lambda_S^2}\varepsilon_S \right)\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_S \right)$ and $\tilde{E}_D(t) \to P^0 r_D^0$.

Thus, if saving and investment $S_I(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then the value of acceleration ε_S causes an unrestricted increase (decrease) of the rate of nominal demand $E_D(t)$. The rate of nominal demand $E_D(t)$ is continuously increasing (decreasing) with the passage of time. It is clearly different from if the saving and investment is done a constant rate $\delta_S > 0$, which produces a limitary change of the rate of nominal demand constrained by the finite value $\frac{\delta_S}{\lambda_S} r_D^0 > 0$.

We can estimate an increase (decrease) $e_D(t)$ in the rate of nominal demand where $e_D(t) \equiv \frac{dE_D(t)}{dt}$, i.e., the increase (decrease) in the rate of nominal demand for the product, which roughly represents increase (decrease) in the product earning on the market.

It takes place, for $t \to +\infty$, $e_D(t) \to \frac{\varepsilon_S}{\lambda_S}\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_S \right)$. Therefore, the variable $e_D(t)$ has maximal limitary value when $\varepsilon_S = \frac{\lambda_S}{2\lambda_D} r_D^0$. Then, for $t \to +\infty$, $\max\{e_D(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$.

Therefore, the maximal increase in the rate of nominal demand $\max\{e_D(t)\}$ for the product on market, which roughly represents the maximal increase of the product earning or a maximal economic growth, is equal for time $t \to +\infty$, $\max\{e_D(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$.

Another observation can be made is if saving and investment $S_I(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then it causes an unrestricted increase (decrease) of the rate of nominal demand $E_D(t)$ with a constant acceleration (deceleration) $\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_S \right)$. If $\varepsilon_S \le \frac{\lambda_S}{\lambda_D} r_D^0$ then $E_D(t)$ increases with a constant acceleration $\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_S \right) \ge 0$. And if $\varepsilon_S > \frac{\lambda_S}{\lambda_D} r_D^0$ then $E_D(t)$ decreases with a constant deceleration $\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 - \frac{\lambda_D}{\lambda_S}\varepsilon_S\right) < 0$. The increase of the rate of nominal demand $E_D(t)$ achieves limitary value of the maximal constant acceleration $\frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$ for value $\varepsilon_S = \frac{\lambda_S}{2\lambda_D} r_D^0$.[3]

For completeness, let me use another variable $E_S(t) \equiv P(t) \times r_S(t)$ where $r_S(t) \equiv \frac{dV_S(t)}{dt}$, i.e., a rate of nominal supply for the product, which roughly represents the product (internal) production.

I make the comparison of variable $E_S(t)$, a rate of nominal supply changed by the saving and investment $S_I(t)$ on the market, with the variable $\tilde{E}_S(t)$, a rate of original nominal supply unchanged by the saving and investment, for $t \to +\infty$, $E_S(t) \to \left(\frac{\varepsilon_S}{\lambda_S}(t - t_0) + P^0 + \frac{\delta_S}{\lambda_S} - \frac{\lambda_D}{\lambda_S^2}\varepsilon_S\right)\left(r_D^0 + \delta_S - \frac{\lambda_D}{\lambda_S}\varepsilon_S\right)$ and $\tilde{E}_S(t) \to P^0 r_D^0$.

Thus, if saving and investment $S_I(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then the value of acceleration ε_S causes an unrestricted increase (decrease) of the rate of nominal supply $E_S(t)$. The rate of nominal supply $E_S(t)$ is continuously increasing (decreasing) with the passage of time, i.e., it takes place a continuous increase (decrease) in the product production or a continuous production growth

[3] If $\varepsilon_S \in (-\infty, 0)$ the limitary value of $e_D(t)$ is negative and monotonically increases. If $\varepsilon_S \in \left(0, \frac{\lambda_S}{2\lambda_D} r_D^0\right)$ the limitary value is positive and monotonically increases. If $\varepsilon_S \in \left(\frac{\lambda_S}{2\lambda_D} r_D^0, \frac{\lambda_S}{\lambda_D} r_D^0\right)$ the limitary value is positive and monotonically decreases. If $\varepsilon_S \in \left(\frac{\lambda_S}{\lambda_D} r_D^0, +\infty\right)$ the limitary value is negative and monotonically decreases.

(decline). It is clearly different from if the saving and investment is done a constant rate $\delta_S > 0$ producing a limitary change of the rate of nominal supply constrained by the finite value $\delta_S \left(P^0 + \frac{r_D^0 + \delta_S}{\lambda_S} \right) > 0$.

We can estimate an increase (decrease) $e_S(t)$ in the rate of nominal supply where $e_S(t) \equiv \frac{dE_S(t)}{dt}$, i.e., the increase (decrease) in the rate of nominal supply for the product, which roughly represents increase (decrease) in the product production on market.

