The Course of the Profit Rate

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THE COURSE OF THE RATE OF PROFIT[1]

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ABSTRACT

In discussions on the rate of profit and its tendency to fall and its role in Marxist theory, a number of phrases are often employed without clarifying what these might really mean. Primary among these are such phrases as 'the rate of profit must ultimately fall' and 'the counter-acting factors cannot possibly offset the tendency in the long run'.

As a result of this ambiguity, and as a result of a legacy of confusion concerning Marx's own ideas on the profit rate beginning with the Western reception of Okishio's (1961) famous theorem, research on the actual mathematical conditions for the profit rate to rise or fall, especially in the long term, has all but ceased.

However there is very strong evidence that the rate of profit has, in fact, been falling in most industrialised economies for some considerable time, and there is good reason to suppose this has at least some bearing on the origins of the present prolonged phase of stagnation in these economies.

The time is therefore ripe to return to a rigorous study of the general mathematical conditions that might govern the long-term movement of the profit rate.

In particular, I will attempt to give mathematical meaning to the two concepts above.

INTRODUCTION

Suppose at any given time t

l is the rate of value creation, that is to say, the magnitude of living labour engaged in production;

v is the wage rate, that is to say, the wage level in value terms;

s is the rate of surplus value, given by

$$s = l - v \tag{1}$$

K is capital advanced

r is the rate of profit, given by

$$r = \frac{s}{K} \tag{2}$$

What are the conditions under which r falls?

METHOD

The question can be phrased, mathematically, as follows: what are the conditions under which $r' < 0$,

[1] This paper considerably updates and revises a paper previously published on www.londonmet.academia.edu/alanfreeman. I am grateful to Andrew Kliman and Manuel Aguilar for their very helpful comments and on an accompanying spreadsheet. All remaining errors are of course my own.

where $r' = \frac{dr}{dt}$

A first step is simply to differentiate (2) with respect to time. This yields

$$r' = \frac{d}{dt}\left(\frac{s}{K}\right) = \frac{s'K - sK'}{K^2} \tag{3}$$

or, dividing through by s/K

$$r \times \left(\frac{s'}{s} - \frac{K'}{K}\right) \tag{4}$$

The condition for r'<0 is then $$\frac{K'}{K} > \frac{s}{s} \tag{5}$$

or, alternatively $$K's > Ks' \tag{5'}$$

Note that in the simple case of constant surplus value, this becomes quite simply

$$K' > 0 \tag{5''}$$

That is to say, the rate of profit continues to fall as long as accumulation is positive.

This conflicts with the findings of Okishio (1961) that the rate of profit must rise if there is cost-reducing technical change and the real wage remains constant, and that this is 'Contrary to Marx's Gesetz des tendenziellen Fall der ProfitRate'. Some confusion exists because of various different conditions on the surplus which can be introduced: Okishio's theorem supposes a constant real wage, above I have assumed constant surplus value, and Marx in his study of the 'Law as Such' (but not in the more general treatment that follows) assumes a constant rate of explotation which in our notation above would be s/(l-s). The above, however, is not the origin of the difference between our findings (5-5'') and Okishio's since many special cases can be deduced from the above in which Okishio's theorem, and the above result, produce different deductions even though all conditions of Okishio's theorem are satisfied.[2]

Okishio himself understood the origin of the difference, but wrongly interpreted Marx's own view, as is clear from Okishio (2000):

> My theorem, the so-called Okishio theorem, is a comparative statics result. Therefore, it has no realistic meaning if capitalist's competition does not establish a new equilibrium following the introduction of a new production method. Marx firmly believed that a new equilibrium was established. In this paper, we investigate the capitalist process without technical change. If we choose the proper parameters and initial conditions, all profit rates converge to zero, Schumpeter's result…

[2] Thus, for example the case where l-s=0, studied by Kliman (2007). A common misunderstanding is evident in Heinrich (2014:footnote 30) which objects that this condition is an 'absurd assumption'. The assumption is not intended, as Kliman et al (2013) point out, as a proof of Marx's theory but as a refutation of Okishio's. This it does: because it shows that under Okishio's assumptions, in a specific case, his result does not hold. It is a straightforward deduction in mathematical logic that, therefore, his theorem cannot be true, because there is at least one case in which it does not apply. There are many other such cases: the case v=0 is chosen for its simplicity.

