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Increasing Returns in Matching and Labour Market Dynamics: Comments on Indeterminacy and Search Theory*

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Abstract

In this note I discuss the condition for indeterminacy in the context of search models with increasing returns in the matching technology. Building on the theoretical framework set forth by Giammarioli (2003), I argue that increasing returns with respect to vacancies at the aggregate level is only a necessary requirement for indeterminate equilibrium paths. Specifically, I show that sunspot equilibria can actually be obtained by imposing an additional condition between the private and the social elasticity of the matching function with respect to unemployment.

JEL Classification: E10, E24, J64.

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1 Introduction

The issue of indeterminacy in the context of the matching framework à la Pissarides (2000) has been addressed by a number of authors. For instance, in a seminal work Burda and Weder (2002) derive indeterminacy in a search and matching economy relying on labour institutions such as payroll taxes, unemployment insurance and social assistance. Moreover, Hashimzade and Ortigueira (2005) find out the possibility of sunspot

*Comments are very welcome.

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equilibria by augmenting the standard matching model with capital accumulation. Furthermore, Zanetti (2006) builds a New Keynesian model with search frictions showing that indeterminacy may be driven by the shape of monetary policy functions. More recently, Krause and Lubik (2010) develop a DSGE model with matching in the labour market by arguing that extrinsic uncertainty can actually cause business cycles, but the required parametrization lies at the boundaries of the region that the empirical literature would consider plausible.

A common feature of the contributions mentioned above is the assumption of constant-returns-to-scale in the matching function. However, it is well-known that indeterminacy has been often framed in the context of increasing returns (cf. Benhabib and Farmer 1994). In confirmation of this, there is a work by Giammarioli (2003) that sets forth a dynamic search model developed in continuous time in which a representative agent chooses the optimal vacancy rate while (un)employment evolves according to the rules of a standard matching framework.

Relying on this theoretical setting, Giammarioli (2003) argues that an aggregate matching function with increasing returns with respect to vacancies is a sufficient requirement for the local indeterminacy of the equilibrium paths followed by the model economy. This dynamic characterization should arise when the stationary solution of the model is given by a sink, i.e., a situation in which the unstable manifold has dimension zero. In this way, sunspot equilibria become possible and the matching economy may justify the 'animal spirits' hypothesis of business cycles (e.g. Farmer and Guo 1994).

In this note, building on the same matching framework described by Giammarioli (2003), I argue that increasing returns with respect to vacancies is only a necessary condition for local indeterminacy; indeed, I show that the required bifurcation is actually obtained by imposing an additional condition between the 'perceived' (or private) and the 'aggregate' (or social) elasticity of the matching function with respect to unemployment.

The note is arranged as follows. Section 2 reviews a centralized matching economy with search. Section 3 analyses the dynamics of the benchmark model. Finally, section 4 concludes.

2 A centralized matching economy

Suppose that L is the employment rate while V is the vacancy rate. Thereafter, the agent charged to set the optimal number of job openings is assumed to solve the following problem:

$$\max_{V} \int_{0}^{+\infty} \exp\left(-\rho t\right) \left(\ln L - V\right) dt \qquad\qquad \rho > 0 \tag{1}$$

s.to

$$\dot{L} = M - \delta L \qquad\qquad 0 < \delta < 1 \tag{2}$$

where ρ is the discount rate, M is the matching function while δ is the instantaneous separation rate.

The matching function has the conventional text-book form. Hence,

$$M = A\left(1 - L\right)^{b} V^{a} \qquad\qquad a + b = 1 \tag{3}$$

where A is a shifting parameter that conveys the efficiency of matching while $(1 - L)$ is the unemployment rate.

The shifting parameter of the matching function, taken as given by the maximizing agent, is defined as

$$A \equiv \left(1 - \overline{L}\right)^{\beta - b} \overline{V}^{\alpha - a} \tag{4}$$

where $\overline{L}$ and $\overline{V}$ are, respectively, the average level of employment and the average number of vacancies in the whole economy.

In a symmetric equilibrium, i.e., whenever $\overline{L} = L$ and $\overline{V} = V$, the matching function in eq. (3) reduces to

$$M = \left(1 - L\right)^{\beta} V^{\alpha} \tag{5}$$

The result in eq. (5) reveals that this framework allows to reconcile a situation of constant returns at the private level, i.e., $a + b = 1$, with the possibility of a matching technology that - on the contrary - displays increasing returns at the aggregate (or social) level, i.e., $\alpha + \beta > 1$. It is worth noting that, in this case, the condition $\alpha > 1$ conveys an aggregate Beveridge curve with a non-monotonic concavity. Specifically, when M has increasing returns with respect to vacancies, the Beveridge curve is convex (concave) at low (high) unemployment rates.[1] Technical details are given in Appendix.

