Problems of utility and prospect theories. A discontinuity of Prelec's function

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A possibility of the existence of a discontinuity of Prelec's (probability weighting) function $W(p)$ at the probability $p = 1$ is discussed. This possibility is supported by the Aczél–Luce question whether Prelec's weighting function $W(p)$ is equal to 1 at $p = 1$, by the purely mathematical restrictions and the "certain–uncertain" inconsistency of the random–lottery incentive experiments. The results of the well-known experiments support this possibility as well.

Introduction

The paper sketches very roughly a future article. It will be included into a group of three articles. Supposed titles of the articles are: "A possible discontinuity of Prelec's function," "Certainty Effect near Certainty" and "A "certain-uncertain" inconsistency of the random-lottery incentive system."

There are a number of theories concerned with one or another concept of utility. They include, e.g., Bernoullian expected utility, von Neumann–Morgenstern expected utility, subjective expected utility, subjectively weighted utility theories (see, e.g., a review by Schoemaker, 1982); prospect theory (see Kahneman and Tversky, 1979) and cumulative prospect theory (see Tversky and Kahneman, 1992) or, in other terminology, original prospect theory and prospect theory; the salience theory of choice under risk (see Bordalo, Gennaioli, Shleifer, 2012); expected uncertain utility theory (Gul and Pesendorfer, 2014); etc.

In the present paper these theories are referred to as utility and prospect theories.

Ever since Bernoulli (1738), the problems with the theory of utility exposed by the Saint Petersburg paradox have been investigated. Von Neumann and Morgenstern (1947) promised the feasibility of a correct and, naturally, rational foundation of economic theory with their book, *Theory of Games and Economic Behavior*. But these promises were dashed by the Allais paradox (Allais, 1953).

Kahneman and Thaler (2006) pointed out that the basic problems of utility and prospect theories, including the paradoxes of Allais (Allais, 1953) and Ellsberg (Ellsberg, 1961), have not yet been adequately solved.

The purpose of the present paper is to consider a possibility of the existence of a discontinuity of Prelec's (probability weighting) function $W(p)$ at the probability $p = 1$ in the context of these problems.

1. The Aczél–Luce question whether *W(1) = 1*

An essential part of problems of utility and prospect theories consists in the problems that are connected with a probability weighting (see, e.g., Tversky and Wakker, 1995). A probability weighting means that subjects treat the probability p by a probability weighting function $W(p)$ which is not equal to p (see also uncertainty perception in (Gul and Pesendorfer, 2014) as an example of perception variety). I define the function $W(p)$ both for uncertain (probable) and certain outcomes. Prelec's weighting function (Prelec, 1998) is one of the most popular probability weighting functions.

One possible way to solve the above problems is to consider the vicinities of the borders of the probability scale, e.g. at $p \sim 1$ (see, e.g., Aczél and Luce, 2007).

Aczél and Luce (2007) emphasized a fundamental question: whether $W(1) = 1$ (whether Prelec's weighting function $W(p)$ (see Prelec, 1998) is equal to 1 at $p = 1$). In this article, I refer to this question as the Aczél–Luce question (or Luce question.

The answer $W(1) \neq 1$ to the Aczél–Luce question may mean the function $W(p)$ has a discontinuity at $p=1$. This is not a quantitative but a qualitative, moreover, a topological feature. So, the answer to the question can qualitatively change the situation in utility and prospect theories, at least in their mathematical aspects.

2. Purely mathematical restrictions

Another possible way to solve the problems of utility and prospect theories has been widely discussed, e.g., in Schoemaker and Hershey (1992); Hey and Orme (1994); Chay, McEwan and Urquiola (2005); Butler and Loomes (2007); Galaabaatar and Karni (2013); Chambers and Hayashi (2014). Its essence consists in a proper attention to noise, uncertainty, imprecision, incompleteness, unforeseen contingencies and other reasons that might cause dispersion, scattering, a spread of the data.

A purely mathematical investigation (see, e.g., Harin, 2010, 2012) has synthesized these two different ways. That is, it considers the dispersion of the data (or the influence of the dispersion of the data) near the borders of the probability scale.

Purely mathematical theorems (see, e.g., Harin, 2010, 2012) prove that the probability p cannot attain 1 under the condition of a non-zero dispersion of the data. The theorems are presented in the form of a sequence of five lemmas and theorems:

1) For a finite non-negative function on an interval $[0, 1]$, an analog of the dispersion D is proved to tend to 0, when the mean M of the function tends to any border of the interval.

2) Hence, if the analog of the dispersion D is not less than a non-zero value, then non-zero restrictions exist on M. Namely, M cannot be closer to any border of the interval than a specific non-zero amount.

This implies that under the condition of a non-zero dispersion D, the mean M cannot attain 1.

