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Lin, Hwan C. and Shampine, L.F.

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Hwan C. Lin* L.F. Shampine†

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Abstract

The statutory patent length is 20 years in most countries. R&D-based endogenous growth models, however, often presume an *infinite* patent length. In this paper, *finite-length* patents are embedded in a non-scale R&D-based growth model, but any patent's effective life may be terminated prematurely at any moment, subject to two idiosyncratic hazards of imitation and innovation. This gives rise to an autonomous system of mixed-type functional differential equations (FDEs). Its dynamics are driven by current, delayed and advanced states. We present an algorithm to solve the FDEs by solving a sequence of standard BVPs (boundary value problems) for systems of ODEs (ordinary differential equations). We use this algorithm to simulate a calibrated U.S. economy's transitional dynamics by making discrete changes from the baseline 20 years patent length. We find that if transitional impacts are taken into account, optimizing the patent length incurs a welfare loss, albeit rather small. This suggests that fine-tuning the world's patent systems may not be a worthwhile effort. [JEL Classification: C63, O31, O34].

Key Words: Patent Length, Innovation, Delay Differential Equation, Advance Differential Equation, Transitional Dynamics, Endogenous Growth

*Corresponding Author: Department of Economics, University of North Carolina at Charlotte, 9201 University City Blvd., Charlotte, NC 28223, U.S.A. Tel: 1-704-687-7646. Email: hwlin@uncc.edu

†Professor Emeritus in the Mathematics Department at Southern Methodist University. Email: lfshampine@aol.com.

1 Introduction

Patents represent a commercial reward system instituted to promote technological innovation with short-lived monopoly power, so as to sustain economic growth in the long run. In most national patent laws, the statutory patent term (or patent length) is twenty years, usually beginning with the patent filing date.[1] The patentee can then command exclusive rights to produce and sell patented products or processes, as long as the associated patent remains legally alive. In the early 1990s, Romer (1990), Segerstrom et al. (1990), Grossman and Helpman (1990a,b) and Aghion and Howitt (1992) pioneered the development of R&D-based endogenous growth models. These authors made path-breaking contributions to endogenous growth theory by explaining persistent per capita growth as a general equilibrium outcome of profit-oriented innovation. With no exception, these seminal works simplified model construction by assuming *infinite-length* patents, while also exhibiting counterfactual scale effects. Though the scale effects feature has been purged from second-generation R&D-based growth models, the infinite patent assumption persists.[2] This assumption makes R&D-based growth models an awkward vehicle for policy analysis of patents and other intellectual property rights (IPR).

The purpose of this paper is to incorporate an empirically-consistent patent system into a continuous-time, non-scale R&D-based growth model for a decentralized perfect-foresight macroeconomy. In this economy, *finite-length* patents are granted to protect innovation of new varieties of capital goods. Besides, as patent protection tends to be imperfect in the real world, our model dictates that all legally live patents confront two distinct idiosyncratic risks at any moment: one refers to an *imitation hazard* of imitative goods coming to underprice some patented goods, while the other pertains to Schumpeterian creative destruction or an *innovation hazard* of innovative goods emerging from research activities that drive some patented and non-patented goods together out of the market. These two hazards can therefore prematurely terminate a finite-length patent's *effective life*, as in the real world.

The contribution of our work is threefold: *First*, for a continuous-time perfect-foresight macroeconomy with *finite-length* patents, we derive a full-fledged autonomous system represented by *functional differential*

[1] The World Trade Organization's TRIPS (Trade-Related Aspects of Intellectual Property Rights) Agreement has been in force since 1995, thereby largely harmonizing patent terms across national laws. The TRIPS Agreement introduced global minimum standards for protecting intellectual property rights (IPR), including those for patents. For instance, Article 33 of the TRIPs Agreement provides, "The term of protection available for patents shall not end before the expiration of a period of twenty years counted from the filing date."

[2] Second-generation endogenous growth models include Jones (1995b), Peretto (1998) , Segerstrom (1998), Young (1998). These works came as a response to Jones (1995a), who criticized Romer (1990) and other seminal R&D-based endogenous growth models as containing empirically inconsistent scale effects.

equations (FDEs) of mixed type.[3] *Second*, we present an iteration algorithm that can be used to solve numerically the FDE system as a sequence of standard boundary value problems (BVPs) for ordinary differential equations (ODEs).[4] *Third*, we calibrate the model to the US economy, respectively, using a benchmark and an alternative parameter set. Both parameter sets contain different features, but each permits the calibrated model to largely mimic the US economy's stylized facts. For each parameter set, we compute the long-run optimal patent length, which maximizes steady-state welfare on the model's balanced-growth path. We also quantify the intertemporal welfare effects of changing current 20 years patent length to the long-run optimal level.

In contrast to Nordhaus (1969), the welfare effects of optimizing the patent length for a dynamic macroeconomy can boil down to a change in the aggregate saving rate and a change in the aggregate income arising from changes in the economy's monopoly-distorted total factor productivity (TFP) and capital stock.[5] We find that the long-run optimal patent length can be either infinite or close to 23.17 years. The infinite patent length is optimal in the long run if we use a (benchmark) parameter set that features a relatively *large* knowledge-spillover externality and a relatively *small* innovation hazard. However, the long-run optimal patent length becomes finite at 23.17 years if we instead use a parameter set that features a relatively *small* knowledge-spillover externality and a relatively *large* innovation hazard. Regardless, we find that optimizing the patent length has only negligible welfare effects under either parameter set. From our transition-dynamics simulations, extending the patent length to the long-run optimal level can promote technological innovation and thus the economy's total factor productivity (TFP), but this innovation-driven productivity gain comes at the expense of short-run physical capital *decumulation* in transition, along with a permanent increase in the economy's saving rate. If such transitional impacts are taken into account, optimizing the patent length actually incurs a welfare loss, albeit rather small, under each of our parameter sets. This suggests that fine-tuning the world's patent systems may not be a worthwhile effort.

[3] A *functional differential equation* (FDE) refers to a differential equation in which the time derivative of the unknown function at the present time depends on the values of the function at (i) previous times or (ii) future times, or (iii) on the value of the function's delayed time derivative. Types (i) - (iii) refer to *delay differential equations* (DDEs), *advance differential equations* (ADEs), and *neutral-type differential equations*, respectively. In contrast, for an ordinary differential equation (ODE), the time derivative of the unknown function depends on the value of the function at the present time only. In the present paper, we deal with a mixed-type FDE system, as it contains differential equations of delayed type, advanced type, and neutral type. For details, see the classical books by Bellman (1963) and Hale (1977).

[4] The spirit of this algorithm is to replace the delayed and advanced arguments in the FDE system with previously computed approximations so as to obtain a BVP for a system of ODEs. It is solved with Matlab's BVP solver **bvp5c**. The solution is used to update the approximations in the FDE system and the process iterated to convergence. The BVP solver is discussed in Kierzenka and Shampine (2008). Matlab is a high-level language developed by The MathWorks, Inc., 3 Apple Hill Dr., Natick, MA 01760.

[5] Using partial equilibrium analysis, Nordhaus (1969) shows that the optimal patent length can be obtained by balancing the dynamic gain of innovation against the static monopoly inefficiency.

Changing the length of patents from *infinity* to a *finite* measure for R&D-based growth models presents a modeling and computational challenge. Inevitably, this change subjects the modeled economy's dynamic evolution to a higher-dimensional FDE system that depends not only on current states, but also on previous and future states. Note that it takes time for a patented monopoly to become a competitive firm. Once the patent length is finite at $T > 0$, as in our model, any patent issued previously exerts a T- period delayed effect on the dynamic evolution of the composition of patented and non-patented goods, despite that this patent's lifespan may be terminated prematurely due to the hazards of product imitation and creative destruction. Hence, past history plays a role in the FDE system. Further, forward-looking agents need to compute the discounted present value of a fresh patent, so as to optimize R&D inputs. This computation requires information about agent's perfect-foresight expectation, for instance, about the cumulative sum of future hazard rates from a T- period horizon. Hence, future states also enter the FDE system. As a result, the autonomous system derived in our paper is composed of mixed-type functional differential equations. Subject to two-point boundary conditions, this system functions in a decentralized economy and determines the dynamic evolution of seven time-varying variables including: (1) aggregate consumption, (2) physical capital stock, (3) knowledge stock (or variety of capital goods), (4) share of monopolistic firms in capital goods (5) value of fresh patents, (6) T- period average of future innovation hazard rates, and (7) T- period average of future interest rates.[6] To the best of our knowledge, there has been no solver available for solving such a nonlinear boundary-value FDE system that includes both *delay* and *advance differential equations*. As noted earlier, the iteration algorithm we present proves quite robust by solving the FDE system as a sequence of standard BVPs for systems of ODEs.

We recognize that whether the patent length is finite or infinite does not essentially matter to endogenous growth theory. Nevertheless, when it comes to the analysis of innovation policies, presuming patents of infinite length for R&D-based growth models becomes awkward or even problematic. For instance, there are problems arising from calibrating such R&D-based growth models to real-world economies, where a patented monopoly has a finite legal life and its effective life can be even shorter in a risky environment. Over the past two decades, there has been a burgeoning interest in the dynamic general equilibrium analysis of patent policy or the North-South IPR protection problem, using analytical frameworks based on various R&D-based growth models. A series of works along this line are inclusive of, but not limited to, Helpman

[6] Of these seven variables, capital stock, knowledge stock, and the share of monopolistic firms are predetermined system states, while the other four are "jump" variables at any moment. Our FDE system therefore presents a boundary value problem (BVP).

(1993), Lai (1998), Kwan and Lai (2003), O'Donoghue and Zweimuller (2004), Grinols and Lin (2006), Furukawa (2007), Chu (2009), Lin (2010), Chu et al. (2012), and Iwaisako and Futagami (2013). However, for the sake of tractability, these works continue to presume *infinite-length* patents. Is it likely that their policy analyses are actually not robust to patent lengths? This remains a largely unexplored question in the innovation and endogenous growth literature.

Some scholars have incorporated a finite-length patent system into R&D-based endogenous growth models. The seminal work of Judd (1985) shows that for a decentralized economy with no capital accumulation, infinite-length patents may achieve the first-best allocation in a dynamic general-equilibrium setting.[7] In contrast, using models with the scale-effect feature, Iwaisako and Futagami (2003), Futagami and Iwaisako (2007) and Lin (2013a) show that the optimal patent length is finite on the balanced-growth path.[8] Like Judd (1985), these works abstract from capital accumulation and do not evaluate the extent to which an economy can gain by optimizing the patent length, when transitional dynamics are taken into account. Chu (2010) incorporates a finite-length patent system in a non-scale R&D-based growth model and allows for interaction between capital and knowledge accumulation. However, his work is confined to a steady state analysis of the effects of patent lengths on R&D along the balanced-growth path.

In a nutshell, scholars have not worked out a full-fledged autonomous FDE system for a dynamic macroeconomy in which finite-length patents are used to protect profit-oriented innovations. The present paper can fill the gap in the endogenous growth literature. True, some scholars have worked to deal with functional differential equations resulting from non-patent factors such as investment gestation lags [e.g., Kalecki (1935), Asea and Zak (1999) and Collard et al. (2008)], or vintage physical capital [e.g., Benhabib and Rustichini (1991), Caballero and Hammour (1994), Boucekkine, Germain, and Licandro (1997), Boucekkine, Licandro, and Paul (1997) and Boucekkine et al. (2005)], or vintage human capital and schooling [e.g., Boucekkine, De la Croix, and Licandro (2002), Cozzi and Impullitti (2010), Cozzi and Galli (2014), and Bambi et al. (2014)]. In contrast to these works, our paper deals with a higher-dimensional boundary value FDE system, which contains differential-difference equations not only of delayed and advanced type, but also of neutral type (see Footnote 3). In this perspective, our work contributes to computational economics by presenting an iteration algorithm that can solve mixed-type FDE boundary value problems.

[7] However, Judd (1985) also shows that if household welfare is represented by non-CES utility functions, infinite-length patents may lead to excessive or deficient innovation in equilibrium.

[8] To be sure, Futagami and Iwaisako (2007) used discrete-time scale and nonscale R&D-based growth models, respectively, to address the optimal patent length problem. But we note that they did not deal with transitional dynamics for the nonscale growth model.

The rest of the paper is organized as follows. Section 2 lays out a non-scale R&D-based growth model of capital goods variety expansion. Section 3 derives an autonomous FDE system and presents an iteration algorithm to solve this system as a sequence of standard boundary value problems for systems of ODEs. Section 4 computes the long-run optimal patent length on a decentralized economy's balance-growth path under two alternative parameter sets. In this section, the iteration algorithm is used to simulate the calibrated model's transition dynamics when the patent length is raised from 20 years to the long-run optimal level. Growth and intertemporal welfare effects are evaluated numerically. Section 6 concludes.

2 The Model

We formulated a patent-regime closed economy using a continuous-time R&D-based growth model with no uncertainty and scale effects. This model has atomistic agents who are forward-looking households or firms with perfect foresight about the course of the economy. Of them, infinitely-lived households optimize intertemporal consumption while firms seek to maximize profits, respectively, in production of final goods, capital goods (or durables) and blueprints (or designs) for making these durables. Innovation of blueprints are patentable, so an individual firm who acquires a patent is granted exclusive rights to produce and market a patented durable. This is a Romer-Jones-type model in that patents create monopolies as an incentive to innovate capital-good varieties, thereby driving the level of total factor productivity to grow persistently in the long run.[9]

What differentiates the present model from the endogenous growth literature is that we introduce a more empirically consistent patent system. First, instead of being granted a perpetual legal life, all patents have an uniformly finite patent term (or patent length), denoted by $T < \infty$. Second, patent protection is imperfect; therefore, even prior to a patent's maturity date, newer innovations may emerge to drive some patented and non-patented firms together out of the market outright, or competitive imitations may come to force patented firms (monopolies) to relegate to competitive firms. Patents can therefore become worthless either expectantly at their maturity dates or prematurely if they cannot fortunately escape the hazards of innovation or imitation, which will be further explained later. Given this empirically plausible patent regime, we will

[9]The model of the paper differs from Romer (1990) in three dimensions. First, patents are finitely lived. Second, as in Rivera-Batiz and Romer (1991), research input is from forgone consumption rather than from labor (human capital). Third, as in Jones (1995a), the R&D (innovation) function displays diminishing marginal returns to the research input, thereby removing counterfactual scale effects. With these modifications, this model is closely related to Jones and Williams (2000) and Eicher and Turnovsky (2001).

show that the Romer-Jones-type endogenous growth model is no longer represented by an autonomous ordinary differential equations (ODEs) system, but by a much more complicated one composed of functional differential equations (FDEs). We lay out the model below.