It takes place, for $t \to +\infty$, $e_S(t) \to \frac{\varepsilon_S}{\lambda_S} \left(r_D^0 + \delta_S - \frac{\lambda_D}{\lambda_S} \varepsilon_S \right)$. Therefore, the variable $e_S(t)$ has maximal limitary value when $\varepsilon_S = \frac{\lambda_S}{2\lambda_D} \left(r_D^0 + \delta_S \right)$. Then, $\max\{e_S(t)\} \to \frac{1}{4\lambda_D} \left(r_D^0 + \delta_S \right)^2 > 0$ for $t \to +\infty$.

Therefore, the maximal increase in the rate of nominal supply $\max\{e_S(t)\}$ for the product on market, which roughly represents the maximal increase of the product production or a maximal production growth, is equal, for time $t \to +\infty$, $\max\{e_S(t)\} \to \frac{1}{4\lambda_D} \left(r_D^0 + \delta_S \right)^2 > 0$.

Another observation can be made is if saving and investment $S_I(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then it causes an unrestricted increase (decrease) of the rate of nominal supply $E_S(t)$ with a constant acceleration (deceleration) $\frac{\varepsilon_S}{\lambda_S} \left(r_D^0 + \delta_S - \frac{\lambda_D}{\lambda_S} \varepsilon_S \right)$. If $\varepsilon_S \le \frac{\lambda_S}{\lambda_D} \left(r_D^0 + \delta_S \right)$ then $E_S(t)$ increases with a constant acceleration $\frac{\varepsilon_S}{\lambda_S} \left(r_D^0 + \delta_S - \frac{\lambda_D}{\lambda_S} \varepsilon_S \right) \ge 0$. And if $\varepsilon_S > \frac{\lambda_S}{\lambda_D} \left(r_D^0 + \delta_S \right)$ then

$E_S(t)$ decreases with a constant deceleration $\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 + \delta_S - \frac{\lambda_D}{\lambda_S}\varepsilon_S\right) < 0$. The increase of the rate of nominal supply $E_S(t)$ has a maximal limitary value with constant acceleration $\frac{1}{4\lambda_D}\left(r_D^0 + \delta_S\right)^2 > 0$ for value $\varepsilon_S = \frac{\lambda_S}{2\lambda_D}\left(r_D^0 + \delta_S\right)$.

Note, if the value $\delta_S = 0$ then both the variable $e_D(t)$ and the variable $e_S(t)$ have maximal limitary values when $\varepsilon_S = \frac{\lambda_S}{2\lambda_D} r_D^0$. Here, $\max\{e_D(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$ and $\max\{e_S(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$ for $t \to +\infty$.

I will talk about economic implications of the results in the next section.

3.3. Economic Considerations

This section outlines how the process of saving and investment in a single-product economy can create a nominal economic growth.

The nominal economic growth is generated through a dual impact of changes in the product prices and changes in the supply-demand equilibrium on the market of product.

After supply-demand equilibrium on the market of product is reached, the nominal economic growth is essentially achieved as a by-product of continuous improvement in the product's quality. To improve the product's quality one has to make an appropriate investment of the product (remember, we are dealing with a single-product economy here). That is done by applying the product saving and investment, i.e., withdrawing an appropriate amount of product from the market. That process creates a product shortage on the market, which interrupts supply-demand equilibrium on the market and drives the product price up. In other words, a higher price for the improved-quality product is caused (from a modeling point of view) by the necessity to break the supply-demand equilibrium on market via a partial withdrawal of the product as the product saving and investment in order to improve the product's quality (note, investment is needed to improve the product's quality).

On the other hand, an increase of the product price decreases the product demand. Depending on particular characteristics of product saving and investment, the dual effect of price increase and demand decrease can cause either a nominal economic growth or a nominal economic decline.

Additionally, a withdrawing of the product from market in the form of saving and investment decreases available amount of the product. Reduced amount of the product on market is compensated by rise of the product supply. Thus, simultaneous withdrawing of the product from market for investment and boosting the product supply have opposite impacts on the market with regard to the supply-demand equilibrium. As a result, the work of two opposite effects may demonstrate itself either as a product surplus or a product shortage on the market at some point in time but the market force will try to bring the market back to new supply-demand equilibrium in the long run.

Moreover, the observations can be made about characteristics of product saving and investment and their impact on the nominal economic growth or the nominal economic decline. If withdrawing of the product from market in the form of saving and investment is performed with a constant rate the effect is a restricted long-term nominal economic growth. If withdrawing of the product from market in the form of saving and investment is done with a constant acceleration of modest value the effect is an unrestricted long-term nominal economic growth. If withdrawing of the product from market in the form of saving and investment is achieved with a constant acceleration of large value the effect is an unrestricted long-term nominal economic decline.

