On the Interaction between Player Heterogeneity and Partner Heterogeneity in Strict Nash Networks

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10 January 2015

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Current Version : December 2014*

Abstract

This paper brings together analyses of Strict Nash networks under exclusive player heterogeneity assumption and exclusive partner heterogeneity assumption. This is achieved through examining how the interactions between these two assumptions influence important properties of Strict Nash networks. Built upon the findings of Billand et al (2011) and Galleotti et al (2006), which assume exclusive partner heterogeneity and exclusive player heterogeneity respectively, I provide a proposition that generalizes the results of these two models by stating that: (i) Strict Nash network consists of multiple non-empty components as in Galleotti et al (2006), and (ii) each non-empty component is a branching or B_i network as in Billand et al (2011). This proposition requires that a certain restriction on link formation cost (called Uniform Partner Rankng), which encloses exclusive partner heterogeneity and exclusive player heterogeneity as a specific case, is satisfied. In addition, this paper shows that value heterogeneity plays a relatively less important role in changing the shapes of Strict Nash networks.

JEL Classification : C72, D85
Keywords : Strict Nash Network, Two-way Flow Network, Information Network, Branching Network, Agent Heterogeneity

1 Introduction

The seminal work of Bala and Goyal (2000), BG henceforth, predicts that in Strict Nash equilibrium two-way flow non-empty network is a center-sponsored star [1]. Such simple

*First draft : December 2014

[1] In BG, there are several versions of Strict Nash networks. The version that is used in this paper and the mentioned literature is the two-way flow model with no decay. That is, information flows to an agent

form of network emerges as a unique Strict Nash network because several simple assumptions are adopted, including agent homogeneity. Naturally, this simplicity has spawned a vast literature that studies properties of Strict Nash networks under the assumption of agent heterogeneity. A strand of this literature assumes that heterogeneity resides in the diversity of link formation cost and value of information that each agent possesses. This strand of literature can be further divided into two sub-categories. One assumes that the diversity of link formation cost and value of information depends exclusively on the identity of link receiver (called exclusive partner hetoregeneity onwards). The other one assumes that such diversity depends exclusively on the identity of link sender (called exclusive player hetoregeneity onwards). However, little is known when such *one-way heterogeneity* is relaxed. This paper aims to fill in this space in the literature by (i) relaxing the one-way heterogeneity assumption and, (ii) generalizing the results on properties of Strict Nash networks found in these two sub-categories of existing literature.

I briefly give an overview of the literature here. For exclusive player heterogeneity, existence of Nash network and full equilibrium characterization of Strict Nash networks (SNNs henceforth) are extensively studied by Galleotti et al (2006), Kamphorst and Laan (2007) and Billand et al (2010). For exclusive partner heterogeneity, full equilibrium characterization and the existence of SNNs are extensively studied by Billand et al (2011) and Billand et al (2012). However, when two heterogeneities are allowed to interact (called two-way heterogeneity henceforth) to my knowledge little is known. We know from Galleotti et al (2006) that SNN under two-way heterogeneity is minimal, and from Haller et al (2007) that it does not always exist. My attempt to understand the shapes of Strict Nash networks under the assumption of two-way heterogeneity is thus a major contribution of this paper to the literature.

By having the goal of bridging the two sub-categories of literature in mind, this paper provides two propositions that show that *the main properties of SNNs found in the models that assume one-way heterogeneity can be preserved* even when two-way heterogeneity is assumed. I elaborate on these main properties as follows:

1. SNN is a center-sponsored star. This property is found in the original model of BG.

2. SNN is a disconnected network, consisting of many center-sponsored star. This property is found in the exclusive player heterogeneity model of Galleotti et al (2006).

3. SNN has a unique component that is a branching or B_{i_0}, where i_0 is the lowest cost agent. This property is found in the exclusive partner heterogeneity model of Billand et al (2011).

