Analogy Making and the Structure of Implied Volatility Skew

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1 October 2014

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This version: December 2014.

An analogy based call option pricing model is put forward. The model provides a new explanation for the implied volatility skew puzzle and is consistent with empirical findings regarding leverage adjusted option returns. It explains puzzling superior performance of covered call writing and worse-than-expected performance of zero-beta straddles. The analogy based stochastic volatility and the analogy jump diffusion models are also developed. The analogy based stochastic volatility model generates the skew even without any correlation between the stock price and volatility processes, whereas, the analogy jump diffusion does not require asymmetric jumps to generate the skew.

JEL Classification: G13, G12

Keywords: Implied Volatility Skew, Implied Volatility Smile, Stochastic Volatility, Jump Diffusion, Covered Call Writing, Zero-Beta Straddle, Leverage Adjusted Option Returns

[1] I am grateful to John Quiggin, Simon Grant, Hersh Shefrin, Emanuel Derman, Don Chance, and the participants in the University of Queensland Economic Theory Seminar for helpful comments and suggestions. All errors and omissions are due to the author.

Analogy Making and the Structure of Implied Volatility Skew

The existence of the implied volatility skew is perhaps one of the most intriguing anomalies in option markets. According to the Black-Scholes model (Black and Scholes (1973)), volatility inferred from prices (implied volatility) should not vary across strikes. In practice, a sharp skew in which implied volatilities fall monotonically as the ratio of strike to spot increases is observed in index options.

The Black-Scholes model assumes that an option can be perfectly replicated by a portfolio consisting of continuously adjusted proportions of the underlying stock and a risk-free asset. The cost of setting up this portfolio should equal the price of the option. Most attempts to explain the skew have naturally relaxed the assumption of perfect replication. Such relaxations have taken two broad directions: 1) Deterministic volatility models 2) Stochastic volatility models without jumps and stochastic volatility models with jumps. In the first category are the constant elasticity of variance model examined in Emanuel and Macbeth (1982), the implied binomial tree models of Dupire (1994), Derman and Kani (1994), and Rubinstein (1994). Dumas, Fleming and Whaley (1998) provide evidence that deterministic volatility models do not adequately explain the structure of implied volatility as they lead to parameters which are highly unstable through time. The second broad category is examined in papers by Chernov et al (2003), Anderson, Benzoni, and Lund (2002), Bakshi, Cao, and Chen (1997), Heston (1993), Stein and Stein (1991), and Hull and White (1987) among others. Bates (2000) presents empirical evidence regarding stochastic volatility models with and without jumps and finds that inclusion of jumps in a stochastic volatility model does improve the model, however, in order to adequately explain the skew, unreasonable parameter values are required. Generally, stochastic volatility models require an unreasonably strong and fluctuating correlation between the stock price and the volatility processes in order to fit the skew, whereas, jump diffusion models need unreasonably frequent and large asymmetric jumps. Empirical findings suggest that models with both stochastic volatility and jumps in returns fail to fully capture the empirical features of index returns and option prices (see Bakshi, Cao, and Chen (1997), Bates (2000), and Pan (2002)).

Highly relevant to the option pricing literature is the intriguing finding in Jackwerth (2000) that risk aversion functions recovered from option prices are irreconcilable with a representative investor. Perhaps, another line of inquiry is to acknowledge the importance of heterogeneous expectations and the impact of resulting demand pressures on option prices. Bollen and Whaley (2004) find that changes in implied volatility are directly related to net buying pressures from public order flows. According to this view, different demands and supplies of different option series affect the skew. Lakonishok, Lee, Pearson, and Poteshman (2007) examine option market activity of several classes of investors in detail and highlight the salient features of option market activity. They find that a large percentage of calls are written as a part of covered call strategy. Covered call writing is a strategy in which a long position in the underlying stock is combined with a call writing position. This strategy is typically employed when one is expecting slow growth in the price of the underlying stock. It seems that call buyers expect higher returns from the underlying stock than call writers, but call writers are not pessimistic either. They expect slow/moderate growth and not a sharp downturn in the price of the underlying stock.

Should expectations regarding the underlying stock matter for option pricing? In the Black-Scholes world where perfect replication is assumed, expectations do not matter as they do not affect the construction of the replicating portfolio or its dynamics. However, empirical evidence suggests that they do matter. Duan and Wei (2009) find that a variable closely related to the expected return on the underlying stock, its systematic risk proportion, is priced in individual equity options.

There is also strong experimental and other field evidence showing that the expected return on the underlying stock matters for call option pricing. Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011) find that participants in laboratory experiments seem to value a call option by equating its expected return to the expected return available from the underlying stock. From this point onwards, we refer to this as the analogy model. In the field, many experienced option traders and analysts consider a call option to be a surrogate for the underlying stock because of the similarity in their respective payoffs.[2] It seems natural to expect that such analogy making/similarity argument influences option valuation, especially when it comes from experienced market professionals. Furthermore, as a call option is defined over some underlying stock, the return on the underlying stock forms a natural benchmark for forming expectations about the option. This article puts forward an analogy based call option pricing model and shows that it provides a new

[2] As illustrative examples, see the following: http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772, http://www.investingblog.org/archives/194/deep-in-the-money-options/, http://www.triplescreenmethod.com/TradersCorner/TC052705.asp, http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

explanation for the implied volatility skew puzzle. The analogy model is also shown to be consistent with recent empirical findings regarding leverage adjusted option returns (Constantinides et al (2013)). The model also explains the superior performance of covered call writing and worse-than-expected performance of zero-beta straddles.

In a laboratory experiment, it is possible to objectively fix the expected return available on the underlying stock and make it common knowledge, however, in the real world; people are likely to have different subjective assessments of the expected return on the underlying stock. An analogy maker expects a return from a call option which is equal to his subjective assessment of the expected return available on the underlying stock. The marginal investor in a call option is perhaps more optimistic than the marginal investor in the corresponding underlying stock. To see this, consider the following: In the market for the underlying stock, both the optimistic and pessimistic beliefs influence the belief of the marginal investor. Optimistic investors influence through demand pressure, whereas the pessimistic investors constitute the suppliers who influence through selling and short-selling. However, highly optimistic investors should favor a call option over its underlying stock due to the leverage embedded in the option. Furthermore, in the market for a call option, covered call writers are typical suppliers (see Lakonishok et al (2007)). Covered call writers are neutral to moderately bullish (and not pessimistic) on the underlying stock. Hence, due to the presence of relatively more optimistic buyers and sellers, the marginal investor in a call option is likely to be more optimistic about the underlying stock than the marginal investor in the underlying stock itself. Hence, the expected return reflected in a call option is likely to be larger than the expected return on the underlying stock. Also, as more optimistic buyers are likely to self-select into higher strike calls, the expected return may rise with strike.

If analogy makers influence call prices, shouldn't a rational arbitrageur make money at their expense by taking an appropriate position in the call option and the corresponding replicating portfolio in accordance with the Black Scholes model? Such arbitraging is difficult if not impossible in the presence of transaction costs. In continuous time, no matter how small the transaction costs are, the total transaction cost of successful replication grows without bound rendering the Black-Scholes argument toothless. It is well known that there is no non-trivial portfolio that replicates a call option in the presence of transaction costs in continuous time. See Soner, Shreve, and Cvitanic (1995). In discrete time, transaction costs are bounded, however, a no-arbitrage interval is created. If analogy price lies within the interval, analogy makers cannot be arbitraged away. We show the conditions under which this happens in a binomial setting. Of course, if the underlying stock dynamics exhibit stochastic volatility or jump diffusion then the Black-Scholes argument does not hold irrespective of transaction costs and/or other limits to arbitrage. Hence, analogy makers cannot be arbitraged away in that case.

Analogy making is complementary to the approaches developed earlier such as stochastic volatility and jump diffusion models. Such models specify certain dynamics for the underlying stock. The idea of analogy making is not wedded to a particular set of assumptions regarding the price and volatility processes of the underlying stock. It can be applied to a wide variety of settings. In this article, first we use the setting of a geometric Brownian motion. Then, we integrate analogy making with jump diffusion and stochastic volatility approaches. Combining analogy making and stochastic volatility leads to the skew even when there is zero correlation between the stock price and volatility processes, and combining analogy making with jump diffusion generates the skew without the need for asymmetric jumps.

This article is organized as follows. Section 1 summarizes existing evidence pointing to analogy based call option pricing. Section 2 builds intuition by providing a numerical illustration of call option pricing with analogy making. Section 3 develops the idea in the context of a one period binomial model. Section 4 puts forward the analogy based option pricing formulas in continuous time. Section 5 shows that if analogy making determines option prices, and the Black-Scholes model is used to back-out implied volatility, the skew arises, which flattens as time to expiry increases. Section 6 shows that the analogy model is consistent with key empirical findings regarding returns from covered call writing and zero-beta straddles. Section 7 shows that the analogy model is consistent with empirical findings regarding leverage adjusted option returns. Section 8 puts forward an analogy based option pricing model when the underlying stock returns exhibit stochastic volatility. It integrates analogy making with the stochastic volatility model developed in Hull and White (1987). Section 9 integrates analogy making with the jump diffusion approach of Merton (1976). Section 10 concludes.

1. The Relevance of Analogy Making for Option Pricing

A call option is commonly considered a surrogate for the underlying stock by investment professionals with decades of experience (see footnote 3 for five examples). A popular strategy advocated by market professionals is called the *stock replacement strategy*. In this strategy, underlying stocks are replaced by the corresponding call options in investment portfolios. The argument underlying this strategy is based on the similarity between call and underlying stock's payoffs, and the fact that a call option is a lot cheaper than buying the underlying stock outright. A careful reading of the investment advice coming from proponents of the *stock replacement strategy*[3] suggests the following: 1) Risk of buying a call option is frequently perceived not to be any larger than the risk of the underlying stock. In fact, often the risk of a call option is perceived to be less than the risk of the underlying stock, as buying a call option requires a smaller cash outlay. 2) Replacing stocks with call options is recommended as long as one expects at least the same return from the call as from the underlying stock.

