A Ponzi scheme exposed to volatile markets

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Abstract. The PGBM model for a couple of counteracting, exponentially growing capital flows is presented: the available capital stock $X(t)$ evolves according to a variant of inhomogeneous geometric Brownian motion (GBM) with time-dependent drift, in particular, to the stochastic differential equation $dX(t) = [pX(t) + \rho_1 \exp(q_1 t) + \rho_2 \exp(q_2 t)]dt + \sigma X(t)dW(t)$, where $W(t)$ is a Wiener process. As a paragon, we study a continuous-time model for a nine-parameter Ponzi scheme with an exponentially growing number of investors. Investors either maintain their investment or withdraw it after some fixed investment span and quit the system. The first two moments of the process and hence a closed-form solution for the mean path are given. The capital stock exhibits a dynamic lognormal probability distribution as long as the system remains solvent. The assumed stochastic performance allows for earlier or later collaps of the investment system as compared to the deterministic analogy ($\sigma = 0$). Allowing also for negative capital values the system's default probability can be calculated at any time by numerically solving the corresponding Kolmogorov forward equation. We use the finite difference method and obtain results in accordance with those of simple Monte-Carlo simulations. Finally, a minor modification of the payout function provides a toy model for a social security system exhibiting critical behaviour. Depending on whether some parameter value violates a weak no-Ponzi game condition or not, the system represents either a non-lasting Ponzi game or a lasting no-Ponzi game in the weak sense.

1 Introduction

"[N]obody knew with the least precision what Mr. Merdle's business was, except that it was to coin money" (Dickens 1857). The career of the admired but devious financier Merdle in Charles Dickens novel *Little Dorrit* culminates as that of "the man of this time" with a "world-wide repute"; but it takes a fatal end when he is recognised as a "master of humbug" and as "the greatest Thief" because his shadowy investment system defaults leaving many investors ruined and at least one bank destroyed. The attentive reader may remember busting contemporary Ponzi schemes well-covered by the media. A Ponzi scheme is a scam investment system that offers seasoned investors attractive yields that are paid exclusively or to a large extent from the money deposited by subsequent investors; one may speak of a Ponzi system in its *narrow* or its *broad* sense, respectively. Such systems fail as soon as new deposits plus some eventual additional earnings (due to compound interests, renewed membership fees or taxes, monetary gifts by donators, dividend yields or any successful asset management) cannot match payments anymore. This we call a Ponzi system in its *strong* sense. However, if this total external cash inflow is large enough with respect to an accordingly modest promised return on investment, there's no cash-flow deficit, and the investment system will not crash as long as there is a non-empty reservoir of new investors. We call this a *weak* Ponzi system. This article covers all of these modes of Ponzi systems, the emphasis being placed on weak Ponzi systems in the broad sense.

Two early historical examples of documented Ponzi schemes are Adele Spitzeder's "Dachauer Bank" that operated a couple of years until 1872 in Germany, and William Miller's "Franklin Syndicate" that blew up in 1899 in the USA. Carlo "Charles" Ponzi's spectacular scam investment system was run during the early 1920ies in Boston and became the nomenclature for so-called Ponzi games. Most prominent is Bernard Madoff's New York based "Investment Securities LLC" and its main feeder fund "Fairfield Sentry Ldt" that operated nearly two decades until its collaps

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in 2008. Contemporary Ponzi schemes are observed to flush up world-wide in high number and with many faces. Even pay-as-you-go social security systems are recognized to show Ponzi-game like aspects. A widely cited and seemingly paradoxical statement by Paul Samuelson was that "social security is a Ponzi scheme that works" (Tanner 2011); the formal and quantitative results of the present paper may contribute to clarify the ambivalence of the situation.

The underlying mechanism does per se correspond to a fraudulent investment system due to some necessary limit of participants; however, ignoring this constraint the violation of a weak no-Ponzi game condition (wNPC) due to unfavourable parameter value relationships makes the distinction between a strong Ponzi game that will collapse and a sustained weak Ponzi game. Usually, the investment system is dooming due to the promise of an unrealistically high rate of return that attracts investors, or due to an inappropriate growth rate of new participants, or due to an excessive average participation time, et cetera. Contrariwise, if operated as weak Ponzi games investment systems as modelled in the present paper may (seemingly) have a sustained financial future.

In recent years, several mathematical models for Ponzi schemes have appeared. All of them basically rely on some sort of dynamic budget equation that allows the available capital to grow due to a nominal interest rate and, most characteristically, under the competing effect of inflowing and outflowing money, that is, due typically to deposits of new participants and due to returns on investment or withdrawals, respectively. Artzrouni's (2009) deterministic approach is based on an ordinary first-order linear differential equation (ODE). The model incorporates a withdrawal function that implies exponential decay and allows for an analytical solution. Because the solution consists of the sum of three exponential functions, i.e., having the structure $a\exp(\alpha t) + b\exp(\beta t) + c\exp(\gamma t)$ (where a, b, and c are coefficients that depend on the model parameters, as do the exponential scaling factors α, β, and γ), such a result is called in Parodi (2013) an *abc*-model solution. Cunha, Valente & Vasconcelos (2013) solve Artzrouni's ODE numerically by means of a higher-order Runge-Kutta algorithm that is available within MATLAB. Such a purely numerical approach allows for studying the dynamic effects of sudden parameter changes immediately. The authors perform a rough sensitivity analysis concerning the long-term solvency of Artzrouni's Ponzi scheme and find that sustainability depends on the initial conditions. A very similar result is obtained in Parodi (2013) for a discrete-time *abc*-model of a Ponzi system. The model allows for sudden withdrawals after fixed investment spans when saturated investors quit the system. On the one hand the model relies on difference equations and therefore is well-suited for spreadsheet applications and thus for studying dynamic effects as mentioned above. On the other hand, being an *abc*-model, it has an analytical solution that leads to the formal derivation of weak no-Ponzi-game conditions (wNPC) for four model variants (which differ in their payout functions and thus correspond to a variety of Ponzi schemes). A model represents a doomed weak Ponzi system only if the wNPC is not satisfied; otherwise the system remains solvent and, assuming enough potential participants, sustainable. Mayorga-Zambrano (2011) and Quituisaca-Samaniego, Mayorga-Zambrano & Medina (2013) provide a stochastic simulation of Ponzi pyramidal schemes, based on a set of recursion relations. Stochasticity enters by means of three fundamental parameters that obey normal distributions, namely, the (epidemic) growth factor for new members, the conversion rate, and the nominal interest rate governing the instantaneous growth of the invested capital. Real world applications are given. The present paper takes up the conceptual ideas of Parodi (2013), but now the model is formally based on a linear first-order stochastic differential equation (SDE) where the capital evolves according to geometric Brownian motion (GBM) with inhomogenous drift. This not only allows for an analytical *abc*-model solution with respect to the mean path of capital evolution, but also for a closed-form representation of the probability distribution function for the capital (at least under some restrictions). As expected, the stochastic market-based interest rate produces (quantifiable) uncertainty about the exact point of capital saturation, i.e., about the system's time of collapse.

The paper is organized as follows. Section 2 provides the mathematical setting by means of introducing the PGBM model, i.e., a principal GBM process with a particular inhomogenous drift. This includes the definition of the stochastic process, the derivation of the corresponding first and second moments, and the presentation of the probability distribution together with its evolution equation. In Section 3 a Ponzi scheme is designed and modelled within the PGBM framework, including the formulation of some weak no-Ponzi game condition (wNPC). Section 4 provides the computation of default probabilities by numerically solving the Kolmogorov forward equation (KfE) as well as by performing Monte-Carlo (MC) simulations. Section 5 is devoted to a toy model social security system that is based on a variant of the paragon model used in Section 3; the system's critical behaviour is exemplified. Finally, in section 6, some conclusions that point out pathways for future work are drawn.

2 Principal geometric Brownian motion

2.1 Stochastic capital dynamics

In the continuous-time model to be presented the capital stock $X(t)$ evolves according to an infinitesimal budget equation

$$dX(t) = \Big[pX(t) + \dot{X}_{\text{IN}}(t) - \dot{X}_{\text{OUT}}(t)\Big]\; dt + \sigma X(t)\, dW(t) \tag{1}$$

with exponentially growing earnings and spendings. The first term reflects the infinitesimal net capital increase (including a market-based nominal interest or performance rate p) and the second term allows for a standard Wiener process $W(t)$ in order to model market volatility (as measured by σ). In particular, we assume that after specification of the rates $\dot{X}_{\text{IN}}(t)$ and $\dot{X}_{\text{OUT}}(t)$ the evolution of the capital is governed by the linear first-order stochastic differential equation (SDE)

$$dX(t) = \left[pX(t) + \rho_1 e^{q_1 t} + \rho_2 e^{q_2 t}\right]\; dt + \sigma X(t)\, dW(t), \tag{2}$$

where the time-dependent drift exhibits two, possibly counteracting exponential inhomogeneities ($\rho_1 < 0 < \rho_2$ or vice versa). This SDE may be considered as a prototypical consumption-and-saving model that incorporates a number of financial and economic processes with competitive contributions. For example, the mean-reverting exponential Brownian motion model analysed by Zhao (2009) is recovered for $p = -\lambda$, $\rho_1 = \lambda\mu$, $\rho_2 = q_1 = q_2 = 0$; the model of Huang, Milevsky & Wang (2003) for an actuarial application within the context of lifetime ruin probabilites is recovered for $p = \mu$, $\rho_1 = -1$, $\rho_2 = q_1 = q_2 = 0$; and the evolution of government debt (or dept-to-GDP ratio) may be approached by a bugdet equation like the one above, see Blanchard (2011) for a non-stochastic, discrete-time analogon. Because of this broad context and because equation (2) reduces to the SDE of geometric Brownian motion (GBM) in the case of disappearing inhomogeneities ($\rho_1{=}\rho_2{=}0$), we henceforth refer to it as principal GBM (PGBM for short). In section 3 the process described by (2) and with ρ_1 and ρ_2 being properly specified, PGBM will serve as a mathematical model for a paragon Ponzi scheme, called Ponzi GBM and abbreviated by PGBM, too.