> Many people have criticised the Okishio theorem (Okishio, 1961). These criticisms have not persuaded me, because, given the assumptions, the theorem is valid. However, I now think my assumptions were inappropriate

There is, however, no evidence that Marx held this belief as is shown by scrupulous exegetical work, demonstrating the contrary, such as Kliman (2007). Therefore, to conclude this introduction, two important research questions have remained not only unanswered, but more or less unexamined, even by Marxist scholars, for many years and, in fact, decades:

(1) What can actually be said about the long-run course of the rate of profit, if Okishio's error is corrected and we approach the study using the method specified in equations (1)-(4)?
(2) What was Marx's own view on the course of the rate of profit?

This short paper gives some initial responses to question (1). It then briefly examines whether these responses tend to confirm, or conflict with, Marx's own written views on the rate of profit. The reason for this double enquiry is, first and foremost, to re-initiate the scientific study of the course of the profit rate. However, it is necessary also to consider the relation of these results to Marx's theory because of the confusion created by repeated claims to the effect that the profit rate cannot possibly fall in the way predicted by Marx, who provided the most advanced explanation of this fall so far available.

A MORE RECENT DEBATE

Two issues have arisen in discussions between Heinrich (2014), Kliman et al (2014) and Carchedi and Roberts (2014). These are: is the fall in the profit rate *inevitable* (and did Marx make this claim) and if it is not inevitable *under what cirumstances does it fall*? Without repeating the arguments already advanced in these three papers, I proceed directly to the mathematical issues involved.

If we suppose as does Heinrich (2014) that s and K are independent functions, little more can be said about the course of r or the condition for r' to be greater or less than 0. Thus if for example K does not grow at all, then obviously, any rising rate of surplus value will yield a rising rate of profit.

In fact, however, K is a function of s because the surplus in part or whole is re-invested. In order to determine the real connection between r, K and s, we must take account of the functional relation

$$K=K(s) \tag{6}$$

In general, K grows through accumulation. The surplus s is accumulated, that is, it is added to K. However, not all of it is accumulated. Some is consumed by the bourgeoisie. Some may go to waste. Some may be accumulated in hoards that do not enter the formation of the rate of profit. Let us designate the rate of production of the part that is *not* accumulated as *b.*

Then
$$0 < b < s \tag{7}$$

And
$$K' = s - b \tag{8}$$

We can now rewrite (5') independent of K, as

$$s(s-b) < Ks' \tag{9}$$

We have still not eliminated K from the righthand side, but we should note that it is not an independent function. We can simplify matters, therefore, by dividing through by K and using the original relation (2) to give

$$r(s\text{-}B)<s' \qquad (10)$$

or

$$r < \frac{s'}{s-b} \qquad (11)$$

This is a general condition for the rate of profit to fall.

This still does not fully remove the dependence on K – that is, the functional relation between K on the one hand, and *s, b* on the other. This is because K is 'hidden' in the definition of *r.* However it is quite a helpful way to study the relation between the variables.

In order to eliminate K completely, we need to do a bit more work and probably use some Lyapunov or other trajectory analysis. I will first use (11) to illuminate a couple of special cases which are key to the literature, and then suggest an approach to the general problem.

THE LAW AS SUCH

If s is a constant,

$$s'=0 \qquad (12)$$

This yields

$$r'<0 \;\; \text{iff}\, r>0 \qquad (13)$$

that is, r falls to zero. This does not give the trajectory of *r;* however this is not difficult to express

We have

$$K=K_{0\_}+ts \qquad (14)$$

Hence

$$r=\frac{s}{K_0+ts} \qquad (15)$$

- a hyperbolic function.

Marx's own 'law as such' is expressed under the slightly different assumption that s/v (the rate of exploitation) is a constant, which is a more general case. It suspect it is relatively easy to show that under Marx's assumption, the rate of profit must fall if *K* is a monotonic function of *s*. Readers are welcome to attempt the exercise. My intention here is to show that there exist simplifying assumptions, which are not at all unreasonable, which make sense of such statements as 'the rate of profit falls without limit' and begin to allow us to establish the circumstances under which this might be so.

