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Abstract

Certain commodity producers face uncertain output and price, but can trade financial derivatives on price. I consider how best to use a put option on price. I introduce the variance surface, which is a data visualization technique that shows the level of variance across a grid of values for the two choice variables, quantity of options and strike price. The variance-minimizing hedge has strike deep in the money and optimal quantity close to expected output, but the variance surface shows there are near-best choices that are less expensive.

Keywords: Variance-minimizing hedge, put option, simulation, data visualization.

JEL Classification: C01, C63, G22, G32, Q14.

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1. Introduction

Wheat farmers and gold miners both face a unique circumstance where the quantity and price of their output is uncertain, but they can trade financial derivatives on price. Such producers face a risk management problem with multiple sources of uncertainty. I use simulation to explore a basic version of this problem: how to use put options on price to minimize the variance of revenue. I use a brute force search algorithm to identify the optimal quantity of options and strike price, but I also use data visualization to show that there are second best positions that receive most of the benefit of the optimal hedge at lower cost.

Research on risk management with multiple sources of uncertainty arguably begins in agricultural economics with McKinnon (1967), who uses the mean-variance framework to analyze the optimal use of forward contracts. McKinnon finds that the optimal quantity of forward contracts is equal to expected output when price and quantity are uncorrelated, less than expected output under negative correlation, and greater than expected output under positive correlation. Losq (1982) challenges these results with a general expected utility approach where he shows it is optimal to hedge less than expected output whenever the third derivative of the utility function is positive, which is associated with skewness preference. There is much further debate on the topic in agricultural economics (Rolfo, 1980; Lapan & Moschini, 1994), but the literature is generally limited to hedging with forward contracts.

Brown and Toft (2002) show how to go beyond basic forward contracts in a model where risk management affects firm value. Brown and Toft's model allows the producer to design an exotic derivative to maximize expected value of the firm net of fixed, variable, and financial distress costs. Oum and Oren (2010) demonstrate how to create an exotic derivative in the mean-variance framework. Exotic derivatives allow the producer to customize the entire payoff function, which is a sophisticated problem in functional analysis that generally requires a numerical solution.

Although the literature covers forward contracts and exotic derivatives, there has been little attention given directly to put options. This may be because put options are a special case of exotic derivatives: the optimal exotic derivative must be at least as good as the optimal put option. Although exotic derivatives are superior, they require some sophistication from the producer to implement because they are not exchange traded. In contrast, put options are exchange traded and the producer faces a relatively simple optimization problem with two variables, quantity of options and strike price.

Since a position in put options is characterized by two variables, I use a data visualization technique to assist with identification of the optimal put option. Data visualization is an important part of research and practice in finance (Lemieux, 2013). I refer to the visualization as the variance surface because it shows level sets of variance across option parameters. I use it to identify the variance-minimizing hedge, show that the minimum is well defined, and explore near-best options.

2. Simulation Experiment

2.1 Data Generating Processes

The probability model for quantity of output Q and price P are described by Equation (1), (2), and (3). I assume the change in quantity D_Q and log price D_P are generated from a multivariate normal distribution, but the quantity is normal and the price is lognormal. This does not yield an immediate analytic solution, but it can be analyzed with simulation.

(1) $Q = Q_0 + D_Q$.

(2) $P = P_0 \exp(D_P)$.

(3) $[D_P, D_Q] \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

The following parameters are fixed throughout the analysis. The initial quantity is one, $Q_0=1$, and price one hundred, $P_0=100$. The change in price and quantity both have zero mean, $\boldsymbol{\mu}=[0,0]$. The variance of change in price is constant throughout, $\boldsymbol{\Sigma}_{1,1}=\sigma_P^2$ with $\sigma_P=0.1$. I calculate the variance with different put options based on a large sample of observations, $n=10^6$, from the joint distribution of $[D_P, D_Q]$.