Not only investment professionals with decades of experience consider a call option to be a surrogate for the underlying stock, participants in a series of laboratory experiments seem to think so too. Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011) find that participants in laboratory experiments seem to value a call option by equating its expected return to the expected return available from the underlying stock. The similarity between their respective payoffs leads subjects in the experiment to co-categorize a call option in the same mental account as the underlying stock. Consequently, a call option is valued in analogy with its underlying stock by equating expected returns. Rockenbach (2004) shows this in a binomial setting. Siddiqi (2012) and Siddiqi (2011) show that this finding regarding call options is robust to adding more states and assets.

Apart from opinions of professionals and experimental evidence, there is also empirical evidence suggesting that the expected return on the underlying matters for pricing options. Duan and Wei (2009) find that a variable closely related to the expected return on the underlying stock, its systematic risk proportion, is priced in individual equity options.

Coval and Shumway (2001) find that expected option returns are too low given their systematic risk in the Black-Scholes/CAPM framework. Perhaps, this is due to analogy making as

[3] As illustrative examples of this advice generated by investment professionals, see the following: http://finance.yahoo.com/news/stock-replacement-strategy-reduce-risk-142949569.html http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772, http://www.investingblog.org/archives/194/deep-in-the-money-options/ http://www.triplescreenmethod.com/TradersCorner/TC052705.asp, http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

valuing a call option in analogy with its underlying stock significantly lowers the option expected return, when compared with the expected return in the Black-Scholes/CAPM framework.

The analogy between a call and its underlying stock is widely perceived in the field by experienced professionals. Furthermore, the subjects in laboratory experiments are found to value a call option in analogy with its underlying stock. Shouldn't such analogy making influence the value of call options in real option markets? If so, what are the implications for option pricing? This article explores this question and puts forward an analogy based call option pricing model.

If analogy making influences the value of call options, what are the implications for put options? A call option can be converted into a put option by combining it with the short underlying stock plus a long position in the risk free bond (in accordance with put-call parity). In this article, the analogy based pricing formula for put options is deduced from analogy based call pricing formula by using put-call parity. Even though analogy is only explicitly made between a call option and its underlying stock, there are strong implications for put option pricing due to the model-free restriction of put-call parity. Put-Call parity does not depend on any specific option pricing model, such as the Black Scholes model. All option pricing models should satisfy this restriction.

How important is analogy making to human thinking process? It has been argued that when faced with a new situation, people instinctively search their memories for something similar they have seen before, and mentally co-categorize the new situation with the similar situations encountered earlier. This way of thinking, termed analogy making, is considered the core of cognition and the fuel and fire of thinking by prominent cognitive scientists and psychologists (see Hofstadter and Sander (2013)). Hofstadter and Sander (2013) write, *"[…] at every moment of our lives, our concepts are selectively triggered by analogies that our brain makes without letup, in an effort to make sense of the new and unknown in terms of the old and known."*
(Hofstadter and Sander (2013), Prologue page1).

The analogy making argument has been made in the economic literature previously in various contexts. Prominent examples that appeal to analogy making in different contexts include the coarse thinking model of Mullainathan et al (2008), the case based decision theory of Gilboa and Schmeidler (2001), and the analogy based expectations equilibrium of Jehiel (2005). This article adds another dimension to this literature by exploring the implications of analogy making for option valuation. Clearly, a call option is similar to the stock over which it is defined, and, as pointed out earlier, this similarity is perceived and highlighted by market professionals with decades of experience who consider a call option to be a surrogate for the underlying stock. As discussed earlier, subjects in laboratory experiments also seem to value call options in analogy with their underlying stocks. Given the importance of analogy making to human thinking in general, it seems natural to consider the possibility that a call option is valued in analogy with 'something similar', which is: the underlying stock. This article carefully explores the implications of such analogy making, and shows that analogy making provides a new explanation for the implied volatility skew puzzle. The analogy model also provides new explanations for the puzzling historical profitability of covered call strategy, and negative returns from zero-beta straddles. Furthermore, the model is consistent with recent empirical findings regarding leverage adjusted option returns (see Constantinides et al (2013)).

2. Analogy Making: A Numerical Illustration

Consider an investor in a two state-two asset complete market world with one time period marked by two points in time: 0 and 1. The two assets are a stock (S) and a risk-free zero coupon bond (B). The stock has a price of \$140 today (time 0). Tomorrow (time 1), the stock price could either go up to \$200 (the red state) or go down to \$94 (the blue state). Each state has a 50% chance of occurring. There is a riskless bond (zero coupon) that has a price of \$100 today. Its price stays at \$100 at time 1 implying a risk free rate of zero. Suppose a new asset "A" is introduced to him. The asset "A" pays \$100 in cash in the red state and nothing in the blue state. How much should the investor be willing to pay for this new asset?

Finance theory provides an answer by appealing to the principle of no-arbitrage: *assets with identical state-wise payoffs must have the same price* or equivalently *assets with identical state-wise payoffs must have the same state-wise returns.* Consider a portfolio consisting of a long position in 0.943396 of S and a short position in 0.886792 of B. In the red state, 0.943396 of S pays \$188.6792 and one has to pay \$88.6792 due to shorting of 0.886792 of B earlier resulting in a net payoff of \$100. In the blue state, 0.943396 of S pays \$88.6792 and one has to pay \$88.6792 on account of shorting 0.886792 of B previously resulting in a net payoff of 0. That is, payoffs from 0.943396S-0.886792B are identical to payoffs from "A". As the cost of 0.943396S-0.886792B is \$43.39623, it follows that the no-arbitrage price for "A" is \$43.39623.

When simple tasks such as the one described above are presented to participants in a series of experiments, instead of the no-arbitrage argument, they seem to rely on analogy-making to figure out their willingness to pay. See Rockenbach (2004), Siddiqi (2011), and Siddiqi (2012). Instead of trying to construct a replicating portfolio which is identical to asset "A", people find an actual asset similar to "A" and price "A" in analogy with that asset. They rely on the principle of analogy: *assets with similar state-wise payoffs should offer the same state-wise returns on average,* or equivalently, *assets with similar state-wise payoffs should have the same expected return.*

Asset "A" is similar to asset S as their payoffs are strongly related. In fact, asset "A" is equivalent to a call option on "S" with a strike price of $100. Expected return from S is 1.05 $\left(\frac{0.5\times200+0.5\times94}{140}\right)$. According to the principle of analogy, A's value should be such that it offers the same expected return as S. That is, analogy makers value "A" at $47.61905.

In the above example, there is a gap of $4.22281 between the no-arbitrage price and the analogy price. Rational investors should short "A" and buy "0.943396S-0.886792B". However, transaction costs are ignored in the example so far.

Let's see what happens when a symmetric proportional transaction cost of only 1% of the price is applied when assets are traded. That is, both a buyer and a seller pay a transaction cost of 1% of the price of the asset traded. Unsurprisingly, the composition of the replicating portfolio changes. To successfully replicate a long call option that pays $100 in cash in the red state and 0 in the blue state with transaction cost of 1%, one needs to buy 0.952925 of S and short 0.878012 of B. In the red state, 0.952925S yields $188.6792 net of transaction cost $\left(200\times0.952925\times(1-0.01)\right)$, and one has to pay $88.6792 to cover the short position in B created earlier $\left(0.878012\times100\times(1+0.01)\right)$. Hence, the net cash generated by liquidating the replicating portfolio at time 1 is $100 in the red state. In the blue state, the net cash from liquidating the replicating portfolio is 0. Hence, with a symmetric and proportional transaction cost of 1%, the replicating portfolio is "0.952925S-0.878012B". The cost of setting up this replicating portfolio inclusive of transaction costs at time 0 is $47.82044, which is larger than the price the analogy makers are willing to pay: $47.61905. Hence, arbitrage profits cannot be made at the expense of analogy makers by writing a call and buying the replicating portfolio. The given scheme cannot generate arbitrage profits unless the call price is greater than $47.82044

Suppose one in interested in doing the opposite. That is, buy a call and short the replicating portfolio to fund the purchase. Continuing with the same example, the relevant replicating portfolio (that generates an outflow of $100 in the red state and 0 in the blue state) is "-0.934056S +0.89575B". The replicating portfolio generates $41.1928 at time 0, which leaves $38.98937 after time 0 transaction costs in setting up the portfolio are paid. Hence, in order for the scheme to make money, one needs to buy a call option at a price less than $38.98937.

Effectively, transaction costs create a no-arbitrage interval $(38.98937, 47.82044)$. As the analogy price lies within this interval, arbitrage profits cannot be made at the expense of analogy makers in the example considered.

2.1 Analogy Making: A Two Period Binomial Example with Delta Hedging

Consider a two period binomial model. The parameters are: Up factor=2, Down factor=0.5, Current stock price=$100, Risk free interest rate per binomial period=0, Strike price=$30, and the probability of up movement=0.5. It follows that the expected gross return from the stock per binomial period is 1.25 $(0.5 \times 2 + 0.5 \times 0.5)$.

The call option can be priced both via analogy as well as via no-arbitrage argument. The no-arbitrage price is denoted by C_R whereas the analogy price is denoted by C_A. Define $x_R = \frac{\Delta C_R}{\Delta S}$ and $x_A = \frac{\Delta C_A}{\Delta S}$ where the differences are taken between the possible next period values that can be reached from a given node.