The rate of change given by (2) is a special case of the more general setting

$$dX(t) = \Big[A(t)X(t) + a(t)\Big]dt + \Big[B(t)X(t) + b(t)\Big]dW(t), \tag{3}$$

with

$$A(t) = p,\;\; a(t) = \rho_1 e^{q_1 t} + \rho_2 e^{q_2 t},\;\; B(t) = \sigma,\;\; b(t) = 0. \tag{4}$$

Such a linear SDE has the solution (see textbooks on SDEs, e.g., Geering et al. 2011)

$$X(t) = \Phi(t)\Big[X(0) + \int_0^t \Phi^{-1}(s)\, a(s)\, ds\Big], \tag{5}$$

where $d\Phi(t) = A(t)\Phi(t)dt + B(t)\Phi(t)dW(t)$ with initial condition $\Phi(0) = 1$. In our case we have $d\Phi(t){=}p\Phi(t)dt{+}\sigma\Phi(t)dW(t)$, i.e., a geometric Brownian motion (GBM), with the familiar solution $\Phi(t) = \Phi(0)\exp[(p - \sigma^2/2)t + \sigma W(t)]$. Hence we are readily left with the solution

$$X(t) = X(0)e^{(p-\frac{1}{2}\sigma^2)t+\sigma W(t)}\ \chi(t), \tag{6a}$$

$$\chi(t) \doteq 1 + \frac{1}{X(0)}\left(\rho_1 \int_0^t e^{-(p-q_1-\frac{1}{2}\sigma^2)t-\sigma W(s)}\, ds + \rho_2 \int_0^t e^{-(p-q_2-\frac{1}{2}\sigma^2)t-\sigma W(s)}\, ds\right). \tag{6b}$$

This result corresponds to a transformed solution of a GBM process, with a time-dependent stochastic transformation factor $\chi(t)$ that involves a couple of exponential functionals. GBM is recovered for $\rho_1{=}\rho_2{=}0$ ($\chi{=}1$).

For the deterministic counterpart with zero noise ($\sigma = 0$), we find (by applying an integrating factor $\exp(-pt)$) the solution of the corresponding linear ordinary differential equation (ODE) $dX(t) = \big[pX(t) + \rho_1 e^{q_1 t} + \rho_2 e^{q_2 t}\big]dt$ to be given by

$$\begin{aligned} X(t) &= \big[X(0) + \frac{\rho_1}{p-q_1} + \frac{\rho_2}{p-q_2}\big]e^{pt} - \frac{\rho_1}{p-q_1}e^{q_1 t} - \frac{\rho_2}{p-q_2}e^{q_2 t} && (7a)\\ &= ae^{pt} + be^{q_1 t} + ce^{q_2 t}, && (7b)\end{aligned}$$

with

$$a \doteq X(0) - b - c, \quad b \doteq -\rho_1/(p - q_1), \quad c \doteq -\rho_2/(p - q_2). \tag{8}$$

For later use we already consider the following additional specification here to the (σ=0)$-$case: beginning at time $t = t_I$ the drift coefficients take new forms and are afterwards labelled $\rho_1^{(t\geq t_I)}$ and $\rho_2^{(t\geq t_I)}$. Again, applying the integrating factor method and integrating from $t = t_I$ to $t > t_I$ we find

$$\begin{aligned} X_{(t\geq t_I)}(t) &= \Big[X(t_I) + \frac{\rho_1^{(t\geq t_I)}}{p-q_1}\, e^{q_1 t_I} + \frac{\rho_2^{(t\geq t_I)}}{p-q_2}\, e^{q_2 t_I}\Big]e^{-pt_I}\, e^{pt} - \frac{\rho_1^{(t\geq t_I)}}{p-q_1}e^{q_1 t} - \frac{\rho_2^{(t\geq t_I)}}{p-q_2}e^{q_2 t} && (9a)\\ &= a_{(t\geq t_I)}e^{pt} + b_{(t\geq t_I)}e^{q_1 t} + c_{(t\geq t_I)}e^{q_2 t}, && (9b)\end{aligned}$$

with

$$a_{(t\geq t_I)} \doteq \Big[X(t_I) - b_{(t\geq t_I)}\, e^{q_1 t_I} - c_{(t\geq t_I)}\, e^{q_2 t_I}\Big]e^{-pt_I}, \quad b_{(t\geq t_I)} \doteq -\frac{\rho_1^{(t\geq t_I)}}{p-q_1}, \quad c_{(t\geq t_I)} \doteq -\frac{\rho_2^{(t\geq t_I)}}{p-q_2}, \tag{10}$$

and where $X(t_I)$ is calculated by means of equation (7).

2.2 The first two moments for the PGBM model

In order to calculate the mean path and the probability distribution for the stochastic process under investigation we need its first two moments. The first moment $E[X(t)]$ (expected value, mean) and the second moment $E[X^2(t)]$ of the stochastic process (3) can be determined by means of the so-called moment equations (e.g., Geering et al. 2011), i.e., by solving the ordinary first order differential equations

$$\frac{d}{dt}E[X(t)] = A(t)E[X(t)] + a(t), \tag{11a}$$

$$\frac{d}{dt}E[X^2(t)] = \Big[2A(t) + B^2(t)\Big]E[X^2(t)] + 2E[X(t)]\Big[a(t) + B(t)b(t)\Big] + b^2(t), \tag{11b}$$

with initial conditions $E[X(0)] = X(0)$ and $E[X^2(0)] = X(0)^2$. Inserting (4), integration is done by means of integrating factors $\exp(-pt)$ and $\exp[-(2p+\sigma^2)t]$ in the case of the first and of the second moment, respectively. In detail, for the first moment one has

$$\int_0^t \frac{d}{d\tau}\big[E(X)e^{-p\tau}\big]d\tau = \int_0^t \Big[\frac{dE[X]}{d\tau} - pE[X]\Big]e^{-p\tau}d\tau = \int_0^t \rho_1 e^{-(p-q_1)\tau} + \rho_2 e^{-p-q_2\tau}\ d\tau,$$
$$E[X]e^{-pt} - X(0) = -\frac{\rho_1}{p-q_1}e^{-(p-q_1)t} + \frac{\rho_1}{p-q_1} - \frac{\rho_2}{p-q_2}e^{-(p-q_2)t} + \frac{\rho_2}{p-q_2},$$

leading to

$$E[X(t)] = \big[X(0) + \frac{\rho_1}{p-q_1} + \frac{\rho_2}{p-q_2}\big]e^{pt} - \frac{\rho_1}{p-q_1}e^{q_1 t} - \frac{\rho_2}{p-q_2}e^{q_2 t} = ae^{pt} + be^{q_1 t} + ce^{q_2 t}. \quad (13)$$

Thus, the expected value $E[X(t)]$ for process (2) corresponds to its solution under the deterministic zero-noise constraint ($\sigma = 0$) as given in (7), i.e., $E[X(t)] = X(t;\sigma=0)$.

For the second moment one has

$$\int_0^t \frac{d}{d\tau}\big[E[X^2]e^{-(2p+\sigma^2)\tau}\big]d\tau = \int_0^t \Big[\frac{dE[X^2]}{d\tau} - (2p+\sigma^2)E[X^2]\Big]e^{-(2p+\sigma^2)\tau}d\tau$$
$$= \int_0^t 2E[X]\big(\rho_1 e^{q_1\tau} + \rho_2 e^{q_2\tau}\big)e^{-(2p+\sigma^2)\tau}\ d\tau$$
$$= \int_0^t 2\big(ae^{p\tau} + be^{q_1\tau} + ce^{q_2\tau}\big)\big(-b(p-q_1)e^{q_1\tau} - c(p-q_2)e^{q_2\tau}\big)e^{-(2p+\sigma^2)\tau}\ d\tau,$$
$$E[X^2]e^{-(2p+\sigma^2)t} - X(0)^2 = 2ab(p-q_1)\frac{e^{-(p-q_1+\sigma^2)t}-1}{p-q_1+\sigma^2}$$
$$+2ac(p-q_2)\frac{e^{-(p-q_2+\sigma^2)t}-1}{p-q_2+\sigma^2} + b^2(2p-2q_1)\frac{e^{-(2p-2q_1+\sigma^2)t}-1}{2p-2q_1+\sigma^2}$$
$$+2bc(2p-q_1-q_2)\frac{e^{-(2p-q_1-q_2+\sigma^2)t}-1}{2p-q_1-q_2+\sigma^2} + c^2(2p-2q_2)\frac{e^{-(2p-2q_2+\sigma^2)t}-1}{2p-2q_2+\sigma^2},$$

yielding

$$E[X^2(t)] = e^{(2p+\sigma^2)t}\big(X(0))^2 + H(t;0)\big). \quad (14)$$

Herein the PGBM specific auxiliary function

$$H(t;t^*) \doteq 2ab\,h(p-q_1,t^*)+2ac\,h(p-q_2,t^*)+2bc\,h(2p-q_1-q_2,t^*)+b^2\,h(2p-2q_1,t^*)+c^2\,h(2p-2q_2,t^*) \quad (15)$$

itself relies on a further auxiliary function

$$h(x,t^*) \equiv h(t;x,t^*,\sigma^2) \doteq \frac{e^{-(x+\sigma^2)t} - e^{-(x+\sigma^2)t^*}}{1+\sigma^2/x}, \quad (16)$$

wherein x, t^*, and σ^2 act as parameters. (These two auxiliary functions are introduced here in order to also simplify the notation in section 3.2). The moments of a pure GBM process are recovered as a special case by setting ρ_1=ρ_2=0 and hence by means of inserting $a = X(0)$, $b = -\rho_1/\,(p-q) = 0$, and $c = -\rho_2/\,p = 0$ into the above equations, leading to $E_{GBM}[X] = X(0)\exp(pt)$ and $E_{GBM}[X^2] = X(0)^2\exp((2p+\sigma^2)\,t)$.