4 Nominal Economic Decline in a Single-Product Economy

An economic decline studied here is fundamentally driven by the external supply of product.

I will show a model where a constant-rate and a constant-acceleration external supply are causing restricted economic decline and unrestricted economic decline correspondingly.

4.1. Model of a Continuous Constant-Rate External Supply

According to this scenario, I assume that amount of the external supply of product $S_X(t)$ on the market increases since time $t = t_0$ according to the following formula,

$$S_X(t) = \begin{cases} 0, & t < t_0 \\ \delta_S (t - t_0), & t \geq t_0 \end{cases} \tag{25}$$

where $S_X(t) = 0$ for $t < t_0$ and $\delta_S > 0$.

Economic forces trying to bring the market into a new equilibrium position are described by the following ordinary differential equations with regard to the product supply $V_S(t)$, demand $V_D(t)$, and price $P(t)$ on the market (see Krouglov, 2006; 2009),

$$\frac{dP(t)}{dt} = -\lambda_P \left(V_S(t) - V_D(t) + S_X(t) \right) \tag{26}$$

$$\frac{d^2 V_S(t)}{dt^2} = \lambda_S \frac{dP(t)}{dt} \tag{27}$$

$$\frac{d^2 V_D(t)}{dt^2} = -\lambda_D \frac{d^2 P(t)}{dt^2} \tag{28}$$

In Eqs. (26) – (28) above the values $\lambda_P, \lambda_S, \lambda_D \geq 0$ are constants.

Let me use new variable $D(t) \equiv (V_S(t) - V_D(t) + S_X(t))$ representing the volume of product surplus (or shortage) on the market. Therefore, behavior of $D(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 D(t)}{dt^2} + \lambda_P \lambda_D \frac{dD(t)}{dt} + \lambda_P \lambda_S D(t) = 0 \tag{29}$$

with the initial conditions, $D(t_0) = 0$, $\frac{dD(t_0)}{dt} = \delta_S$.

Similar to Eq. (29), the product price $P(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P(t)}{dt^2} + \lambda_P \lambda_D \frac{dP(t)}{dt} + \lambda_P \lambda_S \left(P(t) - P^0 + \frac{\delta_S}{\lambda_S} \right) = 0 \tag{30}$$

with the initial conditions, $P(t_0) = P^0$, $\frac{dP(t_0)}{dt} = 0$.

Let me introduce another variable $P_1(t) \equiv P(t) - P^0 + \frac{\delta_S}{\lambda_S}$ to simplify an analysis of the product price behavior. Then behavior of the variable $P_1(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dP_1(t)}{dt} + \lambda_P \lambda_S P_1(t) = 0 \tag{31}$$

with the initial conditions, $P_1(t_0) = \frac{\delta_S}{\lambda_S}$, $\frac{dP_1(t_0)}{dt} = 0$.

The behavior of solutions for both $D(t)$ and $P_1(t)$ described by Eqs. (29) and (31) depends on the roots of the corresponding characteristic equations. Also Eqs. (29) and (31) have the same characteristic equation.

When the roots of characteristic equation are complex-valued (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$) both the variable $D(t)$ and variable $P_1(t)$ experience damped oscillations for time $t \geq t_0$. When the roots of characteristic equation are real and different (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$) both the variable $D(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$. When the roots of characteristic equation are real and equal (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$) both the variable $D(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$ as well.

It takes place $D(t) \to 0$ and $P_1(t) \to 0$ for $t \to +\infty$ if the roots of characteristic equation are complex-valued ($\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$), real and different ($\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$), or real and equal ($\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$).

Therefore, it takes place for the product price $P(t)$, for the product demand $V_D(t)$, for the product supply $V_S(t)$, and for the external supply of product $S_X(t)$ if $t \to +\infty$,

$$P(t) \to P^0 - \frac{\delta_S}{\lambda_S} \tag{32}$$

$$V_D(t) \to r_D^0 (t - t_0) + V_D^0 + \frac{\lambda_D}{\lambda_S} \delta_S \tag{33}$$

$$V_S(t) \to \left(r_D^0 - \delta_S\right)(t - t_0) + V_D^0 + \frac{\lambda_D}{\lambda_S} \delta_S \tag{34}$$

$$S_X(t) = \delta_S (t - t_0) \tag{35}$$

For analyze of the economic decline I use the variable $E_D(t) \equiv P(t) \times r_D(t)$ where $r_D(t) \equiv \frac{dV_D(t)}{dt}$, i.e., a rate of nominal demand for the product, which roughly represents the product earning on the market.

I make the comparison of variable $E_D(t)$, a rate of nominal demand changed by the external supply of product $S_X(t)$ on the market, with the variable $\tilde{E}_D(t)$, a rate of original nominal demand unchanged by the external supply of product for $t \to +\infty$, $E_D(t) \to \left(P^0 - \frac{\delta_S}{\lambda_S}\right) r_D^0$ and $\tilde{E}_D(t) \to P^0\, r_D^0$.