Figure 1 shows the binomial tree and the corresponding no-arbitrage and analogy prices. Two things should be noted. Firstly, in the binomial case considered, before expiry, the analogy price is always larger than the no-arbitrage price. Secondly, the delta hedging portfolios in the two cases $Sx_R - C_R$ and $Sx_A - C_A$ grow at different rates. The portfolio $Sx_A - C_A$ grows at the rate equal to the expected return on stock per binomial period (which is 1.25 in this case). In the analogy case, the value of delta-hedging portfolio when the stock price is 100 is 17.06667 $(100 \times 0.98667 - 81.6)$. In the next period, if the stock price goes up to 200, the value becomes 21.33333 $(200 \times 0.98667 - 176)$. If the stock price goes down to 50, the value also ends up being equal to 21.33333 $(50 \times 0.98667 - 28)$. That is, either way, the rate of growth is the same and is equal to 1.25 as $17.06667 \times 1.25 = 21.33333$. Similarly, if the delta hedging portfolio is constructed at any other node, the next period return remains equal to the expected return from stock. It is easy to verify that the portfolio $Sx_R - C_R$ grows at a different rate which is equal to the risk free rate per binomial period (which is 0 in this case).

The fact that *the delta hedging portfolio under analogy making grows at a rate which is equal to the perceived expected return on the underlying stock* is used to derive the analogy based option pricing formulas in continuous time in section 4. In the next section, the corresponding discrete time results are presented. Note, as discussed earlier, the marginal investor in a call option is likely to be more optimistic than the marginal investor in the underlying stock. In the context of the example presented, this would mean that they perceive different binomial trees. Specifically, they would perceive different up and down factors as up and down factors are a function of distribution of returns.

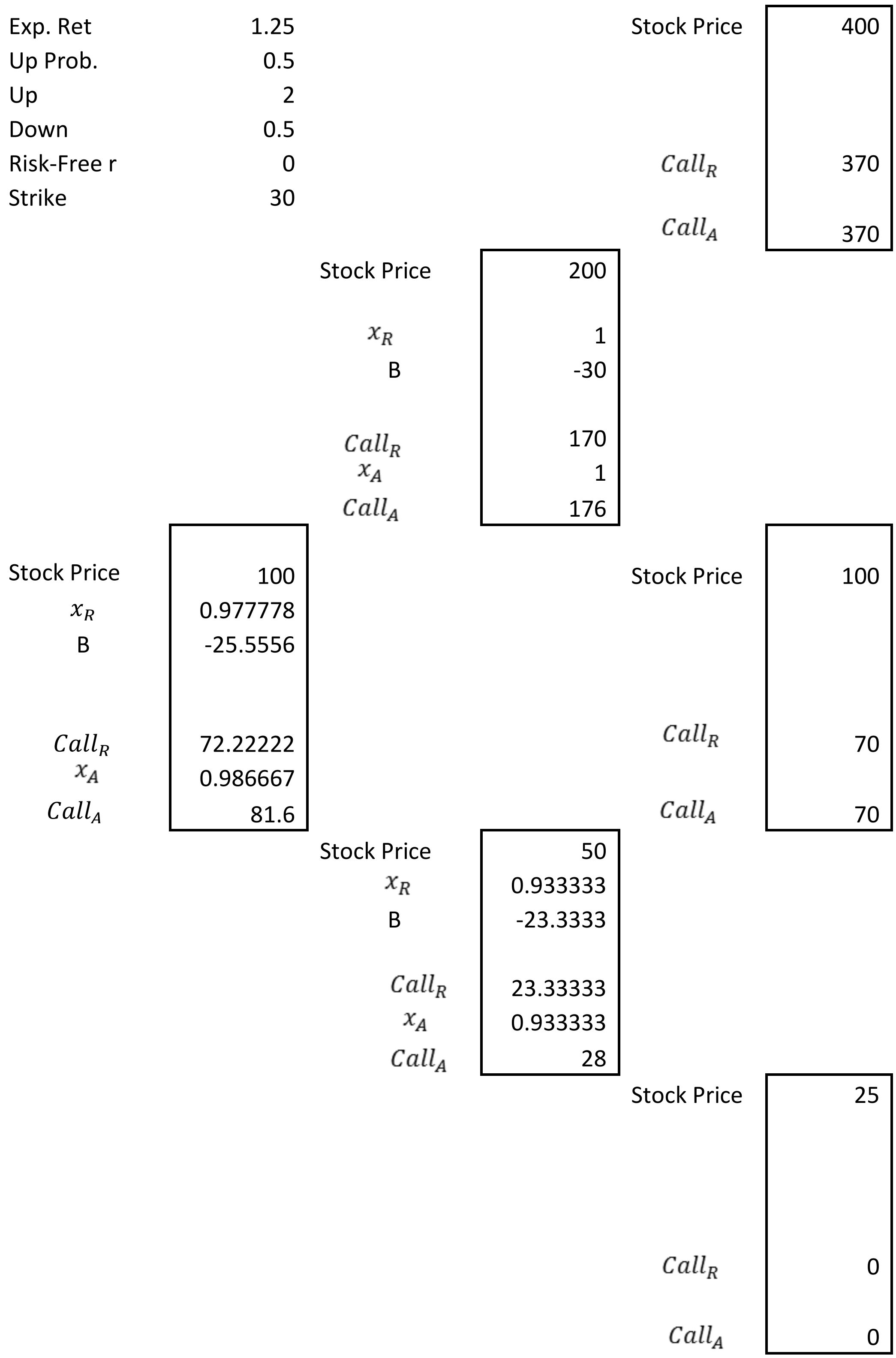
Exp. Ret 1.25
Up Prob. 0.5
Up 2
Down 0.5
Risk-Free r 0
Strike 30
Stock Price 400
$Call_R$ 370
$Call_A$ 370
Stock Price 200
x_R 1
B -30
$Call_R$ 170
x_A 1
$Call_A$ 176
Stock Price 100
x_R 0.977778
B -25.5556
$Call_R$ 72.22222
x_A 0.986667
$Call_A$ 81.6
Stock Price 100
$Call_R$ 70
$Call_A$ 70
Stock Price 50
x_R 0.933333
B -23.3333
$Call_R$ 23.33333
x_A 0.933333
$Call_A$ 28
Stock Price 25
$Call_R$ 0
$Call_A$ 0

Figure 1

3. Analogy Making: The Binomial Case

Consider a two state world. The equally likely states are Red, and Blue. There is a stock with prices $X_1, and\ X_2$ corresponding to states Red, and Blue respectively, where $X_1 > X_2$. The state realization takes place at time T. The current time is time t. We denote the risk free discount rate by r. That is, there is a riskless zero coupon bond that has a price of B in both states with a price of $\frac{B}{1+r}$ today. For simplicity and without loss of generality, we assume that $r = 0$ and $T - t = 1$. The current price of the stock is S such that $X_1 > S > X_2$. We further assume that $S < \frac{X_1+X_2}{2}$. That is, the stock price reflects a positive risk premium. In other words, $S = f \cdot \frac{X_1+X_2}{2}$ where $f = \frac{1}{1+r+\delta}$.[4] δ is the risk premium reflected in the price of the stock.[5] As we have assumed $r = 0$, it follows that $f = \frac{1}{1+\delta}$.

Suppose a new asset which is a European call option on the stock is introduced. By definition, the payoffs from the call option in the two states are:

$$C_1 = max\{(X_1 - K), 0\}, C_2 = max\{(X_2 - K), 0\} \tag{3.1}$$

Where K is the striking price, and $C_1, and\ C_2,$ are the payoffs from the call option corresponding to Red, and Blue states respectively.

How much is an analogy maker willing to pay for this call option?

There are two cases in which the call option has a non-trivial price: 1) $X_1 > X_2 > K$ and 2) $X_1 > K > X_2$

The analogy maker infers the price of the call option, P_c , by equating the expected return from the call to the return he expects from holding the underlying stock:

$$\frac{\{C_1 - P_c\} + \{C_2 - P_c\}}{2 \times P_c} = \frac{\{X_1 - S\} + \{X_2 - S\}}{2 \times S} \tag{3.2}$$

[4] In general, a stock price can be expressed as a product of a discount factor and the expected payoff if it follows a binomial process in discrete time (as assumed here), or if it follows a geometric Brownian motion in continuous time.

[5] If the marginal call investor is more optimistic than the marginal stock investor, they would perceive different values of $X_1$and X_2 so that their assessment of δ is different accordingly.

For case 1 ($X_1 > X_2 > K$), one can write:

$$P_c = \frac{C_1 + C_2}{X_1 + X_2} \times S$$

$$=> P_c = \left(1 - \frac{2K}{X_1 + X_2}\right) S \tag{3.3}$$

Substituting $S = f \cdot \frac{X_1+X_2}{2}$ in (3.3):

$$P_c = S - Kf \tag{3.4}$$

The above equation is the one period analogy option pricing formula for the binomial case when call remains in-the-money in both states.

The corresponding no-arbitrage price P_r is (from the principle of no-arbitrage):

$$P_r = S - K \tag{3.5}$$

For case 2 ($X_1 > K > X_2$), the analogy price is:

$$P_c = S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f \tag{3.6}$$

And, the corresponding no-arbitrage price is:

$$P_r = \frac{X_1 - K}{X_1 - X_2}(S - X_2) \tag{3.7}$$

Proposition 1 ***The analogy price is larger than the corresponding no-arbitrage price if a positive risk premium is reflected in the price of the underlying stock and there are no transaction costs.***

Proof.

See Appendix A ■

Suppose there are transaction costs, denoted by "c", which are assumed to be symmetric and proportional. That is, if the stock price is S, a buyer pays $S(1+c)$ and a seller receives $S(1-c)$. Similar rule applies when the bond or the option is traded. That is, if the bond price is B, a buyer pays $B(1+c)$ and a seller receives $B(1-c)$. We further assume that the call option is cash settled. That is, there is no physical delivery.