Consequently, the literature confirms that (1) exclusive player heterogeneity *cannot* alter the shape of SNN, yet it splits the connected SNN in BG into many components,

through a link even if he does not sponsor it. Moreover information does not decay even if the path of information flow contains multiple links and (2) conversely, exclusive partner heterogeneity *cannot* increase the quantity of components in SNN, yet it can alter the shape and increase the set of SNNs. This raises the question of whether both conclusions hold true when the two heterogeneities are allowed to interact. The result of Proposition 1 in this paper confirms that this is indeed the case, so long as link formation cost satisfies a certain restriction called *Uniform Partner Ranking*. I remark that exclusive partner heterogeneity and exclusive player heterogeneity satisfy this restriction. Consequently my propositions generalize the Proposition 3.1 of Galleotti et al (2006) and Proposition 1 of Billand et al (2011).

To further elaborate on this result, I remark that Uniform Partner Ranking, UPR henceforth, is a sufficient but not necessary condition to predict that every non-empty component of SNN is a branching or B_i network. Indeed, what warrants a non-empty component of SNN being a branching or B_i is the existence of a *common best partner;* an agent that generates a lowest link formation cost to every other agent in the component if chosen as a partner. This fact is formally stated as Lemma 4, which becomes the most important building block of the two propositions in this paper. In Proposition 1, UPR is simply a restriction on link formation cost that guarantees that a common best partner exists in every component of SNN. This in turn guarantees that every component of SNN is a branching or B_i network. I remark that this result is striking, since it shows that the shape of a component in SNN can be predicted by having a condition (existence of Common Best Partner in this case) that is much weaker than most of the conditions in the existing literature such as exclusive partner heterogeneity and exclusive player heterogeneity.

Following these findings, this paper seeks to establish Proposition 2 in order to further eliminate a major difference between Proposition 1 mentioned above and Proposition 1 in Billand et al (2011). This difference is the fact that Uniform Partner Ranking predicts that non-empty components of SNN are multiple while exclusive partner heterogeneity predicts that the non-empty component of SNN is unique. To eliminate this difference this paper proposes a stronger restriction on the structure of link formation cost called Single Community condition. Using this condition, I establish Proposition 2, which states that under Single Community condition SNN contains at most one non-empty component that is a branching or B_i network in which where i is i_0, the agent that gives rise to the lowest link formation cost. I remark further that exclusive partner heterogeneity does satisfy Single Community condition so that Proposition 2 is also generalization Proposition 1 in Billand et al (2011).

The paper proceeds as follows. The model and relevant notations are described in Section 2. In Section 3 I introduce a lemma that shows that if a common best partner exists then a component of SNN is a branching or B_i network. In Section 4 these lemmas are put in use to establish Proposition 1 and 2, which are the main results of this paper. Finally, Section 5 concludes.

2 The Model

Let $N = \{1, ..., n\}$ be a set of agents and let i and j be typical members of this set. Each agent possesses a nonrival distinct piece of information that is valuable both to himself and anyone who has an entry to it. There are two ways to which a pair of agents can have an entry to each other's information: there is a pairwise link between i and j or there is a series of links where the two ends are i and j.

Link establishment and individual's strategy. Link establishment is costly and one-sided. A strategy of i is $g_i = \{g_{i,j} : j \in N, j \neq i\}$, where $g_{i,j} = 1$ if i establishes a link with j and $g_{i,j} = 0$ otherwise. If $g_{i,j} = 1$, I say that *i accesses j* or j receives the link from i. Since all links form the network, I write $g = \{g_i : i \in N\}$ to represent both a strategy profile and a network.

Network representation. In this paper a node depicts an agent, and an arrow from j to i represents that j receives a link from i. If all arrows are removed, the network merely represents how information flows among agents. This structure of information flow is denoted by $\bar{g} = \{\bar{g}_{i,j} : i, j \in N, i \neq j\}$, where $\bar{g}_{i,j} = 1$ if $g_{i,j} = 1$ or $g_{j,i} = 1$ or both, and $\bar{g}_{i,j} = 0$ otherwise.