Finally, the variance of the stochastic process can be calculated by means of

$$Var[X(t)] = E[X^2(t)] - E[X(t)]^2. \quad (17)$$

2.3 The probability distribution for a positiv definite variable

The PGBM model can be regarded as a transformed GBM model, as highlighted in section 2.1. The probability density function (pdf) of geometric Brownian motion follows a lognormal distribution. Likewise, the PGBM exhibits a lognormal probability density function

$$P(X,t) = \frac{1}{\sqrt{2\pi\,\sigma^2(t)}\,X} \exp\left\{-\frac{(\ln(X)-\mu(t))^2}{2\sigma^2(t)}\right\}, \tag{18}$$

where the time-dependent scale and shape parameters are given by

$$\mu(t) = \ln(E[X]) - \frac{1}{2}\sigma^2(t) = \ln\left(\frac{E[X]^2}{\sqrt{E[X^2]}}\right), \tag{19a}$$

$$\sigma^2(t) = \ln\left(1+\frac{Var[X]}{E[X]^2}\right) = \ln\left(\frac{E[X^2]}{E[X]^2}\right) \tag{19b}$$

and hence depend on the moments $E[X]$ and $E[X^2]$ as derived in the previous subsection. These relations result after calculating the expected value $E[X]{=}\int_0^\infty X\,P(X,t;\,\mu(t),\sigma^2(t))\;dX = \exp(\mu(t)$ $+\sigma^2(t)/2)$ and the second moment $E[X^2]{=}\int_0^\infty X^2\,P(X,t;\,\mu(t),\sigma^2(t))\;dX = \exp(2\mu(t)+2\sigma^2(t))$. Note that due to the logarithmic dependency on X distribution (18) is only defined for a positive definite variable $X>0$.

The parameter functions of the pdf for the pure GBM process are recovered by inserting the corresponding first and second moments (see previous section) into equation (19), leading to $\mu_{GBM}(t) = \ln(X(0)) + (p-\frac{1}{2}\sigma^2)\,t$ and $\sigma^2_{GBM}(t) = \sigma^2\,t$, implying $E_{GBM}[X] = X(0)\exp(pt)$. Equivalently, given relation (6), i.e., $X(t) = \chi(t)\,\Phi(t)$ with $\Phi(t)$ being the solution of a GBM process with $\Phi(0) = X(0)$, the pdf for GBM is recovered for a constant transformation factor $\chi(t) = 1$.

Because the solution of the PGBM process is a transformed solution of the GBM process the parameter functions (19) can be expressed by an excellent approximate analytic specification that avoids making use of the bulky second moment. Denoting the pdf for GBM by $P_\Phi(\Phi(t),t)$, then according to the change-of-variable rule for probability distributions, one has $P(X(t),t) = P_\Phi(\Phi(t),t)\,|\partial\Phi/\partial X| = P_\Phi(X(t)/\chi(t),t)\,/\,|\chi(t)|$. See also the theorem provided by Rujivan (2011) about coinciding transition densities of affine transformations of Itô processes. Together with the relation $\sigma^2(t) = 2\,(\,\ln(E[X]) - \mu(t)\,)$ the parameter functions become

$$\mu(t) = \mu_{GBM}(t) \,+\, \ln(\chi(t)), \tag{20a}$$

$$\sigma^2(t) = \sigma^2_{GBM}(t) + 2\ln\left(\,E[X]/(\chi(t)\,E_{GBM}[X])\,\right). \tag{20b}$$

For ease of computational tractability of the stochastic transformation factor $\chi(t)$, which involves a couple of stochastic integrals in the form of exponential functionals, we replace it by its *adjusted* expected value

$$\chi(t) \approx \tilde{E}[\chi(t)] = 1 - \frac{1}{X(0)}\left(\rho_1\,\frac{e^{-\left(p-q_1-\alpha\sigma^2\right)\,t}-1}{p-q_1-\alpha\sigma^2} + \rho_2\,\frac{e^{-\left(p-q_2-\alpha\sigma^2\right)\,t}-1}{p-q_2-\alpha\sigma^2}\right), \tag{21}$$

with the adjustment parameter being set to $\alpha = 1/2$. Theoretically, the expected value of each of the exponential functionals demands $\alpha = 1$ (see, e.g., Huang, Milevsky & Wang (2004) for a detailed derivation of the first (and second) moment of such *stochastic present value* random variables). However, using this value produced results that were inconsistent with the results obtained either by means of equation (19) or with Monte-Carlo samplings or with the outcomes of numerically integrating the evolution equation as discussed later. We henceforth rely on this adjusted expected value. Similar to before, the pdf for pure GBM is recovered for $\rho_1{=}\rho_2{=}0$ and hence for $\tilde{E}[\chi] = 1$ and $E[X] = E_{GBM}[X]$.

Equation (19) will later be used to graph the pdfs in figures 1 and 2; however, equations (20)-(21) provide a feasible and handy alternative.

2.4 The evolution of the pdf

Having found the pdf of the process, we also want to consider its differential evolution with respect to time. The resulting partial differential equation will be crucial for the numerical approach dealt with in section 3.2 below. The probability density function $P(X,t)$ for a process X obeying SDE (3) is known to evolve according to the one-dimensional Kolmogorov forward equation (Fokker-Planck equation)

$$\frac{\partial}{\partial t}P(X,t) = \frac{1}{2}\frac{\partial^2}{\partial X^2}\Big([B(X,t)\,X(t)+b(t)]^2\,P(X,t)\Big) - \frac{\partial}{\partial X}\Big([A(X,t)\,X(t)+a(t)]\,P(X,t)\Big) \quad (22)$$

(see, e.g., Gardiner 1985). Inserting the values attributed in (4), PGBM obeys the Kolmogorov forward equation (KfE)

$$\frac{\partial}{\partial t}P(X,t) = \frac{1}{2}\frac{\partial^2}{\partial X^2}\Big(\sigma^2 X^2 P(X,t)\Big) - \frac{\partial}{\partial X}\Big(\Big[p\,X-\hat{X}(t)\Big]\,P(X,t)\Big), \quad (23)$$

where $\hat{X}(t) \equiv a(t) = \rho_1 \exp(q_1 t) + \rho_2 \exp(q_2 t)$ and with an initial probability density $P(X,0) = \delta(X - X(0))$ ($X \in R_+$, using the Kronecker-δ). For ρ_1=ρ_2=0 and hence for $\hat{X}(t) = 0$ we get the pdf evolution equation for GBM. It would be a cumbersome exercise indeed to verify that function (18) satisfies equation (23); here we consider the coincidence between this analytical result and the numerical results obtained later as convincing enough (cf. figures 1 and 2).

Unfortunately, pdf (18) only holds as long as $X > 0$, $E[X] > 0$, and $\tilde{E}[\chi] > 0$, due to their logarithmic dependencies. However, as will be evident from figures 1 to 3, starting at some first passage time there are trajectories occuring at X-values below zero, and the analytical pdf will necessarily fail. Therefore we will rely on numerical calculations in the following sections, i.e., numerically integrate the above KfE (23) in order to provide the probability density function (pdf) and the related cumulative distribution function (cdf). Computational details with respect to the adopted numerical method are relegated to the Appendix.

3 Ponzi geometric Brownian motion

The principal GBM (PGBM) model developed in the previous section will now serve as the framework for an idealized Ponzi system. The following construction of an investment system illustrates the design process for generating the Ponzi scheme: after setting up the particularities of the scheme in section 3.1 (like determining the presumed evolution of the number of members and like fixing some withdrawal modalities), the model is recognized as a PGBM process in section 3.2 that can be handled accordingly. A (weak) no-Ponzi game condition is proposed in section 3.3. Modifications and variations of the setup can be dealt with in a similar fashion, and we will take advantage of this later in section 5.