Introduction of the transaction cost does not change the analogy price as the expected returns on call and on the underlying stock are proportionally reduced. However, the cost of replicating a call option changes. The total cost of successfully replicating a long position in the call option by buying the appropriate replicating portfolio and then liquidating it in the next period to get cash (as call is cash settled) is:

$$\left(\frac{X_1-K}{X_1-X_2}\right)\left\{\frac{S}{1-c}-\frac{X_2}{1+c}\right\}+c\left\{\frac{S}{1-c}+\frac{X_2}{1+c}\right\}\ \ if\ X_1>K>X_2 \tag{3.8}$$

$$\left\{\frac{S}{1-c}-\frac{K}{1+c}\right\}+c\left\{\frac{S}{1-c}+\frac{K}{1+c}\right\}\ \ if\ X_1>X_2>K \tag{3.9}$$

The corresponding inflow from shorting the appropriate replicating portfolio to fund the purchase of a call option is:

$$\left(\frac{X_1-K}{X_1-X_2}\right)\left\{\frac{S}{1+c}-\frac{X_2}{1-c}\right\}-c\left\{\frac{S}{1+c}+\frac{X_2}{1-c}\right\}\ \ if\ X_1>K>X_2 \tag{3.10}$$

$$\left\{\frac{S}{1+c}-\frac{K}{1-c}\right\}-c\left\{\frac{S}{1+c}+\frac{K}{1-c}\right\}\ \ if\ X_1>X_2>K \tag{3.11}$$

Proposition 2 shows that if transaction costs exist and the risk premium on the underlying stock is within a certain range, the analogy price lies within the no-arbitrage interval. Hence, riskless profit cannot be earned at the expense of analogy makers.

Proposition 2 ***In the presence of symmetric and proportional transaction costs, analogy makers cannot be arbitraged out of the market if the risk premium on the underlying stock satisfies:***

$$\mathbf{0 \leq \delta \leq \frac{(1-c)(1+c)}{(1-c)^2 - 2\frac{S}{K}c(1+c)} - 1} \qquad \boldsymbol{if\ X_1 > X_2} > K \tag{3.12}$$

$$\mathbf{0 \leq \delta \leq}$$

$$\mathbf{\frac{\frac{K\left(X_1^2 - X_2^2\right)(1-c^2)}{2}}{X_2(X_1-K)(X_1+X_2)(1-c)^2 - S\left\{(1+c)^2\left(X_1^2-X_2^2\right) - X_1(X_1-X_2)(1-c^2)\right\}} - 1}$$

$$\boldsymbol{if\ X_1} > K > \boldsymbol{X_2} \tag{3.13}$$

Proof.

See Appendix B

■

Intuitively, when transaction costs are introduced, there is no unique no-arbitrage price. Instead, a whole interval of no-arbitrage prices comes into existence. Proposition 2 shows that for reasonable parameter values, the analogy price lies within this no-arbitrage interval in a one period binomial model. As more binomial periods are added, the transaction costs increase further due to the need for additional re-balancing of the replicating portfolio. In the continuous limit, the total transaction cost is unbounded. Reasonably, arbitrageurs cannot make money at the expense of analogy makers in the presence of transaction costs ensuring that the analogy makers survive in the market.

It is interesting to consider the rate at which the delta-hedged portfolio grows under analogy making. Proposition 3 shows that under analogy making, the delta-hedged portfolio grows at a rate $\frac{1}{f} - 1 = r + \delta$. This is in contrast with the Black Scholes Merton/Binomial Model in which the growth rate is equal to the risk free rate, r.

Proposition 3 ***If analogy making determines the price of the call option, then the corresponding delta-hedged portfolio grows with time at the rate of*** $\frac{1}{f}-1$.

Proof.

See Appendix C

■

Corollary 3.1 ***If there are multiple binomial periods then the growth rate of the delta-hedged portfolio per binomial period is*** $\frac{1}{f}-1$.

In continuous time, the difference in the growth rates of the delta-hedged portfolio under analogy making and under the Black Scholes/Binomial model leads to an option pricing formula under analogy making which is different from the Black Scholes formula. The continuous time formula is presented in the next section.

4. Analogy Making: The Continuous Case

We maintain all the assumptions of the Black-Scholes model except one. We allow for transaction costs whereas the transaction costs are ignored in the Black-Scholes model. As is well known, introduction of the transaction costs invalidates the replication argument underlying the Black Scholes formula. See Soner, Shreve, and Cvitanic (1995). As seen in the last section, transaction costs have no bearing on the analogy argument as they simply reduce the expected return on the call and on the underlying stock proportionally.

Proposition 4 shows the analogy based partial differential equation under the assumption that the underlying follows geometric Brownian motion, which is the limiting case of the discrete binomial model. We also explicitly allow for the possibility that different marginal investors determine prices of calls with different strikes. This is reasonable as call buying is a bullish strategy with more optimistic buyers self-selecting into higher strikes.

Proposition 4 ***If analogy makers set the price of a European call option, the analogy option pricing partial differential Equation (PDE) is***

$$(r + \delta_K)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r + \delta_K)S + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2}$$

Where δ_K is the risk premium that a marginal investor in the call option with strike 'K' expects from the underlying stock.

Proof.

See Appendix D

■

Just like the Black Scholes PDE, the analogy option pricing PDE can be solved by transforming it into the heat equation. Proposition 5 shows the resulting call option pricing formula for European options without dividends under analogy making.

Proposition 5 ***The formula for the price of a European call is obtained by solving the analogy based PDE. The formula is $C = SN(d_1) - Ke^{-(r+\delta_K)}N(d_2)$ where $d_1 = \frac{ln(S/K)+(r+\delta_K+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = \frac{ln\left(\frac{S}{K}\right)+\left(r+\delta_K-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$***

Proof.

See Appendix E.

■

Corollary 5.1 ***The formula for the analogy based price of a European put option is***

$$Ke^{-r(T-t)}\left\{1 - e^{-\delta_K(T-t)}N(d_2)\right\} - SN(-d_1)$$

Proof. Follows from put-call parity. ■

As proposition 5 shows, the analogy formula is exactly identical to the Black Scholes formula except for the appearance of δ_K, which is the risk premium that a marginal investor in the call option with strike K expects from the underlying stock. Note, that full allowance is made for the possibility that such expectations vary with strike price as more optimistic investors are likely to self-select into higher strike calls.

5. The Implied Volatility Skew

If analogy making determines option prices (formulas in proposition 5), and the Black Scholes model is used to infer implied volatility, the skew is observed. Table 1 shows two examples of this. In the illustration titled "IV-Homogeneous Expectation", the perceived risk premium on the underlying stock does not vary with the striking price. The other parameters are: $r = 2\%,\ \sigma = 20\%, T - t = 30\ days, and\ S = 100$. In the illustration titled "IV-Heterogeneous Expectations", the risk premium on the underlying stock is varied by 40 basis points for every 0.01 change in moneyness. That is, for a change of \$5 in strike, the risk premium increases by 200 basis points. This captures the possibility that more optimistic investors self-select into higher strike calls. Other parameters are kept the same.

Table 1

The Implied Volatility Skew

IV-Heterogeneous Expectations						**IV-Homogeneous Expectations**		
K/S	Risk Premium	Black Scholes	Analogy Price	Implied Vol.	Implied Vol. – Historical Vol.	Risk Premium	Implied Vol.	Implied Vol. – Historical Vol.
0.9	10%	10.21	10.93	36.34%	16.34%	10%	36.34%	16.34%
0.95	12%	5.69	6.47	29.33%	9.33%	10%	27.87%	7.87%
1.0	14%	2.37	2.985	25.4%	5.4%	10%	23.78%	3.78%
1.1	18%	0.129	0.231	22.74%	2.74%	10%	21.46%	1.46%

As Table 1 shows, the implied volatility skew can be observed with both homogeneous and heterogeneous expectations. It also shows that the difference between implied volatility and realized volatility is higher with heterogeneous expectations. It is easy to see that higher the dispersion in beliefs, greater is the difference between implied and realized volatilities (as long as more optimistic investors self-select into higher strike calls). This is consistent with empirical evidence that shows that higher the dispersion in beliefs, greater is the difference between implied and realized volatilities (see Beber A., Breedan F., and Buraschi A. (2010)). Figure 2 is a graphical illustration of Table 1.