2.1 Households

First, the economy is populated with a continuum of identical households of measure one. Each household has the mass of $L(t)$ identical workers at time t and it keeps growing at a constant rate of $n > 0$. Hence, the economy's time t population or labor force is equal to $L(t) = e^{nt}$, given that $L(0)$ is normalized to one. Starting at $t = 0$, the representative household's decision problem is to maximize her family lifetime utility (1a), subject to the flow budget constraint (1b):

$$\max_c U = \int_0^\infty \left(\frac{c(t)^{1-\gamma} - 1}{1-\gamma} \right) e^{-(\rho-n)t} dt, \ \rho > 0, \ \gamma > 0 \tag{1a}$$

$$s.t.: \quad \dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \tag{1b}$$

In (1a), ρ and γ are two preference parameters with ρ measuring the rate of time preference and $\frac{1}{\gamma}$ the elasticity of intertemporal substitution. As is well known, the utility flow of $\frac{c^{1-\gamma}-1}{1-\gamma}$ reduces to $\ln(c)$ if $\gamma = 1$. In the familiar flow budget constraint (1b), a is per-worker asset stock, $\dot{a} \equiv \frac{da}{dt}$ is the time derivative of a at a point in time, r is the risk-free rate of interest, $(r-n)a$ is per-worker net capital income, w is per-worker wage, and c is per-worker consumption. Households take as given all market prices such as w and r in their optimization problems. Solving the representative household's optimization problem yields the familiar Euler condition:

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\gamma} \equiv g_c(t), \ \text{ or } \ \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\gamma} + n \equiv g_C(t). \tag{2}$$

where $C(t) = c(t)L(t)$ measure the rate of the economy's aggregate consumption at time t. In solving household's decision problem, we have noticed that: (i) for the family lifetime utility to be bounded in the steady state, the parameter constraint of $\rho > n + (1-\gamma)g_c^*$ must to be invoked, where $g_c^* \equiv \frac{\dot{c}(\infty)}{c(\infty)}$ represents a constant steady-state growth rate of per-worker consumption; (ii) asset stock a consists of physical capital and vintage patents; and (iii) in the present model with no uncertainty and with well functioning financial markets, bond finance and equity finance are equivalent, and households must earn the same return on each type of asset under no-arbitrage conditions. Indeed, as will be seen later, the rate of interest r is equal to

the net return on physical capital, which in turn must be tied to the risk-adjusted rate of return on each vintage patent. Taking (iii) into account, we can formulate the flow budget constraint in terms of (1b) while neglecting the household's asset portfolio.[10]

2.2 Final goods

In the final-good sector, there are identical firms of measure one engaging in perfect competition. Final goods are consumables and can be saved to finance physical capital and R&D investments, based on household's intertemporal optimization. Let Y measure the economy's aggregate final output at a point in time produced with all available labor L and capital goods $X(i)$, $i \in [0,V]$, with V denoting the variety of capital goods that have been invented and have not yet been creatively destructed. The aggregate Y production function is given by

$$Y = (hL)^{1-\alpha}\left(\int_0^V X(i)^{\sigma\alpha}di\right)^{1/\sigma},\ 0<\sigma<\frac{1}{\alpha} \tag{3}$$

where h is exogenous and can be interpreted as a non-R&D driven efficiency term, hL is a measure of effective labor, and i refers to a specific type of capital goods, α is the capital share, $1-\alpha$ is the labor share, and σ is a positive parameter reflecting the degree of substitutability. Empirically, since the change in total factor productivity is not entirely driven by R&D, introducing the non-R&D driven efficiency term, h, permits better model calibration (Comin 2004). We assume that term h grows at a constant rate $g_h > 0$ so that $h(t) = e^{g_h t}$. Hence $h(t)L(t) = e^{(n+g_h)t}$ represents effective labor at time t. We chose final goods as the numeraire. Profit maximization by competitive final-good producers yields the economy's aggregate demands for labor and individual durables

$$wL = (1-\alpha)Y \tag{4}$$

$$p(i)X(i) = \alpha Y\cdot\left(\frac{X(i)^{\sigma\alpha}}{\int_0^V X(i)^{\sigma\alpha}di}\right),\ i\in[0,V] \tag{5}$$

where $1-\alpha$ is the labor share, α is the capital share, $p(i)$ is the price of type i capital good, and $X(i)$ is the quantity demanded of the type-i capital good. Later, we will discuss how the capital share α is split between capital rental income and monopoly profit. (5) implies that the price elasticity of demand for an individual

[10] Futagami and Iwaisako (2007) use a discrete-time endogenous growth model with finite patent length $T > 0$. They formulate the household's flow budget constraint by taking account of vintage patents portfolio in a countable set $[1,2,\ldots,T]$. Our flow budget constraint is much simpler, even with a continuum of vintage patents in a closed interval $[0,T]$. Later, in subsection 2.3.2, we will show how risk-adjusted returns on vintage patents can be equalized.

durable is equal to $\varepsilon \equiv \frac{1}{1-\sigma\alpha} > 1$ due to $\sigma\alpha < 1$.[11].

Final output Y is allocated for consumption C, capital investment I_K, and R&D investment I_V. Hence, the market equilibrium condition for final goods at any moment is given by

$$Y = C + I_K + I_V, \text{ or } s \equiv 1 - \frac{C}{Y} = s_K + s_V \tag{6}$$

where s is the aggregate saving rate, $s_K \equiv \frac{I_K}{Y}$ is the capital investment rate, and $s_V \equiv \frac{I_V}{Y}$ is the economy's R&D intensity. Yet, R&D is expensed in accounting practice, so the aggregate saving rate is represented by s_K in the real world. Investment in capital goods accumulates the stock of physical capital K, while investment in R&D creates new designs for capital goods and thus raises the variety of these goods V, which is taken as a proxy of knowledge stock. Without loss of generality, we assume that one unit of any type of capital goods requires one unit of forgone consumption. So, the stock of physical capital can be measured by $K = \int_0^V X(i)di$. Stocks K and V are predetermined at any moment and they evolve over time according to

$$\dot{K} \equiv \frac{dK}{dt} = I_K - \delta K = s_K Y - \delta K, \ \ \delta > 0 \tag{7a}$$

$$(1+\psi)\dot{V} \equiv \frac{dV}{dt} = \xi I_V = \xi s_V Y, \ \ \psi > 0 \tag{7b}$$

where δ is a constant rate of capital depreciation, ψ is a constant coefficient of Schumpeterian creative destruction, and ξ is an endogenous measure of research productivity. These latter two terms (ψ, ξ) relate to innovation of capital goods, based on Jones and Williams (2000). Their roles are discussed in order. First, in terms of (7b), the R&D input of I_V, along with the research productivity of ξ, can produce the mass of new capital-good designs equal to $(1+\psi)\dot{V}$ at a point in time. Yet, at the same time, these innovations create new rivals and destruct $\psi\dot{V}$ existing firms immediately in the capital-good sector. Accordingly, the innovation of designs merely leads to a net increase in V equal to $\dot{V}$ rather than $(1+\psi)\dot{V}$. Creative destruction is thus endogenized: when more innovative designs arrive, more of existing capital goods are driven out of the market. Later, we will explain how the creative-destruction coefficient ψ can result in an innovation hazard rate confronting incumbent firms at any moment, and how this hazard rate can affect over time the composition of patented and non-patented firms under a patent system that grants finite-length patents. Next,

[11] If parameter σ were set equal to one, production function (3) would reduce to that of Romer (1990). As such, durables would become neither substitutable nor complementary and the markup $\eta \equiv \frac{1}{\sigma\alpha}$ would reduce to $\frac{1}{\alpha}$. In this paper, σ is less than $1/\alpha$ so that the markup η is greater than one and does not have to be fixed at the inverse of the capital share (α), as in Jones and Williams (2000) and Eicher and Turnovsky (2001).

consider research productivity. This productivity is endogenously determined in terms of $\xi = \mu V^{\phi_1} I_V^{\phi_2 - 1}$, where μ, ϕ_1 and ϕ_2 are three positive parameters: (i) $\mu > 0$ is an exogenous technical term; (ii) ϕ_1 is assumed to satisfy $0 < \phi_1 < \phi_{max} \equiv 1 - \frac{\phi_2}{\varepsilon\sigma(1-\alpha)} < 1$ so that there is a positive externality of knowledge spillovers while ensuring a bounded long-run innovation rate in the model;[12] (iii) and ϕ_2 is assumed to satisfy $0 < \phi_2 < 1$, so as to capture the effect of research congestion, for example, because of too many research firms conducting similar projects in patent races. That is, with $0 < \phi_2 < 1$, research input I_V displays diminishing social marginal return on research productivity ξ.

2.3 Capital goods, R&D, and the patent system

We now turn to capital-good and R&D firms and formulate how they operate when patents have a finite and uniform term, as mandated by most patent laws.

2.3.1 Entry, patent length, hazards of innovation and imitation, and short-lived monopoly

There is an unit-measure of small and symmetric R&D firms. At the outset, one must notice that as soon as a new design is invented, the innovator receives a term-T patent and a new capital-good firm is established immediately either by the innovator or by whoever has paid a sunk cost for this patent. The sunk entry cost, denoted by υ, is the price of newly issued patents (called "fresh patents"). In equilibrium, the price of fresh patents must equal the marginal R&D cost of inventing a new design. That is, $\upsilon = 1/\xi$ at any moment when perfectly competitive research firms optimize their research inputs.[13] Next, once a new firm has been established in the capital-good sector, it becomes a patented monopoly producing and selling a new type of durables at a monopoly price. To produce one unit of durables, any firm must borrow from households one unit of foregone consumption at the prevailing interest rate r. Since forgone consumption depreciates at a constant rate of δ, the marginal cost of producing one unit of any type of durables is equal to $r + \delta$. Therefore, the price that any patented monopoly charges must uniformly be set at $p_p = \eta(r + \delta)$, given the markup rate of $\eta = \frac{\varepsilon}{\varepsilon - 1} = \frac{1}{\sigma\alpha} > 1$.[14] As well, given demand functions (5), any patented monopoly must

[12] From (7b) and (15), the instantaneous innovation rate is given by $g_V(t) \equiv \frac{\dot{V}(t)}{V(t)} = \mu V(t)^{\phi_1 - 1}(s_V(t)Y(t))^{\phi_2}$. On a balanced-growth path, it can be shown that the long-run steady-state innovation rate is determined by $\lim_{t\to\infty} g_V(t) = \frac{\phi_2(n+g_h)}{1-\phi_1-\phi_2/\varepsilon\sigma(1-\alpha)}$, which is bounded and positive if and only if ϕ_1 is less than an upper bound denoted by $\phi_{max} \equiv 1 - \frac{\phi_2}{\varepsilon\sigma(1-\alpha)} < 1$.

[13] Any individual research firm is atomistic and views research productivity ξ as constant in (7b). Thus, ξ is the marginal R&D productivity and the inverse of ξ is the marginal R&D cost of inventing a new design.

[14] Recall that $\varepsilon \equiv \frac{1}{1-\sigma\alpha}$ is the price elasticity of demand for any individual durable; see (5).

produce the same output level (X_p) and earn the same monopoly profit flow equal to $\pi = (\eta - 1)(r + \delta)X_p$ at a point in time, irrespective of the vintage of the patent it holds. However, the profit flow is time-varying or state-dependent and cannot last forever for each of the following three reasons:

1. **Finite patent length** (T): if a new patent is issued at time t, the monopolistic firm who holds this patent must reduce to a perfectly competitive firm by the maturity date of $t+T$ and can only henceforth sell its capital good at a common competitive price $p_{np} = r + \delta$. This is because when a patent expires, the patented design becomes freely accessible.

2. **Innovation hazard rate** (λ): At a point in time, there are $V(t)$ types of capital goods available on the market. As noted earlier (see equation (7b)), in an instant dt, there are $(1+\psi)\dot{V}(t)dt$ new designs invented (new capital-good firms established), and at the same time, $\psi\dot{V}(t)dt$ types of existing capital goods are driven out of the market. Thus, any capital-good firm confronts the same innovation hazard of creative destruction. The probability that an incumbent firm, monopolistic or competitive, is to be driven out of the market in an instant is equal to $\frac{\psi\dot{V}(t)dt}{V(t)} = \psi g_V(t)dt \equiv \lambda(t)dt > 0$, where $\lambda(t) \equiv \psi g_V(t)$ is the instantaneous innovation-hazard rate. This hazard rate is endogenously determined by the time-varying innovation rate $g_V(t)$, given a constant creative-destruction coefficient of $\psi > 0$. Hence, with an ongoing innovation process ($g_V(t) > 0$), a patented monopoly could be displaced by newer innovations in any instant with probability $\lambda(t)dt$, even if the patent is legally alive.

3. **Imitation hazard rate** (λ_m): The hazard of imitation by competitive firms is another factor that may force any patented monopoly to relegate prematurely to another competitive firm. This hazard prevails because patent protection is imperfect, so that a patented firm can lose its monopoly power at any moment before the patent expires. Following Helpman (1993), we assume an exogenous imitation hazard rate, denoted by $\lambda_m > 0$, so that $\lambda_m dt$ is the probability that a patented monopoly is to relegate to a competitive firm in an instant. There are $V_p(t)$ patented monopolies out of $V(t)$ capital-good firms at a point in time. In an instant, there will be $\lambda_m V_p(t)dt$ patented monopolies that are forced to become competitive firms prematurely.

2.3.2 The survival function, life expectancy, and value of patents

As discussed above, patented monopolies confront two idiosyncratic risks (i.e., innovation hazard rate $\lambda(t)$ and imitation hazard rate λ_m) at any point in time. Thus, with T patent length, the probability that a

monopoly that holds a legally live patent of vintage τ at time $t \geq \tau$ is to survive to the time point of $t' \geq t$ can be measured by the survival function,

$$\mathbb{S}(t,t' \mid \tau \leq t) = e^{-(\int_t^{t'} [\lambda(s)+\lambda_m]ds)} \text{ for } \tau \leq t \leq t' \leq \tau + T \tag{8}$$

where $\int_t^{t'} \lambda(s)ds = \int_t^{t'} \psi g_V(s)ds$ and $\int_t^{t'} \lambda_m ds = (t'-t)\lambda_m$ are innovation-induced and imitation-induced aggregate hazards, respectively, between t and t'. Note that with the patent length of $T \in (0,\infty)$, a patented monopoly's survival probability at time t must be forced to jump down to zero if $t > \tau + T$; that is, no monopoly is viable if holding a legally outdated patent. So, given that $\mathbb{S}(t,t' \mid \tau \leq t)$ is a continuous function of $t' \in (t, \tau + T)$ but discontinuous at maturity date $t' = \tau + T$, a vintage-τ patent's lifetime distribution function, denoted by $F(t,t' \mid \tau \leq t) = 1 - \mathbb{S}(t,t' \mid \tau \leq t)$, is also continuous for $t' \in (\tau, \tau + T)$, but discontinuous at $t' = \tau + T$.[15] Hence, $F(t,t' \mid \tau \leq t)$ is a mixed probability distribution function, and the life expectancy for a fresh patent issued at time t (that is, $\tau = t$) can therefore be determined by $T^e = \int_t^{t+T} (t'-t)dF(t,t' \mid \tau = t) + \mathbb{S}(t,t' = T \mid \tau \leq t) \cdot T$. Certainly, with no innovation and imitation hazards ($\psi = \lambda = \lambda_m = 0$), we observe that $\mathbb{S}(t,t' \mid \tau \leq t) = 1$ is fixed at one and jumps down to zero when the patent expires. In this case, a fresh patent's life expectancy is always equal the patent; i.e, $T^e = T$. However, in the economic environment where both innovation and imitation hazards prevail, and it can be shown that in a steady state equilibrium, a fresh patent's life expectancy is given by

$$T^e = \frac{1}{\psi g_V^* + \lambda_m} \left(1 - e^{-[\psi g_V^* + \lambda] \cdot T}\right) \equiv T^{e*} \tag{9}$$

As implied, a fresh patent's expected lifespan T^e must be shorter than the patent length T, and if patent terms are infinite ($T \to \infty$), the life expectancy is fixed at $\frac{1}{\psi g_V^* + \lambda_m}$, which is the inverse of combined hazard rates. Suppose that the sum of hazard rates is $\psi g_V^* + \lambda_m = 0.05$ and the patent length is $T = 20$ years. Then from (9), a patent's life expectancy is computed at 12.6 years only. The equation for computing T^e will prove useful for model calibration in the paper.