3.1 Member evolution and capital flow

An investment system is founded by N_{00} initiators which together contribute a total initial capital X_0. They start with N_0 initial investors and continuously recruit more investors according to the rate

$$\dot{N}_{new}(t) = \Delta N_0\, e^{qt}, \quad (24)$$

where ΔN_0 provides the initial number increase of new investors, q is an exponential growth factor, time t is measured in time units appropriate to the process (e.g., in years or weeks), and the dot represents the first derivative with respect to time. Therefore, within time intervals of unit length, the number of new participants is also exponentially growing by an amount $\int_{t-1}^{t} \dot{N}_{new}(\tau)\; d\tau =$

$N_0^* e^{qt}$, with $N_0^* := \Delta N_0(1-e^{-q})/q$, in reminiscence to the evolution of members in the discrete-time model of Parodi (2013). Each investor stays invested only for a fixed investment span t_I: if she entered the system at time $t-t_I$, she will quit it again at time t. Thus, the number of members (including the initiators) at time t is $N(t) = N_{00} + N_0 + \int_0^t \dot{N}_{new}(\tau)\, d\tau$ (if $t < t_I$) or $N(t) = N_{00} + \int_{t-t_I}^t \dot{N}_{new}(\tau)\, d\tau$ (if $t \geq t_I$), yielding

$$N(t) = N_{00} + \begin{cases} N_0 + \frac{\Delta N_0}{q}(e^{qt}-1) & (t < t_I) \\ \frac{\Delta N_0}{q}(1-e^{-qt_I})\, e^{qt}. & (t \geq t_I) \end{cases} \tag{25}$$

The initial number of members is $N(0) = N_{00} + N_0$, the number of initiators remains constant at N_{00}, and the instantaneous number of investors is $N(t) - N_{00}$. Note that $N(t)$ counts the number of members *after* all the entries and exits have taken place at a time t. In particular, at $t{=}t_I$, the number of members $N(t_I)$ excludes the N_0 very first investors due to their resignations at this moment. Each new investor lends a single amount I_0 to the initiators when entering the system. Thus, the cash inflow rate to the system due to new members is (for $t{>}0$)

$$\dot{X}_{\mathrm{IN}}(t) = I_0 \dot{N}_{new}(t) = I_0\, \Delta N_0\, e^{qt}. \tag{26}$$

The initiators pay interest $I_0 r$ to each investor as a return on investment (ROI), with a constant periodical interest rate r. Similarly, each initiator is compensated with a periodic amount $I_{00} r_{00}$, where $I_{00} = X_0/N_{00}$ is the average funding by an initiator and r_{00} is the respective interest rate. While the initiators always keep participating in the system, an investor withdraws his investment I_0 and quits the system after an investment period t_I. Accordingly, the infinitesimal payments leaving the system are $\dot{X}_{\mathrm{OUT}}(t) = N_{00} I_{00} r_{00} + [N(t) - N_{00}] I_0 r$ (if $t < t_I$) or $\dot{X}_{\mathrm{OUT}}(t) = N_{00} I_{00} r_{00} + [N(t) - N_{00}] I_0 r + \dot{N}_{new}(t-t_I)\, I_0$ (if $t \geq t_I$), with the very last term taking into account the full withdrawal of those investors who joined at time $t - t_I$ and who drop out again at time t. We arbitrarily neglect an additional last ROI for the departing investors, $\dot{N}_{new}(t-t_I)\, I_0 r$, as well as subtracting the interests for the newest participants, $\dot{N}_{new}(t)\, I_0 r$; both these terms can be added if considered relevant. The capital outflow rate now simply is (for $t{>}0$)

$$\dot{X}_{\mathrm{OUT}}(t) = N_{00} I_{00} r_{00} + \begin{cases} \left(N_0 + \Delta N_0 \frac{e^{qt}-1}{q}\right) I_0 r & (t < t_I) \\ \Delta N_0 I_0 \left[\frac{r}{q} + \left(1 - \frac{r}{q}\right) e^{-qt_I}\right] e^{qt}. & (t \geq t_I) \end{cases} \tag{27}$$

Table 1: THE PARAMETERS OF THE PONZI SCHEME

N_{00}	number of initiators	r_{00}	promised rate of return to initiators
N_0	initial number of investors	r	promised rate of return to investors (ROI)
ΔN_0	initial number of new investors	q	growth rate of the number of new investors
I_{00}	average capital of an initiator	p	market-based asset performance rate
I_0	investment of an investor	σ	financial market constant volatility
t_I	investment time span		

Auxiliary definitions: $X_{00}{:=}N_{00}I_{00}$, $X_0{:=}N_0 I_0$, $\Delta X_0{:=}\Delta N_0 I_0$

3.2 The first two moments of Ponzi geometric Brownian motion

Inserting the earnings and spendings as provided by equations (26) and (27), respectively, into the capital growth equation (1), the SDE (2) as specified for the PGBM model uprises, with assembled coefficients

$$\rho_1 = \Delta X_0\Big(1-\frac{r}{q}\Big)\Big(1-\mathbf{1}_{[t\geq t_I]}e^{-qt_I}\Big) \tag{28a}$$

$$\rho_2 = \Big(\frac{\Delta X_0}{q}-X_0\Big)\, r\,\big(1-\mathbf{1}_{[t\geq t_I]}\big) - X_{00}r_{00}. \tag{28b}$$

Herein, for the sake of a more compact notation, the indicator function $\mathbf{1}_{[t\geq t_I]} = \begin{cases} 0 & t<t_I \\ 1 & t\geq t_I \end{cases}$ enforces the case distinction between times before and after the beginning of member drain. This drain can be avoided by choosing the investment span t_I large enough (for practical purposes) or even infinite (for formal reasons). The initial values suited for the two regimes are $X(0)=X_{00}+X_0$ (if $t<t_I$) or $X(t_I)=X_{(t<t_I)}(t_I)-X_0$ (if $t\geq t_I$).

The expected value $E[X(t)]$ is, according to equation (13) with relation (28) inserted, given by

$$E[X(t)] = ae^{pt}+be^{qt}+c, \tag{29a}$$

$$a = \Big(X_0-\frac{\Delta X}{q-p}\Big)\Big(1-\frac{r}{p}\Big)\big(1-\mathbf{1}_{[t\geq t_I]}e^{-pt_I}\big)+X_{00}\Big(1-\frac{r_{00}}{p}\Big), \tag{29b}$$

$$b = \frac{\Delta X}{q-p}\Big(1-\frac{r}{q}\Big)\big(1-\mathbf{1}_{[t\geq t_I]}e^{-qt_I}\big), \tag{29c}$$

$$c = \Big(X_0-\frac{\Delta X}{q}\Big)\frac{r}{p}\big(1-\mathbf{1}_{[t\geq t_I]}\big)+\frac{X_{00}r_{00}}{p}. \tag{29d}$$

Note that $E[X(t)]$ may exhibit a jump at regime switching time t_I. We call this form of solution an *abc-model solution in continuous time* for a Ponzi scheme, due to analogous results provided by the discrete-time model of Parodi (2013).

In order to calculate the second moment of the stochastic process, we rely on equations (14), (28), and (8) and find

$$E[X^2(t)] = e^{(2p+\sigma^2)t}\times\begin{cases}\big((X_{00}+X_0)^2+H(t;0)\big), & (t<t_I)\\ \big(E[X^2(t_I)]e^{-(2p+\sigma^2)\,t_I}+H(t;t_I)\big), & (t\geq t_I).\end{cases} \tag{30}$$

Herein the auxiliary function $H(t;t^*)$ is defined in equation (15)f and depends on the coefficients a, b, and c given in eqs. (29b)-(29d) (and which therefore have to be chosen according to whether $t<t_I$ or $t\geq t_I$). A technical note in addition: the quantity $E[X^2(t_I)]$ immediately after regime switching can be calculated by requiring momentary constancy of variance, i.e. $Var[X(t_I)] = \lim_{\Delta t\to 0} Var[X(t_I-\Delta t)]$, hence by setting $E[X^2(t_I)] = E[X(t_I)]^2 + \lim_{\Delta t\to 0}\big(E[X^2(t_I-\Delta t)]-E[X(t_I-\Delta t)]^2\big)$.

The outcome of a typical PGBM process is illustrated in Figures 1 and 2 for nearly identical sets of parameter values, they only differ with respect to the investment span. In figure 1, the investment span is larger than the chosen time domain, hence no withdrawals occur. Shown in the top diagram are the mean path $E[X(t)]$ (solid curve), that crosses the (X=0)-level at an expected default time t=21.5 periods, and the 0.1%- and 95%-percentile paths (dashed lines), as determined by means of a Monte-Carlo simulation (see section 4). First passage of the (X=0)-level (for the 0.1%-percentile path) is at about t=15.5 periods. The distribution of the paths at two selected time points is presented in the histogramms at the bottom of Figure 1. Before first passage time,

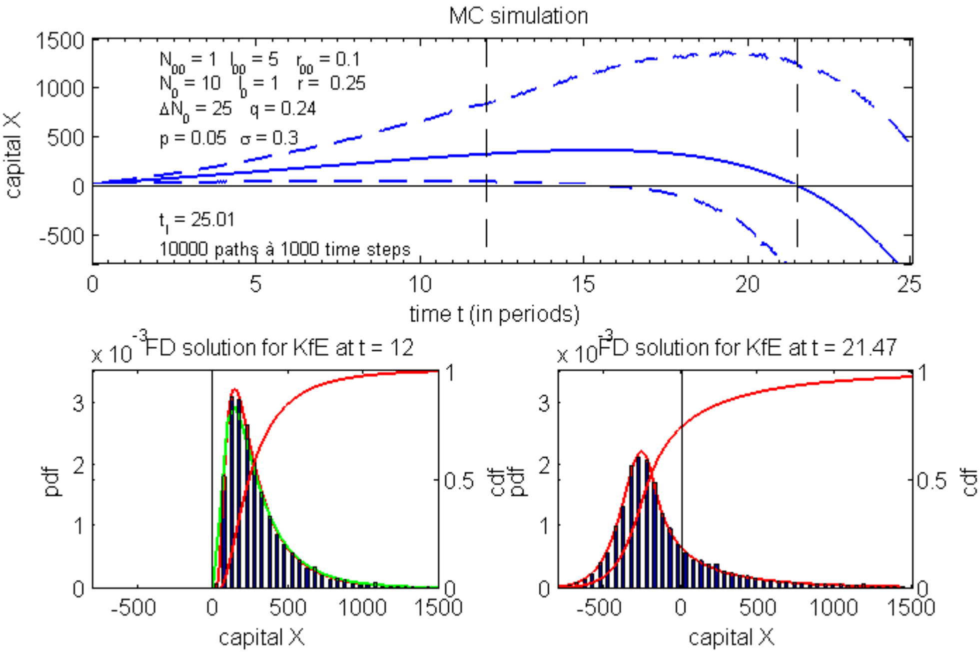

Figure 1: The evolution of a Ponzi system with infinite investment span, i.e., without quitting participants and hence without final withdrawals. For practical purposes, i.e., to avoid withdrawals, the investment span is a finite value larger than the time window of interest; here it is set to t_I=25.01.