The inverse problem to this study is the following question: under what circumstances is r'>0? To this we now turn.

COUNTERACTING FACTORS

We can now consider some of the 'counteracting factors' discussed by Marx, as well as others. It is clear that if b can be any arbitrary quantity b, then there will always be some value of b such that r'>0.

Note that b can be greater than *s.* In this situation (disaccumulation) the capital stock is actually used up in consumption, or waste. In parenthesis, b can also be less than zero if hoarded non-employed capital is brought into use. There are therefore a wide range of possibilities if we consider circumstances external to the direct relation between surplus production and accumulation. A sufficiently large magnitude of b can always offset the fall in the rate of profit.

However, to suppose that b can be arbitrarily large is to suppose that the capitalists cease to accumulate, which is another way of saying that capitalism ceases to be capitalism. To the extent that special circumstances suspend accumulation (crisis, war, state intervention, etc) therefore, the tendency of the rate of profit to fall can always be offset. However, if these factors are external, properly speaking they cannot be considered counteracting factors since they do not arise from the accumulation process *per se* – one can always offset the law of gravity with a sufficiently large lifting machine but this does not negate the law.

This said, one of the consequences of crisis is a rise in b for a variety of reasons which we do not need to discuss – for example, liquidation of physical assets, state consumption, or simple hoarding. An issue to investigate is why this appears insufficient to offset the TRPF in most of capitalist history. This is a concrete empirical question, not an issue of the law itself. It does however refute the argument (Robert-Carchedi) that the rate of profit must inevitably fall.

We first consider the cases when b is constant. First, for simplicity, suppose $b=0$

Suppose first that s does rise.

We can consider this under two general headings:

(1) l is constant, but v falls (rising rate of exploitation)

(2) l is rising, and v may behave in a number of ways

If *l* rises sufficiently fast, it can of course offset any fall in the rate of profit. My reading of the argument proposed by Marx is that, however, it is constrained by biological factors since it is limited by the growth rate of the population absolutely, and relatively by the proportion of the population that is economically active (employed)

First let us suppose constant *l*. In that case we can write

$$S = el,\ [0 < e < 1] \tag{16}$$

where e is the rate of exploitation.

We then have $s' = e'l$

and we can rewrite (11) as

$$r < \frac{e'}{e} \tag{17}$$

This is as far as I have reached without moving to a more general treatment but there may be something to be had from this useful simple form of the condition for r to fall

THE LONG RUN

When we study the rate of profit we are interested in long-run processes. Unfortunately the notion of ‘long run’ is poorly defined in economics. This creates space for many fantastical ideas, because it is always possible to imagine scenarios in which, for a short period of time, any proposal that one cares to mention may possibly come to pass. At an instant when a body is thrown in the air, it is static, just before it falls. This does not allow us to deduce that bodies can in general escape the law of gravity, yet it is extremely hard to define, philosophically, exactly in what sense the law of gravity does not apply to a body at this exact moment of stasis. Moreover, there is no limit to the amount of time that a body may spend ascending, provided it is thrown with sufficient force. Yet we feel no discomfort in such anodyne statements as ‘what goes up, must come down’.

The way to settle all such discussions is to study those relations between time and the rate of profit which must necessarily hold, regardless not merely of the initial motion of the economy, but of all possible motions that it may subsequently pass through, from one moment of time to the next. In the terminology of state space analysis, we wish to know what relations are invariant with respect to trajectory.[3] We can illustrate this with two curves, each of which represents a different way in which a variable y=y(t) may move from its value at time t=0 to another value at time t=1

We can distinguish two kinds of economic variables:

(1) those that depend on how the curve moves over time
(2) those that do not

Variables of the first type, in general, will have different values in the first trajectory from those they arrive at in the second trajectory. They are, in the language of economics, path-dependent.

Even when variables are path-dependent, there are relations between these variables which may be path-independent. The best-known such cases are those which in mechanics, and more generally in physics, are *conserved* magnitudes, sometimes called constants of motion. One such is potential energy, V, which is the energy that a body possesses by virtue of its position relative to other bodies, under the influence of gravitation. When a body moves ‘up’ – away from the earth, for example – its potential increases, and when it moves ‘down’ this potential decreases. This is independent of the way it moves from one position to another.