Thus, we can clearly see that if the external supply of product $S_X(t)$ on the market is brought with a constant rate $\delta_S > 0$ then the quantitative value of that rate δ_S has a restricted impact on the rate of nominal demand $E_D(t)$.

Therefore, the change in the rate of nominal demand $\Delta E_D(t) \equiv E_D(t) - \tilde{E}_D(t)$ for the product on market, which roughly represents a difference of the product earning or an economic decline, is equal for time $t \to +\infty$, $\Delta E_D(t) \to -\frac{\delta_S}{\lambda_S} r_D^0$.

Another observation that can be made is that eventual change of the rate of nominal demand $\Delta E_D(t)$ is constrained by the value $\frac{\delta_S}{\lambda_S} r_D^0 > 0$. Therefore, the external supply of product to the market with a constant rate $\delta_S > 0$ doesn't produce a profound terminal effect on the rate of nominal demand $E_D(t)$.

Nevertheless, the model shows the amount of eventual rate of nominal demand adjustment $-\frac{\delta_S}{\lambda_S} r_D^0 < 0$ is directly proportional to the rate of the external supply of product $\delta_S > 0$ to the market (the limitary value of $\Delta E_D(t)$ is also directly proportional to the initial rate of demand $r_D^0 > 0$ and, correspondingly, inversely proportional to a supply inducement constant $\lambda_S \geq 0$ inherent to the economic system).

For completeness, let me introduce another variable $E_S(t) \equiv P(t) \times r_S(t)$ where $r_S(t) \equiv \frac{dV_S(t)}{dt}$, i.e., a rate of nominal supply for the product, which roughly represents the product (internal) production.

I make the comparison of variable $E_S(t)$, a rate of nominal supply changed by the external supply of product $S_X(t)$ on market, with the variable $\tilde{E}_S(t)$, a rate of original nominal supply unchanged by the external supply of product, for $t \to +\infty$, $E_S(t) \to \left(P^0 - \frac{\delta_S}{\lambda_S} \right) \left(r_D^0 - \delta_S \right)$ and $\tilde{E}_S(t) \to P^0 \, r_D^0$.

Therefore, the difference in the rate of nominal supply $\Delta E_S(t) \equiv E_S(t) - \tilde{E}_S(t)$ of the product on market, which roughly represents a difference of the product production or a production growth, is equal, for time $t \to +\infty$, $\Delta E_S(t) \to -\delta_S \left(P^0 + \frac{r_D^0 - \delta_S}{\lambda_S} \right)$.

Therefore, the variable $\Delta E_S(t)$ has minimal limitary value when $\delta_S = \frac{1}{2} \left(P^0 \lambda_S + r_D^0 \right) > 0$. Then,

$\min\{\Delta E_S(t)\} \to -\frac{\lambda_S}{4} \left(P^0 + \frac{r_D^0}{\lambda_S} \right)^2 < 0$ for $t \to +\infty$.

I will analyze economic implications of the obtained results in a following section.

4.2. Model of a Continuous Constant-Acceleration External Supply

According to this scenario, I assume that amount of the external supply of product $S_X(t)$ on the market increases since time $t = t_0$ according to the following formula,

$$S_X(t) = \begin{cases} 0, & t < t_0 \\ \delta_S \left(t - t_0\right) + \dfrac{\varepsilon_S}{2} \left(t - t_0\right)^2, & t \geq t_0 \end{cases} \tag{36}$$

where $S_X(t) = 0$ for $t < t_0$, $\delta_S \geq 0$, and $\varepsilon_S > 0$.

Economic forces trying to bring the product supply $V_S(t)$, demand $V_D(t)$, and price $P(t)$ on the market into a new equilibrium position are described by Eqs. (26) – (28).

Let me use again the variable $D(t) \equiv \left(V_S(t) - V_D(t) + S_X(t)\right)$ representing the volume of product surplus (or shortage) on the market. The behavior of $D(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 D(t)}{dt^2} + \lambda_P \lambda_D \frac{dD(t)}{dt} + \lambda_P \lambda_S D(t) - \varepsilon_S = 0 \tag{37}$$

with the initial conditions, $D(t_0) = 0$, $\dfrac{dD(t_0)}{dt} = \delta_S$.

If one uses a new variable $D_1(t) \equiv D(t) - \dfrac{\varepsilon_S}{\lambda_P \lambda_S}$, then Eq. (37) becomes,

$$\frac{d^2 D_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dD_1(t)}{dt} + \lambda_P \lambda_S D_1(t) = 0 \tag{38}$$

with the initial conditions, $D_1(t_0) = -\dfrac{\varepsilon_S}{\lambda_P \lambda_S}$, $\dfrac{dD_1(t_0)}{dt} = \delta_S$.