Top: The mean path (solid blue line) for the Ponzi scheme under investigation is shown together with the 0.1%-percentile and the 95%-percentile (lower and upper blue dashed lines, respectively) as resulting from a Monte-Carlo simulation using the eleven indicated parameter values. Because the investment span t_I is at least as large as the time domain shown, no withdrawals have occured yet. The mean path may formally be described by $E[X(t)] = ae^{pt} + be^{qt} + c$, with $a = 481.316$, $b = -5.483$, and $c = -460.833$. Bust time for the mean path is at t=21.5 (see the position of the right vertical dashed line). However, due to volatility the first passage time at bust level $X(t)$=0 already occurs around 15 periods. For the times indicated by vertical dashed lines, the probability functions for $X(t)$ are given in the lower panels of the figure.

Bottom: The distributions of the simulated paths are shown at times t=12 and t=21.5 by means of normalized histograms for Monte-Carlo simulation data (black bars, representing centered bins with widths of 50 capital units). Overlaid are the graphs for the probability density functions (pdf, left scales) as determined by numerically solving the Kolmogorov forward equation (KfE, red envelopes). The green graph in the left panel is the analytical solution for the pdf involving the lognormal distribution (equations (18) and (19)) that is valuable as long as the capital remains positive definite; therefore no such solution is available for the graph in the right panel. Also shown in red are the corresponding cumulative distribution functions (cdf, right scales), from which the system's instantaneous probability of default (PD) can be read. For example, the cdf has a value of about 0.752 at the mean path default time t=21.5, hence the probability of default up to this time is about PD=75% and, correspondingly, the survival probability is about 1-PD=25%.

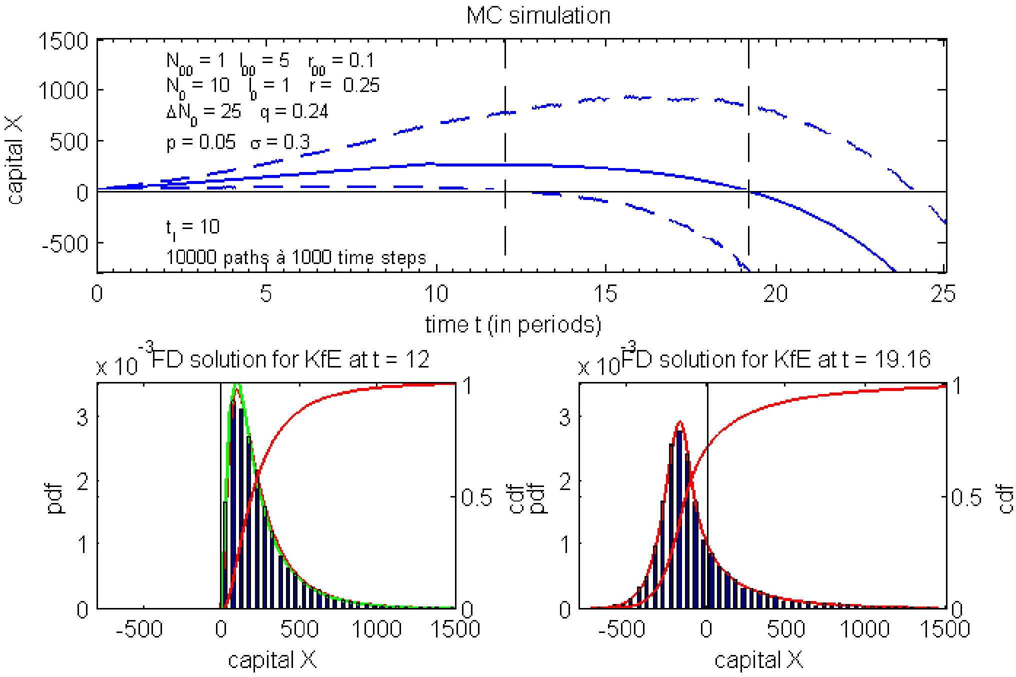

Figure 2: THE EVOLUTION OF A PONZI SYSTEM WITH FINITE INVESTMENT SPAN, i.e., with participants quitting after an investment period of $t_I = 10$ time units and at the same time withdrawing their initial deposits. All other parameter values and designations are as with figure 1. The overall qualitative behaviour of the systems shown in figures 1 and 2 is alike. However, with the onset of withdrawals after the first investment span at $t = t_I = 10$, the available capital undergoes a (small) jump and declines with a higher pace. After $t = t_I$ the mean path follows $E[X(t)] = ae^{pt} + be^{qt} + c$, with $a = 186.350$, $b = -4.985$, and $c = 10.0$. Due to the higher amount of migrated capital (i.e., including withdrawals), the system defaults earlier than the comparative system shown in figure 1 with infinite investment span: first passage time for the 0.1%-percentile path now is around 12 periods and for the mean path the bust already occurs at about t=19.2 periods. As can clearly be seen in the right bottom panel histogramm, at this expected bust time the majority of paths has negative capital values; nevertheless, from the cdf, that has a value of PD=0.764 at $X = 0$, one can read that the system's survival probability up to this point still is about 1-PD=24%.

for example at t=12 periods, all paths are still above the bust level X=0; their envelope exhibits the shape of a lognormal distribution (i.e., it is calculated by means of equations (18) and (19)). However, after the first passage time increasingly many paths enter the negative capital domain ($X < 0$) and therefore correspond to crashed Ponzi systems. At the expected default time t=21.5 periods about 25% of the paths are still in the positive capital domain ($X > 0$); these systems will survive somewhat longer.

3.3 No-Ponzi game condition

In a typical Ponzi system the available capital grows during a first phase when new members contribute their deposits and the other members still get a comparatively small total amount of the return on investments (ROIs). After a while, when there are many participants that receive ROIs, the system's total available capital reaches a peak that is followed by a second phase characterized by a pronounced demise of the capital, finally leading to the collapse of the system at some bust

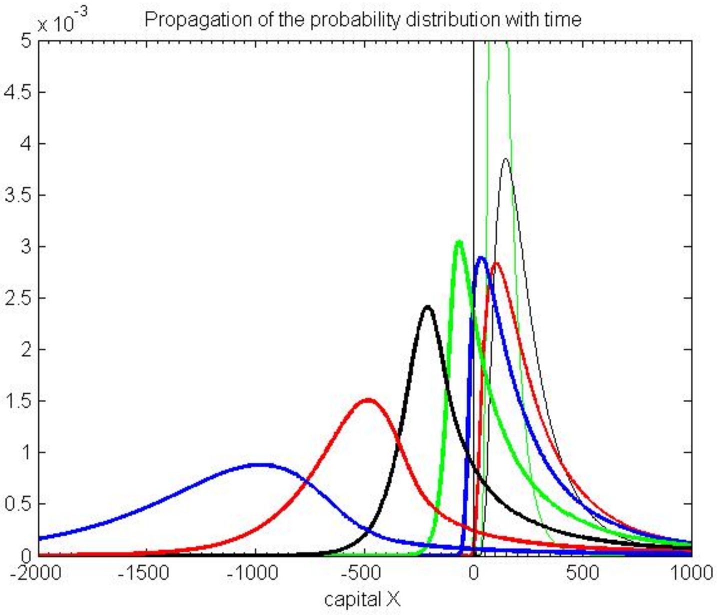

Figure 3: PROPAGATION OF THE PROBABILITY DISTRIBUTION. Snapshots of the numerically determined pdfs for the system represented in figure 1 are shown for times t=5, 10, 15, 17, 19, 21, 23, 25 time units. The thinnest and tallest distribution corresponds to t=5 (shown in green), the fat and broad distribution to the left corresponds to t=25 (shown in blue). Due to the underlying diffusion process, the pdfs smudge with time. The distribution is of lognormal type only as long as the probability for negative capital values is zero (t=5, 10, 15) and can be expressed in closed form by means of equation (18). For later times, or more precisely, starting with the first passage time around t=15, the distributions are no longer lognormal and exhibit increasingly large wings above the negative branch of the X-axis.

time t_{BUST}. This time point of saturation $X(t_{\text{BUST}}) = 0$ for the capital mean path usually has to be determined numerically. However, if coefficient c in the abc-model solution (29a) is comparably small with respect to the other two coeffiecients, i.e., if $|a|$, $|b| \gg |c|$, it may be ignored for an approximation. Then, using equation (29), condition $E[X(t)] = 0$ leads to the estimation

$$t_{\text{BUST}} \approx \frac{1}{q-p} \ln\left(-\frac{a}{b}\right), \qquad (|a|,\ |b| \gg |c|) \tag{31}$$

where either $q > p, -a/b > 1$ or $p > q, -b/a > 1$ must hold.