We now turn to the value of patents. With well-functioning financial markets, the no-arbitrage principle dictates that the equilibrium price υ of fresh patents is tied not only to the marginal R&D costs of $1/\xi$, as

[15] It can be shown that $\lambda(t') + \lambda_m = -\frac{dS(t,t'|\tau \leq t)/dt'}{S(t,t'|\tau \leq t)}$. This implies that hazard rates are conditional "death" probability densities. See Lin (2013a) for detailed discussion of the relationships among a patent's hazard rate, survival function and lifetime distribution function.

noted earlier, but also to the discounted present value of the future profit stream that a typical monopoly expects to earn under perfect foresight. That is, for fresh patents issued at time t (therefore, $\tau = t$), it holds that

$$\upsilon(t) = \int_t^{t+T} \pi(s)\mathbb{S}(t,s \mid \tau = t)e^{-\int_t^s r(t')dt'}ds = \int_t^{t+T} \pi(s)e^{-\int_t^s (r(t')+\lambda(t')+\lambda_m)dt'}ds \tag{10}$$

Using Leibniz's rule to differentiate $\upsilon[t]$ with respect to t yields:

$$r(t) = \frac{\dot{\upsilon}(t)}{\upsilon(t)} + \left(\frac{\pi(t)}{\upsilon(t)} - \frac{\pi(t+T)e^{-T\cdot[\bar{r}(t)+\bar{\lambda}(t)+\lambda_m]}}{\upsilon(t)}\right) - (\lambda(t)+\lambda_m) \tag{11}$$

where $\bar{r}(t)$ is a T periods forward average of future interest rates and $\bar{\lambda}(t)$ is a T periods forward average of future hazard rates of innovation. They are defined below:

$$\bar{r}(t) = \frac{1}{T}e^{-\int_t^{t+T} r(t')dt'} \text{ and } \dot{\bar{r}}(t) = \frac{1}{T}[r(t+T) - r(t)] \tag{12}$$

$$\bar{\lambda}(t) = \frac{1}{T}e^{-\int_t^{t+T} \lambda(t')dt'} \text{ and } \dot{\bar{\lambda}}(t) = \frac{1}{T}[\lambda(t+T) - \lambda(t)] \tag{13}$$

Eq.(11) is a no-arbitrage condition that governs private equity investment in newly issued patents. It says that in the economic environment with no uncertainty, the risk-free interest rate $r(t)$ must be equal to the risk-adjusted rate of return on equity investment. As indicated by the right-hand side of (11), the risk-adjusted return on fresh-patent equity investment is the sum of capital gains rate $\frac{\dot{\upsilon}(t)}{\upsilon(t)}$ and net dividend yield $\frac{\pi(t)}{\upsilon(t)} - \frac{\pi(t+T)e^{-T\cdot[\bar{r}(t)+\bar{\lambda}(t)+\lambda_m]}}{\upsilon(t)}$ minus this investment's risk premium, which is determined by the innovation-hazard rate $\lambda(t)$ plus the imitation-hazard rate λ_m. For private equity investment in vintage patents, the no-arbitrage condition reduces to $r(t) = \frac{\dot{\upsilon}_\tau(t)}{\upsilon_\tau(t)} + \frac{\pi_\tau(t)}{\upsilon_\tau(t)} - (\lambda(t)+\lambda_m)$, where $\pi_\tau(t) = \pi(t)$ for any vintage patent and $\upsilon_\tau(t)$ is the time-t price of patents of vintage $\tau < t$.[16] Certainly, with a longer expected monopoly duration, newly issued patents are more valuable than vintage ones [i.e., $\upsilon(t) > \upsilon_\tau(t)$]. Regardless, investing in either fresh or vintage patents, households can always earn the same risk-adjusted rate of return, as this return must be tied to the risk-free interest rate, r, as shown above. This is not a surprise, because the hazards of innovation and imitation are two idiosyncratic risks. As such, we can neglect household's asset portfolio and use eq. (1b) to formulate the flow budget constraint.

[16] This condition can easily be obtained patents by taking the time derivative of $\upsilon_\tau(t) = \int_t^{\tau+T} \pi(t')e^{-\int_t^{t'}[r(s)+\lambda(s)+\lambda_m]ds}dt'$, where the upper bound $\tau+T$ is independent of t.

2.3.3 Dynamics of the composition of patented and non-patented durables

In this subsection, we show how the patent length (T), the innovation hazard rate (λ), and the imitation hazard rate (λ_m) interact to determine the structural dynamics of the capital-good sector. First, there are V capital-good firms at a time and these incumbent firms can be split into (i) V_p patented monopolies, (ii) $V_{p'}$ competitive firms whose patents are legally alive but worthless, and (iii) V_{np} competitive firms whose patents have expired. Hence, the identity of $V(t) = V_p(t) + V_{p'}(t) + V_{np}(t)$ must hold at any moment. Differentiating this identity with time t, it holds that $\dot{V}(t) = \dot{V}_p(t) + \dot{V}_{p'}(t) + \dot{V}_{np}(t)$, where $\dot{V}_{p'}(t) = \lambda_m V_p(t)$ due to the hazard of imitation and $\dot{V}_{np}(t) = (1+\psi)\dot{V}(t-T)e^{-\int_{t-T}^{t}\lambda(t')dt'}$ due to the hazard of innovation (creative destruction).[17] Next, using these relationships, we can differentiate the fraction of patented monopolies, denoted by $\zeta(t) \equiv V_p(t)/V(t)$, to obtain the equation of motion for $\zeta(t)$:

$$\dot{\zeta}(t) = [1-\zeta(t)]\frac{\dot{V}(t)}{V(t)} - \lambda_m \zeta(t) - (1+\psi)\frac{\dot{V}(t-T)}{V(t)}e^{-T\bar{\lambda}(t-T)} \tag{14}$$

where $\bar{\lambda}(t-T) = \frac{1}{T}\int_{t-T}^{t}\lambda(t')dt'$ based on (13). From this equation, the fraction of patented monopolies tends to increase with current innovation $\dot{V}(t)$. However, previous innovation $\dot{V}(t-T)$ tends to decrease this fraction with a delayed effect, subject to the patent length T. Consistent with out intuition, the hazards of innovation and imitation also tend to shrink the relative size of patented monopolies in the capital-good sector.

2.4 GDP, TFP and gross capital income

We close the model by deriving a reduced-form final-good production function. Consider physical capital stock $K = \int_0^V X[i]di = V_p X_p + V_{p'} X_{p'} + V_{np} X_{np}$ and capital goods variety $V = V_p + V_{p'} + V_{np}$, where X_p, $X_{p'}$ and X_{np} are representative output levels per type of capital goods, respectively, produced by V_p patented monopolies, $V_{p'}$ patented competitive firms and V_{np} non-patented competitive firms. Since output X_p is priced at $p_p = \eta(r+\delta)$ and outputs $X_{p'}$ and X_{np} sell at $p_{np} = r+\delta$, demand functions of capital goods (5) imply $X_{np} = X_{p'} = \eta^{\varepsilon} X_p$ and thus $K = V X_{np}(\eta^{-\varepsilon}\zeta + 1 - \zeta)$. With these relationships, the economy's

[17] To see how the hazard of innovation determine $\dot{V}_{np}(t)$, consider the issuing of $(1+\psi)\dot{V}(t-T)$ new patents at time $t-T$. Some of these then fresh patents will be destructed prematurely, and any surviving patents must expire at time t. With a time-varying innovation-hazard rate, $\lambda(t)$, the mass of surviving vintage patents at time t is given by $(1+\psi)\dot{V}(t-T)e^{-\int_{t-T}^{t}\lambda(t')dt'}$. Patented monopolies of these surviving patents then turn into competitive firms at time t, so we have $\dot{V}_{np}(t) = (1+\psi)\dot{V}(t-T)e^{-\int_{t-T}^{t}\lambda(t')dt'}$.

final-good production function (3) can reduce to

$$Y(t) \quad = \quad z(t)[A(t)h(t)L(t)]^{1-\alpha}K(t)^{\alpha}, \tag{15}$$

where A and z are defined below:

$$A(t) \equiv V(t)^{\frac{1}{\varepsilon\sigma(1-\alpha)}}, \quad z(t) \equiv \frac{\left(\eta^{1-\varepsilon}\zeta(t)+1-\zeta(t)\right)^{1/\sigma}}{\left(\eta^{-\varepsilon}\zeta(t)+1-\zeta(t)\right)^{\alpha}}. \tag{16}$$

In equation (15), A is an endogenous technology term, z is a monopoly-distortion term, and $zA^{1-\alpha}h^{1-\alpha}$ is the level of monopoly-distorted TFP (total factor productivity). The instantaneous growth rate of TFP is then given by

$$g_{TFP}(t) = \frac{\dot{z}(t)}{z(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)} + (1-\alpha)g_h \tag{17}$$

where $(1-\alpha)\frac{\dot{A}(t)}{A(t)}$ and $(1-\alpha)g_h$ are R&D-driven and non-R&D driven TFP growth rates, respectively. As will become clear, the monopoly-distortion component does not have permanent effects on the economy's TFP growth, since both $\frac{\dot{\zeta}(t)}{\zeta(t)}$ and $\frac{\dot{z}(t)}{z(t)}$ must converge to zero along a balanced growth path. It is also important to note that the economy's gross capital income, αY, can be split into monopoly profit $V_p\pi$ and capital rental income $(r+\delta)K$ according to the following equations:[18]

$$V_p(t)\pi(t) = m(t)\alpha Y(t), \tag{18}$$

$$(r(t)+\delta)K(t) = (1-m(t))\alpha Y(t) \quad \text{or} \quad \frac{K}{Y} = \frac{(1-m(t))\alpha}{r(t)+\delta} \tag{19}$$

$$m(t) \equiv \frac{\zeta(t)(\eta-1)}{(1-\zeta(t))\eta^{\varepsilon}+\zeta(t)\eta} \in (0,1) \tag{20}$$

where term m is an endogenous fraction of gross capital income earned by patented monopolies. These equations say that if all durables were priced at marginal cost (i.e., $\eta = 1$) or if there were no monopolies ($\zeta = 0$), then the fraction term, m, would be zero and all capital income would come in the form of rents for foregone consumption. Thus far, we have built a dynamic general-equilibrium R&D-based growth model

[18] Equation (15) implies that the flow of gross capital income is given by $\alpha Y = \int_0^V p[i]X[i]di$. As well, αY can be split into gross rental income $(r+\delta)K$ and aggregate profit $\Pi = V_p\pi = \zeta V\pi$, where $V_p = \zeta V$ is the number of patented firms and $\pi = (\eta-1)(r+\delta)X_p$. Using the relative output relationship, $X_{np} = X_{p'} = \eta^{\varepsilon}X_p$, we can then derive the profit/rent ratio as $\frac{\Pi}{(r+\delta)K} = \frac{(\eta-1)\zeta}{\zeta+(1-\zeta)\eta^{\varepsilon}}$. Define $m \equiv \frac{\Pi}{\alpha Y}$. Then $1-m = \frac{(r+\delta)K}{\alpha Y}$ and $\frac{m}{1-m} = \frac{(\eta-1)\zeta}{\zeta+(1-\zeta)\eta^{\varepsilon}}$, which can solve for m, as in (20).

for a decentralized macroeconomy in which patents have a finite legal life. The next focus is on the model's computational aspects.

3 Autonomous System and Numerical Algorithm

In this section we discuss the model's autonomous system and present an iteration algorithm to solve it.

3.1 Functional differential equations (FDEs) of mixed type

The model we have presented above include seven equations of motion given by (2), (7a), (7b), (11), (12), (13), and (14). These equations constitute an autonomous system that drives the modeled economy to evolve over time. This system is given below:

$$\dot{C}(t) = C(t)\left(\frac{r(t)-\rho}{\gamma}+n\right) \tag{21a}$$

$$\dot{K}(t) = s_K(t)Y(t) - \delta K(t) \tag{21b}$$

$$\dot{V}(t) = \left(\frac{\mu}{1+\psi}\right)V(t)^{\phi_1}(s_V(t)Y(t))^{\phi_2} \tag{21c}$$

$$\dot{\upsilon}(t) = [r(t)+\lambda(t)+\lambda_m]\upsilon(t) - \pi(t) + \pi(t+T)e^{-T[\bar{r}(t)+\bar{\lambda}(t)+\lambda_m]} \tag{21d}$$

$$\dot{\bar{r}}(t) = \frac{1}{T}[r(t+T)-r(t)] \tag{21e}$$

$$\dot{\bar{\lambda}}(t) = \frac{1}{T}[\lambda(t+T)-\lambda(t)] \tag{21f}$$

$$\dot{\zeta}(t) = (1-\zeta(t))\frac{\dot{V}(t)}{V(t)} - \lambda_m\zeta(t) - (1+\psi)\frac{\dot{V}(t-T)}{V(t)}e^{-T\bar{\lambda}(t-T)} \tag{21g}$$

where the auxiliary variables of $Y(t')$, $s_K(t')$, $s_V(t')$, $r(t')$, $\pi(t')$, and $\lambda(t')$ are the functions of $C(t')$, $K(t')$, $V(t')$, $\zeta(t')$ and $\upsilon(t')$, with $t' = t$ or $t+T$, and these functions have been given in Section 2.[19]

Equations (21a) - (21g) represent a complicated nonlinear system composed of mixed-type functional differential equations (FDEs). In this system, (21a) - (21c) are three ordinary differential equations (ODEs),

[19]For instance, $Y(t)$ is a function of $K(t)$, $V(t)$ and $\zeta(t)$ based on (15) and (16), and so are $r(t) = \frac{(1-m(t))\alpha Y(t)}{K(t)} - \delta$ and $\pi(t) = \frac{m(t)\alpha Y(t)}{V(t)}$ in terms of (18)-(20). As well, we have shown $\lambda(t) = \psi \cdot \frac{\dot{V}(t)}{V(t)}$ in subsection 2.3.1 and we can write $s_V(t) = \left(\upsilon(t)\cdot\mu\cdot V(t)^{\phi_1}\right)^{1/(1-\phi_2)}/Y(t)$ based on the R&D equilibrium condition of $\upsilon(t) = 1/\xi(t)$. Therefore, $\lambda(t)$ and $s_V(t)$ are each the function of $K(t)$, $V(t)$, $\zeta(t)$ and $\upsilon(t)$. Certainly, $s_K(t) = 1 - \frac{C(t)}{Y(t)} - s_V(t)$ is the function of $C(t)$, $K(t)$, $V(t)$, $\zeta(t)$ and $\upsilon(t)$.