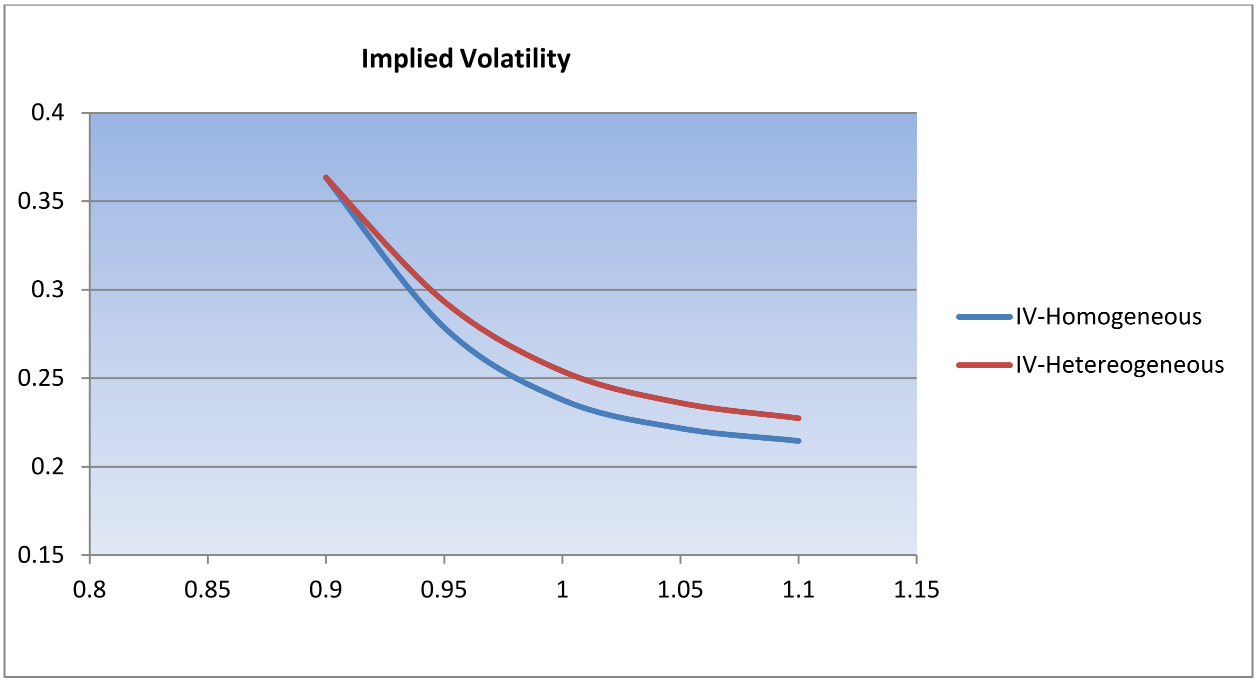

Figure 2

It is easy to illustrate that, with analogy making, the implied volatility skew gets flatter as time to expiry increases. As an example, with underlying stock price=$100, volatility=20%, risk premium on the underlying stock=5%, and the risk free rate of 0, the flattening with expiry can be seen in Figure 3. Hence, the implications of analogy making are consistent with key observed features of the structure of implied volatility skew.

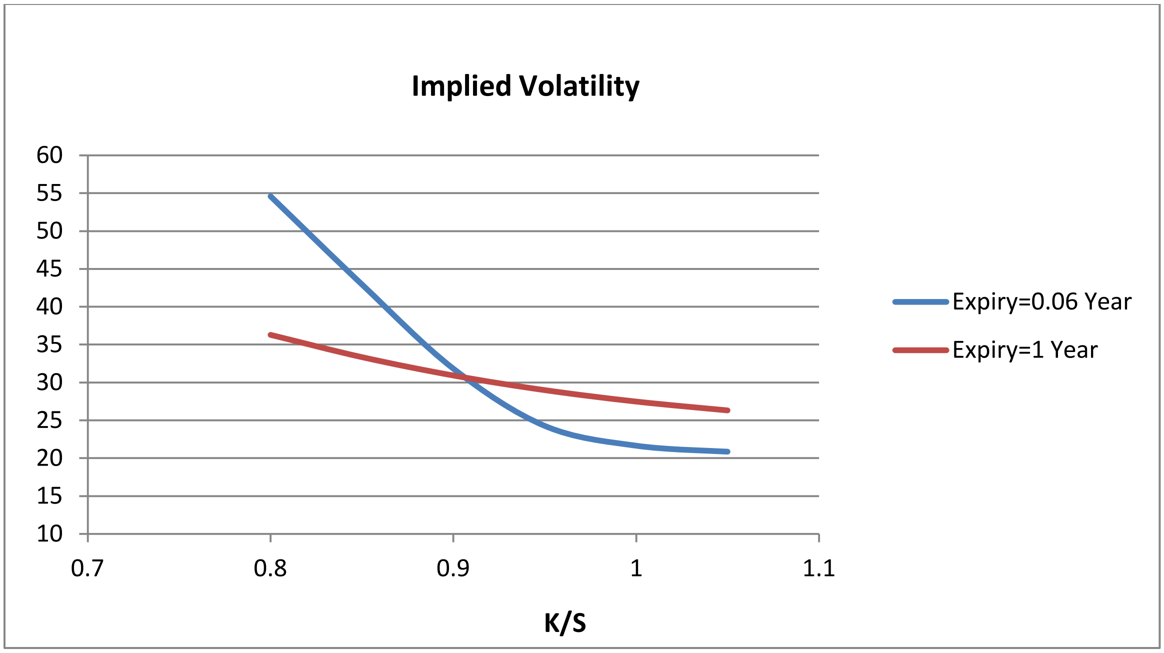

Figure 3

As an illustration of the fact that implied volatility curve flattens with expiry, Figure 4 is a reproduction of a chart from Fouque, Papanicolaou, Sircar, and Solna (2004) (Figure 2 from their paper). It plots implied volatilities from options with at least two days and at most three months to expiry. The flattening is clearly seen.

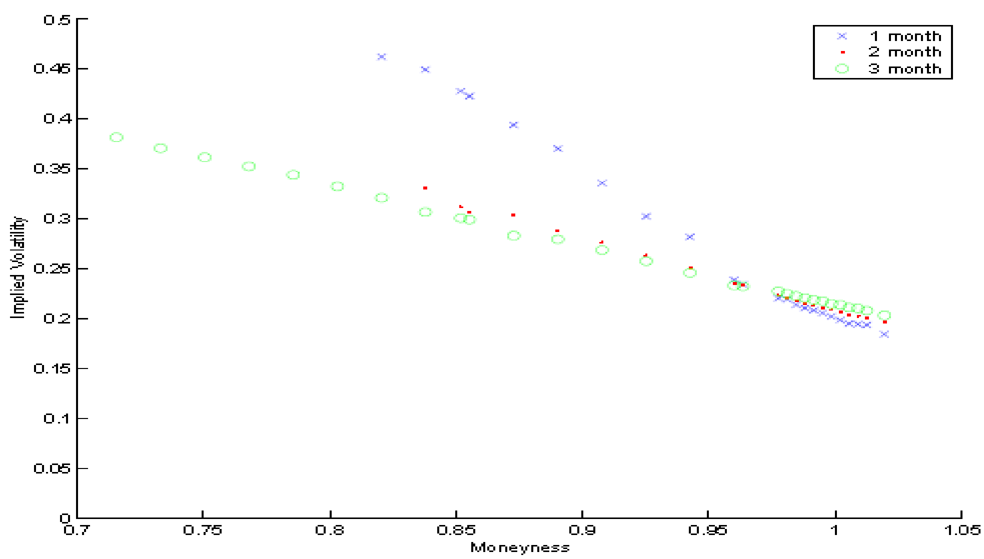

Figure 4 *Implied volatility as a function of moneyness on January 12, 2000, for options with at least two days and at most three months to expiry.*

So far, we have only considered analogy making as the sole mechanism generating the skew. Stochastic volatility and jump diffusion are other popular methods that give rise to the skew. In sections 8 and 9, we show that analogy making is complementary to stochastic volatility and jump diffusion models by integrating analogy making with the models of Hull and White (1987) and Merton (1976) respectively. In the next section, the profitability of covered call writing and zero-beta straddles is examined in the analogy model.

6. The Profitability of Covered Call Writing with Analogy Making

The profitability of covered call writing is quite puzzling in the Black Scholes framework. Whaley (2002) shows that BXM (a Buy Write Monthly Index tracking a Covered Call on S&P 500) has significantly lower risk when compared with the index, however, it offers nearly the same return as the index. Similar conclusions are reached in studies by Feldman and Roy (2004) and Callan Associates (2006). In the Black Scholes framework, the covered call strategy is expected to have lower risk as well as lower return when compared with buying the index only. See Black (1975). In fact, in an efficient market, the risk adjusted return from covered call writing should be no different than the risk adjusted return from just holding the index.

The covered call strategy (S denotes stock, C denotes call) is given by:

$$V = S - C$$

With analogy making, this is equal to:

$$V = S - \left\{ SN(d_1^A) - Ke^{-(r+\delta_K)(T-t)} N(d_2^A) \right\}$$

$$=> V = \left(1 - N(d_1^A)\right)S + N(d_2^A)Ke^{-(r+\delta_K)(T-t)} \tag{6.1}$$

The corresponding value under the Black Scholes assumptions is:

$$V = \left(1 - N(d_1)\right)S + N(d_2)Ke^{-r(T-t)} \tag{6.2}$$

A comparison of 6.1 and 6.2 shows that covered call strategy is expected to perform much better with analogy making when compared with its expected performance in the Black Scholes world. With analogy making, covered call strategy creates a portfolio with a weight of $1 - N(d_1^A)$ on the stock and a weight of $N(d_2^A)$ on a hypothetical risk free asset with a return of $r + \delta_K$. The stock has a return of $r + \delta$ plus dividend yield. This implies that, with analogy making, the return from covered call strategy is expected to be comparable to the return from holding the underlying stock only. The presence of a hypothetical risk free asset in 6.1 implies that the standard deviation of covered call returns is lower than the standard deviation from just holding the underlying stock. Hence, the superior historical performance of covered call strategy is no mystery if call prices are determined via analogy making.

6.1 The Zero-Beta Straddle Performance with Analogy Making

Another empirical puzzle in the Black-Scholes/CAPM framework is that zero beta straddles lose money. Goltz and Lai (2009), Coval and Shumway (2001) and others find that zero beta straddles earn negative returns on average. This is in sharp contrast with the Black-Scholes/CAPM prediction which says that the zero-beta straddles should earn the risk free rate. A zero-beta straddle is constructed by taking a long position in corresponding call and put options with weights chosen so as to make the portfolio beta equal to zero:

$$\theta \cdot \beta_{Call} + (1 - \theta) \cdot \beta_{Put} = 0$$

$$=> \theta = \frac{-\beta_{Put}}{\beta_{Call} - \beta_{Put}}$$

Where $\beta_{Call} = N(d_1) \cdot \frac{Stock}{Call} \cdot \beta_{Stock}$ and $\beta_{Put} = (N(d_1) - 1) \cdot \frac{Stock}{Put} \cdot \beta_{stock}$

It is straightforward to show that with analogy making, where call and put prices are determined in accordance with proposition 5, the zero-beta straddle earns a significantly smaller return than the risk free rate with returns being negative for a wide range of realistic parameter values. Hence, the observed empirical performance of zero-beta straddle is no puzzle with analogy based option pricing. Intuitively, with analogy making, both call and put options are more expensive when compared with Black-Scholes prices. Hence, the returns are smaller.

