Dimensional Analysis of Production and Utility Functions in Economics

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Abstract. This paper explores dimensional analysis of production and utility functions in economics. As raised by Barnett, dimensional analysis is important in consistency checks of economics functions. However, unlike Barnett's dismissal of CES and Cobb-Douglas production functions, we will demonstrate that under constant return-to-scale and other assumptions, production function can indeed be justified dimensionally. And then we consider utility functions.

1. Introduction

In usual economics analysis, there is not much concern for dimensional analysis. One justification for not doing dimensional analysis is that economics is statistical studies. In statistics, regression analysis does not consider dimensional problems. However, when economics escape its statistical query, economics equations do require dimensional analysis. For example, if we assume, or learn of, real or hypothetical economy such that every production process follows Cobb-Douglas production function and all parameters are pre-calibrated, then Cobb-Douglas function is no longer of statistical nature. Furthermore, productivity and technology factor A must, in this hypothetical reality, have consistent and fixed dimensional unit. Otherwise, we no longer can even measure this factor, as we even do not know dimension. The recourse to statistics is not allowed, as this is not a problem of regression analysis.

The whole problem is indeed raised by Barnett (2003), but Barnett also argues that CES and Cobb-Douglas function is meaningless. We will argue that this is not the case with some restrictive but reasonable assumptions. Also, we will consider utility functions and how dimensional analysis may be performed.

2. CES production function and dimensional analysis

CES production function is usually given by the following:

$$Y = A\left[\alpha K^{r} + (1-\alpha)\, L^{r}\right]^{\frac{1}{r}}$$

where K is stock amount of capital and L is flow amount of labor and α, r being dimensionless parameters. K and L have dimensions of [*capital*] and [*labor*]. Y has dimension of [*good*]. For production function, distinction of flow and stock does not matter much as long as we define Y appropriately. Barnett argues that because capital and labor has different units, they cannot be added together naively. This is fundamentally true. But notice that what we want for K and L is amount of capital and labor. If amount of K can be measured in the labor unit, or vice versa, then the problem disappears.

Let K and L have the same unit [*labor*]. Then, the unit of $\left[\alpha K^{r} + (1-\alpha)\, L^{r}\right]$ is given by [$labor^{r}$]. $\left[\alpha K^{r} + (1-\alpha)\, L^{r}\right]^{\frac{1}{r}}$ is given by unit [*labor*]. Thus productivity or technology factor A has consistent unit and dimension all the time: A is of dimensional unit [*good/labor*], regardless of r. Thus the problem disappears completely.

One may question that labor is defined in hours worked (time) and therefore the above resolution is flawed. However, this comes from great misunderstanding of L. The reason why we use time as reference measure is because in economic theory, with homogeneous labor assumption, the hours worked are proportional to the works labor do. Thus, if both labor and capital can be measured with single input unit measure, then dimensional problem disappears. The resolution still leaves out the question of what measure should be used to evaluate both capital and labor, and this will not be discussed here.

This implies that constant return-to-scale Cobb-Douglas production function $Y = AK^{\alpha}L^{1-\alpha}$ is also dimensionally consistent.

It is true that the resolution above leaves out the case of Cobb-Douglas function with increasing returns-to-scale or decreasing returns-to-scale. The use of permittivity factor κ, in form of $Y = A\kappa K^{\alpha}L^{\beta}$ to solve dimensional problem is not allowed, since then κ does not have fixed dimensional units. (κ's dimension changes when α and β change.) Thus, this implies that Cobb-Douglas function leaves out some details required to implement increasing returns-to-scale or decreasing returns-to-scale.

3. Utility functions and dimensional analysis

Now we look at some utility functions. First, we look at Cobb-Douglas utility function, as in Prescott (1986). The utility function is given by the following:

$$U(C,L) = \frac{\left[C^{1-\sigma}\left(1-L\right)^{\sigma}\right]^{1-\chi}}{1-\chi}$$

where C is consumption with unit [*consumption*] and σ, χ are dimensionless parameters. $1-L$ is also considered to be in unit [*labor*]. We first do naive dimensional analysis. Then U is in the unit [$consumption^{(1-\sigma)(1-\chi)} labor^{\sigma(1-\chi)}$]. Even if we make this unit convertible into some dimension [*utility*] by adding multiplicative permittivity factor κ to U, the fact that σ and χ are change-able, both in theory and reality (they are assumed to be constants in an economics model, but theoretically they can change) make conversions meaningless. After all κ will not have fixed dimension.
Then what would be a way to solve this problem? We instead relax assumption on the dimensional unit of U. For goods Y, we know that we can measure goods by the unit as kilogram or gram, if goods are all homogeneous. Same with labor input, assuming labor is homogeneous. However, for the unit measure of utility, it is unclear what the measure must be. It is not clearly volume, but it is not clearly area. How can fix the dimension of utility? We cannot. Thus, while C and L must have fixed dimensional units, U does not. We exploit this fact to solve dimensional problems in utility. While the above utility form can still be used, we modify it to illustrate our purpose.

$$U(C,L) = \frac{\left[\left(\kappa_1 C\right)^{1-\sigma}\left(\kappa_2\left(1-L\right)\right)^{\sigma}\right]^{1-\chi}}{1-\chi}$$

κ_1 and κ_2 convert unit of consumption goods and unit of leisure into initial utility basket, given by dimensional unit [*iutility/good*] and [*iutility/labor*]. Then utility U in general has form of [$iutility^{1-\chi}$].
Now, let us evaluate different utility function, as in Galí (2008).

$$U(C,L) = \mu\frac{C^{1-\sigma}}{1-\sigma} - \chi\frac{L^{1+\varphi}}{1+\varphi}$$

Convert into:

$$U(C,L) = \mu\frac{\left(\kappa_1 C\right)^{1-\sigma}}{1-\sigma} - \chi\frac{\left(\kappa_2 L\right)^{1+\varphi}}{1+\varphi}$$

Here, κ_1 has dimension $[iutility_K/good]$ and κ_2 has dimension $[iutility_L/labor]$. This suggests that $\frac{(\kappa_1 C)^{1-\sigma}}{1-\sigma}$ is of dimension $[ituility_K^{1-\sigma}]$, while $\frac{(\kappa_2 L)^{1+\varphi}}{1+\varphi}$ is of dimension $[ituility_L^{1+\varphi}]$. Let $\mu = 1$. Then, χ has dimension of $[iutility_K^{1-\sigma}/iutility_L^{1+\varphi}]$. It thus becomes clear from dimensional analysis that χ is not an meaningful economic parameter, but a utility comparator parameter. Thus, utility function can be converted to:

$$U(C, L) = \frac{(\kappa_1 C)^{1-\sigma}}{1-\sigma} - \frac{(\kappa_2 \chi_2 L)^{1+\varphi}}{1+\varphi}$$

where χ_2 is also used to change steady labor input in an economics model, as in Kim (2014), demonstrating that economic parameter σ and φ do not have any influence on steady labor input when initial technology factor sans stochastic elements is 1 and technology factor does not grow in long-run. Note that χ can also be interpreted as showing equivalence of 1 $[iutility_L^{1+\varphi}]$ with χ $[iutility_K^{1-\sigma}]$. Because utility is not dimensionally fixed, this equivalence is allowed, even though φ and σ are not guaranteed to be fixed.

4. Conclusion

This paper addresses concerns of Barnett (2003) regarding dimensional problems of production and utility functions. While concerns remain regarding consistent and uniform measure for both amount of capital and labor, increasing and decreasing returns-to-scale and messy nature of utility measurement, it is nevertheless not much as serious as Barnett (2003) seems to show.

References

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