Information flow. Information of j flows to i directly through a link between i and j, regardless to who sponsors it. Alternatively, Information of j can also flow to i through a *chain*. Formally, a chain between i and j ($i \neq j$) is a sequence $j_0, ..., j_m$ such that $\bar{g}_{j_l, j_{l+1}} = 1$ for $l = 0, ..., m-1$ and $j_0 = i$ and $j_m = j$. In this case, I say that *i observes j* and vice versa.

Costs and benefits If i accesses j, then i pays $c_{i,j}$. If i observes j, he finds that the information of j has the value equal to $V_{i,j}$ [2].

Cost heterogeneity Let $C = \{c_{i,j} : i, j \in N, i \neq j\}$ be a *cost structure*, C is said to assume cost homogeneity if $c_{i,j} = c$ for all $c_{i,j} \in C$, and cost heterogeneity if it holds true that $c_{i,j} \neq c_{k,l}$ for some $c_{i,j}, c_{k,l} \in C$. Cost heterogeneity can be further classified as follows. If $c_{i,j} = c_i$ for all i ($c_{i,j} = c_j$ for all j), C is said to assume exclusive player (partner) heterogeneity. Finally, if $c_{i,j} \neq c_i$ for some i and $c_{i,j} \neq c_j$ for some j, C is said to assume two-way heterogeneity.

The payoffs. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be such that $\pi(x, y)$ is strictly increasing in x and strictly decreasing in y. The payoff of player i is given by:

$$\pi_i(g) = \pi\left(\sum_{j \in N_i(\mathbf{g}) \setminus \{i\}} V_{ij}, \sum_{j \in N_i(\mathbf{g}) \setminus \{i\}} \mathbf{g_{i,j}} c_{ij}\right) \tag{1a}$$

where $N_i(\mathbf{g}) \setminus \{i\}$ is the set of all agents that i observes.

Network-related Notations. Recall from the above that a chain from i to j is a sequence of distinct players $j_0, ..., j_m$ such that $\bar{g}_{j_l, j_{l+1}} = 1$ for $l = 0, ..., m-1$ and $j_0 = i$ and $j_m = j$, a path is defined similarly except that link sponsorship matters. A path from j to

[2] In the model of BG, $V_{i,i} = V_{i,j} = K$, where K is a constant. However, in this paper $V_{i,j}$ is left undefined since our concern is on the appearance of SNN and not on who receives a higher payoff than who. Consequently I do not include $V_{i,i}$ in the payoff function. I remark that my reasoning here is in line with that of Billand et al (2011)

i is a sequence $j_0, ..., j_m$ such that $g_{j_l, j_{l+1}} = 1$ for $l = 0, ..., m-1$ and $j_0 = i$ and $j_m = j$. A cycle is defined in the same fashion as a chain, except that $j_0 = i$ and $j_m = i$ and all other players in the sequence are distinct. I use these notations to define the following terms. A network is connected if there is a chain for every distinct $i, j \in N$. A subnetwork of g is a network g' such that $g' \subset g$. A component of g is a maximal connected subgraph of g. A component is denoted by $D(g)$. A component is said to be minimal if it contains no cycle. In a minimal component, every distinct pair of agents is connected by a chain so that a removal of a link $g_{i,j}$ splits the component into two components - one containing i and the other one containing j. I denote these two modified components by $D^i(g_{i,j})$ and $D^j(g_{i,j})$ respectively.

Consider an agent i, $I_i(g)$ and $O_i(g)$ are defined as the set of all links of i that are not sponsored by i and the set of all links that i establishes respectively [3]. If $I_i(g) = O_i(g) = 0$, then i is said to be an isolated agent. If every agent in a network is isolated, then the network is an empty network. If either $I_i(g) = 1$ or $O_i(g) = 1$ (but not both), then i is said to be a terminal agent.