(21d) - (21f) are three advance differential equations (ADEs), and (21g) is a delay differential equation (DDE) of neutral type, since its delayed arguments $[\dot{V}(t-T), \bar{\lambda}(t-T)]$ contain a lagged derivative term.[20] In short, the perfect-foresight FDE system's dynamic evolution is driven by its current, previous and future states in an infinite time horizon. To the best of our knowledge, this paper is the first that shows how finite patent terms can give rise to a full-fledged autonomous FDE system for an R&D-based growth model.[21]

Patent length plays a critical role here. If patents have a finite statutory term, vintage patents must prevail, as in our R&D-based growth model. With the hazards of imitation and innovation, these vintage patents must therefore exert effects in a risk-adjusted delayed manner on the structural dynamics of the innovating capital-good sector, as discussed in subsection 2.3.3 and as formulated in (21g). As well, to optimize R&D inputs, forward-looking agents must calculate a fresh patent's discounted present value in a T periods forward manner, as reflected in (10), (11) and (21d). Hence, like vintage capital models (see, for instance, Benhabib and Rustichini 1991, Cozzi and Galli 2014 and Bambi et al. 2014), vintage patents must be in tandem with functional differential equations. Indeed, if the patent length T is set to be infinite, both (21e) and (21f) become redundant due to $e^{-T\bar{\lambda}(t)} \to 0$ in (21d) and $e^{-T[\bar{r}(t)+\bar{\lambda}(t)+\lambda_m]} \to 0$ in (21g). As such, the seven-dimensional FDE system of (21a) - (21g) must collapse into a simpler five-dimensional ODE system represented by (21a) - (21d) and (21g), where (21d) and (21g) reduce to $\dot{\upsilon}(t) = [r(t)+\lambda(t)+\lambda_m]\upsilon(t) - \pi(t)$ and $\dot{\zeta}(t) = (1-\zeta(t))\frac{\dot{V}(t)}{V(t)} - \lambda_m\zeta(t)$, respectively.

3.2 Transforming into a stationary dynamic system

In this paper, we present an iteration algorithm that can solve numerically the autonomous FDE system of (21a) - (21g). To solve it, however, we must first transform this system into a stationary one. This is because the FDE system includes *non-stationary* variables such as $K(t)$, $V(t)$, $Y(t)$, $C(t)$, $\upsilon(t)$, and $\pi(t)$, which can continue to grow over time along the long-run balanced-growth path, as opposed to this system's *stationary* variables such as interest rate $\bar{r}(t)$, $r(t)$, $\bar{\lambda}(t)$, $\lambda(t)$, $s_V(t)$, $s_K(t)$ and $\zeta(t)$, which must converge to their respective steady state levels in the long run. For those non-stationary variables, their constant steady-state

[20] See Footnote 3 for detailed definitions of different types of functional differential equations.

[21] Judd (1985) is the first work that shows functional differential equations can arise from finite-length patents in a dynamic general-equilibrium growth model. However, his simplified model does not allow to derive a full-fledged dynamic system for a decentralized macroeconomy.

growth rates, labeled by an asterisk, are given below (see Appendix A for derivations):

$$g_K^* = g_Y^* = g_C^* = \theta_K(n+g_h) \tag{22a}$$

$$g_V^* = \theta_V(n+g_h) \tag{22b}$$

$$g_A^* = \theta_A(n+g_h) \tag{22c}$$

$$g_\upsilon^* = g_\pi^* = \theta_\upsilon(n+g_h) \equiv (\theta_K - \theta_V)(n+g_h) \tag{22d}$$

where θ_x, $x \in \{V, A, K, \upsilon\}$, is a structural composite of growth-relevant parameters (α, σ, ε, ϕ_1, ϕ_2) :

$$\theta_V = \frac{\phi_2}{1-\phi_1-\phi_2/[\varepsilon\sigma(1-\alpha)]};\ \ \theta_A = \frac{\phi_2}{(1-\phi_1)[\varepsilon\sigma(1-\alpha)]-\phi_2};\ \ \theta_K = \theta_A + 1 = (\frac{1-\phi_1}{\phi_2})\theta_V. \tag{23}$$

These structural composites are coined "*growth kernels*" in Lin (2013b), since they translate a constant effective-labor growth rate, $n+g_h$, into an associated variable's long-run growth rate. Some features merit attention. First, with a common steady-state growth rate, physical capital stock K, finial output Y, and aggregate consumption C share the same growth kernel of θ_K in terms of (22a). Second, a fresh patent's price υ and monopoly profit π have the same steady-state growth rate, so they share the common growth kernel of $\theta_K - \theta_V$ according to (22d).[22] Third, a positive long-run innovation rate g_V^* requires that $0 < \phi_1 < 1 - \frac{\phi_2}{\varepsilon\sigma(1-\alpha)}) < 1$, as implied by (22b) and (23). That is, for the economy to deliver a constant long-run GDP growth rate, the externality of knowledge spillovers, captured by ϕ_1, must be positive but cannot be too much significant. The size of the economy and patent terms have no effects on long-run growth in our non-scale R&D-based growth model, however.

Long-run growth makes non-stationary variables grow exponentially at different paces in the long run. We therefore use more than one growth-kernel powered labor force — $(hL)^{\theta_K}$, $(hL)^{\theta_V}$, $(hL)^{\theta_A}$ and $(hL)^{\theta_K-\theta_V}$ — to serve as normalization factors. For instance, C, K, V, A and υ can be normalized to $\widetilde{C} \equiv C/(hL)^{\theta_K}$, $\widetilde{K} \equiv K/(hL)^{\theta_K}$, $\widetilde{V} \equiv V/(hL)^{\theta_V}$, $\widetilde{A} \equiv A/(hL)^{\theta_A}$, and $\widetilde{\upsilon} \equiv \upsilon/(hL)^{\theta_\upsilon}$, respectively. In so doing, each variable, labeled by a tilde, is a bounded scale-adjusted quantity.[23] In this manner, the non-stationary system of (21a)

[22] This growth kernel, denoted by $\theta_K - \theta_V$, makes intuitive sense. In steady state, final output growth ($g_Y^* = \theta_K(n+g_h)$) expands the market for capital goods while the arrival of newer capital goods ($g_V^* = \theta_V(n+g_h)$) dilutes this market.

[23] Similar normalization factors are used in Eicher and Turnovsky (2001) and Lin (2013b).

- (21f) can be transformed into a stationary one as follows:

$$\dot{\widetilde{C}}(t) = \left(\frac{r(t)-\rho}{\gamma} + n\right)\widetilde{C}(t) - g_C^*\widetilde{C}(t) \tag{24a}$$

$$\dot{\widetilde{K}}(t) = s_K(t)\widetilde{Y}(t) - \delta\widetilde{K}(t) - g_K^*\widetilde{K}(t) \tag{24b}$$

$$\dot{\widetilde{V}}(t) = \left(\frac{\mu}{1+\psi}\right)\widetilde{V}(t)^{\phi_1}(s_V(t)\widetilde{Y}(t))^{\phi_2} - g_V^*\widetilde{V}(t) \tag{24c}$$

$$\dot{\widetilde{\upsilon}}(t) = [r(t)+\lambda(t)+\lambda_m]\widetilde{\upsilon}(t) - \widetilde{\pi}(t) + \widetilde{\pi}(t+T)e^{-T[\bar{r}(t)+\bar{\lambda}(t)+\lambda_m+g_V^*-g_K^*]} - (g_K^* - g_V^*)\widetilde{\upsilon}(t) \tag{24d}$$

$$\dot{\bar{r}}(t) = \frac{1}{T}[r(t+T) - r(t)] \tag{24e}$$

$$\dot{\bar{\lambda}}(t) = \frac{1}{T}[\lambda(t+T) - \lambda(t)] \tag{24f}$$

$$\dot{\zeta}(t) = (1-\zeta(t))\left(\frac{\dot{\widetilde{V}}(t)}{\widetilde{V}(t)} + g_V^*\right) - \lambda_m\zeta(t) - (1+\psi)\left(\frac{\widetilde{V}(t-T)}{\widetilde{V}(t)}\right)\left(\frac{\dot{\widetilde{V}}(t-T)}{\widetilde{V}(t-T)} + g_V^*\right)e^{-T(\bar{\lambda}(t-T)+g_V^*)} \tag{24g}$$

We now start to think how to solve the normalized FDE system. First, it is assumed that this system has been in a steady state associated with the baseline patent length $T = T_0 = 20$ years. Now, at the present time of $t = 0$, we perturb the initial steady-state by a permanent change in the patent length from T_0 to T_1. Denote the initial steady state by $\boldsymbol{u}_0 = [\widetilde{C}_0^*, \widetilde{K}_0^*, \widetilde{V}_0^*, \widetilde{\upsilon}_0^*, \bar{r}_0^*, \bar{\lambda}_0^*, \zeta_0^*]$ for $t \leq 0$ and a new steady state by $\boldsymbol{u}_1 \equiv \boldsymbol{u}(t \to \infty) = [\widetilde{C}_1^*, \widetilde{K}_1^*, \widetilde{V}_1^*, \widetilde{\upsilon}_1^*, \bar{r}_1^*, \bar{\lambda}_1^*, \zeta_1^*]$, where subscript 0 (1) indicates variables associated with patent length T_0 (T_1).[24] The task is to compute the dynamic trajectory of $\boldsymbol{u}(t) \equiv [\widetilde{C}(t), \widetilde{K}(t), \widetilde{V}(t), \widetilde{\upsilon}(t), \bar{r}(t), \bar{\lambda}(t), \zeta(t)]$ for $t \in [0, \infty)$ along which the economy transitions from $\boldsymbol{u}_0$ to $\boldsymbol{u}_1$. To this end, we need to solve the FDE system of (24a) - (24g), subject to the following boundary conditions:

$$\text{Left-side boundaries at } t \leq t_L = 0\text{:} \widetilde{K}(0) = \widetilde{K}_0^*;\ \widetilde{V}(0) = \widetilde{V}_0^*;\ \zeta(0) = \zeta_0^* \tag{25a}$$

$$\text{Right-side boundaries at } t \to t_R = \infty\text{:} \widetilde{C}(\infty) = \widetilde{C}_1^*;\ \widetilde{\upsilon}(\infty) = \widetilde{\upsilon}_1^*;\ \bar{r}(\infty) = \bar{r}_1^*;\ \bar{\lambda}(\infty) = \bar{\lambda}_1^* \tag{25b}$$

This is a two-point boundary value problem (BVP) for the FDE system. Left-side boundary conditions (25a) dictate that stock variables $\widetilde{K}(t)$, $\widetilde{V}(t)$ and $\widetilde{\zeta}(t)$ are predetermined at any moment, while right-side boundary conditions (25b) imply that jump variables $\widetilde{C}$(t), $\widetilde{\upsilon}(t)$, $\bar{r}(t)$ and $\bar{\lambda}(t)$ are allowed to make discrete changes at any moment in response to changes in any parameter including patent length T.[25] From our patent-length

[24] See Appendix B for analytical solutions of the model's steady state equilibrium associated with $T \in (0, \infty)$.

[25] Note that the interest rate $r(t)$ is predetermined at any moment in terms of (19) and (20) because $K(t)$, $Y(t)$, $V(t)$, and $\zeta(t)$

policy experiments below, we expect to see a discrete change at time $t = 0$ in each of these jump variables.

3.3 Iteration algorithm

To the best of our knowledge, there are no solvers readily available to solve boundary value problems associated with nonlinear functional differential equations of mixed type. Recently, Collard et al. (2008) presented an iteration algorithm to solve numerically a two-dimensional boundary value FDE system of *delay* and *advance* types. They solved the the FDE boundary value problem by solving a sequence of initial value problems (IVPs) for systems of delay differential equations (DDEs). Their algorithm combines *forward shooting* and the *method of steps*.[26] However, in the present paper, the FDE system is much more complicated. It is seven dimensional and includes mixed functional differential equations of *delay*, *advance* and *neutral* types. Using shooting methods to solve BVPs for such a high-dimensional FDE system as a sequence of IVPs is often impractical because the IVPs are not stable.

The iteration algorithm we present below is simple and appears robust in solving the boundary-value FDE system we have derived for an infinite-horizon, perfect-foresight innovating economy. Whenever one attempts to solve infinite-horizon dynamic systems, it is necessary to truncate the infinite horizon $[0, \infty)$ into a finite horizon like $[0, d]$.[27] Thus, the time-distance d must be large enough to minimize the truncation error to a satisfying degree. Some experimentation with our model showed $d = 500$ to be more than adequate.

Algorithm For purposes of illustration, the FDE system (24a) - (24g) is henceforth represented by $\dot{\boldsymbol{u}}(t) = \boldsymbol{F}(\boldsymbol{u}(t), \boldsymbol{u}(t-T), \dot{\boldsymbol{u}}(t-T), \boldsymbol{u}(t+T))$, where $\boldsymbol{F} : \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$, $t \in [0, \infty)$, is a seven-dimensional vector of real-valued functions containing *current-time*, *delayed* and *advanced* arguments. The basic idea of this algorithm is to treat $\dot{\boldsymbol{u}} = \boldsymbol{F}(.)$ as a standard ODE system by fixing its lags and leads. Linear interpolation of the known initial and new steady states $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ provides approximations good enough to start the

are all predetermined state variables. In contrast, $\bar{r}(t)$ is a jump variable, since it reflects agents' perfect foresight about $r(t')$ for $t' \in [t, t+T)$. Both $\lambda(t)$ and $\bar{\lambda}(t)$ are jump variables, however, since the instantaneous innovation-hazard rate is determined by parameter ψ and the innovation rate $g_V(t) \equiv \frac{\dot{V}(t)}{V(t)}$, which is a jump variable.

[26] For initial value problems associated with linear DDEs, one can apply the Laplace transform to obtain analytical solutions. However, it is complicated to explore stability properties of DDEs in that their characteristic equations always involve transcendental functions so that there is an infinite number of complex characteristic roots even for a simple linear delay differential equation; see Judd (1985) or the classical works of Bellman (1963) and Hale (1977). For initial value problems associated with nonlinear DDEs, one can use the well-known method of steps, which solves a DDE system's IVP by solving IVPs for each successive sub-period. This type of initial-value DDE systems can be solved numerically with solvers such as Matlab's **DDE23**.