Analogy based option pricing not only generates the implied volatility skew, it is also consistent with key empirical findings regarding option portfolio returns such as covered call writing and zero-beta straddles.

7. Leverage Adjusted Option Returns with Analogy Making

Leverage adjustment dilutes beta risk of an option by combining it with a risk free asset. Leverage adjustment combines each option with a risk-free asset in such a manner that the overall beta risk becomes equal to the beta risk of the underlying stock. The weight of the option in the portfolio is equal to its inverse price elasticity w.r.t the underlying stock's price:

$\beta_{portfolio} = \Omega^{-1} \times \beta_{call} + (1 - \Omega^{-1}) \times \beta_{riskfree}$

where $\Omega = \frac{\partial Call}{\partial Stock} \times \frac{Stock}{Call}$ (i.e price elasticity of call w.r.t the underlying stock)

$\beta_{call} = \Omega \times \beta_{stock}$

$\beta_{riskfree} = 0$

$=> \beta_{portfolio} = \beta_{stock}$

Constantinides, Jackwerth and Savov (2013) uncover a number of puzzling empirical facts regarding leverage adjusted index option returns. They find that over a period ranging from April 1986 to January 2012, the average percentage monthly returns of *leverage-adjusted index call and put options are decreasing in the ratio of strike to spot.* They also find that *leverage adjusted put returns are larger than the corresponding leverage adjusted call returns*. The empirical findings in Contantinides et al (2012) are inconsistent with the Black-Scholes/CAPM framework, which predicts that the leverage adjusted returns should be equal to the return from the underlying index. That is, they should not fall with strike, and the leverage adjusted put option returns should not be any different than the leverage adjusted call returns.

If analogy making determines call prices, then the behavior of leverage adjusted call and put returns should be a lot different than their predicted behavior under the Black-Scholes assumptions. For call options:

$$\Omega^{-1} \cdot \frac{1}{dt}\left[\frac{E[dC]}{C}\right] + (1 - \Omega^{-1})r \tag{7.1}$$

where $$E[dC] = \left((r + \delta)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt \tag{7.2}$$

According to the analogy based PDE:

$$(r+\delta)C = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r+\delta)S + \frac{\partial^2 C}{\partial S^2}\frac{\sigma^2 S^2}{2} \tag{7.3}$$

Substituting (7.3) and (7.2) in (7.1) and simplifying leads to:

$$Leverage\ Adjusted\ Call\ Return = \Omega^{-1}(\delta) + r \tag{7.4}$$

Ω rises as the ratio of strike to spot increases. So, the leverage adjusted call option return must fall as the ratio of strike to spot increases. Hence, analogy based option pricing is consistent with empirical evidence. The corresponding leverage adjusted put option return with analogy based option pricing is:

$$Leverage\ Adjusted\ Put\ Return = r + \delta\left\{\frac{S-C}{S\left(1-N(d_1)\right)}\right\} \tag{7.5}$$

The term in brackets, $\frac{S-C}{S\left(1-N(d_1)\right)}$, falls as the ratio of strike to spot increases. Furthermore, (7.5) is larger than (7.4). That is, the leverage adjusted put returns must fall as the ratio of strike to spot increases and are also larger than the corresponding leverage adjusted call returns. Hence, empirical findings in Constantinides et al (2012) regarding leverage adjusted option returns are consistent with analogy based option pricing.

8. Analogy based Option Pricing with Stochastic Volatility

In this section, we put forward an analogy based option pricing model for the case when the underlying stock price and its instantaneous variance are assumed to obey the uncorrelated stochastic processes described in Hull and White (1987):

$$dS = \mu S dt + \sqrt{V} S dw$$

$$dV = \varphi V dt + \varepsilon V dz$$

$$E[dwdz] = 0$$

Where $V = \sigma^2$ (Instantaneous variance of stock's returns), and φ and ε are non-negative constants. dw and dz are standard Guass-Weiner processes that are uncorrelated. Time subscripts in S and V are suppressed for notational simplicity. If $\varepsilon = 0$, then the instantaneous variance is a constant, and we are back in the Black-Scholes world. Bigger the value of ε, which can be interpreted as the volatility of volatility parameter, larger is the departure from the constant volatility assumption of the Black-Scholes model.

Hull and White (1987) is among the first option pricing models that allowed for stochastic volatility. A variety of stochastic volatility models have been proposed including Stein and Stein (1991), and Heston (1993) among others. Here, we use Hull and White (1987) assumptions to show that the idea of analogy making is easily combined with stochastic volatility. Clearly, with stochastic volatility it does not seem possible to form a hedge portfolio that eliminates risk completely. This is because there is no asset which is perfectly correlated with $V = \sigma^2$.

If analogy making determines call prices and the underlying stock and its instantaneous volatility follow the stochastic processes described above, then the European call option price (no dividends on the underlying stock for simplicity) must satisfy the partial differentiation equation given below (see Appendix F for the derivation):

$$\frac{\partial C}{\partial t} + (r+\delta)S\frac{\partial C}{\partial S} + \varphi V\frac{\partial C}{\partial V} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\varepsilon^2 V^2\frac{\partial^2 C}{\partial V^2} = (r+\delta)C \tag{8.1}$$

Where δ is the risk premium that a marginal investor in the call option expects to get from the underlying stock.

By definition, under analogy making, the price of the call option is the expected terminal value of the option discounted at the rate which the marginal investor in the option expects to get from investing in the underlying stock. The price of the option is then:

$$C(S_t, \sigma_t^2, t) = e^{-(r+\delta)(T-t)} \int C(S_T, \sigma_T^2, T) p(S_T | S_t, \sigma_t^2) dS_T \tag{8.2}$$

Where the conditional distribution of S_T as perceived by the marginal investor is such that $E[S_T | S_t, \sigma_t^2] = S_t e^{(r+\delta)(T-t)}$ and $C(S_T, \sigma_T^2, T)$ is $max(S_T - K, 0)$.

By defining $\bar{V} = \frac{1}{T-t}\int_t^T \sigma_\tau^2 d\tau$ as the means variance over the life of the option, the distribution of S_T can be expressed as:

$$p(S_T|S_t, \sigma_t^2) = \int f(S_T|S_t, \bar{V})\, g(\bar{V}|S_t, \sigma_t^2) d\bar{V} \tag{8.3}$$

Substituting (8.3) in (8.2) and re-arranging leads to:

$$C(S_t, \sigma_t^2, t) = \int \left[e^{-(r+\delta)(T-t)} \int C(S_T) f(S_T|S_t, \bar{V}) dS_T \right] g(\bar{V}|S_t, \sigma_t^2) d\bar{V} \tag{8.4}$$

By using an argument that runs in parallel with the corresponding argument in Hull and White (1987), it is straightforward to show that the term inside the square brackets is the analogy making price of the call option with a constant variance $\bar{V}$. Denoting this price by $Call_{AM}(\bar{V})$, the price of the call option under analogy making when volatility is stochastic (as in Hull and White (1987)) is given by (proof available from author):

$$C(S_t, \sigma_t^2, t) = \int Call_{AM}(\bar{V}) g(\bar{V}|S_t, \sigma_t^2)\, d\bar{V} \tag{8.5}$$

Where $Call_{AM}(\bar{V}) = SN(d_1^M) - Ke^{-(r+\delta)(T-t)}N(d_2^M)$

$$d_1^M = \frac{ln\left(\frac{S}{K}\right) + \left(r+\delta+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\,;\; d_2^M = \frac{ln\left(\frac{S}{K}\right) + \left(r+\delta-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Equation (8.5) shows that the analogy based call option price with stochastic volatility is the analogy based price with constant variance integrated with respect to the distribution of mean volatility.

8.1 Option Pricing Implications

Stochastic volatility models require a strong correlation between the volatility process and the stock price process in order to generate the implied volatility skew. They can only generate a more symmetric U-shaped smile with zero correlation as assumed here. In contrast, the analogy making stochastic volatility model (equation 8.5) can generate a variety of skews and smiles even with zero correlation. What type of implied volatility structure is ultimately seen depends on the parameters δ and ε. It is easy to see that if $\varepsilon = 0$ and $\delta > 0$, only the implied volatility skew is generated, and if

$\delta = 0$ and $\varepsilon > 0$, only a more symmetric smile arises. For positive δ, there is a threshold value of ε below which skew arises and above which smile takes shape. Typically, for options on individual stocks, the smile is seen, and for index options, the skew arises. The approach developed here provides a potential explanation for this as ε is likely to be lower for indices due to inbuilt diversification (giving rise to skew) when compared with individual stocks.

9. Analogy based Option Pricing with Jump Diffusion

In this section, we integrate the idea of analogy making with the jump diffusion model of Merton (1976). As before, the point is that the idea of analogy making is independent of the distributional assumptions that are made regarding the behavior of the underlying stock. In the previous section, analogy making is combined with the Hull and White stochastic volatility model to illustrate the same point.

Merton (1976) assumes that the stock returns are a mixture of geometric Brownian motion and Poisson-driven jumps:

$$dS = (\mu - \gamma\beta)Sdt + \sigma Sdz + dq$$

Where dz is a standard Guass-Weiner process, and $q(t)$ is a Poisson process. dz and dq are assumed to be independent. γ is the mean number of jump arrivals per unit time, $\beta = E[Y - 1]$ where $Y - 1$ is the random percentage change in the stock price if the Poisson event occurs, and E is the expectations operator over the random variable Y. If $\gamma = 0$ (hence, $dq = 0$) then the stock price dynamics are identical to those assumed in the Black Scholes model. For simplicity, assume that $E[Y] = 1$.