The time of occurence of the peak, t_{PEAK}, can be computed by means of the condition $dE[X(t)]/dt = 0$, implying

$$t_{\text{PEAK}} = \frac{1}{q-p} \ln\left(-\frac{ap}{bq}\right). \tag{32}$$

In order to avoid a bust destiny one has to choose the nine parameter values of the parameter vector $\phi \doteq \{t_I, X_{00}, X_0, \Delta X_0, r_{00}, r, q, p, \sigma\}$ in such a way that there is only monotonous capital growth and no occurence of a peak time. Therefore, requiring a non-existing logarithmic value in (32) corresponds to requiring a negative argument of the logarithm, and also assuming positive rates q and p leads to a simple *weak* no-Ponzi game condition

$$\text{wNPC}: \qquad \left.\frac{a}{b}\right|_{\phi} > 0. \qquad (p, q > 0) \tag{33}$$

The condition is called *weak* because an unlimited pool of potential participants is supposed. A *strong* no-Ponzi game condition (NPC) would include the above wNPC as well as some maximum number of participants, $N_{\max}$, i.e., the additional condition $N(t) \leqq N_{\max}$. This paper puts its focus on weak Ponzi games. Correspondingly, if condition (33) holds for a given set of parameter values, the system represents a no-Ponzi game as long as we may forget about the number limit condition, hence it's a Ponzi game in its weak sense. If on the other hand condition (33) is violated, the system is identified as a Ponzi system that will collapse due to other reasons than reaching a number limit. For example, the systems shown in figures 1 and 2 satisfy $a/b \approx -88$ and -37, respectively, identifying them as true Ponzi systems. In order to study the critical behaviour of a system, one may trigger the value of a single parameter, say $\phi_i \in \phi$, while keeping the other parameter values $\phi_j \in \phi$ $(j \neq i)$ fixed, until the system tilts from a weak Ponzi game (no Ponzi game) to a strong Ponzi game, or vice versa. Formally, this is equivalent to finding the zeros or the singularities of the function $f(\phi_i) \doteq a(\phi_i)/b(\phi_i)$. In section 5 an illustrated example for numerically determining such single critical parameter values will be given.

4 Default probability

Despite the restricted domain of definition of the lognormal pdf (i.e., to positive definite capital values), we want to calculate the probability of default (written PD or P_D) at times when this restriction is violated. The leading question is: how likely is it, that the system will collaps before some time T? Stated differently, what is the system's default probability before a given time T? We are now going to describe two numerical procedures that allow for consistent results.

I. Monte-Carlo simulation. Starting with an initial value $X(0) = X_{00} + X_0$ we construct a multitude of N independent paths, each by approximating equation (2) according to the discrete-time evolutionary rule

$$X(t + \Delta t) = X(t) + \left[pX(t) + \rho_1 e^{qt} + \rho_2\right] \Delta t + \sigma X(t) \epsilon \sqrt{\Delta t}, \tag{34}$$

where the last term is a well known approach with respect to geometric Brownian motion (GBM), including a random sample ϵ from a normal distribution with a zero mean and a standard deviation of 1.0 (see, e.g., Geering et al. 2011). Selecting all the values at a specified time T, we construct a histogram with K bins such that the entries $H(k,T)$ in bins $k = 1, 2, ..., k_0$ correspond to negative values of $X(T)$ and those in bins $k = k_0 + 1, k_0 + 2, ..., K$ to positive values of $X(T)$. The histogramm is normalized by the total number $N = \sum_{k=1}^{k=K} H(k,T)$ of simulated paths and thus traces the corresponding probability distribution function for the Ponzi system wealth values X at time T. As soon as the wealth of a single Ponzi system lays below zero, this system has collapsed. Therefore the probability for a default before a time T is estimated by the cumulative sum

$$P_D^{MC}(t < T) = \frac{1}{N} \sum_{k=1}^{k_0} H(k,T) \tag{35}$$

This actually is a lower boundary, because a few paths that have hit the level $X(t) = 0$ before time T may have recovered to $X(T) > 0$ and are therefore not counted. However, performing enough simulations the effect turns out to be neglectable. Examples of histograms accompaning Monte-Carlo simulations and created at selected times are given in figures 1 and 2, with the corresponding PD values revealed in the figure caption.

II. Kolmogorov forward equation (Fokker-Planck equation). In order to quantify the probability for a collaps of the investment system before a particular time T, we now numerically

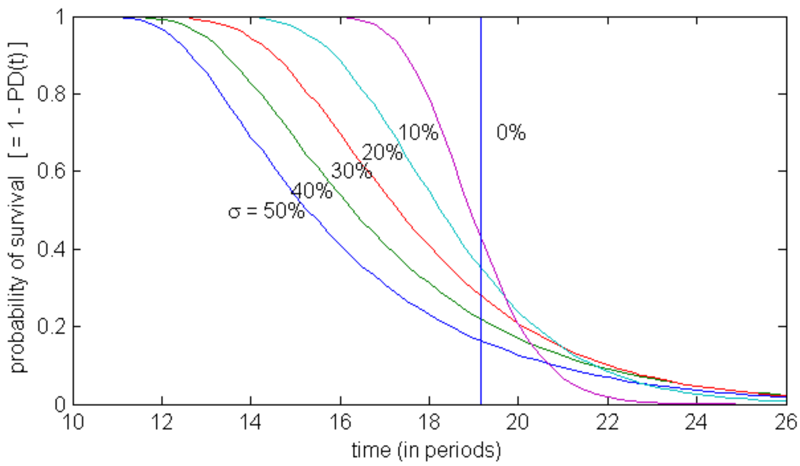

Figure 4: SURVIVAL PROBABILITY. The survival probability $1 - PD$ (i.e., the complement to the default probability PD) for the Ponzi system represented in figure 2 is shown for different constant market volatilities σ. The curves are interpolated solutions of the Kolmogorov forward equation for the PGBM model, with total integrated evolution times $t = 10, 10.25, 10.5, ..., 26$ periods. A volatility of 0% corresponds to the deterministic PGBM model which is represented by the mean path in figure 2 and which hits the zero capital line at bust time $t = 19.17$ periods (blue vertical line): before that time point, survival probability is one, afterwards zero. Exposition to increasing volatilities has the effect of correspondingly lower (higher) survival probabilities at some given pre-bust (post-bust) time. For example, the system of figure 2 with $\sigma = 30\%$ has a survival probability of only about 30% until deterministic bust time (red graph). An otherwise similar system but exposed to a 50%-volatility has a survival probability of only 50% up to about 15 periods and less than 20% until deterministic bust time, but up to about 23 periods there still is a 5%-survival probability (blue graph).

solve the Kolmogorov forward equation (23) by means of a finite difference method to get the discretized pdf $P(X_i, T)$ for equally spaced grid values $X_i = i\,\Delta X$, $i \in \{i_{min}, \ldots, -2, -1, 0, 1, 2, \ldots, i_{max}\}$. The sophisticated computational details, including a proper variable substitution that allows for negative values of X, are relegated to the Appendix. Because of the requirement $\sum_{i=-i_{min}}^{i_{max}} P(X_i, T)\Delta X = 1$, the default probability of the system is then measured by means of the cumulative distribution function (cdf)

$$P_D^{PDE}(t < T) = \sum_{i=-i_{min}}^{0} P(X_i, T)\Delta X, \tag{36}$$

i.e., by numerically integrating the pdf. Example results are shown in figures 1 and 2 where the pdfs provide smooth envelopes to the histograms and where the graphs for the corresponding cdfs allow to directly read off the PDs.

5 Toy model for a social security system

Within the context of social security the investment system defined in section 3 may be reinterpreted as a pay-as-you-go pension scheme as follows: At retirement the retiree pays a unique amount of I_0, called the deposit, to the insurance company. This asset corresponds to all the social security payments during work life, including compounds. For some time span t_I, representing a retiree's mean life time after retirement, each beneficiary receives periodically an annuity $I_0 r$, where r is the conversion rate. Hence, equations (24), (25), and (26) for the number of new retirees entering the system, the number of retirees in total, and the capital inflow rate, respectively, remain unchanged. However, after the decease of a beneficiary, i.e. after the span t_I, neither the original deposit I_0 nor a possibly remaining asset value $I_0\,(1 - r\,t_I)$ of the retiree will be withdrawn but remain in the system. No withdrawals are performed. Therefore the system's payout function is no longer given by equation (27), but modified to

$$\dot{X}_{\mathrm{OUT}}(t) = X_{00} r_{00} + \begin{cases} \Big(X_0 + \Delta X_0 \frac{e^{qt}-1}{q}\Big)\, r & (t < t_I) \\ \Delta X_0 \frac{r}{q}\big(1 - e^{-qt_I}\big) e^{qt}. & (t \geq t_I) \end{cases} \tag{37}$$

Correspondingly, the coefficients within the drift term in the SDE for the PGBM process change from equation (28) to

$$\rho_1 = \Delta X_0 \Big[1 - \frac{r}{q}\Big(1 - \mathbf{1}_{[t \geq t_I]} e^{-qt_I}\Big)\Big] \tag{38a}$$

$$\rho_2 = \Big(\frac{\Delta X_0}{q} - X_0\Big)\, r\,\big(1 - \mathbf{1}_{[t \geq t_I]}\big) - X_{00} r_{00}. \tag{38b}$$

Actually, only the first coefficient, ρ_1, has changed. The *abc*-model coefficients entering the formulae for the expected value (equation (13)) and for the second moment (equation (14)) are now available for plain evaluation or for further formal calculations. For example, requiring initial values $X(0) = X_{00} + X_0$ (if $t < t_I$) and $X(t_I) = X(t; t < t_I)$ (if $t \geq t_I$) (that is, without additional withdrawals X_0 due to the initial group of retirees) and relying directly on equations (7) and (9) we get an expected value

$$E[X(t)] = a e^{pt} + b e^{qt} + c, \tag{39a}$$

$$a = \left(X_0 - \frac{\Delta X}{q-p}\right)\Big(1 - \frac{r}{p}\big(1 - \mathbf{1}_{[t \geq t_I]} e^{-pt_I}\big)\Big) + X_{00}\left(1 - \frac{r_{00}}{p}\right), \tag{39b}$$

$$b = \frac{\Delta X}{q-p}\Big(1 - \frac{r}{q}\big(1 - \mathbf{1}_{[t \geq t_I]} e^{-qt_I}\big)\Big), \tag{39c}$$

$$c = \left(X_0 - \frac{\Delta X}{q}\right)\frac{r}{p}\big(1 - \mathbf{1}_{[t \geq t_I]}\big) + \frac{X_{00} r_{00}}{p}. \tag{39d}$$