One conserved magnitude is value added. No matter how the economy moves from time t=0 to time t=1, it remains true that the total new value created, during this time, is equal to the total labour expended. So over any year, it doesn’t matter if there is a two-shift system, a three-shift system, or when Easter falls, the only thing that counts is how many hours the workers have worked, during the year. We can express this as an integral relation:

$$L = \int_0^1 l(t)dt \tag{18}$$

where L is total labour expended and l(t) is the size of the labour force at any given time.

[3] In what follows we make only one assumption, which is that of continuity. Technically we suppose that the movement of the economy lies on a differentiable manifold.

In general, we have to be more aware of what all variables really mean. In the equations so far given, all except K are *flow* variables, that is, they represent the instantaneous rate at which the quantity is changing. So (for example) at any given time t, $l(t)$ tells us 'how much value per hour' is being added to the product, that is, what is the size of the labour force. Since these are instantaneous variables, we didn't put the time subscript on them but it is always there, implicitly.

To study path dependence, and hence to answer questions about long-run tendencies in capitalism, we must distinguish between these instantaneous magnitudes and (where relevant) their accumulated sums over time, taking into account everything that adds to them and everything that subtracts from them.

A hydraulic analogy suggests itself: if we wish to know how full the bath is, we have to subtract, over time, the amount of water that falls out through the plug from the amount that comes in through the tap. Or, substituting a lake for a bath, we have to subtract the water lost through evaporation, fixation by biological consumption, and riverine outflow, from the water gained by riverine inflow, rainfall, and biological excretion. What matters here is not the details of these gains and losses, but where the lake is at the end of the process. The answer is that it will fall by an amount equal to the difference between what goes out and what comes in, or, if that latter magnitude exceeds the former, it will correspondingly rise.[4]

Various authors have attempted to study path-dependent behaviour by presupposing some given rate of flow of inputs, outputs, or both (see for example Foley 1982). This at least addresses the problem. But it does not provide the kind of answer we are looking for, because we don't wish to know 'what will happen if we suppose a particular pattern of flows in and out' but 'what remains true, regardless of flows in and flows out?'

To address this question in a general way, we will adopt a notational convention. We use L in capital letters to distinguish the total labour expended, over a definite period, from the instantaneous rate. Similarly

$$K_1 = K_0 + \int_0^1 (s-b)dt \tag{19}$$

and so on. To clarify a minor point, what is the time suffix of the variables thus generated? Following the temporal notation we suppose that the time suffix to be used in talking of any flow variable in a period t=[0,1] is the beginning of the period, hence $L_{t[0]}$ or simply L_0: by this is meant the labour added between t_0 and t_1. In general this is given by

$$L_0 = \int_0^1 l dt \tag{18'}$$

[4] As Mirowski (1991) notes, conservation faded from the common discourse of economists with the rise of the marginal school. However, it did not fade from physics and is if anything more central than ever. Emmy Noether established the now decisive link between conservation and parity through which, in physics itself, theories of conservation and with them the study of invariants of motion, are not regarded as opposed to field theory but integral to it. It is the economists of 1890, and their successors, who have in effect pronounced such concepts worthless: not, I humbly submit, a good reason for ignoring these concepts.

Following the same notational convention, the time assigned to any *instantaneous* variable is (somewhat obviously, but the point needs to be made) the time at which this is measured. Thus the price of any commodity j at any time t is *p* (j,t), and the rate of profit at any time t is *r(t)* or $r_{t,}$ and so on.

The question can now be posed in the following way: between t=0 and t=1, what are the conditions for the rate of profit r_1 to be less than r_0?

This is not a path-independent relation. If, shortly after t=0, the rate of exploitation rises to 1 and stays at this level until just before t=1, and if all surplus is accumulated and the labour force remains constant, then almost all the labour discharged in this time will be accumulated and, consequently, K will rise by an amount almost equal to the size of the labour force. If, alternatively, the rate of exploitation falls to zero and stays at this level until just before t=1 then no matter what the size of the labour force, there will be no accumulation, K will not change, and so the instantaneous rate of profit at time t=1 will depend exclusively on the instantaneous rate of exploitation at that moment, on the basis of which any conceivable Sraffian supposition can be imposed.