[1] Changes in the concavity of the Beveridge curve are discussed, inter alia, by Benati and Lubik (2014), Lubik and Rhodes (2014), Bonthuis et al. (2013), Lubik (2013).

Whenever the Hosios (1990) condition is met, i.e., when bargaining power of workers coincides with private elasticity of the matching function with respect to unemployment, the problem in eq.s (1) and (2) can be decentralized as a competitive equilibrium by assuming that wages are determined through a Nash bargaining mechanism. As a consequence, the efficient dynamics of a decentralized solution is strictly equivalent to the dynamic patterns implied by the centralized matching economy described above.

3 Dynamics of the benchmark model

The system of autonomous differential equations derived from the solution of the optimization problem in eq.s (1) and (2) is given by

$$\dot{l} = \exp\left(\left(a_1 - 1\right) l + a_2\lambda + K\right) - \delta \tag{6}$$

$$\dot{\lambda} = \delta + \rho - \exp\left(-l - \lambda\right) + b\frac{\exp\left(a_1 l + a_2\lambda + K\right)}{1 - \exp\left(l\right)} \tag{7}$$

where $l \equiv \ln L$, λ is the logarithm of the co-state variable associated to the state variable L, $a_1 \equiv -\beta L\left(\left(1-\alpha\right)\left(1-L\right)\right)^{-1}$, $a_2 \equiv \alpha\left(1-\alpha\right)^{-1}$ and $K \equiv a_2 \ln\left(a\right)$.

The Jacobian matrix (J) of the linearized system is the following:

$$J \equiv \begin{bmatrix} \left(a_1 - 1\right)\delta & a_2\delta \\ \left(\rho + \delta + b\delta u\right) + ba_1 u\delta + u^2 b\delta & a_2 ub\delta + \left(\rho + \delta + b\delta u\right) \end{bmatrix} \tag{8}$$

where $u \equiv L\left(1-L\right)^{-1}$ (cf. Giammarioli 2003, p. 22).

The trace of J is given by

$$\mathrm{TR}\left(J\right) = a_1\delta + a_2 ub\delta + \rho + b\delta u \tag{9}$$

Substituting for a_1 and a_2 leads to

$$\mathrm{TR}\left(J\right) = \frac{\alpha ub\delta - u\beta\delta}{1-\alpha} + \rho + b\delta u \tag{10}$$

Resorting to numerical simulations, Gimmarioli (2003, tables 1 and 2, pp. $29 - 30$) sets $\beta = b$. As a consequence, eq. (10) reduces to

$$\mathrm{TR}\left(J\right) = \rho \tag{11}$$

Taking into account the result in eq. (11), it is possible to conclude that whenever $\beta = b$, the trace of J is exactly equal to the discount rate of the maximizing agent. Considering the hypothesis of positive discounting, ρ is always positive. Therefore, whenever $\beta = b$, the trace of J is positive no matter the value of α.

Since the trace of J provides the sum of the two eigenvalues - say r_1 and r_2 - associated to the dynamic system in eq.s (6) and (7), it seems hard to configure the possibility of indeterminate equilibrium paths, unless admitting an implausible negative discounting; indeed, such a dynamic characterization requires a negative value of $\mathrm{TR}(J)$. This statement is confirmed by the numerical results in tables 1 and 2 in which - keeping constant ρ and δ - the quantitative extent of r_1 and r_2 is retrieved for different values of a,b, α and β.[2]

$\alpha = a$	$\beta = b$	r_1	r_2	DYNAMICS
0.70	0.30	-1.737	1.767	*Saddle*
0.65	0.35	-1.473	1.503	*Saddle*
0.60	0.40	-1.290	1.320	*Saddle*
0.55	0.45	-1.156	1.186	*Saddle*
0.50	0.50	-1.058	1.088	*Saddle*
0.45	0.55	-0.986	1.016	*Saddle*
0.40	0.60	-0.932	0.962	*Saddle*
0.35	0.65	-0.894	0.924	*Saddle*
0.30	0.70	-0.869	0.899	*Saddle*