[27] Certainly, one can somehow transform an infinite horizon $[0, \infty)$ into a finite time interval such as $[0, 1]$. However, the dynamic system of interest must therefore contain a singularity at the right end of this interval. Thus, one has yet to truncate the finite time interval close to its right end.

iteration. We then solve the resulting ODE system with Matlab's BVP solver (**bvp5c**). We use the solution $\boldsymbol{u}(t)$ thus obtained to update the lag and lead terms, so as to obtain an updated ODE system. This system is re-solved in the same fashion to obtain another updated system. An "updating-solving" iteration therefore keeps going until a predetermined error tolerance between two successive solutions is satisfied. In a nutshell, by successively updating the FDE system's lags and leads, the iteration algorithm is to solve a sequence of standard BVPs for systems of ODEs on a finite interval:

$$\dot{\boldsymbol{u}}(t)^{(k)} = \boldsymbol{F}(\boldsymbol{u}(t)^{(k)}, \boldsymbol{u}(t-T)^{(k-1)}, \dot{\boldsymbol{u}}(t-T)^{(k-1)}, \boldsymbol{u}(t+T)^{(k-1)}), \;\; t \in [0,d], k = 1, 2, \ldots \tag{26}$$

where k is the number of iterations. Before the iteration is started, the linear interpolation is applied so that $\dot{\boldsymbol{u}}(t-T)^{(0)} \approx \left(\frac{\boldsymbol{u}_1-\boldsymbol{u}_0}{d}\right)$, $\boldsymbol{u}(t-T)^{(0)} = \boldsymbol{u}_0 + \left(\frac{\boldsymbol{u}_1-\boldsymbol{u}_0}{d}\right)\cdot(t-T)$, and $\boldsymbol{u}(t+T)^{(0)} = \boldsymbol{u}_0 + \left(\frac{\boldsymbol{u}_1-\boldsymbol{u}_0}{d}\right)\cdot(t+T)$, where d is a chosen time-distance large enough for the modeled economy to work out transitional dynamics in a finite horizon. For $k \geq 1$, $\boldsymbol{u}(t-T)^{(k-1)}$, $\dot{\boldsymbol{u}}(t-T)^{(k-1)}$ and $\boldsymbol{u}(t+T)^{(k-1)}$ are updated recursively by a sequence of solutions obtained by repeatedly solving (26) as a regular ODE system. It is crucial that the BVP solver provides approximate solutions that can be evaluated conveniently anywhere in $[0, d]$. The Matlab solver **bvp5c** has this capability. The FDE system can thus be solved approximately if the maximum absolute error between successive solutions, calculated by $\max_{t\in[0,d]}\left\{\mid \boldsymbol{u}(t)^{(k+1)} - \boldsymbol{u}(t)^{(k)} \mid\right\}$, is less than a predetermined error tolerance.

Model Calibration Excising the iteration algorithm will follow right after we calibrate the model to match a set of empirical data regarding the U.S. economy. The procedure of model calibration to be described later permits to obtain a benchmark parameter set in Table 1, so that our calibrated model can deliver an initial steady state that largely mimics the U.S. economy's stylized facts, as summarized in Table 2. On the macroeconomic front, these stylized facts refer to the steady-steady real GDP growth rate of $g_Y^* = 0.032$, the steady-state real per capita GDP growth rate of $g_y^* = 0.022$, the steady-state TFP growth rate of $g_{TFP}^* = 0.014$, the steady-state capital/output ratio of $\frac{K}{Y} = 3$, the steady-state physical capital investment rate of $s_K^* = 0.23$, and the steady state R&D intensity of $s_V^* = 0.026$.[28]

[28] Based on data from U.S. Bureau of Labor Statistics, the U.S manufacturing TFP annual growth rate from 1987 to 2012 is equal to 1.81%. The annual TFP growth rate is set at 1.25% in Jones and Williams (2000) using 1948-1997 data from U.S Bureau of Labor Statistics. To be conservative, we set $g_{TFP}^* = 0.014$ largely in the middle. The steady state physical capital investment rate of $s_K^* = 0.23$ we set is slightly more than the average annual physical capital investment rate of about 0.21 based on the data of 1990-2004 from the Penn World Table 6.2. By definition, the capital investment rate is given by $s_K = (\dot{K} + \delta K)/Y = (\frac{\dot{K}}{K} + \delta)\frac{K}{Y}$. Hence, the capital depreciation rate δ must be calibrated to 0.037 to force s_K to be 0.21, given $\frac{K}{Y} = 3$ and $g_K = 0.032$ in steady

On the microeconomic front, the calibrated model can deliver a fresh patent's steady-state life expectancy equal to $T^{e*} = 12$ years and the steady-state capital goods average markup rate of $\bar{\eta}^* = 1.11$, where $\bar{\eta}^* \equiv \eta\zeta^* + (1-\zeta^*)$, to be in line with average markup estimates from Norrbin (1993). As regards a patented product's lifespan, it can be as short as 5 years or as long as 20 from various empirical studies discussed in Jones and Williams (2000). We calibrate parameters to make $T^{e*} = 12$ years stay about in the middle range and this implies a plausible risk premium $(\lambda^* + \lambda_m)$ of about 5% for equity investment in newly issued patents.[29] What follows detail how we can use US economy's stylized facts to calibrate model parameters:

i) The prevailing patent length is 20 years in the U.S. So, we set $T_0 = 20$ to be the baseline level. A typical value of time preference is $\rho = 0.025$. Given $g_C^* = g_Y^* = 0.032$, one can solve (2) for parameter $\gamma = 2$ approximately.

ii) Following Jones and Williams (2000) and other works, we set the capital share at $\alpha = 0.36$ and this implies a labor share of $1-\alpha = 0.64$. As well, it holds that $(1-\alpha)(g_A^* + g_h) = g_{TFP}^* = 0.014$ due to (17). Further, Comin (2004) shows that the contribution of R&D to productivity growth in the U.S. is less than 5 tenths of 1% point. Accordingly, we set the ratio of R&D-driven TFP growth $((1-\alpha)g_A^*)$ to TFP growth (g_{TFP}^*) equal to 0.45. This ratio requires that the parameter of g_h (non-R&D-driven efficiency growth rate) be calibrated to 0.0125, along with the annual U.S. population growth rate being set at at $n = 0.01$.

iii) Given $g_K^* = g_Y^* = 0.032$, $\frac{K^*}{Y^*} = 3$ and $s_K^* = 0.23$, we can use (19) to calibrate the capital depreciation rate δ to 0.045 (see footnote 28).

iv) Using $\frac{K^*}{Y^*} = 3$, $s_K^* = 0.23$, $r^* = 0.07$, $\alpha = 0.36$ and $\delta = 0.045$, one can obtain $m = 0.042$ based on (19). Now given a set of stylized steady state values ($m^* = 0.042$, $s_V^* = 0.026$, $T^{e*} = 12$, $g_K^* = 0.032$ and $g_A^* = \frac{g_{TFP}^*}{1-\alpha} - g_h = 0.0094$), and noting that $\eta = \frac{1}{\sigma\alpha}$, $\varepsilon = \frac{1}{1-\sigma\alpha}$ and $\lambda^* = \psi g_V^*$, one can solve the seven equations of (9), (22a), (22b), (22c), (B.4) (B.7), (B.9) for seven unknowns, including the five parameters of $\sigma = 1.75$, $\phi_1 = 0.47$, $\phi_2 = 0.48$, $\psi = 0.75$ and $\lambda_m = 0.03$ as well as two steady-state values (innovation rate $g_V^* = 0.029$ and monopoly fraction $\zeta^* = 0.18$). Hence, a patented monopoly's markup rate is parametrized

state. We consider $\delta = 0.037$ to be too small, so we set δ equal to 0.045 in the present paper. In Grossmann et al. (2013), g_K is set at 0.03, so setting δ equal to 0.04 can deliver the steady-state capital investment rate of $s_K^* = 0.21$. In other calibration exercises for non-scale R&D-based growth models, Eicher and Turnovsky (2001) set δ equal to 0.05 and their calibrated model delivers an even greater steady state capital investment rate equal to 0.26 and a relatively large steady-state capital/output ratio of 3.5, whereas Jones and Williams (2000) assume a zero capital depreciation rate.

[29] Given the baseline patent length of $T = 20$ years, it is clear from (9) that the sum of hazard rates $\lambda + \lambda_m (= \psi g_V + \lambda)$ must be about 0.05 for T^{e*} to equal 12 years. Jones and Williams (2000) set T^{e*} equal to 10 years (given $T \to \infty$), implying a risk premium of 10%, while in other calibrated models [e.g., Grinols and Lin (2006) and Grossmann et al. (2013)], T^{e*} is set at 20 years (given $T \to \infty$), also implying a risk premium of 5%.

Final Goods Production & Depreciation		Design Production		Innovation & Imitation Hazards		Preference & Baseline Patent Length		Population & Labor Efficiency	
α	0.36	ϕ_1	0.47	ψ	0.75	ρ	0.025	n	0.01
σ	1.75	ϕ_2	0.48	λ_m	0.03	γ	2	g_h	.0125
δ	0.045	μ	0.2176			T_0	20		

Table 1: Benchmark parameter set: normalizing the initial steady state level $\widetilde{V}$ to unity requires calibrating μ to 0.2176, subject to rounding errors.

g_Y^*	g_y^*	g_{TFP}^*	s_V^*	s_K^*	$\frac{K^*}{Y^*}$	r^*	T^{e*}
0.032	0.022	0.014	0.026	0.23	3.0	0.07	12

Table 2: Initial steady state under benchmark parameter set: Given the risk premium of 0.052 (due to $\psi g_V^* = 0.022$ and $\lambda_m = 0.03$) and patent length $T = 20$, a fresh patent's steady-state life expectancy is calculated at $T^{e*} \approx 12$ years in terms of (9). The initial steady-state solution for (24a) - (24g) is given by $\boldsymbol{u}_0 = [\widetilde{C}_0^*, \widetilde{K}_0^*, \widetilde{V}_0^*, \zeta_0^*, \widetilde{\upsilon}_0^*, \bar{r}_0^*, \bar{\lambda}_0^*] = [1.36, 5.61, 1.0, 0.18, 0.95, 0.07, 0.022]$, subject to rounding errors.

at $\eta = \frac{1}{\sigma\alpha} = 1.59$. As such, the steady state sectoral average of markup rates for capital goods is calculated at $\bar{\eta}^* = 1.59 \times \zeta^* + 1 \times (1 - \zeta^*) = 1.11$. This is in line with Norrbin (1993), as noted above.

v) Lastly, we leave a free technical parameter μ to be determined. This parameter is in the design production function (21c), but it does not appear in the above six-equation system. Normalize the initial steady state of $\widetilde{V}_0^*$ to unity requires calibrating this parameter to $\mu = 0.2176$ using (B.3) and the other parameters that have been calibrated above. We thus obtain a benchmark parameter set in Table 1.

Examples With the iteration algorithm described above and Matlab's **bvp5c**, we successfully solved the calibrated FDE system (24a) - (24g) as a sequence of standard BVPs for systems of ODEs, respectively, associated with three new patent terms $T_1 = 25, 30, 50$ that take effect for any innovations starting at $t = 0$.[30] For each of these new patent terms, the algorithm takes 24, 20, and 14 iterations, respectively, to solve the FDE system, if the maximum absolute error tolerance is set at 10^{-8}. We observed that the number of iterations increases if the patent length is decreased. As indicated in Figure 1, if the new patent length is as small as 12 years, the algorithm requires 41 iterations to finish. Figure 2 reports solutions based on the new patent length T_1 equal to 25, 30 or 50 years. Jumps in some solution components at $t = 0$ and delayed arguments in the FDE system lead to jumps in the first derivatives of some components at

[30] In implementing the algorithm , we chose a finite horizon $[0, d = 500]$ as a proxy of $[0, \infty)$ as noted earlier. In the iteration process, to solve each ODE system in the sequence with **bvp5c**, we set the absolute error at 10^{-8} and the relative error at 10^{-4}. The iteration is set to stop when the maximum error, calculated by $\max_{t\in[0,d]} \left\{ \mid \boldsymbol{u}(t)^{(k+1)} - \boldsymbol{u}(t)^{(k)} \mid \right\}$, is less than 10^{-8}. A copy of Matlab code is available from the corresponding author.

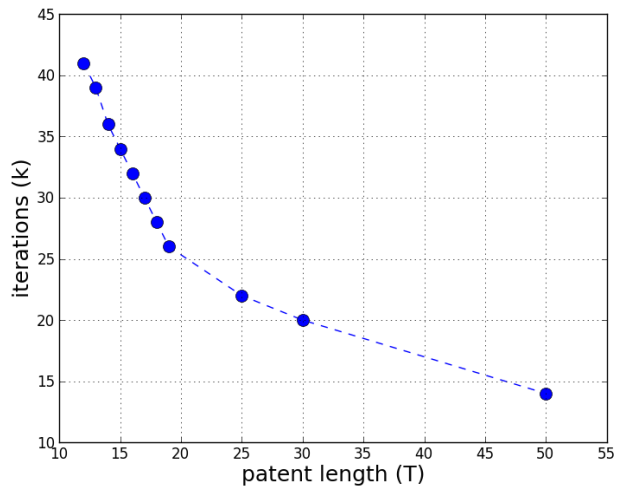

Figure 1: Patent Length and Iterations: k is the number of iterations obtained by solving a sequence of ODE systems (26) for switching to selected patent lengths, with the absolute error tolerance of 10^{-8}.

$t = T = T_1 \in \{25, 30, 50\}$. These discontinuities in the first derivative are quite visible in Figure 2(d), (e) and (f). This figure plots the immediate impacts (jumps in $\widetilde{C}$, $\widetilde{\upsilon}$, $\bar{r}$, $\bar{\lambda}$) at $t = 0$ and the dynamic trajectory of $\boldsymbol{u}(t) = [\widetilde{C}(t), \widetilde{K}(t), \widetilde{V}(t), \widetilde{\upsilon}(t), \bar{r}(t), \bar{\lambda}(t)]$ at $t \in (0, d = 500)$ that leads the calibrated economy to transition to the new steady state $\boldsymbol{u}_1$ (associated with T_1). The market mechanisms for these immediate and transitional impacts are analyzed below:

First of all, raising the patent length from the $T_0 = 20$ baseline level at $t = 0$ extends a newly established monopoly's expected lifespan, making the price $\widetilde{\upsilon}$ of newly issued patents jump up immediately [see Figure 2(e)]. This increases R&D demand for foregone consumption from households and thereby drives the expected T_1 periods average interest rate to jump up at the same moment [see Figure 2(f)].[31] Plus, with more forgone consumption redirected into innovative activities, the scale-adjusted innovation rate $\frac{\dot{\widetilde{V}}}{\widetilde{V}}$ jumps up at $t = 0$, so does the expected T_1 periods average innovation-hazard rate $\bar{\lambda}$ [see Figure 2(c) & (g)].[32] However, the scale-adjusted quantity of consumption $\widetilde{C}$ may jump up or down at $t = 0$, depending on the new patent length (T_1).[33] Recall that there are two activities — physical capital accumulation ($\dot{\widetilde{K}}$) and knowledge

[31] The immediate jumps at $t = 0$ is too small to visualize for each new patent length: $\bar{r}(0) = 0.06922433, 0.06921155, 0.06921608$ for $T_1 = 25, 30, 50$, respectively, compared to $\bar{r}(0) = 0.006921154$ for $T_0 = 20$.

[32] As indicated in panel (c) of Figure 2, the value of $\widetilde{V}(t)$ at $t \leq 0$ is constant at the steady-state level $\widetilde{V}_0^* = 1$, and $\widetilde{V}(t)$ becomes a concave, increasing function of $t > 0$ in transition. This implies a jump-up in the innovation rate $\frac{\dot{V}}{V}$ (or $\frac{\dot{\widetilde{V}}}{\widetilde{V}}$) at $t = 0$ (Recall $\frac{\dot{V}}{V} = \frac{\dot{\widetilde{V}}}{\widetilde{V}} + g_V^*$). After the initial jump, the innovation rate continues to decline toward its initial steady state level in the long run.