Similar to Eq. (37), the product price $P(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P(t)}{dt^2} + \lambda_P \lambda_D \frac{dP(t)}{dt} + \lambda_P \lambda_S \left(P(t) - P^0 + \frac{\delta_S}{\lambda_S} + \frac{\varepsilon_S}{\lambda_S} \left(t - t_0\right) \right) = 0 \tag{39}$$

with the initial conditions, $P(t_0) = P^0$, $\dfrac{dP(t_0)}{dt} = 0$.

Let me introduce a variable $P_1(t) \equiv P(t) - P^0 + \frac{\delta_S}{\lambda_S} + \frac{\varepsilon_S}{\lambda_S}(t - t_0) - \frac{\lambda_D}{\lambda_S^2}\varepsilon_S$ to simplify an analysis of the product price behavior. The behavior of variable $P_1(t)$ is described by the following equation for $t > t_0$,

$$\frac{d^2 P_1(t)}{dt^2} + \lambda_P \lambda_D \frac{dP_1(t)}{dt} + \lambda_P \lambda_S P_1(t) = 0 \tag{40}$$

with the initial conditions, $P_1(t_0) = \frac{\delta_S}{\lambda_S} - \frac{\lambda_D}{\lambda_S^2}\varepsilon_S$, $\frac{dP_1(t_0)}{dt} = \frac{\varepsilon_S}{\lambda_S}$.

The behavior of solutions for $D_1(t)$ and $P_1(t)$ described by Eqs. (38) and (40) depends on the roots of the corresponding characteristic equations. Again Eqs. (38) and (40) have the same characteristic equation.

As before, when the roots of characteristic equation are complex-valued (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$) the variable $D_1(t)$ and variable $P_1(t)$ experience damped oscillations for time $t \geq t_0$. When the roots of characteristic equation are real and different (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$) the variable $D_1(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$. When the roots are real and equal (i.e., $\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$) the variable $D_1(t)$ and variable $P_1(t)$ don't oscillate for time $t \geq t_0$.

It occurs $D_1(t) \to 0$ and $P_1(t) \to 0$ for $t \to +\infty$ if the roots of characteristic equation are complex-valued ($\frac{\lambda_P^2 \lambda_D^2}{4} < \lambda_P \lambda_S$), real and different ($\frac{\lambda_P^2 \lambda_D^2}{4} > \lambda_P \lambda_S$), or real and equal ($\frac{\lambda_P^2 \lambda_D^2}{4} = \lambda_P \lambda_S$).

It takes place for the product surplus (shortage) $D(t)$, for the product price $P(t)$, for the product demand $V_D(t)$, for the product supply $V_S(t)$, and for the external supply of product $S_X(t)$ if $t \to +\infty$,

$$D(t) \to \frac{\varepsilon_S}{\lambda_P \lambda_S} \tag{41}$$

$$P(t) \to -\frac{\varepsilon_S}{\lambda_S}(t-t_0) + P^0 - \frac{\delta_S}{\lambda_S} + \frac{\lambda_D}{\lambda_S^2}\varepsilon_S \tag{42}$$

$$V_D(t) \to \left(r_D^0 + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right)(t-t_0) + V_D^0 + \frac{\lambda_D}{\lambda_S}\delta_S - \frac{\lambda_D^2}{\lambda_S^2}\varepsilon_S \tag{43}$$

$$V_S(t) \to \left(r_D^0 - \delta_S + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right)(t-t_0) - \frac{\varepsilon_S}{2}(t-t_0)^2 + V_D^0 + \frac{\lambda_D}{\lambda_S}\delta_S + \frac{\varepsilon_S}{\lambda_P \lambda_S} - \frac{\lambda_D^2}{\lambda_S^2}\varepsilon_S \tag{44}$$

$$S_X(t) = \delta_S (t-t_0) + \frac{\varepsilon_S}{2}(t-t_0)^2 \tag{45}$$

For analyze of the economic growth I use the variable $E_D(t) \equiv P(t) \times r_D(t)$ where $r_D(t) \equiv \frac{dV_D(t)}{dt}$, i.e., a rate of nominal demand for the product, which roughly represents the product earning on market.

I make the comparison of variable $E_D(t)$, a rate of nominal demand changed by the external supply of product $S_X(t)$ on market, with the variable $\tilde{E}_D(t)$, a rate of original nominal demand unchanged by the external supply, for $t \to +\infty$, $E_D(t) \to \left(-\frac{\varepsilon_S}{\lambda_S}(t-t_0) + P^0 - \frac{\delta_S}{\lambda_S} + \frac{\lambda_D}{\lambda_S^2}\varepsilon_S\right)\left(r_D^0 + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right)$ and $\tilde{E}_D(t) \to P^0 r_D^0$.