While the model presented in section 3 describes a Ponzi scheme with withdrawals (done by quitting investors), the variant offered here represents a simple social security system that corresponds to a scheme without withdrawals at the end of an insurance span (when the retiree leaves the system due to death). To visually illustrate the kinship of the two models, their mean paths due to equations (29) (with t_I) and (39), otherwise using the same parameter values, are shown in figure 5 as dot-dashed and solid black lines, respectively. Up to time t_I, the formulas and therefore the graphs are identical. Afterwards, the no-withdrawal requirement delays but doesn't prevent the crash. For the social security model several different mean path scenarios are shown *ceteris paribus*: to create a graph only one parameter value was changed with respect to the comparative values denoted in the upper left corner of the figure. The system is sensible with regard to critical parameter values that decide whether the system will default or survive. For example, if

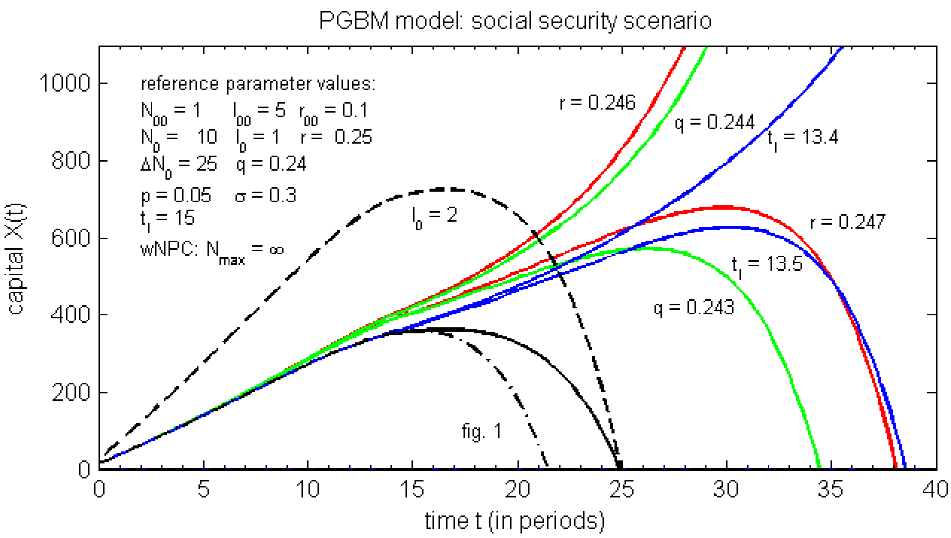

Figure 5: The PGBM model representing a social security system. For the particular social security model introduced in the text different mean path scenarios are shown *ceteris paribus*: the graph given by the black solid line corresponds to the parameter values denoted in the upper left corner, called reference parameter set, while the paired colored graphs result from changing single parameter values by minor amounts, as indicated. The system may react quite sensibly to changes of parameter values. For example, a small shrinkage of the periodical annuity rI_0 received by the retiree (corresponding to a rather modest decrease of the promised rate of return on investment, say, for example, from $r = 0.25$ to $r = 0.247$) has a large impact on the system's longevity (bust time increases from 25 to 38 periods). Most intriguing, however, is that depending on the values of the parameters relative to critical values (which can be determined by means of the wNPG condition), a system represents either a non-lasting Ponzi game or a lasting no-Ponzi game (in the weak sense). The longevity of a basically non-lasting social security system is longer the closer the single parameter value lies to the corresponding single critical value. Hence, a seemingly stable system may rather suddenly come to an abrupt end.

the investor's return on investment rate is r=0.247 (lower red graph) the system collapses rather abruptly, while for r=0.246 (upper red graph), where the payouts are only slightly smaller, the system will survive. Actually, the critical parameter value is $r \approx 0.24675$. Similar observations hold for the single critical values of the parameters q and t_I (green and blue lines, respectively), with critical values $q \approx 0.2435$ and $t_I \approx 13.41$. The critical behaviour of the system with respect to changing single parameter values is illustrated in figure 6: the parameters r, q, and t_I act as single variables ϕ_i for the function $f(\phi_i) = a(\phi_i)/b(\phi_i)$ given by equation (33). Positive values correspond to no-Ponzi games (actually, to sustained weak Ponzi games), negative values to (strong) Ponzi games. For the three parameters shown, the function exhibits singularites with a flipping sign, thus marking the critical values. However, not all of the components of the parameter vector are exposed to critical behaviour. For example, changing the height of an investor's initial deposit I_0 will neither change the type of the game nor alter the bust time (dashed black line in figure 5). In general, a system represents either a non-lasting Ponzi game or a lasting no-Ponzi game (in the sense of a sustained weak Ponzi game), depending on the values of the parameters relative to critical values. The closer a parameter value lies to the corresponding single critical value, the longer the system will survive. Based on this short sensitivity analysis we assume that a seemingly doomed system may possibly be saved by means of initiating an advantageous parameter value transition. The opposite could possibly happen as well: a seemingly safe system is doomed after

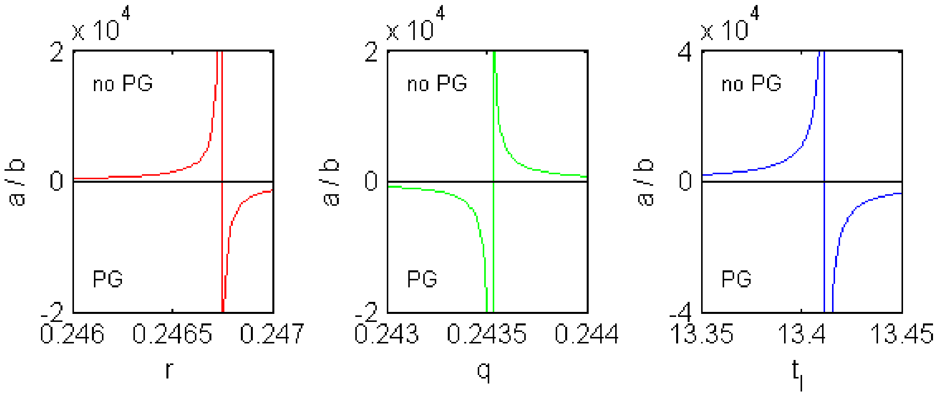

Figure 6: Determination of single parameter critical values. Some graphs for the wNPC function a/b are shown, where a and b are the parameter-dependent coefficients of the *abc*-model solution as applied to the social security scenario presented in figure 5. For illustration, the parameters r, q, and t_I are chosen to act as single variables. Positive values correspond to no-Ponzi games (no PG) in the weak sense (i.e., to weak Ponzi games), negative values to strong Ponzi games (PG). For the three parameters shown, the function exhibits singularities with a flipping sign, thus identifying the critical values.

an unfavourable shift of a single parameter value. However, the model investigated so far keeps the parameter values constant, no sudden *a posteriori* changes are considered. Closer inspection of the effects on a running system's fate due to single or multiple parameter value changes remains a topic for future research.

6 Conclusions

To summarize, the nine parameter PGBM model for a Ponzi scheme presented in this paper is part of the broad family of consumption-and-saving models and is formulated within the well-known framework of geometric Brownian motion (GBM) with one stochastic variable (namely, the market-based interest rate). Particularly, the inhomogenous drift term includes two exponential terms. The formal model solution corresponds to a transformed solution of the familiar GBM process. The first moment of the process, i.e., the expected value for the available capital, is given as the solution of an *abc*-model. Also considering the second moment, the probability density function for the available capital is either given analytically by means of a dynamic lognormal distribution (requiring positive funding) or numerically (allowing for negative capital values as well). The numerical results are used to calculate Ponzi scheme bust probabilities at any time of capital evolution. Depending on the values of the model parameters relative to their critical values, this evolution may exhibit either a peak followed by a final saturation of the capital (representing a non-lasting Ponzi game) or an uprising trend (representing a sustained no-Ponzi game). This ambivalence is due to a weak no-Ponzi game condition (wNPC) that ignores a limit on the number of participants. However, the number of participants cannot realistically exceed a maximum number. Therefore, even if a system may satisfy the wNPC, at some time point the reservoir of new participants is exhausted and the system *as designed here* (i.e., with an exponentially increasing number of new members in each period) won't work anymore.

Future work may cover model improvements as well as applications and hence include for example the following topics:

(i) The Ponzi scheme model introduced here optionally allows for a variant with or without any withdrawals after some investment span. Withdrawals are, however, expected to be *discrete*. Some modification is wished for that implements stepwise discrete (as in Parodi 2013) or continuous withdrawals, too. The latter variant would faciliate a direct comparison of the capital dynamics as exhibited by our model with that of Artzrouni (2009).

(ii) The lognormal probability distribution function is only appropriate for the stochastic PGBM process as long as one considers time intervals before the first passage time, i.e., before the probability for negative capital values becomes non-zero. However, when determined numerically or by means of Monte-Carlo simulations the distribution smoothly passes through this critical time point (cf. figure 3). As some tests showed (but not discussed in this paper), these post-first-passage-time distributions are no more of lognormal type. Attempts by the author to perform a "right-wing approximation", i.e., to fit only the right wings of lognormal distributions to those of the simulated distributions, were at best moderately promising. As seen from a theoretical viewpoint, finding an analytical expression for the first passage time may be more helpful (but may not be mandatory) when tackling this question concerning the breakdown of the lognormal pdf description and some analytical continuation.