[33] The immediate jump is too small to visualize: $\widetilde{C}(0) = 1.361586, 1.361596, 1.362942$ for $T_1 = 25, 30, 50$, respectively, compared to $\widetilde{C}(0) = 1.361958$ for $T_0 = 20$. That is, $\widetilde{C}$ jumps down at $t = 0$ for $T_1 = 25$ or 30, but it jumps up for $T_1 = 50$.

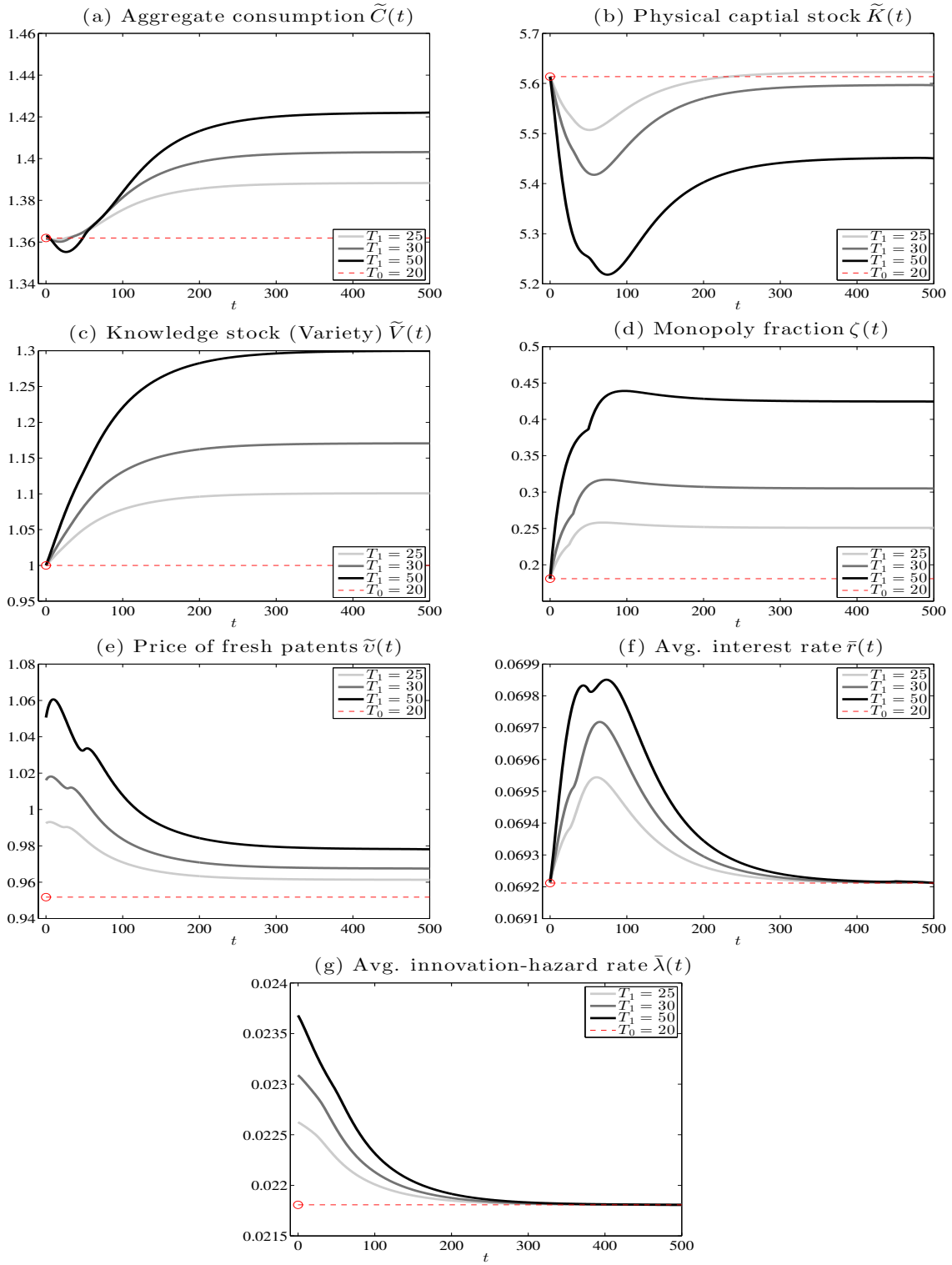

Figure 2: Solutions of FDE system (24a) - (24g): as patent length is raised from T_0 to T_1 at $t = 0$, patent price, innovation-hazard rate, and interest rate all jump up instantaneously. Consumption jumps down with $T_1 = 25$ or 30 and it jumps up with $T_1 = 50$. Empty circle denotes an initial steady state.

accumulation ($\dot{\widetilde{V}}$) — competing for foregone consumption in the economy. The expansion of one activity may crowd out the other. As Figure 2(b) indicates, compared to $T_1 = 25$ or $T_1 = 30$, the new patent length of $T_1 = 50$ results in more significant decumulation of physical capital, thereby allowing households to save less (i.e., $\widetilde{C}$ jumps up) at $t = 0$ while maintaining sufficient foregone consumption to sustain R&D expansion.

Next, raising the patent length promotes innovation-induced growth in transition. Though the growth effect peters out eventually in the long run, they generate permanent level effects. As indicated in Figure 2, the greater the patent length, the more significant are the permanent level effects on scale-adjusted consumption ($\widetilde{C}$), scale-adjusted knowledge stock ($\widetilde{V}$), scale-adjusted patent price ($\widetilde{\upsilon}$) and monopoly fraction (ζ). The permanent level effect on physical capital stock, however, is ambiguous. As Figure 2(b) indicates, extending the patent length increases the stock of capital permanently for $T_1 = 25$, but it does the opposite for $T_1 = 30$ or 50.

4 The long-run optimal patent length and intertemporal welfare

In this section, we make a policy experiment by the switch from $T_0 = 20$ to the economy's long-run optimal patent length, denoted by $T_1 = T^{OPT}$, under the the benchmark parameter set (Table 1). The purpose is to evaluate whether or not an innovative economy that currently protects innovations with 20 years patent length can gain instead by implementing the long-run optimal patent length. In order to highlight the roles of Schumpeterian creative destruction and knowledge spillovers, we also do the same welfare evaluation using an alternative parameter set provided in Table 3, which mainly features a much greater innovation-hazard coefficient of $\psi = 3.6$ (the benchmark is 0.75) and a much smaller knowledge spillover coefficient of $\phi = 0.20$ (the benchmark is 0.47). As will be shown below, the long-run optimal patent length is infinitely large ($T^{OPT} = \infty$) under the benchmark parameter set, whereas it is finite at $T^{OPT} \approx 23.17$ years under the alternative.

4.1 Determining optimal patent length

The long-run optimal patent length T^{OPT} is such that maximizes the economy's steady-state scale-adjusted consumption $\widetilde{C}^* = (1 - s^*)\widetilde{Y}^*$ along a balanced-growth path. Maximizing the economy's steady-state lifetime utility via T^{OPT} is tantamount to obtain the long-run optimal patented-monopoly fraction, denoted by ζ^{OPT}, that maximizes steady-state consumption $\widetilde{C}^*$. As we explain below, ζ^* is a strictly increasing function

Final Goods Production & Depreciation		Design Production		Innovation & Imitation Hazards		Preference & Baseline Patent Length		Population & Labor Efficiency	
α	0.36	ϕ_1	0.20	ψ	3.6	ρ	0.025	n	0.01
σ	1.75	ϕ_2	0.48	λ_m	0.03	γ	2	g_h	.0125
δ	0.06	μ	0.3034			T_0	20		

Table 3: Alternative Parameter Set: In this set, the values of $\phi_1 \psi$, and δ are different from those in the benchmark parameter set (1), and normalizing $\widetilde{V}_0^*$ to unity requires re-calibrating μ to 0.3034, subject to rounding errors.

g_Y^*	g_y^*	g_{TFP}^*	s_V^*	s_K^*	$\frac{K^*}{Y^*}$	r^*	T^{e*}
0.028	0.018	0.012	0.03	0.26	3	0.06	9

Table 4: Initial steady state under alternative parameter set: Given the risk premium of 0.098 (due to $\psi g_V^* = 0.068$ and $\lambda_m = 0.03$) and patent length $T = 20$, a fresh patent's steady-state life expectancy is calculated at $T^{e*} \approx 9$ years in terms of (9). The initial steady-state solution for (24a) - (24g) is given by $\boldsymbol{u}_0 = [\widetilde{C}_0^*, \widetilde{K}_0^*, \widetilde{V}_0^*, \zeta_0^*, \widetilde{\upsilon}_0^*, \bar{r}_0^*, \bar{\lambda}_0^*] = [1.3, 5.5, 1.0, 0.008, 0.75, 0.06, 0.068]$, subject to rounding errors.

of patent length T and $\widetilde{C}^*$ can be expressed as a function of ζ^*. Therefore, once ζ^{OPT} is obtained via the maximization of $\widetilde{C}^*$, we can compute T^{OPT} via the one-to-one relationship between ζ^{OPT} and T^{OPT}:

$$T = G(\zeta) \equiv -\frac{1}{(1+\psi)g_V^*} \ln(\frac{1}{1+\psi}[1 - \frac{\zeta}{\zeta_{max}}]) \tag{27}$$

where $\frac{dG(.)}{d\zeta} > 0$, $T \to \infty$ if $\zeta = \zeta_{max}$, and $\zeta_{max} \equiv \frac{g_V^*}{g_V^* + \lambda_m}$ is the greatest possible monopoly fraction that prevails if $T \to \infty$.[34]

To find ζ^{OPT}, we differentiate the natural logarithm of $\widetilde{C}^* = (1 - s^*)\widetilde{Y}^*$ with respect to ζ^*. This yields $\mathscr{E}_{\widetilde{C}^*} = \mathscr{E}_{\widetilde{Y}^*} - \left(\frac{s^*}{1-s^*}\right)\mathscr{E}_{s^*}$ in elasticity terms, where $s^* = s_V^* + s_K^*$ is the aggregate saving rate, and $\mathscr{E}_{\widetilde{C}^*} \equiv \frac{d\ln(\widetilde{C})}{d\ln(\zeta^*)}$, $\mathscr{E}_{\widetilde{Y}^*} \equiv \frac{d\ln(\widetilde{Y})}{d\ln(\zeta^*)}$ and $\mathscr{E}_{s^*} \equiv \frac{d\ln(s^*)}{d\ln(\zeta^*)}$ are the steady-state elasticities of consumption $\widetilde{C}^*$, income $\widetilde{Y}^*$ and saving rate s^* with respect to ζ^*. These elasticity terms are the functions of ζ^*, as we have derived in Appendix C. Hence, along a balanced growth path, the consumption-maximized monopoly fraction ζ^{OPT} and thus the optimal patent length $T^{OPT} = G(\zeta^{OPT})$ must be such that satisfy $\mathscr{E}_{\widetilde{C}^*} = 0$ or

$$(1 - s^*)\mathscr{E}_{\widetilde{Y}^*} - s^*\mathscr{E}_{s^*} = 0 \tag{28}$$

The long-run optimal patent length T^{OPT} can therefore be obtained at the margin where a differential change in the patent length has no effects on steady-state scale-adjusted consumption. In our model, the steady-

[34] Based on (B.4), we can obtain equation (27) and ζ_{max}.

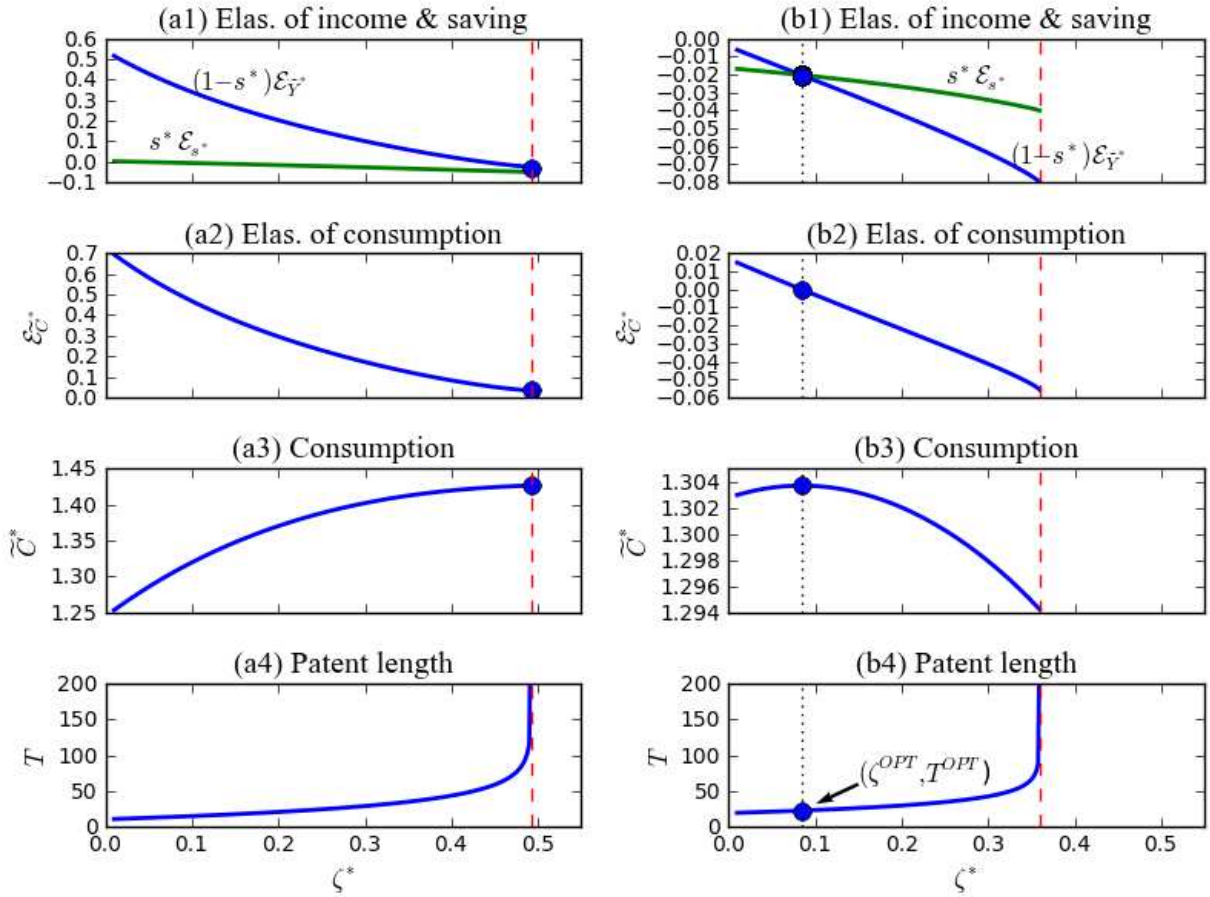

Figure 3: Determining Optimal patent length T^{OPT} in steady state: (i) panels (a1) - (a4) are under benchmark parameter set and panels (b1) - (b4) under the alternative; (ii) the dashed line indicates the maximum monopoly fraction (ζ_{max}) attainable at $T \to \infty$; (iii) panel (a4) indicates $(\zeta^{OPT}, T^{OPT}) = (\zeta_{max}, \infty) \approx (0.49, \infty)$ and panel (b4) refers to $(\zeta^{OPT}, T^{OPT}) \approx (0.085, 23.17)$.

state welfare effects of optimizing the patent length boil down to a change in the aggregate saving rate and a change in the aggregate income arising from changes in the economy's monopoly-distorted total factor productivity (TFP) and capital stock. This is in contrast to Nordhaus (1969), where the optimal patent length is obtained by balancing dynamic gains from innovation against static inefficiency from monopoly distortions.