Thus, if the external supply of product $S_X(t)$ on market is done with a constant acceleration $\varepsilon_S > 0$ then the value of acceleration ε_S causes an unrestricted increase (decrease) of the rate of nominal demand $E_D(t)$. The rate of nominal demand $E_D(t)$ is continuously increasing (decreasing) with the passage of time. It is clearly different from if the external supply of product is brought a constant rate $\delta_S > 0$, which produces limitary change of the rate of nominal demand constrained by the finite value $\frac{\delta_S}{\lambda_S} r_D^0 > 0$.

We can estimate an increase (decrease) $e_D(t)$ in the rate of nominal demand where $e_D(t) \equiv \frac{dE_D(t)}{dt}$, i.e., the increase (decrease) in the rate of nominal demand for the product, which roughly represents increase (decrease) in the product earning on market.

It takes place, for $t \to +\infty$, $e_D(t) \to -\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 + \frac{\lambda_D}{\lambda_S} \varepsilon_S \right) < 0$ when $\varepsilon_S > 0$.

Note that limitary value of variable $e_D(t)$ doesn't have extremal points in the region $\varepsilon_S > 0$. In fact, the variable $\Delta E_S(t)$ has maximal limitary value when $\varepsilon_S = -\frac{\lambda_S}{2\lambda_D} r_D^0$. Then, respectively for $t \to +\infty$,

$$\max\{e_D(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0.$$

Therefore, the variable $e_D(t)$ is always negative in the region $\varepsilon_S > 0$. Changes in the rate of nominal demand in the region $\varepsilon_S > 0$ have negative limitary values. Note, the maximal increase in the rate of nominal demand $\max\{e_D(t)\}$ for the product on market happens outside of the region $\varepsilon_S > 0$ and is equal, for time $t \to +\infty$, $\max\{e_D(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$.

Another observation can be made is if the external supply of product $S_X(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then it causes in the region $\varepsilon_S > 0$ an unrestricted decrease of the rate of nominal demand $E_D(t)$ with a constant deceleration $-\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right) < 0$. The increase of the rate of nominal demand $E_D(t)$ can achieve limitary value of the maximal constant acceleration $\frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$ for value $\varepsilon_S = -\frac{\lambda_S}{2\lambda_D} r_D^0 < 0$ outside of the region $\varepsilon_S > 0$.[4]

For completeness, let me use another variable $E_S(t) \equiv P(t) \times r_S(t)$ where $r_S(t) \equiv \frac{dV_S(t)}{dt}$, i.e., a rate of nominal supply for the product, which roughly represents the product (internal) production.

I make the comparison of variable $E_S(t)$, a rate of nominal supply changed by the external supply of product $S_X(t)$ on market, with the variable $\tilde{E}_S(t)$, a rate of original nominal supply unchanged by the external supply, for $t \to +\infty$, $E_S(t) \to \left(-\frac{\varepsilon_S}{\lambda_S}(t - t_0) + P^0 - \frac{\delta_S}{\lambda_S} + \frac{\lambda_D}{\lambda_S^2}\varepsilon_S\right)\left(r_D^0 - \delta_S + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right)$ and $\tilde{E}_S(t) \to P^0\, r_D^0$.

Thus, if the external supply of product $S_X(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then the value of acceleration ε_S causes an unrestricted increase (decrease) of the rate of nominal supply $E_S(t)$. The rate of nominal supply $E_S(t)$ is continuously increasing (decreasing) with the passage of

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[4] The limitary value of $e_D(t)$ is negative and monotonically increases if $\varepsilon_S \in \left(-\infty, -\frac{\lambda_S}{2\lambda_D} r_D^0\right)$. The limitary value is positive and monotonically increases if $\varepsilon_S \in \left(-\frac{\lambda_S}{\lambda_D} r_D^0, -\frac{\lambda_S}{2\lambda_D} r_D^0\right)$. The limitary value is positive and monotonically decreases if $\varepsilon_S \in \left(-\frac{\lambda_S}{2\lambda_D} r_D^0, 0\right)$. The limitary value is negative and monotonically decreases if $\varepsilon_S \in (0, +\infty)$.

time, i.e., it takes place a continuous increase (decrease) in the product production or a continuous production growth (decline). It is clearly different from if the external supply of product is done with a constant rate $\delta_S > 0$, which produces a limitary change of the rate of nominal supply constrained by the finite value $-\delta_S \left(P^0 + \frac{r_D^0 - \delta_S}{\lambda_S} \right)$.

We can estimate an increase (decrease) $e_S(t)$ in the rate of nominal supply where $e_S(t) \equiv \frac{dE_S(t)}{dt}$, i.e., the increase (decrease) in the rate of nominal supply for the product, which roughly represents increase (decrease) in the product production on market.