The stock price dynamics then become:

$$dS = \mu Sdt + \sigma Sdz + dq$$

Clearly, with jump diffusion, the Black-Scholes no-arbitrage technique cannot be employed as there is no portfolio of stock and options which is risk-free. However, with analogy making, the price of the option can be determined as the return on the call option demanded by the marginal investor is equal to the return he expects from the underlying stock.

If analogy making determines the price of the call option when the underlying stock price dynamics are a mixture of a geometric Brownian motion and a Poisson process as described earlier, then the following partial differential equation must be satisfied (see Appendix G for the derivation):

$$\frac{\partial C}{\partial t} + (r+\delta)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \gamma E[C(SY,t) - C(S,t)] = (r+\delta)C \tag{9.1}$$

If the distribution of Y is assumed to log-normal with a mean of 1 (assumed for simplicity) and a variance of v^2 then by using an argument analogous to Merton (1976), the following analogy based option pricing formula for the case of jump diffusion is easily derived (proof available from author):

$$Call = \sum_{j=0}^{\infty} \frac{e^{-\gamma(T-t)}\left(\gamma(T-t)\right)^j}{j!} Call_{AM}\left(S,(T-t),K,r,\delta,\sigma_j\right) \tag{9.2}$$

$$Call_{AM}\left(S,(T-t),K,r,\delta,\sigma_j\right) = SN(d_1^M) - Ke^{-(r+\delta)(T-t)}N(d_2^M)$$

$$d_1^M = \frac{ln\left(\frac{S}{K}\right) + \left(r+\delta+\frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j\sqrt{T-t}} \qquad d_2^M = \frac{ln\left(\frac{S}{K}\right) + \left(r+\delta-\frac{\sigma_j^2}{2}\right)(T-t)}{\sigma_j\sqrt{T-t}}$$

$$\sigma_j = \sqrt{\sigma^2 + v^2\left(\frac{j}{T-t}\right)} \text{ and } v^2 = \frac{f\sigma^2}{\gamma}$$

Where f is the fraction of volatility explained by jumps.

The formula in (9.2) is identical to the Merton jump diffusion formula except for one parameter, δ, which is the risk premium that a marginal investor in the call option expects from the underlying stock.

9.1 Option Pricing Implications

Merton's jump diffusion model with symmetric jumps (jump mean equal to zero) can only produce a symmetric smile. Generating the implied volatility skew requires asymmetric jumps (jump mean becomes negative) in the model. However, with analogy making, both the skew and the smile can be generated even when jumps are symmetric. In particular, for low values of δ, a more symmetric smile is generated, and for larger values of δ, skew arises.

Even if we one assumes an asymmetric jump distribution around the current stock price, Merton formula, when calibrated with historical data, generates a skew which is a lot less pronounced (steep) than what is empirically observed. See Andersen and Andreasen (2002). The skew generated by the analogy formula (with asymmetric jumps) is typically more pronounced (steep) when compared with the skew without analogy making. Hence, analogy making potentially adds value to a jump diffusion model.

If prices are determined in accordance with the formula given in (9.2) and the Black Scholes formula is used to back-out implied volatility, the skew is observed. As an example, Figure 5 shows the skew generated by assuming the following parameter values:

$(\mathrm{S} = 100, \mathrm{r} = 5\%, \gamma = 1 \text{ per year}, \delta = 5\%, \sigma = 25\%, \mathrm{f} = 10\%, \mathrm{T} - \mathrm{t} = 0.5 \text{ year})$.

In Figure 5, the x-axis values are various values of strike/spot, where spot is fixed at 100. Note, that the implied volatility is always higher than the actual volatility of 25%. Empirically, implied volatility is typically higher than the realized or historical volatility. As one example, Rennison and Pederson (2012) use data ranging from 1994 to 2012 from eight different option markets to calculated implied volatility from at-the-money options. They report that implied volatilities are typically higher than realized volatilities.

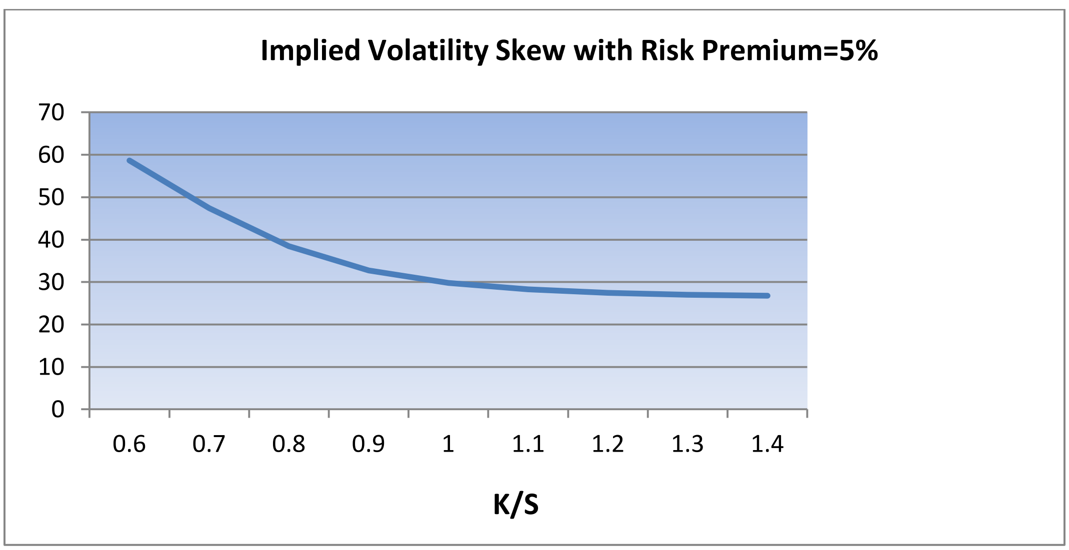

Figure 5

In general, in the jump diffusion analogy model, the skew generated turns into a smile as the risk premium on the underlying falls (approaches the risk-free rate). Figure 6 shows one instance when the risk premium is 1% and fraction of volatility due to jumps is 40% (all other parameters are kept the same).

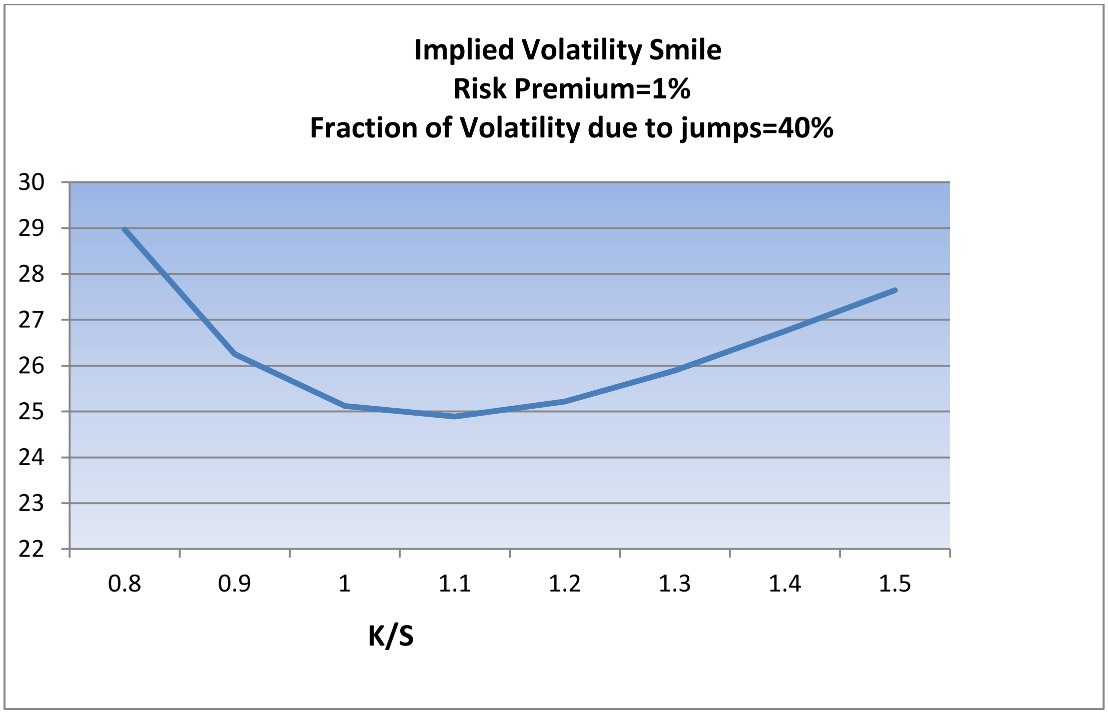

Figure 6

10. Conclusions

The observation that people tend to think by analogies and comparisons has important implications for option pricing that are thus far ignored in the literature. Prominent cognitive scientists argue that analogy making is the way human brain works (Hofstadter and Sander (2013)). There is strong experimental evidence that a call option is valued in analogy with the underlying stock (see Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011)). A call option is commonly considered to be a surrogate for the underlying stock by experienced market professionals, which lends further support to the idea of analogy based option valuation. In this article, the notion that a call option is valued in analogy with the underlying stock is explored and the resulting option pricing model is put forward. The analogy option pricing model provides a new explanation for the implied volatility skew puzzle. The analogy based explanation complements the existing explanation as it is possible to integrate analogy making with stochastic volatility and jump diffusion approaches. The paper does that and puts forward analogy based option valuation models with stochastic volatility and jumps respectively. In contrast with other stochastic volatility and jump diffusion models in the literature, analogy making stochastic volatility model generates the skew even when there is zero correlation between the stock price and volatility processes, and analogy based jump diffusion can produce the skew even with symmetric jumps.