Figure (3) demonstrates our computation of the long-run optimal patent length based on (27)and (28). This figure contains panels (a1) - (a4) under the benchmark parameter set and panels (b1) - (b4) under the alternative. Figure 3(a1) indicates that the steady-state income effect, $(1-s^*)\mathscr{E}_{\tilde{Y}^*}$, of raising the steady-state monopoly fraction ζ^* has never been less than the associated saving effect, $s^*\mathscr{E}_{s^*}$, for $\zeta^* \in (0, \zeta_{max}]$. As

such, The steady-state elasticity of consumption $\mathscr{E}_{\widetilde{C}^*}$ with respect to ζ^* is always positive in sign (Figure 3(a2)) till the patent length rises to infinity (Figure 3(a4)). Hence, under the benchmark parameter set, the steady-state consumption is maximized at $T^{OPT} \to \infty$ or equivalently at $\zeta^{OPT} = \zeta_{max} \approx 0.49$.

In stark contrast, under the alternative parameter set, the economic environment calls for a finite long-run optimal patent length equal to $T^{OPT} \approx 23.17$ years, along with the long-run optimal monopoly fraction of $\zeta^{OPT} \approx 0.085$, as demonstrated in Figure 3(b1) - (b4). Compared to the benchmark environment, the alternative features a relatively *insignificant* positive externality via the knowledge spillovers channel ($\phi_1 = 0.20$) while entailing a relatively *large* negative externality via the creative-destruction channel ($\psi = 3.6$). Therefore, the alternative environment calls for a finite steady-state consumption-maximizing patent length. Our simulation results show that an economy's long-run optimal patent length can either finite or infinite, in contrast to Judd (1985), Iwaisako and Futagami (2003), Lin (2013a) and Iwaisako and Futagami (2013).

4.2 Dynamics and welfare evaluation

We now proceed to evaluate the change in intertemporal welfare resulting from extending the baseline patent length of $T_0 = 20$ to the long-run optimal level T^{OPT}.

4.2.1 Benchmark economic environment

The first policy experiment extends the patent length to the long-run optimal level $T^{OPT} \to \infty$ for the benchmark economy (calibrated by the parameters of Table 1). As noted earlier in Section 3.3, we chose the time duration of $d = 500$ as a proxy of infinity. So, we set $T^{OPT} \approx d$ and solve the FDE system (24a) - (24g) under the benchmark parameter set. The solution thus obtained is shown in Figure 4(a) - (g), and the indicated transition paths of $\widetilde{C}(t)$, $\widetilde{K}(t)$, $\widetilde{V}(t)$, $\zeta(t)$, $\widetilde{\upsilon}(t)$, $\bar{r}(t)$, and $\bar{\lambda}(t)$ are qualitatively similar to those corresponding to the example of raising the patent length to $T_1 = 50$ (Figure 2). Here, we provide more computational results for the following analysis:

First of all, increasing the patent length from 20 to $T^{OPT} \approx d = 500$ years at $t = 0$ pushes the price of fresh patents to jump up immediately (Figure 4(e)). This increases R&D demand for forgone consumption, thereby making R&D intensity s_V jump up at the same moment (Figure 4(j)). On the other hand, as the improved R&D incentive works to crowd out physical capital investment to some extent, the capital investment rate s_K jumps down concurrently (Figure 4(k)). Further, as indicated, the instantaneous *decrease* in

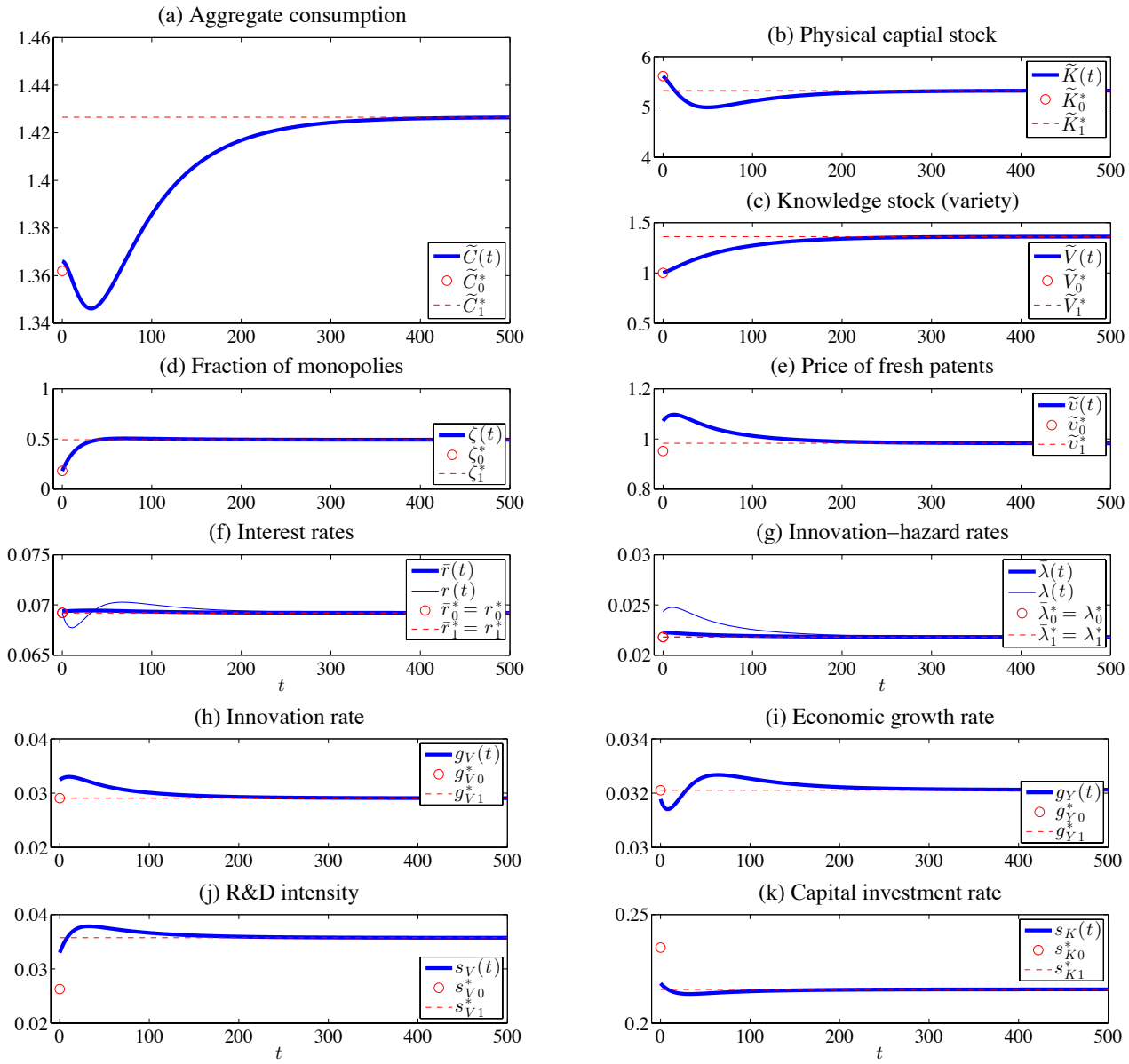

Figure 4: Dynamic solutions of raising T from 20 to $T^{OPT} \to \infty$ (approximately, $T^{OPT} \approx 500$) under benchmark parameter set

s_k outweighs the instantaneous *increase* in s_V. Therefore, households are able to finance the instantaneous increase in the innovation rate g_V, while also enjoying an instantaneous increase (though very small) in consumption $\widetilde{C}$ at $t = 0$.

In transition, however, the economy experiences *decumulation* of the capital stock (Figure 4(b)) and slowdowns of economic growth (Figure 4(i)), even though ongoing product innovation can still sustain *accumulation* of the knowledge stock (Figure 4(c) & (h)). Given the decumulation-accumulation dynamics in transition, we see households have to sacrifice consumption for a substantial period of time to finance expanding R&D activities, although the economy's R&D intensity eventually has to fall toward a new steady state level. In the long run, the economy therefore transitions to a new balanced-growth steady state with a higher knowledge stock vs. a lower capital stock and with a greater R&D intensity vs. a smaller capital investment rate, compared to the initial steady state. All these together, households can enjoy a higher steady-state living standard. As indicated, the new steady-state consumption of $\widetilde{C}_1^* \approx 1.43$ is about 5% above the initial steady-state level $\widetilde{C}_0^* \approx 1.36$ (see Figure 4(a)).

However, households suffer consumption sacrifices in the early transition stage. As indicated in Figure 4(a) , the initial divergence of consumption from the new steady state level lasts 31 years approximately and it takes about 33 years to recover household's initial steady-state consumption. As well, starting from this recovered consumption level, the half-life of convergence is about 36 years.[35] Thus, if the time profile of consumption is taken into account, optimizing the patent length to infinity actually results in a mild loss to household's lifetime welfare. This intertemporal welfare loss is equivalent to a 0.33% *decrease* in per-capital consumption against the initial balanced-growth steady state. This can be computed by the measure of Ω using the following equation,

$$\int_0^d \frac{[(1+\Omega)c_0(t)]^{1-\gamma}-1}{1-\gamma} e^{-(\rho-n)t} dt = \int_0^d \frac{c_1(t)^{1-\gamma}-1}{1-\gamma} e^{-(\rho-n)t} dt, \tag{29}$$

where consumption per capita is given by $c(t) = \frac{C(t)}{L(t)} = \frac{\widetilde{C}(t)e^{(n+g_h)t}}{e^{nt}} = \widetilde{C}(t)e^{g_h t}$ and index 0 (1) labels the patent

[35] In one-sector neoclassical growth models, all variables converge at the same constant rate. In contrast, our numerical solution for (24a) - (24g) implies a three-dimensional stable manifold because there are three stock variables ($\widetilde{V}$, $\widetilde{K}$, ζ), and therefore variables converge at different rates across times and across sectors; see Eicher and Turnovsky (2001) for detailed discussions of two-dimensional stable manifolds in R&D-based growth models. From the computed transition paths in Figure 4, we find that: (i) for the knowledge stock, the half-life of convergence is about 50 years; (ii) for the capital stock, the initial decumulation-divergence phase takes about 45 years to finish, and then starting from the bottom level, the half-life is about 73 years; and (iii) for the monopoly fraction, the half-life is only 10 years. It is well-known that an economy's convergence to the steady state is typically slow in R&D-based growth models; see Jones (1995a) and Steger (2003).

regime of $T_0 = 20$ ($T_1 = T^{OPT}$). For the benchmark economy, Ω is equal to -0.0033 approximately. We do some robustness checks by changing some parameters such as ψ (creative destruction), ϕ_2 (knowledge spillover), λ_m (imitation hazard rate) and ρ (time preference), while ensuring a perturbed economic environment that still calls for an infinitely large patent length. And we find that while the intertemporal welfare effect (measured by Ω) can turn out to be positive if the creative destruction coefficient ψ is small enough, it is at most negligible.

4.2.2 Alternative economic environment

For the second policy experiment, we change parameters ψ and ϕ_1 in a significant way so that the perturbed economic environment (calibrated by the alternative parameter set; Table 3) instead calls for a long-run optimal patent length equal to $T^{OPT} \approx 23.17$ years. In order for better model calibration, the alternative parameter set includes a larger capital depreciation rate equal to $\delta = 0.06$ so that the initial steady state, as shown in Table 4, remains largely consistent with US economy's stylized facts. As noted earlier, this parameter set characterizes a much smaller knowledge spillovers externality ($\phi_1 = 0.20$) and a much larger creative-destruction coefficient ($\psi = 3.6$) than the benchmark. These features lead to a much smaller long-run optimal patent length, about 3 years longer than the baseline level $T_0 = 20$ years.

We solve the FDE system (24a) - (24g) under the alternative parameter set, and the solution is provided in Figure 5. As indicated, the transition paths are qualitatively similar to those (Figure 4) based on the benchmark parameter set, except for the dynamic trajectories of consumption $\widetilde{C}(t)$ and expected average interest rate $\bar{r}(t)$. In the second policy experiment, $\widetilde{C}$ and $\bar{r}$ jumps down at $t = 0$ and $\widetilde{C}(t)$ experiences falls in transition for a prolonged period before turning around to drift up to a new steady state level of $\widetilde{C}_1^* = 1.3037$, which is merely about 0.06% above the initial steady-state consumption $\widetilde{C}_0^* \approx 1.3029$. Further, the initial divergence of consumption from the new steady state level lasts 68 years (compared to 31 years under benchmark) approximately and it takes about 110 years (compared to 33 years under benchmark) to recover household's initial steady-state consumption. As well, starting from this recovered consumption level, the half-life of convergence is about 45 years (compared to 36 years under benchmark). Clearly, in the alternative economic environment, a larger innovation hazard of creative destruction and a smaller externality of intertemporal knowledge diffusion can combine to force the economy to devote more resources to sustain R&D activities at the expense of consumption and capital investment, thereby resulting a much slower economic transition to the new steady state than in the benchmark economic environment. Solving

(29) for Ω under the alternative parameter set, we find that extending the patent length to the long-run optimal level ($T^{OPT} = 23.17$) incurs a small welfare loss equivalent to a per-capita consumption loss of 0.21% ($\Omega = -0.0021$) against the initial balanced-growth steady state.

Our numerical simulations under the two alternative parameter sets suggest that the world's long-run optimal patent length should be above 20 years. However, the switch from 20 years to the long-run optimal patent length tends to incur a welfare loss, albeit rather small, if decreased consumption in transition is taken into account.