I vary other parameters to explore sensitivity of the results. The variance in quantity, $\boldsymbol{\Sigma}_{2,2}=\sigma_Q^2$, represents unhedged risk. I report results for a small level of unhedged risk, $\sigma_Q=0.01$, and a large level, $\sigma_Q=0.05$. The covariance between change in quantity and price is $\boldsymbol{\Sigma}_{1,2}=\rho\sigma_P\sigma_Q$. The correlation, ρ, represents whether output provides a natural hedge on price ($\rho<0$) or not ($\rho>0$). I report results for negative, zero, and positive correlation, $\rho=-0.5$, 0.0, or +0.5.

I use the Black Scholes formula for the option premium, as in Equation (4) and (5). The premium, $O(k)$, depends on strike price, k. I assume the interest rate is zero, $r=0$, and the option expires after one time step, $\tau=1$, to reflect a static trading strategy in the put option.

(4) $d_1=(1/\sigma_P\sqrt{\tau})(\log(S_0/k)+(r+½\sigma_P^2)\tau), d_2=d_1-\sigma_P\sqrt{\tau}$.

(5) $O(k)=ke^{-r\tau}\Phi(-d_2)-S_0\Phi(-d_1)$.

I calculate net revenue, $N(q,k)$, as in Equation (6). The probability distribution of net revenue depends on strike price, k, and quantity of options, q.

$$(6) \qquad N(q,k)=PQ+q(\max[k-P,0]-O(k)e^{r\tau}).$$

I calculate the variance of net revenue, V(q,k), as in Equation (7).

$$(7) \qquad V(q,k)=Var(N(q,k))$$

I use a brute force search for to identify the variance-minimizing hedge. I define the search set as **S**=**q** x **k**, where **q**={0, 0.02, …, 2} and **k**={50, 51, …,150}, which is chosen to cover the range of possible values for price and quantity of output. I calculate the variance V(q,k) at each point in the search set based on the random sample of observations for [D_P, D_Q] and identify the variance-minimizing hedge directly. I also use the search set to build the variance surface.

2.2 Variance Surface

Figure 1 shows the variance surface when price and quantity independent, ρ=0, and unhedged risk is large, σ_Q=0.05. The axes are defined by the search set **S** and the height of the surface shows the variance of net revenue with that quantity of options and strike price.