It takes place, for $t \to +\infty$, $e_S(t) \to -\frac{\varepsilon_S}{\lambda_S} \left(r_D^0 - \delta_S + \frac{\lambda_D}{\lambda_S} \varepsilon_S \right)$. Therefore, the variable $e_S(t)$ has maximal limitary value when $\varepsilon_S = \frac{\lambda_S}{2\lambda_D} \left(-r_D^0 + \delta_S \right)$. Then, $\max\{e_S(t)\} \to \frac{1}{4\lambda_D} \left(r_D^0 - \delta_S \right)^2 > 0$ for $t \to +\infty$.

Therefore, the maximal increase in the rate of nominal supply $\max\{e_S(t)\}$ for the product on market, which roughly represents the maximal increase of the product production or a maximal production growth, is equal, for time $t \to +\infty$, $\max\{e_S(t)\} \to \frac{1}{4\lambda_D} \left(r_D^0 - \delta_S \right)^2 > 0$.

Another observation can be made is if the external supply $S_X(t)$ on the market is done with a constant acceleration $\varepsilon_S > 0$ then it causes an unrestricted increase (decrease) of the rate of nominal supply $E_S(t)$ with a constant acceleration (deceleration) $-\frac{\varepsilon_S}{\lambda_S} \left(r_D^0 - \delta_S + \frac{\lambda_D}{\lambda_S} \varepsilon_S \right)$. If $\varepsilon_S \le \frac{\lambda_S}{\lambda_D} \left(-r_D^0 + \delta_S \right)$ then

$E_S(t)$ increases with a constant acceleration $-\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 - \delta_S + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right) \geq 0$ (if $\delta_S \leq r_D^0$ then resultant ε_S is negative, i.e., $\varepsilon_S \leq \frac{\lambda_S}{\lambda_D}\left(-r_D^0 + \delta_S\right) \leq 0$). If $\varepsilon_S > \frac{\lambda_S}{\lambda_D}\left(-r_D^0 + \delta_S\right)$ then $E_S(t)$ decreases with a constant deceleration $-\frac{\varepsilon_S}{\lambda_S}\left(r_D^0 - \delta_S + \frac{\lambda_D}{\lambda_S}\varepsilon_S\right) < 0$. Increase of the rate of nominal supply $E_S(t)$ has maximal limitary value with constant acceleration $\frac{1}{4\lambda_D}\left(r_D^0 - \delta_S\right)^2 > 0$ if value $\varepsilon_S = \frac{\lambda_S}{2\lambda_D}\left(-r_D^0 + \delta_S\right)$.

If value $\delta_S = 0$ then both the variable $e_D(t)$ and the variable $e_S(t)$ have maximal limitary values when $\varepsilon_S = -\frac{\lambda_S}{2\lambda_D} r_D^0$, which is located outside of the region $\varepsilon_S > 0$. Here, $\max\{e_D(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$ and $\max\{e_S(t)\} \to \frac{1}{4\lambda_D}\left(r_D^0\right)^2 > 0$ for $t \to +\infty$.

I will talk about economic implications of the results in the next section.

4.3. Economic Considerations

This section outlines how the external supply of product in a single-product economy can create a nominal economic decline.

The nominal economic decline is generated through a dual impact of changes in the product prices and changes in the supply-demand equilibrium on the market of product.

The nominal economic decline presented here is caused by the oversupply of product that violates the previously achieved supply-demand equilibrium on the market of product. To capitalize on the market of product an external producer supplies its production on the market. The external supply of product creates a product surplus on the market, which interrupts supply-demand equilibrium on the market and drives the product price down. In other words, a lower price of the product on market is caused by an interruption of supply-demand equilibrium via the extra supply of product from an external producer.

On the other hand, a decrease of the product price increases the product demand. Depending on particular characteristics of the external supply of product, the dual effect of price decrease and demand increase can cause either a nominal economic growth or a nominal economic decline.

Additionally, an external supply of product on the market increases the available amount of product. Increased amount of the product on market is compensated by decrease of the product supply from an internal producer. Thus, simultaneous providing extra supply of the product on market from an external producer and reducing the product supply from an internal producer have the opposite impacts on the market with regard to supply-demand equilibrium. As a result, the work of two opposite effects may demonstrate itself either as a product surplus or a product shortage on the market at some points in time but the market forces will try to bring the market to a new supply-demand equilibrium position in the long run.

Moreover, the observations can be made about characteristics of the external supply of product and their impact on the nominal economic growth or the nominal economic decline. If the external supply of product on market is performed with a constant rate the effect is a restricted long-term nominal economic decline. Such restricted nominal economic decline can be called a *secular stagnation*. If the external supply of product on market is brought with a constant acceleration the effect is an unrestricted long-term nominal economic decline. Such unrestricted nominal economic decline can be called a *secular decline*.