Table 1: Matching function with constant returns ($\rho = 0.03$ and $\delta = 0.1$)

$\beta = b$	a	α	r_1	r_2	DYNAMICS
0.40	0.60	0.60	-1.290	1.320	*Saddle*
0.40	0.60	0.70	-1.265	1.295	*Saddle*
0.40	0.60	0.80	-1.351	1.381	*Saddle*
0.40	0.60	0.90	-1.702	1.732	*Saddle*
0.40	0.60	0.99	-4.931	4.961	*Saddle*
0.40	0.60	1.01	$0.015 - 4.850i$	$0.015 + 4.850i$	*Source*
0.40	0.60	1.05	$0.015 - 4.850i$	$0.015 + 4.850i$	*Source*
0.40	0.60	1.10	$0.015 - 1.413i$	$0.015 + 1.413i$	*Source*
0.40	0.60	1.20	$0.015 - 0.920i$	$0.015 + 0.920i$	*Source*

[2] MAT LAB™ 6.5 codes used to derive those results are available from the author.

Table 2: Matching function with increasing returns ($\rho = 0.03$ and $\delta = 0.1$)

The numerical results in tables 1 and 2 convey two broad conclusions. First, as long as the private and the social matching functions display constant returns to scale, the stationary solution of the dynamic system is given by a saddle point. In this case, the equilibrium path of the model economy is determinate since there exists a unique trajectory that verifies the first-order conditions (FOCs) of the maximum problem in eq.s (1) and (2). Moreover, when the aggregate matching function displays increasing returns, determinacy fails but - as long as $\beta = b$ - the stationary solution is given by a source. In this case, the trajectories generated by eq.s (6) and (7) cannot be optimal because their explosive tendency will lead to the violation of the transversality condition that enters the FOCs of the problem in eq.s (1) and (2).[3]

In order to find the requirements for indeterminacy, I consider the more general case in which β is different from b. Hence,

$$\text{TR}\,(J) = \frac{u\delta\,(\alpha b - \beta)}{1-\alpha} + \rho + b\delta u \tag{12}$$

Whenever $\alpha > 1$, the condition under which $\text{TR}(J)$ is negative is the following:

$$\alpha b - \beta > 0 \quad \text{or} \quad \alpha > \frac{\beta}{b} \tag{13}$$

This is exactly the condition provided by Giammarioli (2003, p. 23). However, (13) necessary but not sufficient; indeed, it must also happen that

$$\left|\frac{u\delta\,(\alpha b - \beta)}{1-\alpha}\right| > \rho + b\delta u \tag{14}$$

Straightforward algebra suggests that whenever $\alpha > 1$, the inequality in (14) is verified if and only if $b > \beta$. This means that a sink, i.e., indeterminacy of equilibrium paths, is actually obtained when the aggregate matching function displays increasing returns with respect to vacancy while, at the private level, the reactivity of the matching function to unemployment is higher than its aggregate counterpart. This statement is confirmed by the numerical results in table 3.

The figures in table 3 reveals another interesting features of the indeterminate model economy, i.e., the fact that when the stationary solution is indeterminate the eigenvalues of the system in eq.s (6) and (7) becomes complex. As a consequence, adjustments towards equilibrium occur through convergent oscillations.

[3]Such a transversality condition is given by $\lim_{t\to\infty} \exp(-\rho t)\exp(\lambda(t)) = 0$.

β	b	a	α	r_1	r_2	DYNAMICS
0.40	0.50	0.60	0.60	-1.168	1.455	*Saddle*
0.40	0.50	0.60	0.70	-1.134	1.435	*Saddle*
0.40	0.50	0.60	0.80	-1.191	1.552	*Saddle*
0.40	0.50	0.60	0.90	-1.441	2.022	*Saddle*
0.40	0.50	0.60	0.99	-3.0765	7.8553	*Saddle*
0.40	0.50	0.60	1.01	$-2.285 - 4.244i$	$-2.285 + 4.244i$	*Sink*
0.40	0.50	0.60	1.05	$-0.4179 - 2.034i$	$-0.417 + 2.034i$	*Sink*
0.40	0.50	0.60	1.10	$-0.1861 - 1.393i$	$-0.186 + 1.393i$	*Sink*
0.40	0.50	0.60	1.20	$-0.0725 - 0.913i$	$-0.072 + 0.913i$	*Sink*

Table 3: Matching function with increasing returns and $b > \beta$ ($\rho = 0.03$ and $\delta = 0.1$)

The analytical and numerical findings derived above are not surprising. For instance, in their seminal article on indeterminacy and increasing returns, Benhabib and Farmer (1994) show that in the optimal-growth model indeterminacy requires increasing returns with respects to the control variable (labour) and an aggregate output elasticity with respect to the stock variable (capital) higher then its private counterpart.