Some important patterns of networks. There are some patterns of networks that are often referred to, since they emerge as Strict Nash Equilibria. I define them here[4]. For a generic set $X \subset N$, let $Q_X = X \cup \{j|$ there exists a path from j to i for every $i \in X\}$. I say that X is a contrabasis of a network g if it is a minimal set with respect to the property that $Q_X = N$. X is said to be an i-point contrabasis if every $j \in X$ accesses i. Furthermore, if i is a point contrabasis of g and $|I_i| \geq 2$ but $|I_j| < 2$ for all $j \neq i$ and $j \in N$, then g is said to be a B_i network. Observe that in B_i network i is the only agent that receives more than one link [5]. Conversely, if in a network there is no agent that receives more than one link and there is a unique agent i that receives no link, then the network is called branching network rooted at i.

Strict Nash Equilibrium. Let g_{-i} denote a strategy profile of all agents except i, ie., $g_i \cup g_{-i} = g$. A best response of an agent i is g_i such that $\pi_i(g_i \cup g_{-i}) \geq \pi_i(g'_i \cup g_{-i})$ for every g'_i that is a strategy of i. A strategy profile or a network g is Nash if every agent plays his best response. A Nash network is a Strict Nash network if the best response of every agent is unique.

2.1 Cost Structure - Single Community and Uniform Partner Ranking

In the main results section, two propositions that fully characterize the shapes of SNN are proven, given that SNN satisfies certain restrictions. Here I introduce these restrictions - *uniform partner ranking* and *single community*. Recall that the cost structure C is defined as $C = \{c_{ij}\}_{i,j \in N, i \neq j}$

Definition 1 (Better Partner). *Consider a set $X \subset N$ and agents $j, k \in X$, j is at least as good a partner as k with respect to the set X if $c_{i,j} \leq c_{i,k}$ for any $i \in X, i \neq j \neq k$. Moreover, if the*

[3] I for incoming and O for outgoing

[4] The definitions of B_i network and branching network are borrowed from Billand et al (2011)

[5] Intuitively, B_i is a network such that every agent in the network can be reached through a path to an agent that accesses i.

inequality is strict then j is said to be a better partner than k with respect to the set X.

Intuitively, this definition states that if we choose a set $X \subset N$ and $i \in X$, then i can 'rank' all agents in this set except himself according to link formation cost that he potentially has to bear, and such ranking is universal regardless to the identity of agent $i \in X$. The definition Uniform Partner Ranking below simply adds that the set $X = N$

Definition 2 (Uniform Partner Ranking)**.** *A cost structure C is said to satisfy Uniform Partner Ranking property if for any distinct pair $j, k \in N$ it holds true that j is at least as good a partner as k or k is at least as a good a partner as j with respect to the set N.*

Remarks 1. *In what follows, this paper lets the set N be permuted such that the permuted set $I(N) = \{i_0, i_1, ..., i_{n-1}\}$ is such that i_x is at least as good a partner as i_y with respect to the set $I(N)$ for $x < y$.*

The example below shows that exclusive partner heterogeneity as in Billand et al (2011)) also satisfies the Uniform Partner Ranking.

Example 1. *Let $c_{ij} = c_j$ (ie., we assume exclusive partner heterogeneity as in Billand et al (2011)). Specifically, let $N = \{1, .., 5\}$ and $C = \{c_1 = 5, c_2 = 4, c_3 = 3, c_4 = 2, c_5 = 1\}$, then clearly C satisfies Uniform Partner Ranking Property.*

It is important to note that if C satisfies the Uniform Partner Ranking Condition, then the agent i_0 can be considered as a *common best partner* among the set of agents N in the sense that every agent (except i_0) agrees that i_0 is the partner that incurs the lowest link formation cost. In more formal terms,

Definition 3. *Let $X \subset N$ be a set of agents, then $i^* \in X$ is said to be a Common Best Partner among all agents in X if $c_{ii^*} \leq c_{ij}$ for all $i, j \in X$ and $i \neq j \neq i^*$.*

In the same manner as the term "better partner" and "at least as good a partner" are defined, I define the term "better player" as follows.