The analogy model differs minimally from the Black Scholes model due to the introduction of one additional variable. The additional variable captures the risk premium (subjective) that the marginal call option investor expects to get from the underlying stock. It is surprising that such a minimally different model can explain a wide variety of phenomena such as implied volatility skew, leverage adjusted option returns, superior performance of covered call writing, and worse-than-expected performance of zero-beta straddles.

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Appendix A

Proof of Proposition 1

For case 1, when $X_1 > X_2 > K$, the results follow from a direct comparison of (3.4) and (3.5).

For case 2, when $X_1 > K > X_2$, the spectrum of possibilities is further divided into three sub-classes and the results are proved for each sub-class one by one. The three sub-classes are: (i) $K = \frac{X_1+X_2}{2}$, (ii) $X_2 < K < \frac{X_1+X_2}{2}$, and (iii) $X_1 > K > \frac{X_1+X_2}{2}$.

Case 2 sub-class (i): $K = \frac{X_1+X_2}{2}$

If we assume that $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2}(S - X_2)$, we arrive at a contradiction as follows:

Substitute $S = f \cdot \frac{X_1+X_2}{2}$ and $K = \frac{X_1+X_2}{2}$ above and simplify, it follows that $f \geq 1$, which is a contradiction as $f < 1$ if the risk premium is positive.

Case 2 sub-class (ii): $\boldsymbol{X_2} < K < \frac{\boldsymbol{X_1+X_2}}{\boldsymbol{2}}$ **or equivalently** $\boldsymbol{K} = \boldsymbol{g}\frac{\boldsymbol{X_1+X_2}}{\boldsymbol{2}}$ **where** $\frac{\boldsymbol{2X_2}}{\boldsymbol{X_1+X_2}} < g < 1$

If we assume that $S \cdot \frac{X_1}{X_1+X_2} - \frac{K}{2} \cdot f \leq \frac{X_1-K}{X_1-X_2}(S - X_2)$, we arrive at a contradiction as follows:

Substitute $S = f \cdot \frac{X_1+X_2}{2}$ and $K = g\frac{X_1+X_2}{2}$ above and simplify, it follows that $X_1 \leq X_2$, which is a contradiction.

Case 2 sub-class (iii): $\boldsymbol{X_1} > K > \frac{\boldsymbol{X_1+X_2}}{\boldsymbol{2}}$ **or equivalently** $\boldsymbol{K} = \boldsymbol{g}\frac{\boldsymbol{X_1+X_2}}{\boldsymbol{2}}$ **where** $\boldsymbol{1} < g < \frac{\boldsymbol{2X_1}}{\boldsymbol{X_1+X_2}}$

Similar logic as used in the case above leads to a contradiction: $X_1 \leq X_2$.

Hence, the analogy price must be larger than the no-arbitrage price if the risk premium is positive and there are no transaction costs.

Appendix B

Proof of Proposition 2

If $X_1 > X_2 > K$ then there is no-arbitrage if the following holds:

$$\left\{\frac{S}{1+c} - \frac{K}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{K}{1-c}\right\} \leq S - Kf \leq \left\{\frac{S}{1-c} - \frac{K}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{K}{1+c}\right\}$$

Realizing that $S - Kf \geq S - K > \left\{\frac{S}{1+c} - \frac{K}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{K}{1-c}\right\}$ $if\ \delta \geq 0$ and simplifying

$S - Kf \leq \left\{\frac{S}{1-c} - \frac{K}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{K}{1+c}\right\}$ leads to inequality (3.12).

If $X_1 > K > X_2$ then there is no-arbitrage if the following holds:

$$\left(\frac{X_1 - K}{X_1 - X_2}\right)\left\{\frac{S}{1+c} - \frac{X_2}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{X_2}{1-c}\right\} \leq S \cdot \frac{X_1}{X_1 + X_2} - \frac{K}{2} \cdot f$$
$$\leq \left(\frac{X_1 - K}{X_1 - X_2}\right)\left\{\frac{S}{1-c} - \frac{X_2}{1+c}\right\} + c\left\{\frac{S}{1-c} + \frac{X_2}{1+c}\right\}$$

Realizing that

$$\left(\frac{X_1 - K}{X_1 - X_2}\right)\left\{\frac{S}{1+c} - \frac{X_2}{1-c}\right\} - c\left\{\frac{S}{1+c} + \frac{X_2}{1-c}\right\} \leq$$

$$\frac{X_1-K}{X_1-X_2}(S-X_2) \le S\cdot\frac{X_1}{X_1+X_2}-\frac{K}{2}\cdot f \;\; if\ \delta \ge 0$$

And simplifying $S\cdot\frac{X_1}{X_1+X_2}-\frac{K}{2}\cdot f \;\le \left(\frac{X_1-K}{X_1-X_2}\right)\left\{\frac{S}{1-c}-\frac{X_2}{1+c}\right\}+c\left\{\frac{S}{1-c}+\frac{X_2}{1+c}\right\}$ leads to (3.1).

Appendix C

Proof of Proposition 3

Case 1: $X_1 > X_2 > K$

Delta-hedged portfolio is $Sx-C$. In this case, $x=1, S=f\cdot\frac{X_1+X_2}{2}$, and $C=S-Kf$

If the red state is realized, $S-C$ changes from Kf to K. If the blue state is realized $S-C$ also changes from Kf to K. Hence, the growth rate is equal to $\frac{1}{f}-1$ in either state.

Case 2: $X_1 > K > X_2$

Delta-hedged portfolio is $Sx-C$. In this case, $x=\frac{X_1-K}{X_1-X_2}, S=f\cdot\frac{X_1+X_2}{2}$, and

$$C=S\cdot\frac{X_1}{X_1+X_2}-\frac{K}{2}\cdot f$$

Consider three sub-classes and prove the result for each: (i) $K=\frac{X_1+X_2}{2}$, (ii) $X_2<K<\frac{X_1+X_2}{2}$, and (iii) $X_1>K>\frac{X_1+X_2}{2}$. For the first sub-class the delta-hedged portfolio changes from the initial value of $f\frac{X_2}{2}$ to $\frac{X_2}{2}$ in both the red and the blue states. Hence, the growth rate is equal to $\frac{1}{f}-1$ in either state. For the second and third sub-classes, the delta-hedged portfolio changes from $\frac{f\{(2-g)X_1X_2-gX_2^2\}}{2(X_1-X_2)}$ to $\frac{\{(2-g)X_1X_2-gX_2^2\}}{2(X_1-X_2)}$ in both red and blue states. Hence, the growth rate is equal to $\frac{1}{f}-1$.

Appendix D

In the binomial analogy case, the delta-hedged portfolio $S\frac{\Delta C}{\Delta S}-C$ grows at the rate $r+\delta_K$. Divide $[0,T-t]$ in n time periods, and with $n\to\infty$, the binomial process converges to the geometric Brownian motion. To deduce the analogy based PDE consider:

$$V = S\frac{\partial C}{\partial S} - C$$

$$\Rightarrow dV = dS\frac{\partial C}{\partial S} - dC$$

Where $dS = uSdt + \sigma SdW$ and by Ito's Lemma $dC = \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW$

$$\Rightarrow (r + \delta_K)Vdt = (uSdt + \sigma SdW)\frac{\partial C}{\partial S} - \left(uS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt - \sigma S\frac{\partial C}{\partial S}dW$$

$$(r + \delta_K)Vdt = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

$$\Rightarrow (r + \delta_K)\left(S\frac{\partial C}{\partial S} - C\right) = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right)$$

$$(r + \delta_K)C = (r + \delta_K)S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} \qquad (D1)$$

The above is the analogy based PDE.

Appendix E

The analogy based PDE derived in Appendix D can be solved by converting to heat equation and exploiting its solution. The steps are identical to the derivation of the Black Scholes model with the risk free rate r, replaced with $r + \delta$.

Appendix F

Start by considering the value of a delta hedged portfolio:

$\pi_t = S_t\Delta - C_t$.

Over a small time interval, dt:

$$d\pi_t = dS_t\Delta - dC_t \qquad \text{(F1)}$$

By Ito's Lemma (time subscript is suppressed for simplicity):

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial V} dV + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} dt + \frac{1}{2} V^2 \varepsilon^2 \frac{\partial^2 C}{\partial V^2} dt \quad \text{(F2)}$$

Substituting (F2) in (F1) and re-arranging:

$$d\pi = \left[\Delta - \frac{\partial C}{\partial S}\right] dS - \left[\frac{\partial C}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} V^2 \varepsilon^2 \frac{\partial^2 C}{\partial V^2}\right] dt - \frac{\partial C}{\partial V} dV \quad \text{(F3)}$$

Choosing $\Delta = \frac{\partial C}{\partial S}$, and realizing that, with analogy making, $E[d\pi] = (r + \delta)\pi dt$, (F3) becomes:

$$(r + \delta)\pi dt = -\left[\frac{\partial C}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} V^2 \varepsilon^2 \frac{\partial^2 C}{\partial V^2}\right] dt - \varphi V \frac{\partial C}{\partial V} dt \quad \text{(F4)}$$

(F4) simplifies to:

$$\frac{\partial C}{\partial t} + (r + \delta) S \frac{\partial C}{\partial S} + \varphi V \frac{\partial C}{\partial V} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \varepsilon^2 V^2 \frac{\partial^2 C}{\partial V^2} = (r + \delta) C \quad \text{(F5)}$$

Appendix G

By following a very similar argument as in appendix F, and using Ito's lemma for the continuous part and an analogous Lemma for the discontinuous part, the following is obtained:

$$\frac{\partial C}{\partial t} + (r + \delta) S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \gamma E[C(SY, t) - C(S, t)] = (r + \delta) C$$