(iii) The PGBM model introduced in this paper assumes frozen parameter values that are given *a priori*. An analytical model extension will include and study the reaction of the system to sudden *a posteriori* changes of single or multiple parameters. For example, in the social security model a lurking deficit could be eliminated through a reform that encompasses an increased retirement age (corresponding to increasing I_0 and lowering t_I), benefit reductions (lowering r) and increasing revenues (higher q and I_0). However, for practical reasons such a sensitivity analysis could of course easily be performed numerically, that is, simply by chosing some initial conditions and iterating the evolution equation (2).

(iv) More realistic volatility models than GBM may be considered as the basic market process, for example models with stochastic volatility or other jump processes. However, with respect to analytical tractability this quickly becomes rather demanding or even unrealizable. Introducing stochasticity numerically in the manner of Mayorga-Zambrano (2011) and Quituisaca-Samaniego et al. (2013), i.e., describing capital evolution by means of recursion relations that include stochastic parameters (e.g., with normally distributed values), is an efficient alternative.

(v) Last but not least, theory profits when it meets with reality, and vice versa. While the PGBM model's clarity and analytical tractability is of didactic value, finding some proper applications would increase the model's usefulness. Realms of possible applications include for example real scam investment systems, the handling of governmental debts, or possibly micro credit systems considered as some kind of reversed Ponzi games (i.e., with negative investment $I_0<0$ corresponding to loans offered by wealthy initiators). Of particular contemporary interest is the comparison of possible social security model outcomes with fact data based on real systems. Official examples of real social security systems that exhibit peaked graphs for the predicted capital evolution (resembling those shown in figure 5) are not hard to find in the internet. Instead of providing hard-won percentiles often several graphs are simultaneously displayed for the mean paths of a best-, mid-, and worst-case scenario. If models like the one presented in this paper (or more sophisticated ones including more demographic and financial details) matching real data would be satisfyingly achievable, the discussion about Ponzi-like aspects of classical social security systems would gain a refined quality. Possibly there would be more agreement upon a statement like "social security may work as a weak Ponzi scheme".

Appendix: Numerical solution of the Kolmogorov forward equation for the evolution of the PGBM-pdf

In order to numerically compute equation (23), we use techniques offered by computational finance. In particular, we proceed in close analogy to a standard procedure as suggested for example in Schwab & Hilber (2007) for solving the Black-Scholes partial differential equation, with modifications where appropriate. For that purpose we apply finite differences in space and the Crank-Nicolson scheme in time, as now disclosed in detail.

(i) *Variable transformation.* Applying the chain rule for derivations and replacing the variable for the capital, X, by a grid space variable x, we write equation (23) in the form

$$\partial_t P(x,t) = \frac{\sigma^2}{2} x^2 \partial_x^2 P(x,t) + \left[\left(2\sigma^2 - p\right) x + p\hat{x}(t)\right] \partial_x P(x,t) + \left(\sigma^2 - p\right) P(x,t)\,, \quad (40)$$

where ∂_t denotes the first partial derivative with respect to time t and where ∂_x and ∂_x^2 denote the first and the second partial derivatives with respect to the space variable x, respectively. Numerical computation of such a parabolic partial differential equation (in our case possibly depending on large absolute values of x) both is computationally more efficient and allows for negative values of x (which is crucial for the PGBM model), if one performs a variable transformation $x = e^y + \overset{*}{x}$ (with some real $\overset{*}{x} \leq 0$ chosen such that $P(\overset{*}{x}, t) \approx 0$ for all relevant times). We accordingly replace $\partial_x = e^{-y}\partial_y$ and $x^2\partial_x^2 = \left(1 + \overset{*}{x}e^{-y}\right)^2 \left(\partial_y^2 - \partial_y\right)$ in equation (40) and demand $P(x,t) = Q(y,t)$. One obtains the transformed Kolmogorov forward equation

$$\partial_t Q(y,t) = \alpha(y)\, \partial_y^2 Q(y,t) + \beta(y,t)\, \partial_y Q(y,t) + \gamma\, Q(y,t) \quad (41a)$$

$$\alpha(y) = \frac{\sigma^2}{2}\left(1 + \overset{*}{x}e^{-y}\right)^2 \quad (41b)$$

$$\beta(y,t) = -\frac{\sigma^2}{2}\left(1 + \overset{*}{x}e^{-y}\right)^2 + \left(2\sigma^2 - p\right)\left(1 + \overset{*}{x}e^{-y}\right) + p\hat{x}(t)e^{-y} \quad (41c)$$

$$\gamma = \sigma^2 - p, \quad (41d)$$

with an initial probability density $Q(y,0) = \delta(y - \ln[X_0 - \overset{*}{x}])$ ($\forall y \in R$). Note that the variable dependent coefficients $\alpha(y)$ and $\beta(y,t)$ are *non-constant*, the latter also being explicitely *time-dependent*, whereas γ is a real constant.

(ii) *Discretization.* Equation (41) is discretized by means of finite differences (FD) in y and the θ-scheme in time. In particular, the restriction of y to a bounded domain $[a,b]$ with mesh size $\Delta y = (b-a)/(N+1)$ (i.e., $y_i{=}a{+}i\Delta y$, $i{=}0,1,2,...,N{+}1$) and to discrete time with steps $\Delta t = T/M$ (i.e., $t_m = m\Delta t$, $m = 0,1,2,...,M$) provides a column vector $\underline{Q}^m = (Q_1^m, Q_2^m, ..., Q_i^m, ..., Q_N^m)^\mathsf{T}$), with boundaries Q_0^m and Q_{N+1}^m, that undergoes time-stepping according to the θ-scheme as follows:

- implicit Euler discretization:

$$\frac{Q_i^{m+1} - Q_i^m}{\Delta t} = \alpha\frac{Q_{i+1}^{m+1} - 2Q_i^{m+1} + Q_{i-1}^{m+1}}{(\Delta y)^2} + \beta\frac{Q_{i+1}^{m+1} - Q_{i-1}^{m+1}}{2\Delta y} + \gamma Q_i^{m+1} =: f_{m+1}\ ;$$

- explicit Euler discretization:

$$\frac{Q_i^{m+1} - Q_i^m}{\Delta t} = \alpha\frac{Q_{i+1}^{m} - 2Q_i^{m} + Q_{i-1}^{m}}{(\Delta y)^2} + \beta\frac{Q_{i+1}^{m} - Q_{i-1}^{m}}{2\Delta y} + \gamma Q_i^{m} =: f_{m}\ ;$$

- Θ-scheme: $\frac{Q_i^{m+1} - Q_i^m}{\Delta t} = \Theta f_{m+1} + (1-\Theta) f_m$. Rearranging as $\frac{Q_i^{m+1}}{\Delta t} - \Theta f_{m+1} = \frac{Q_i^m}{\Delta t} + (1-\Theta) f_m$ leads to

$$\begin{aligned}&\frac{1}{\Delta t}Q_i^{m+1} - \Theta\alpha_{FD}^- Q_{i-1}^{m+1} - \Theta\beta_{FD}Q_i^{m+1} - \Theta\alpha_{FD}^+ Q_{i+1}^{m+1}\\ &= \frac{1}{\Delta t}Q_i^m + (1-\Theta)\alpha_{FD}^- Q_{i-1}^m + (1-\Theta)\beta_{FD}Q_i^m + (1-\Theta)\alpha_{FD}^+ Q_{i+1}^m,\end{aligned}$$

with discretization coefficients

$$\alpha_{FD}^{\pm} = \frac{\alpha}{(\Delta y)^2} \pm \frac{\beta}{2\Delta y}, \qquad \beta_{FD} = -\frac{2\alpha}{(\Delta y)^2} + \gamma, \tag{42}$$

being approximately determined at mesh points $y_i = a + i\Delta y$ in space and $t_m = m\Delta t$ in time. Hence, one actually has discrete, non-constant coefficients $\alpha = \alpha(y)$ and $\beta = \beta(y,t)$, and accordingly coefficients $\alpha_{FD}^{\pm}$ and β_{FD}. The above result corresponds to a linear system of equations that has to be solved at every time step. Written in matrix form, one has for example at time $m\Delta t$

$$\underline{Q}^{m+1} = \big(\mathbb{I} - \Delta t\, \Theta \mathbb{A}^m\big)^{-1}\big(\mathbb{I} + \Delta t\,(1-\Theta)\mathbb{A}^m\big)\underline{Q}^m\,, \tag{43}$$

where $\mathbb{I}$ is the diagonal $N \times N$ identity matrix and $\mathbb{A}^m$ is the tridiagonal $N \times N$ matrix

$$\mathbb{A}^m = \begin{pmatrix} \beta_{FD} & \alpha_{FD}^{+} & 0 & \dots & 0 \\ \alpha_{FD}^{-} & \beta_{FD} & \alpha_{FD}^{+} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \alpha_{FD}^{-} & \beta_{FD} & \alpha_{FD}^{+} \\ 0 & \dots & 0 & \alpha_{FD}^{-} & \beta_{FD} \end{pmatrix} \tag{44}$$

We set $\Theta = 1/2$ (Crank-Nicolson Θ-scheme). Note that a superscript involving m is not to be read as an exponent but as an index of time. Note furthermore that —unlike with the most basic procedure for solving the Black-Scholes equation with constant discretization coefficients— iteration of equation (43) requires the space- and time-dependent matrix $\mathbb{A}^m$ to be recalculated at every time step m, a rather time-consuming computational side effect. The above formalism was implemented into a MATLAB 7.0.1-routine (release 14, code available from the author upon request) and was run on a common laptop in order to provide the numerical results presented in figures 1 to 4.

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