Definition 4 (Better Player)**.** *Consider a set $X \subset N$ and agents $i, j \in X$, i is at least as good a player as j with respect to the set X if $c_{i,k} \leq c_{j,k}$ for any $k \in X, i \neq j \neq k$. Moreover, if the inequality is strict then i is said to be a better player than j with respect to the set X.*

Having defined all these terms, I define a restriction that is stronger than Uniform Partner Ranking as follows.

Definition 5 (Single Community Condition)**.** *A cost structure C is said to satisfy Single Community Condition if the set N can be permuted into the set $I(N) = \{i_0, i_1, ..., i_{n-1}\}$ such that i_x is at least as good a partner as i_y if and only if i_x is at least as good a player as i_y with respect to the set $I(N)$ for any $x < y$.*

Example 2. *Let $c_{ij} = c_i$ (ie., we assume exclusive player heterogeneity as in Billand et al (2011)). Specifically, let $N = \{1, .., 5\}$ and $C = \{c_1 = 5, c_2 = 4, c_3 = 3, c_4 = 2, c_5 = 1\}$, then we have that C satisfies Single Community Property.*

I note how Single Community Condition (SCC onwards) may illustrate a realism. Intuitively, SCC states that if an agent i happens to pay a lower link formation cost than j regardless to the identity of the agent who behaves as a partner, then every other agent also prefers to form a link with i than with j in the sense that i incurs a lower information cost than j. If we think of link formation cost as the physical efforts of an agent in communicating, and we assume further that this effort depends on language skills. Then if i has better language skills than j, it is natural to assume that i spends less effort contacting other agents than j, and other agents also are also likely to prefer contacting i than j because they know that communication with i is likely to be smoother.

3 Useful Lemmas

In this section I build up several lemmas that are used to prove the main results. The primary goal is to prove that if a common best partner exists in a component of SNN, then this component is a branching or B_{i^*}, where i^* is a common best partner. This fact is further used to prove Proposition 1 and 2 in the next section.

The first lemma guarantees that SNN is minimal, a result that is prevalent in the literature.

Lemma 1 (SNN is minimal)**.** *A component in an SNN is minimal.*

Proof of Lemma 1. Suppose not. Consider a cycle in a non-minimal component, observe that all agents in it have at least two chains through which they observe one another. In this cycle, consider an agent who establishes at least one link. If he removes the link, the component remains unbroken so that he still observes all other agents in the component. Thus, he is better off removing the link in order to reduce his link formation cost, a contradiction. □

Next, Lemma 2 and 3 show that in a component of SNN a common best partner is unique, if it exists. Recall that $D^j\left(g_{i,j}\right)$ is defined as a split component that contains j if the link $g_{i,j}$ is removed.

Lemma 2 (The existence of individual's best partner)**.** *In an SNN, if i accesses j, then* $c_{i,j} < c_{i,k}$ *for any agent k that is contained in* $D^j\left(g_{i,j}\right)$ *and* $k \neq j$.

The proof is trivial and is omitted. Intuitively, if i decides to access an agent in $D^j\left(g_{i,j}\right)$, then he chooses an agent that incurs the lowest link formation cost. The fact that our equilibrium prediction criterion is SNN further necessitates that this agent is unique and the above inequality is strict. In the proof of the lemma below, the agent j in $D^j\left(g_{i,j}\right)$ is called unique best partner of i.