Figure 1 -- Variance Surface

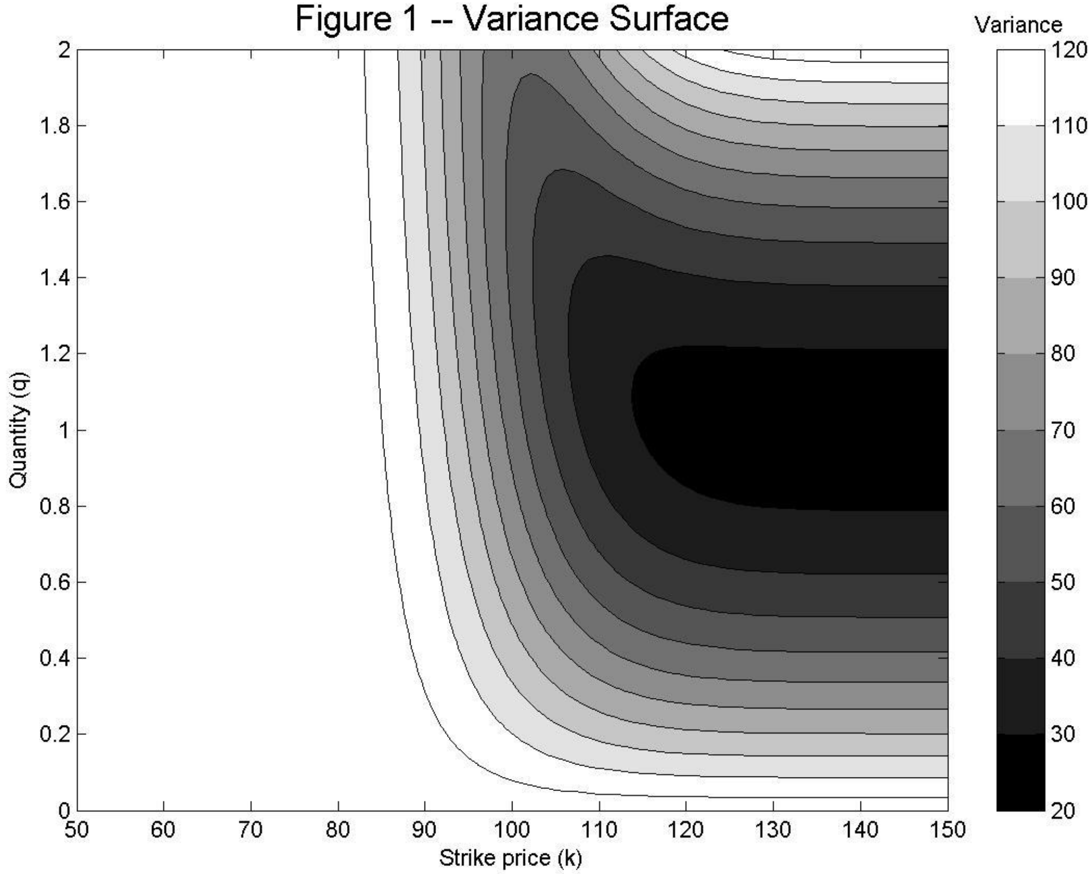

Figure 1 suggests that the variance surface is a convex function, which is a desirable property because it suggests the variance-minimizing hedge may be unique. My estimate of the variance-minimizing hedge is, indeed, unique and it is also a corner solution: the variance-minimizing hedge has (q*,k*)=(1.00, 150) and V(q*,k*)=25.5. The quantity of options equals expected output E(Q)=1 and the strike price is far above the initial price P_0=100, which means the option is deep in the money. Since this option is almost surely in the money, it functions like a forward contract with artificially high forward price. This option is expensive because of high intrinsic value and does not exploit the convexity of the put option payoff function.

Although the variance-minimizing hedge is unique, Figure 1 shows that many other put options can reduce variance to near-minimum. For example, a put option that is only slightly in the money, say k=110, and quantity larger than one, say q=1.2, will cause variance to equal approximately 30. Such an option is similar to the variance-minimizing hedge but is less expensive because it has lower intrinsic value.

2.3 Sensitivity Analysis

I report how the location of the variance-minimizing hedge changes with the level of unhedged risk, σ_Q, and the size of correlation between quantity and price, ρ.

Table 1: Optimal quantity of put options, strike price, and variance of revenue for different risk structures

	Negative correlation (ρ=-0.5)	Zero correlation (ρ=0.0)	Positive correlation (ρ=+0.5)
Small quantity risk	(q*,k*)=(0.94, 150)	(1.00, 150)	(1.06, 150)
(σ_Q=0.01)	V(q*,k*)=0.77	1.01	0.77
Large quantity risk	(1.00, 150)	(1.00, 150)	(1.26, 150)
(σ_Q=0.05)	1.02	25.5	19.8

Table 1 shows the optimal strike price is deep in the money across different risk structures, but there are small changes in the optimal quantity of options. The optimal quantity of options are in line with McKinnon (1967). The optimal quantity of options is slightly below expected output when there is negative correlation between price and output because negative correlation provides a natural hedge, which reduces the need for the put option. The optimal quantity of options is slightly above expected output when there is positive correlation because positive correlation accentuates risk, which increases the need for the put option.

3. Discussion

I find that the variance-minimizing put option generally has strike deep in the money, which means it functions like a forward contract with very high forward price. It may be that the optimal strike diverges to arbitrarily large values, which is unrealistic. It is possible to address this unrealistic problem by including a volatility smile in option pricing, where tail options have higher prices than used in this paper. Volatility smile would not change the variance of revenues because the option premium is constant, but volatility smile would decrease average revenue; a researcher could explore how the optimal hedge changes with volatility smile by using a mean-variance utility surface, rather than the variance surface.

Although I identify the variance-minimizing hedge, the variance surface shows that other options do nearly as well. This demonstrates the value of visual analytics and exploratory data analysis over a blind faith in an optimal solution. More can be done with data visualization in this setting, such as calculating different types of surfaces. For example, the mean-variance utility surface mentioned above. It is also possible to use animations of the surface to further explore the parameter space of the model. For example, an animation of the surface as correlation range from -1 to +1 could reveal interesting structures in the results that are not apparent in static analysis.

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