Note, that the restriction has opposite signs at the opposite borders of the interval. Hence, there is a point in the middle part of the interval, at which the restriction equals zero. Therefore, if the first derivative of the restriction does not change sign, then the restriction has its maximum absolute value at the borders and diminishes to zero at some point within the interval.

3–4) As far as the probability estimation corresponds to such a function and a non-zero dispersion of data takes place, then non-zero restrictions exist on the probability estimation.

This signifies that under the condition of a non-zero dispersion of data, the probability estimation cannot take on the value 1.

5) As far as the probability is the limit of the probability estimation and a non-zero dispersion of data takes place, then non-zero restrictions exist on the probability. Namely, the probability cannot be closer, than by a specific non-zero amount, to a border of the probability scale (see also, e.g., Li, 2013 for some considerations in the presence of data noise and jumps).

This signifies that under the condition of a non-zero dispersion of the data, the probability p cannot take on the value 1, i.e., $p < 1$. It cannot even be arbitrarily close.

As a matter of fact, a non-zero dispersion of data can be caused, e.g., by non-zero noise, which is practically unavoidable in economics.

3. A discontinuity of Prelec's function

There is a deal of evidence for the existence of a qualitative difference between subjects' treatment of the probabilities of uncertain (probable) and certain outcomes (see, e.g., Kahneman and Tversky, 1979; McCord and de Neufville, 1986; Gneezy, List and Wu, 2006; Halevy, 2008). Therefore, in the general case, one should distinguish between the values of the probability weighting function $W(p)$ of a certain outcome and the limit of the probability weighting function $W(p)$ of uncertain outcomes as the probability of uncertain outcomes tends to 1.

Let us additionally define or specify a value $W_{Certain}$ of the probability weighting function $W(p)$ for a certain outcome. At that, $W_{Certain}$ may be assumed to be equal to 1. Otherwise, other values of $W(p)$ may be normalized by $W_{Certain}$.

Let us here additionally specify a value $W(1)$ as the limit of the probability weighting function $W(p)$ for a probable (uncertain) outcome as p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) .$$

If $W(1) = W_{Certain}$, then $W(p)$ is continuous (at $p = 1$). This is usually assumed by default. But this has not been proven for the general case. So, if $W_{Impossible}$ is defined for the impossible case, then, similar to Aczél and Luce (2007),

$$W(p) = \begin{cases} W_{\mathrm{Im}possible} & p = 0 \\ W(p) & p \in]0,1[\\ W_{Certain} & p = 1 \end{cases}$$

and $W(p)$ can be continuous or discontinuous.

One may modify the Aczél–Luce question whether $W(1) = 1$ into the question whether $W(1) = W_{Certain}$ or whether $W(p)$ is continuous at $p = 1$. Note, various aspects of continuity have been discussed in the literature. See, e.g., Aczél and Luce (2007), Kothiyal, Spinu and Wakker (2011), Delbaen, Drapeau and Kupper (2011), Spinu and Wakker (2013).

To answer the modified question and to prove or disprove the continuity of $W(p)$ at $p=1$ one should determine and measure the difference

$$W_{Certain} - W(1) = ?$$

The answer $W(1) \neq W_{Certain}$ to the modified Aczél–Luce question means that the function $W(p)$ has a discontinuity at $p=1$. This is not a quantitative but a qualitative, moreover, a topological feature. So, the answer to the question can qualitatively change the situation in utility and prospect theories, at least in their mathematical aspects.

So, the modified Aczél–Luce question supposes the possibility of the existence of a discontinuity of the probability weighting function $W(p)$ at the probability $p = 1$.

The above purely mathematical theorems predict that, under the condition of a non-zero dispersion of the data, the probability p cannot take on the value 1, i.e., $p < 1$.

One may additionally suppose here, that if the probability weighting function $W(p) \leq p$ at $p > 3/4$ and $W_{Certain} = 1$, then this opens up a possibility of the existence of a discontinuity of $W(p)$ at $p = 1$. So, one may say the theorems predict the possibility of the existence of a discontinuity of the probability weighting function $W(p)$ at the probability $p = 1$, under the condition of a non-zero dispersion of the data.

Note, the theorems and the additional supposition radically contradict the accepted view. But they do not really contradict the majority of the existing utility and prospect theories and models. Moreover, the consequences of the theorems can be added to these theories and models as technical corrections of the probability, due to dispersion in the data or noises. At first, the correction can be roughly chosen as a linear addition, biased down to the probability $p = 0$, according to the hypothesis of an uncertain future (see, e.g., Harin, 2007). Such a correction can seem to be a straight-line approximation of the middle (roughly linear) part of the probability weighting curve from the middle to the borders.