Lemma 3 (Existence of Unique Common Best Partner)**.** *In a non-empty component of SNN, if a common best partner among all agents in the component exists, then this component contains at most one agent that receives more than one link. Moreover, this agent is a unique common best partner among all agents in the component.*

Proof. Let k be an agent that receives more than one link. Let j_1 be an agent that accesses k. By Lemma 2 we know that k is the best partner of j_1 in $D^k\left(g_{j_1,k}\right)$. Let j_2 be another agent that accesses k. By Lemma 2 we know that k is also the best partner of j_2 in $D^k\left(g_{j_2,k}\right)$. Observe that the union of $D^k\left(g_{j_2,k}\right)$ and $D^k\left(g_{j_1,k}\right)$ contains all agents in the non-empty component. Thus, k is a common best partner among the set of all agents in the component.

I now prove that k is a unique common best partner among all agents in the component. Suppose not, let k' be another common best partner. Without loss of generality let us assume that k' is contained in $D^k\left(g_{j_2,k}\right)$. Consequently j_2 is indifferent between accessing k and k' so that this network is not SNN, a contradiction.

Finally, I prove that k is the only agent that receives more than one link. Suppose not, let k' be another agent that receives more than one link. Then from the proof above we know that this agent is a common best partner. However, we have proved earlier that k is a unique common best partner, a contradiction. □

Finally, in what follows I introduce the main lemma of this paper. It characterizes the shape of a non-empty component in SNN given that a common best partner exists.

Lemma 4 (The Prediction that SNN component is either B_{i^*} or branching, through the existence of Common Best Partner)**.** *A non-empty component in SNN is a branching or B_{i^*}, given that a common best partner (denoted by i^*) exists.*

Proof. By Lemma 3, we know that a component of SNN has at most one agent that receives more than one link. Consequently to complete the prove it suffices to show that 1) if a component contains no agent that receives more than one link then this component is branching and, 2) if a component contains exactly one agent that receives more than one link then this component is a B_{i^*} network.

I prove the first part by contradiction. Suppose that the component is not a branching. Recall that branching is defined as a network such that there is exactly one agent that receives no link and all other agents receive exactly one link. Consequently if the network is not a branching there are two cases: (1) the network is such that every agent receives exactly one link and, (2) the network has more than one agent who receive no link and every other agent receives exactly one link.

For the first case, consider a terminal agent i. By our presupposition he receives a link from an agent. Let this agent be $j+1$. Observe that $j+1$ is not a terminal agent, because he also receives a link from another agent. Let this agent be $j+2$. Again, he is not a terminal agent for the same reason. Thus, this algorithmic procedure continues infinitely. It follows that this network has infinite amount of agents, a contradiction.

For the second case, consider agents x and y who receive no link. Since x and y are in the same component there is a chain between x and y. Let the sequence of agents in this chain be $x, j_1, j_2, j_3, j_4, ..., j_K, y$ respectively. Since x is assumed to receive no links, it is the case that that x accesses j_1. Since it is assumed from the beginning the that every agent receives at most one link, it is the case that j_1 accesses j_2, j_2 accesses j_3 and so on. This follows that j_K accesses y, a contradiction to the assumption that y receives no link.

I now prove the second subsections - if a non-empty component of SNN contains an agent that receives more than one link then it is a B_{i_0} network. By Lemma 2, we know that in such a component there is only one agent that receives more than one link, and this agent is i^*, a common best partner. Therefore what remains to prove is that i^* is a point contrabasis of this component. This in turn requires that I prove that: (i) for each agent l in this component there exists a path from l to j for some $j \in I_{i^*}(g)$ and, (ii) if $j \in I_{i^*}(g)$ accesses $k \neq i^*$ then there is no path from k to $j' \in I_{i^*}(g)$ and $j' \neq j$ so that $I_{i^*} \backslash \{j\}$ is not a contrabasis.