Since the rate of profit is not path-independent, the question is 'what path-independent assertions may be made of the rate of profit, given the law of accumulation as expressed in (8)'?

To show the method I suppose b = 0. To clarify what this means let's use the variable S, upper case, to mean the total surplus generated between the two points in time. If *b* = 0 this is all invested, and so K increases by that amount. To see how this can be expressed rigorously, we can now write

$$r_0 = \frac{s_0}{K_0}; r_1 = \frac{s_1}{K_0+S} \tag{20}$$

and, subtracting,

$$\Delta r_0 = \frac{s_0(K_0+S)-s_1K_0}{K_0(K_0+S)} \tag{21}$$

so that the condition for a rising profit rate between these two points in time is

$$s_0(K_0+S) > s_1K_0 \tag{22}$$

Or

$$\frac{s_1}{s_0} > \frac{K_0}{K_0+S} \tag{23}$$

$$s1 > s_0(K_0+S)/K_0 \tag{24}$$

That is, if the rate of profit is not to fall, then the rise in s must in some sense 'outweigh' the addition of S to K_0. The question is then as follows: is there always a period, sufficiently long, such that a rise in the rate of surplus value cannot outweigh it? Or rather, what are the conditions under which it can?

Two assumptions are required to yield this result. They are sufficient, but not necessary:

(1) There should be some upper bound to $s1$
(2) There should be some lower bound to s between time 0 and time 1

To see this, first note that s_0 and K_0 are given: they are simply constants. As regards condition (1) this simply states that the final rate of surplus production cannot be indefinitely large. Obviously, if at time 1, it is allowed that the surplus can be indefinitely large, then it is always possible to choose s1 such that the rate of profit will rise. Once this obvious restriction is accepted, the key is to grasp the relation between s_1 and S.

What is *S,* actually? Actually it is what in economics is sometimes called a 'stock' concept but this is a misnomer, since it does not form a stock. It is the total labour discharged over the time period concerned, less the total value consumed by workers over the same period. This is an integral – a summed total of what happens in each moment throughout the process of getting from time 0 to time 1. We give the name 'trajectory' to any such process.

This total can be reached in a second-order infinity of ways; each possible trajectory of the economy between time 1 and time 2 yields a different way of reaching the final total *S.* So for example, if the period runs from March 1st to April 1st then the magnitude of S will depend on whether a spring holiday falls within these dates, which in turn depends on which year the given period falls, not to mention the religious proclivities of the state authorities and the resolution of any conflict between these proclivities and the urgency of extracting profit. Or, if there is a two-shift system so that the factories close at night, the creation of surplus value will cease at night whereas if there is a three-shift system it will not. S simply adds up all the time that is worked, and subtracts all the wages paid, over the given period.

This makes it quite difficult to define the *general* conditions for the rate of profit to fall, because there are such a variety of possible movements of the economy. For example, at least in theory, s could fall to zero immediately after t=0, and rise to s_1 just before time t=1. In that case we would have S=0 and the rate of profit would rise.

The notion of upper and lower bounds are crucial in studying this question, I believe. Whilst it is almost impossible to establish a completely general law that will cover every possible trajectory of the economy, we can nevertheless establish what must necessarily hold, for an possible trajectory of the economy. That is the purpose of the two conditions specified above. The question can then be posed as follows:

Given K_0, s_0, s_1, and t_1-t_0, under what condition is $r_1 < r_0$? We say 'given' s_1 because of condition 1. I claim that condition (2) is sufficient. In this case, there is a number s^* such that $s(t)$ never falls below s* between t=0 and t=1. In that case, S cannot be less than ts^*. But then, we can always choose $t = t_1 - t_0$ that is large enough, so that the rate of profit will be lower at time t_1 than at time t_0. I think this is quite a good interpretation of Marx's 'auf dei Dauer'.

Note also: this proof explains Andrew Kliman's counter-example to the proposition 'the rate of profit must inevitably fall in the long run', and also Heinrich's. Suppose we invert the specification and take *t* to be given, but pose the question 'is there a trajectory of *s* such that the rate of profit rises?' then the answer is yes, but in order to find this trajectory, we must allow *s* to sink to a level that is low enough. Once there is a lower bound for *s* it is impossible to achieve this. In fact, the longer is the period *t* the lower must be the level that we let *s* shrink to.

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