4. Experiments and a “certain–uncertain” inconsistency

Instead of the predictions of the theorems, experiments seem not to support these predictions.

The prevalent experimental procedure in utility and prospect theories (and particularly in these experiments) is the random-lottery incentive system (see, e.g., Starmer, 2000 and Baltussen et al., 2012; etc.).

Harin (2014) emphasizes that in the random-lottery incentive system, the choices of certain (sure) outcomes are stimulated by uncertain lotteries.

This inconsistency is quite evident but has not yet been mentioned in the literature (see, e.g., von Gaudecker, van Soest and Wengstrom, 2011; Andreoni and Sprenger, 2012; Vossler, Doyon and Rondeau, 2012; Baltussen et al., 2012). The inconsistency was revealed in the recent report Harin (2014). The present article develops this report.

Because of this “certain–uncertain” inconsistency, the deductions from a random-lottery incentive experiment that includes a certain outcome cannot be unquestionably correct, especially at $p \sim 1$.

5. Experimental evidence of Starmer and Sugden

One can see the following in the description of the well-known experiment of Starmer and Sugden (1991):

Page 974: “For groups A and D, this page began with an underlined text stating that question 22 would be played for real. For groups B and C, the corresponding text stated that one of the two questions would be played for real and that which question was to played out would be decided at the end of the experiment in the following way. The subject would roll a six-sided die. If the number on the die was 1, 2, or 3, then question 21 would be played; if the number was 4, 5, or 6, question 22 would be played.”

“One problem, which we shall call P', required a choice between two lotteries R' (for "riskier") and S' (for "safer"). R' gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); S' gave £7.00 for sure.”

So, in the R'-S' problem, R' gives *£10.00*0.2+£7.00*0.75 = £7.25*. S' gives *£7.00*1 = £7.00*. Here *R' = £7.25>S' = £7.00*.

Let us consider the results from table 2 on Page 976, those are of interest here (the **boldface** is my own):

- Group = B, Incentive = Random lottery, R':S' = 19:21
- Group = C, Incentive = Random lottery, R':S' = 22:18
- Group = D, Incentive = **P' real**, R':S' = **13:27**

One can evidently see that the results for **P' real** incentive (**13:27**) differ essentially from those for random lottery incentive (19:21 and 22:18).

Therefore, the experiment shows that the random lottery incentives can essentially modify subjects’ choices in comparison with the real incentives, when these choices include certain outcomes and the probability ($0.2 + 0.75 = 0.95 \sim 1$) of the uncertain choices is near the border of the probability scale.

Let us evaluate the percentage of the subjects choosing the uncertain outcome and the direction of the modification of $W(p)$. The total number of the subjects in each group is equal to $40=19+21=22+18=\mathbf{13+27}$. So, the percentage is equal to *19/40=48%*, *22/40=55%* and ***13/40=33%***. One may see that the modification of $W(p)$ by the random lottery incentives is directed from ***13/40=33%*** to *19/40=48%* and *22/40=55%*. That is it is directed from 0 to 1. So, this supports the feasibility of the random lottery incentives’ hiding a possible discontinuity of $W(p)$ and causing the inverse S-shaped curve of $W(p)$.

So, the experiment of Starmer and Sugden (1991) evidently supports:

1. The “certain–uncertain” inconsistency.
2. The possibility of the existence of a discontinuity of $W(p)$ at the probability $p = 1$.

Conclusions

A possibility of the existence of a discontinuity of Prelec's (probability weighting) function $W(p)$ at the probability $p = 1$ is discussed.

Aczél and Luce (2007) emphasized a fundamental question (the Aczél–Luce question): whether $W(1) = 1$ (whether Prelec's weighting function $W(p)$ (see Prelec, 1998) is equal to 1 at $p = 1$).

In the present paper, two values of the probability weighting function $W(p)$ are specified:

- $W_{Certain}$ for a certain outcome and
- $W(1)$, as the limit of the probability weighting function $W(p)$ for the uncertain outcome when p tends to 1

$$W(1) \equiv \lim_{p \to 1} W(p) \ .$$

The Aczél–Luce question whether $W(1)=1$ is modified to the question whether $W(1)=W_{Certain}$. The answer

$$W(1) \neq W_{Certain}$$

means a discontinuity of $W(p)$ at the probability $p=1$. This is a new topological feature. Therefore, it can qualitatively change the utility and prospect theories, at least in their mathematical aspects.

Purely mathematical theorems (see, e.g., Harin, 2012b), under the condition of a non-zero dispersion of data, prove the probability p cannot attain the value 1 or even approach it arbitrarily closely, and predict the possibility of the existence of a discontinuity of the probability weighting function $W(p)$ at $p = 1$.

The well-known experiment of Starmer and Sugden (1991) supports this possibility.

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