To prove that for each l in this component there exists a path from l to j for some $j \in I_{i^*}(g)$, I introduce the following notations. j' is such that $j' \in O_{i^*}$. k is such that $g_{j',k} = 1$ or $g_{j,k} = 1$. In what follows I consider the following 4 cases depending on the identity of l: (a) $l \neq i^*$ and $l \in D^k(g_{j,k})$, (b) $l \neq i^*$ and $l \in D^k(g_{j',k})$, (c) $l \neq i^*$ and $l \notin D^k(g_{j,k}), D^k(g_{j',k})$ and, (d) $l = i^*$.

For case (a), observe that without $g_{j,k}$ the split component $D^k(g_{j,k})$ is a branching rooted at k since k receives no link and every other agent receives exactly one link. Consequently for any agent $l \in D^k(g_{j,k})$ for some $j \in I_{i^*}$ such that $g_{j,k} = 1$ there is path from l to j (via k if $l \neq k$).

Next, I consider case (b). Similar to case (a), observe that without $g_{j',k}$ the split component $D^k(g_{j',k})$ is a branching rooted at k since k receives no link and every other agent receives exactly one link. Consequently for any such agent l there is a path from l to i^* (via j'). This in turn guarantees that there is a path from l to any $j \in I_{i^*}$ (via j' and i^*).

For case (c), observe that since $l \neq i^*$, $l \neq j$ and $l \notin D^k(g_{j,k}), D^k(g_{j',k})$ it is the case that $l = j'$ for some $j' \in O_{i^*}$. This in turn guarantees that there exists a path from l to j (via i^*).

For case (d), since $l = i^*$ the path from i^* to an agent j is the link from i^* to j.

Finally, I prove that if $j \in I_{i^*}(g)$ accesses $k \neq i^*$ then there is no path from k to $j'' \in I_{i^*}(g)$ for $j'' \neq j$ so that $I_{i^*} \backslash \{j\}$ is not a contrabasis. Suppose by contradiction that there is path from k to j''. Then k receives more than one link so that $k = i^*$, a contradiction. □

4 Main Result - Equilibrium Characterization

In Lemma 4, the existence of a common best partner in a component of SNN guarantees that the component is a branching or B_{i^*}. Proposition 1 below makes use of this lemma in the following way. It imposes UPR to guarantee every component contains a common best partner. This in turn guarantees that every component of SNN is a branching or B_{i^*}

Proposition 1. *Let C satisfy Uniform Partner Ranking Condition and $V_{i,j}$ flow freely, then every non-empty component is a branching or B_{i^*}.*

Proof. Since UPR is satisfied, we know that all agents can be permuted $\{i_0, i_1, ..., i_{n-1}\}$ so that i_x is at least as good a partner as i_y with respect to the set N for $x < y$. Consequently in a non-empty component of SNN there exists i^{x^*} such that $x^* \leq y$ for any i_y that is in the same component. Naturally i^{x^*} is i^*, a Common Best Partner in the component. This fact, which guarantees that every non-empty component of SNN has a Common Best Partner, together with Lemma 4 guarantee that every non-empty component is a branching or B_{i^*}. □

I remark that Proposition 1 in this paper can be considered as a generalization of Proposition 1 in Billand et al (2011) and Proposition 3.1 in Galleotti et al (2006). This is because both exclusive player heterogeneity and exclusive partner heterogeneity in link formation cost satisfy Uniform Partner Ranking condition. A comparison with Proposition 1 of Billand et al (2011) is noteworthy. Specifically, a major similarity between the two propositions is that a non-empty component is a branching or B_i network, and a major difference is that Proposition 1 in Billand et al (2011) predicts that a non-empty component is unique, while the Proposition 1 in this paper predicts that SNN can contain multiple non-empty components. Proposition 2 below aims at eliminating this difference by imposing a stronger restriction on the cost structure. I remark that despite the fact that the restriction is stronger, Proposition 2 remains a generalization of Proposition 1 in Billand et al (2011).