4 Concluding remarks

In this note, building on Giammarioli (2003), I show that within the text-book matching model à la Pissarides (2000) indeterminacy of the equilibrium paths requires increasing returns at the aggregate level with respect to the control variable (vacancies) *and* an aggregate elasticity of the matching function with respect to the state variable (unemployment) higher than its private counterpart. Form the point of view of the theory on dynamic systems, this result is isomorphic to the standard requirements for indeterminacy that hold in the conventional model of optimal growth (cf. Benhabib and Farmer 1994).

5 Appendix

In a symmetric equilibrium, the model described in section 2 implies the following Beveridge curve:

$$V = \left(\frac{\delta(1-U)}{U^{\beta}} \right)^{\frac{1}{\alpha}} \tag{A.1}$$

where $U \equiv 1 - L$ is the unemployment rate.

The expression in eq. (A.1) reveals that the Bevridge curve is a monotonically decreasing function of U for all the eligible values of α, β and δ.[4] In addition, straightforward algebra conveys that the concavity of (A.1) depends on the sign of the following coefficient

$$\Phi \equiv \frac{(1-\alpha)\left(U\left(1-\beta\right)+\beta\right)\left(1+\beta\frac{1-U}{U}\right)}{\alpha\left(1-U\right)} + \frac{2U\left(1-\beta\right)+3\beta}{U^2} \tag{A.2}$$

Specifically, when Φ is positive (negative), the Beveridge curve in eq. (A.1) is convex (concave). Those analytical results can be easily linked with the dynamic findings outlined in section 3. On the one hand, in the case of determinacy, i.e., whenever $\alpha + \beta = 1$, the expression in (A.2) is always positive. As a consequence, the Beveridge curve displays the conventional convex pattern. On the other hand, whenever the conditions for indeterminacy are met and, notably, $\alpha > 1$, the first addend of Φ becomes negative. Since this term follows a reverse-u-shaped pattern with respect to U while the second addend is positive and decreasing with lower bound at $2 + \beta$, Φ may chance its sign as U approaches to 1. Setting $\alpha = 1.2$ and $\beta = 0.4$, this possibility is illustrated in figure A1.

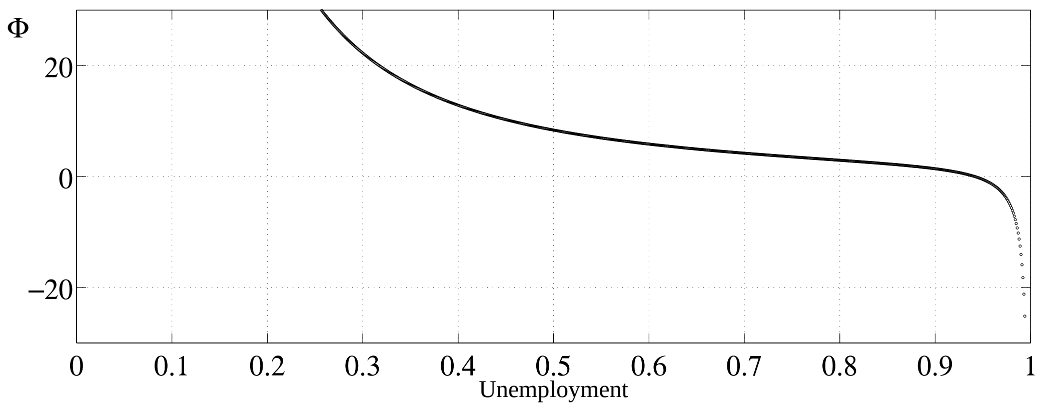

Figure A1: Indeterminacy and the concavity of the Beveridge curve

[4]Indeed, $\partial V/\partial U = -\left(\delta/\alpha\right)\left(\delta(1-U)/U^{\beta}\right)^{(1-\alpha)/\alpha}\left(1+\beta\left(1-U\right)/U\right)/U^2 < 0$ for all the eligibles α, β and δ.

The diagram shows that when the unemployment rate is around 95%, Φ switches from positive to negative. As a consequence, beyond this critical threshold the Beveridge curve becomes concave.

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