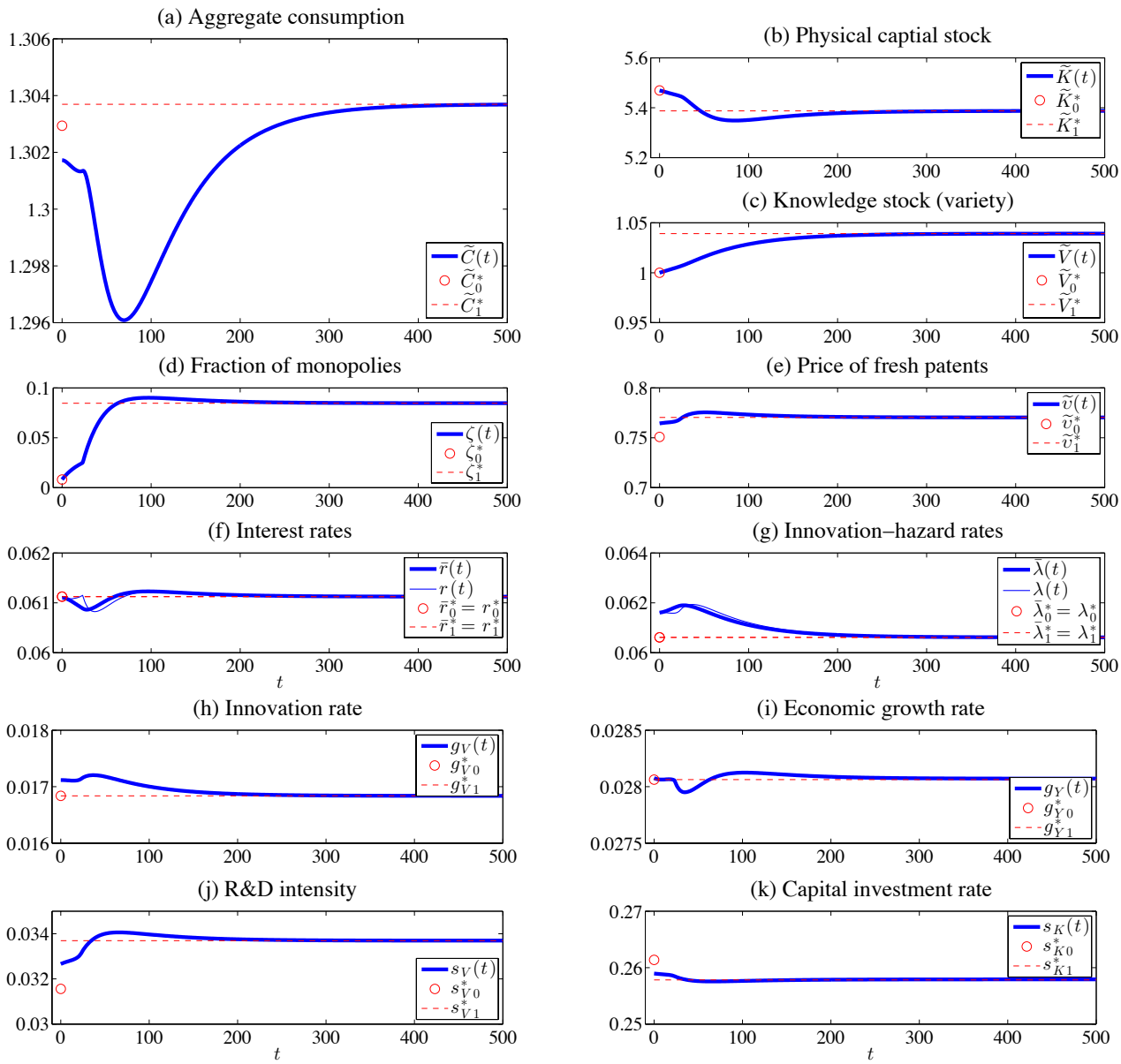

Figure 5: Dynamic solutions of raising T from 20 to $T^{OPT} \approx 23.17$ under alternative parameter set

5 Concluding Remarks

Patents have a finite legal term and their lifespans may be terminated prematurely by various hazards in the real world. This nature is analogous to vintage capital. Vintage capital in growth models gives rise to a dynamic system of functional differential equations (FDE), so do finite-length patents. However, scholars were often reluctant to incorporate finite-length patents into R&D-based growth models, because the resulting FDE system tends to be structurally complicated. Indeed, there is no solver available for solving FDE boundary value problems. In this paper, we work out an autonomous FDE system of delay, advance and neutral types for a non-scale R&D-based growth model, where finite-length patents confront imitation and innovation hazards throughout their legal lifetime duration. If the patent length is instead set to be infinite, the mixed-type FDE system derived in the present paper can revert to a lower-dimensional ODE system, as we usually see in the endogenous growth literature.

We present an iteration algorithm that can solve the FDE system by solving a sequence of standard BVPs for systems of ODEs. Our R&D-based growth model is thus not only computable, but also a more suitable vehicle for the dynamic general-equilibrium analysis of innovation policies such as patent designs or R&D subsidization. The algorithm may render scholars no longer have to assume infinite patent terms for R&D-based growth models. Further, the model we derive have interesting applications and extensions. For instance, how does a finite-patent growth model compare to an infinite-patent growth model in terms of transitional dynamics and economic convergence? Or what is the optimal mix of patent scope and length for an innovative economy? We leave these questions for future studies.

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Appendix

A Steady-state growth rates

Along a balanced-growth path, every variable grows at a constant rate, including zero growth. Steady-state growth rates such as $g_V^* \equiv \frac{\dot{V}(\infty)}{V(\infty)}$, $g_A^* \equiv \frac{\dot{A}(\infty)}{A(\infty)}$, $g_K^* \equiv \frac{\dot{K}(\infty)}{K(\infty)}$ and $g_Y^* \equiv \frac{\dot{Y}(\infty)}{Y(\infty)}$ can be obtained by solving the four equations system:

$$g_Y^* - g_K^* = 0 \tag{A.1a}$$

$$(\phi_1 - 1)g_V^* + \phi_2 g_Y^* = 0 \tag{A.1b}$$

$$g_Y^* - (1-\alpha)g_A^* - \alpha g_K^* = (1-\alpha)(n + g_h) \tag{A.1c}$$

$$g_A^* - \frac{1}{\varepsilon\sigma(1-\alpha)} g_V^* = 0 \tag{A.1d}$$

These equations result from (21b), (21c), (15), and the definition of A. For instance, (21b) implies $g_Y^* = g_K^* = g_C^*$. Solving (A.1a) - (A.1d) yields (22a) - (22d). From the R&D equilibrium condition $\upsilon = 1/\xi$ and

(7b), the patent price equation is given by $\upsilon = s_V Y/\dot{V} = s_V Y/(g_V V)$. Along a balanced-growth path, both s_V and g_V are stationary. Thus, differentiating the patent price equation yields

$$g_\upsilon^* = g_Y^* - g_V^* = (\theta_K - \theta_V)(n + g_h) \tag{A.2}$$

B Derivations of steady state equilibrium with patent length $T \in (0, \infty)$

At the outset, it is conductive to notice that g_V^*, $g_K^* = g_C^* = g_Y^*$, $\lambda^* = \psi g_V^*$, and $r^* = \rho + \gamma(g_C^* - n)$ are the steady-state aggregate growth rates, innovation-hazard rate, and interest rate, and these steady-state values are independent of patent length T; see (22a) - (22d) in the text. Now setting $\dot{\widetilde{C}} = \dot{\widetilde{K}} = \dot{\widetilde{V}} = \dot{\zeta} = \dot{\widetilde{\upsilon}} = 0$ in (21a) - (21d), one can obtain the steady-state equilibrium of $\widetilde{C}$, $\widetilde{K}$, $\widetilde{V}$, ζ and $\widetilde{\upsilon}$:

$$\widetilde{C}^* = (1 - s^*)\widetilde{Y}^* \tag{B.1}$$

$$\widetilde{K}^* = \widetilde{A}^* \left(\frac{z^* s_K^*}{\delta + g_K^*} \right)^{1/(1-\alpha)} \tag{B.2}$$

$$\widetilde{V}^* = \left[z^* s_V^* \left(\frac{z^* s_K^*}{\delta + g_K^*} \right)^{\alpha/(1-\alpha)} \left(\frac{\mu}{(1+\psi) g_V^*} \right)^{1/\phi_2} \right]^{\theta_V} \tag{B.3}$$

$$\zeta^* = \frac{g_V^*}{g_V^* + \lambda_m} \left(1 - (1+\psi) e^{-T(1+\psi) g_V^*} \right), \tag{B.4}$$

$$\widetilde{\upsilon}^* = \frac{\alpha m^*}{\zeta^*} \left(\frac{\widetilde{Y}^*}{\widetilde{V}^*} \right) \left(\frac{1 - e^{-T(r^* + \lambda^* + \lambda_m + g_V^* - g_K^*)}}{r^* + \lambda^* + \lambda_m + g_V^* - g_K^*} \right) \tag{B.5}$$

where our earlier definitions suffice to recognize the steady-state equilibrium values of auxiliary variable:

$$\widetilde{A}^* = \widetilde{V}^{*1/(\varepsilon\sigma(1-\alpha))}, \ \widetilde{Y}^* = z^* \widetilde{A}^{*1-\alpha} \widetilde{K}^{*\alpha}, \ z^* \equiv \frac{\left(\eta^{1-\varepsilon}\zeta^* + 1 - \zeta^*\right)^{1/\sigma}}{\left(\eta^{-\varepsilon}\zeta^* + 1 - \zeta^*\right)^{\alpha}}, \tag{B.6}$$

$$m^* \equiv \frac{\zeta^*(\eta - 1)}{(1 - \zeta^*)\eta^\varepsilon + \zeta^* \eta}, \tag{B.7}$$

$$s_K^* = \frac{K^*}{Y^*}(\delta + g_K^*) = \alpha(1 - m^*) \left(\frac{\delta + g_K^*}{r^* + \delta} \right), \tag{B.8}$$

$$s_V^* = (\lambda^* + g_V^*) \left(\frac{\alpha m^*}{\zeta^*} \right) \left(\frac{1 - e^{-T(r^* + \lambda^* + \lambda_m + g_V^* - g_K^*)}}{r^* + \lambda^* + \lambda_m + g_V^* - g_K^*} \right). \tag{B.9}$$

To derive s_V^*, one needs to use the R&D equilibrium condition $\upsilon(1+\psi)\dot{V} = s_V Y$, which implies $s_V = \upsilon(\lambda + g_V)V/Y = \widetilde{\upsilon}(\lambda + g_V)\widetilde{V}/\widetilde{Y}$ at any moment. Substitution of (B.5) permits to obtain (B.9).

C Optimal Patent Length for a Decentralized Economy

From eq.(B.4), we note that monopolistic fraction ζ must increase with patent length T in the steady state. Differentiating this equation with respect to T yields

$$\frac{\partial \zeta^*}{\partial T} = \left(\frac{g_V^*}{g_V^* + \lambda_m}\right)(1+\psi)^2 g_V^* e^{-(1+\psi)g_V^* T} > 0, \;\; 0 < T < \infty \tag{C.1}$$

This monotonic relationship allows us to to find the long-run optimal patent length by the lifetime utility-maximizing monopolistic fraction. With this convenient property and using (B.7) - (B.9), we can obtain the following steady-state elasticity relationships:

$$\mathscr{E}_{z^*} = \frac{1}{\sigma}\left[\frac{\eta^{1-\varepsilon} - 1}{\eta^{1-\varepsilon}\zeta^* + 1 - \zeta^*}\right] - \alpha\left[\frac{\eta^{-\varepsilon} - 1}{\eta^{-\varepsilon}\zeta^* + 1 - \zeta^*}\right] \tag{C.2}$$

$$\mathscr{E}_{m^*} = \frac{1}{\zeta^*} + \frac{\eta^{\varepsilon} - \eta}{(1-\zeta^*)\eta^{\varepsilon} + \zeta^*\eta} > 1 \tag{C.3}$$

$$\mathscr{E}_{s_V^*} = \mathscr{E}_{m^*} - \frac{1}{\zeta^*} + \left(\frac{(r^* + \lambda^* + \lambda_m + g_V^* - g_Y^*)e^{-(r^*+\lambda^*+\lambda_m+g_V^*-g_Y^*)T}}{1 - e^{-(r^*+\lambda^*+\lambda_m+g_V^*-g_Y^*)T}}\right)\frac{\partial T}{\partial \zeta^*} > 0 \tag{C.4}$$

$$\mathscr{E}_{s_K^*} = -\left(\frac{m^*}{1-m^*}\right)\mathscr{E}_{m^*} < 0 \tag{C.5}$$

where $\mathscr{E}_{x^*} \equiv \frac{d\ln(x^*)}{d\ln(\zeta^*}$ denotes the steady-state elasticity of $x^*$with respect to ζ^*. Now differentiating $\widetilde{C}^* = (1-s^*)\widetilde{Y}^*$ w.r.t. ζ^*yields

$$\mathscr{E}_{\widetilde{C}^*} = \mathscr{E}_{\widetilde{Y}^*} - \left(\frac{s^*}{1-s^*}\right)\mathscr{E}_{s^*} \tag{C.6}$$

where the steady-state elasticity of $\widetilde{Y}^*$w.r.t. ζ^* based on (B.6) and the steady-state elasticity of s^* w.r.t. ζ^* based on $s^* = s_K^* + s_V^*$ are given by

$$\mathscr{E}_{\widetilde{Y}^*} = \mathscr{E}_{z^*} + (1-\alpha)\mathscr{E}_{\widetilde{A}^*} + \alpha\mathscr{E}_{\widetilde{K}^*} \tag{C.7}$$

$$\mathscr{E}_{s^*} = \frac{1}{s^*}\left(s_V^*\mathscr{E}_{s_V^*} + s_K^*\mathscr{E}_{s_K^*}\right) \tag{C.8}$$

Next, based on (B.2), (B.3) and (B.6), we can obtain the following elasticities for $\widetilde{K}$, $\widetilde{V}$ and $\widetilde{A}$:

$$\mathscr{E}_{\widetilde{V}^*} = \theta_V \left[\mathscr{E}_{z^*} + \mathscr{E}_{s_V^*} + \frac{\alpha}{1-\alpha}(\mathscr{E}_{z^*} + \mathscr{E}_{s_K^*}) \right] \tag{C.9}$$

$$\mathscr{E}_{\widetilde{A}^*} = \frac{1}{\varepsilon\sigma(1-\alpha)} \mathscr{E}_{\widetilde{V}^*} \tag{C.10}$$

$$\mathscr{E}_{\widetilde{K}^*} = \mathscr{E}_{\widetilde{A}^*} + \frac{1}{1-\alpha}(\mathscr{E}_{z^*} + \mathscr{E}_{s_K^*}) = \frac{1}{\varepsilon\sigma(1-\alpha)} \mathscr{E}_{\widetilde{V}^*} + \frac{1}{1-\alpha}(\mathscr{E}_{z^*} + \mathscr{E}_{s_K^*}) \tag{C.11}$$

Substituting (C.9) - (C.11) into (C.7) yields

$$\mathscr{E}_{\widetilde{Y}^*} = \frac{1}{1-\alpha}\left(1 + \frac{\theta_V}{\varepsilon\sigma(1-\alpha)}\right)\mathscr{E}_{z^*} + \frac{\alpha}{1-\alpha}\left(1 + \frac{\theta_V}{\varepsilon\sigma(1-\alpha)}\right)\mathscr{E}_{s_K^*} + \frac{\theta_V}{\varepsilon\sigma(1-\alpha)}\mathscr{E}_{s_V^*} \tag{C.7'}$$

where θ_V is defined in (23). Note that both $\mathscr{E}_{\widetilde{Y}^*}$ and $\mathscr{E}_{s^*}$ can be expressed by $\mathscr{E}_{s_V^*}$, $\mathscr{E}_{s_K^*}$, $\mathscr{E}_{z^*}$ and $\mathscr{E}_{m^*}$, which are functions of ζ^*, given the model's structural parameters. We can numerically solve $\mathscr{E}_{\widetilde{C}\zeta} = 0$ or $\mathscr{E}_{\widetilde{Y}^*} = \frac{s^*}{1-s^*}\mathscr{E}_{s^*}$ for the optimal monopolistic fraction ζ^{OPT} and thereby the optimal patent length T^{OPT} in terms of (B.4) or (27) in the text.