Proposition 2 (SNN with value homogeneity, a single community)**.** *Let C satisfy Single Community Condition and $V_{i,j} = V_j$. Then a non-empty Strict Nash network is a minimal network that has a unique non-empty component that is a B_{i_0} or a branching.*

Proof. Since if C satisfies SCC it also satisfies UPR. It holds true that every non-empty component is a B_{i^*} or branching. Consequently what remains to be proven is that the non-empty component is unique, and that $i^* = i_0$

To prove that a non-empty component is unique, I suppose not. Let i_A and j_A be in a component, and i_B and j_B in another component. Assume further that i_A accesses j_A and i_B accesses j_B and, without loss of generality, that i_A is at least as friendly as i_B. This entails that i_A is at least as good a player as i_B so that $c_{i_A,j_B} \leq c_{i_B,j_B}$. Since $V_{ij} = V_j$, the value of information that i_A receives if he accesses j_B is at least equal to the value of information that i_B receives from accessing j_B. This fact, together with the fact that $c_{i_A,j_B} \leq c_{i_B,j_B}$, leads to the conclusion that i_A has strictly positive deviation by accessing j_B if the network is SNN, a contradiction.

To prove that $i^* = i_0$, I first prove that i_0 is not in a non-empty component. To do so suppose by contradiction that i_0 is in a non-empty component. Suppose further that i and j are agents in the non-empty component and that i accesses j0. Since $c_{i_0,j} \leq c_{i,j}$ for any i, j in the network, it follows that i_0 has a positive deviation by accessing j, a contradiction.

Finally, since i_0 is in the non-empty component it follows by Lemma 3 that $i_0 = i^*$. □

I adjourn this section by comparing between Proposition 2 in this paper and Proposition 1 in Billand et al (2011). While both predict that SNN has a unique non-empty component that is a branching or B_i network, they differ in that Proposition 1 in Billand et al (2011) further predicts that the unique non-empty contains all agents in the network so that a non-empty component does not existss. I remark that this difference derives from the fact that UPR allows for the cost structure such that $c_{i,j} \neq c_{k,j}$ for $k \neq j$ while exclusive partner heterogeneity does not. Intuitively, if i finds that accessing j is profitable, then exclusive partner heterogeneity predicts that an isolated agent k finds likewise since

$c_{i,j} = c_{k,j} = c_j$. However, under UPR this reasoning does not apply. Indeed, it can be the case that $c_{j,k}$ and $c_{k,j}$ are both sufficiently large so that both find that accessing one another is not profitable.

5 Concluding Remarks

In this paper, I provide two propositions that aim to understand the interaction between player heterogeneity and partner heterogeneity, and how such interaction influences the properties of SNN. The main conclusions are:

1. Even if $c_{i,j} \neq c_i$ and $c_{i,j} \neq c_j$ so that two-way heterogeneity is assumed, a non-empty component of SNN is a branching or B_i so long as all agents in a component of SNN agree on who the link receiver is that incurs the lowest link formation cost. Consequently in this paper the prediction of the shape of a non-empty component in SNN is similar to that of Billand et al (2011), which assumes exclusive partner heterogeneity. This conclusion is formally stated as Lemma 4.

2. As a result of the conclusion above, if all agents in the network agree on which agent is at least as good a partner (as measured by a lower link formation cost) than which then it can be concluded that every non-empty component in this SNN is a branching or B_i network. This restriction is called Uniform Partner Ranking, and the prediction of SNN is formally stated as Proposition 1 in this paper.

3. Finally, it can be said that value heterogeneity does not predict the shape of each component in SNN. Indeed, when an agent *i* decides whether to form a link in order to access a component, he weights the benefits of accessing this component against his link formation cost with the lowest-cost partner in this component. Therefore it can be concluded that value heterogeneity does not alter his choice of partner.

Naturally, a question that remains is how we can predict the shape and properties of SNN in the absence of Uniform Partner Ranking. This becomes a potential research question to explore.

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