5 Economic Policy Implications

Thus, a nominal economic growth (schematically presented for a single-product economy) is fundamentally caused by the processes of saving and investment. Investment of the product is used to improve quality of the product itself. Saving of the product causes a decrease of quantity of the available product on market, which causes the increase of the product price. And increase of the product price causes a decrease of the demand for the product. A dual effect of the price increase and the demand decrease can cause either a nominal economic growth or a nominal economic decline.

With regard to a nominal economic growth/decline caused by the product investment there are identified three distinct scenarios. If investment of the product is performed with a constant rate the result is shown as a limited long-term nominal economic growth. If investment of the product is done with a modest constant acceleration the result is revealed as an unlimited long-term nominal economic growth. If investment of the product is achieved with a large constant acceleration the result is demonstrated as an unlimited long-term nominal economic decline.[5]

On the other hand, a nominal economic decline (outlined for a single-product economy) is fundamentally caused by the external supply of product to market. The external supply of product violates supply-demand equilibrium on the market and creates a product surplus, which drives the product price lower. In other words, a decline of the product price on market is caused by the interruption of supply-demand equilibrium from the extra supply of product by an external producer. And decrease of the product price causes an increase of the demand for the product. A dual effect of the price decrease and the demand increase can cause either a nominal economic growth or a nominal economic decline.

The extra supply of product on market by an external producer is compensated by the decrease of the product supply from an internal producer. Thus, the extra supply of product to market from an external producer and the reduction of the product supply from an internal producer have opposite impacts on the market's supply-demand equilibrium. The resultant work of two opposite factors may demonstrate itself

[5] I have ignored the effects of international trade and technological advances on the nominal economic growth. One can say that those produce the exaggerating impacts compared with a basic impact of the saving and investment processes.

either as a product surplus or as a product shortage on the market at some points in time but the market forces will bring market to a new supply-demand equilibrium position in the long run.

With regard to a nominal economic growth/decline caused by the external supply of product there are identified two distinct scenarios. If the external supply of product on market is performed with a constant rate the effect is a limited long-term nominal economic decline conventionally called a secular stagnation. If the external supply of product on market is brought with a constant acceleration the effect is an unlimited long-term nominal economic decline. Such unlimited nominal economic decline may be called a secular decline.

Thus, economic policies pertaining to handling of the secular stagnation and ensuing from the presented model can be summarized as follows.

First, policy makers should prevent a transformation of the secular stagnation into the secular decline. It means the markets should be protected from the supply of products from external suppliers brought with a persistently accelerated rate, which would case an unlimited long-term nominal economic decline. The protection measures can be either bureaucratic obstacles or economic tariffs and quotas etc.

Second, impact of the secular stagnation per se can be handled via a two-stage policy. The first stage is to counterbalance an impact on the markets of the external supply of products from external suppliers brought with a constant rate, which would cause a limited long-term nominal economic decline (a.k.a. the secular stagnation). Alleviations can be done by addressing the origin of the problem, namely, a violation of the supply-demand equilibrium on markets by the external supply of products. Actually, best way to equalize the external supply of products on markets is to advance the additional demand for products. However, the additional demand for the products would still alleviate the limited long-term nominal economic decline exhibited in the form of secular stagnation but it would not be able to cause a nominal economic growth itself. The latter is ensured by the application of the second stage of economic policies. Thus, the second stage is to commence the processes of savings and investments in order to stipulate a nominal economic growth. The savings and investments should be done with a modest acceleration, which would result in an unlimited long-term nominal economic growth. If the savings and investments are achieved with a large acceleration, it would result in an unlimited long-term nominal economic decline. If the savings and investments are performed with a constant rate it would result in a limited long-term nominal economic growth. It is worth to note that the aforesaid two-stage policies can be executed concurrently.

6 Conclusions

Presented here is a simplified mathematical model that explains the effects of a nominal economic growth and a nominal economic decline. Among the considered cases are a limited long-term nominal economic growth, an unlimited long-term nominal economic growth, a limited long-tem nominal economic decline (or secular stagnation), and an unlimited long-term nominal economic decline (or secular decline).

Thus, economic policies pertaining to handling of the secular stagnation and ensuing from the presented model can be summarized as follows.

First, measures should be taken to prevent a transformation of the secular stagnation into the secular decline.

Second, the impact of the secular stagnation per se should be handled via a two-stage policy. The first stage is to counterbalance impact on the markets of the external supply of products from external suppliers through a policy of advancing the additional demand for products. The additional demand would alleviate a limited long-term nominal economic decline exhibited in the form of secular stagnation. Besides, the second stage of economic policy is to commence the processes of savings and investments in order to stipulate a nominal economic growth. The savings and investments should be done with a modest acceleration, which would result in an unlimited long-term nominal economic growth. Both stages of alleviating economic policies can be executed concurrently.

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