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Abstract

We consider two-sided matching problems where agents on one side of the market (hospitals) are required to satisfy certain distributional constraints. We show that when the preferences and constraints of the hospitals can be represented by an M$^\natural$-concave function, (i) the generalized Deferred Acceptance mechanism is strategyproof for doctors, (ii) it produces the doctor-optimal stable matching, and (iii) its time complexity is proportional to the square of the number of possible contracts. Furthermore, we provide sufficient conditions for representation by an M$^\natural$-concave function. These conditions are applicable to various existing works and enable new applications as well, thereby providing a recipe for developing desirable mechanisms in practice.

JEL Classification: C78, D61, D63

Keywords: two-sided matching, many-to-one matching, market design, matching with contracts, matching with constraints, M$^\natural$-concavity, strategyproofness, deferred acceptance.

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1 Introduction

The theory of two-sided matching has been extensively developed, and it has been applied to design clearinghouse mechanisms in various markets in practice.[1] As the theory has been applied to increasingly diverse types of environments, however, researchers and practitioners have encountered various forms of distributional constraints. As these features have been precluded from consideration until recently, they pose new challenges for market designers.

The *regional maximum quotas* provide an example of distributional constraints. Under the regional maximum quotas, each agent on one side of the market (who we call a hospital) belongs to a region, and there is an upper bound on the number of agents on the other side (who we call doctors) who can be matched in each region. Regional maximum quotas exist in many markets in practice. A case in point is medical residency matching in Japan. Although the match organizers initially employed the standard Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962), it was criticized as placing too many doctors in urban areas and causing doctor shortage in rural areas. To address this criticism, the Japanese government now imposes a regional maximum quota to each region of the country. Regulations that are mathematically isomorphic to regional maximum quotas are utilized in various contexts, such as Chinese graduate admission, Ukrainian college admission, Scottish probationary teacher matching, among others (Kamada and Kojima, 2012, 2015).

Furthermore, there are many matching problems in which *minimum quotas* are imposed. School districts may need at least a certain number of students in each school in order for the school to operate, as in college admissions in Hungary (Biro, Fleiner, Irving, and Manlove, 2010). The cadet-branch matching program organized by United States Military Academy imposes minimum quotas on the number of cadets who can be assigned to each branch (Sönmez and Switzer, 2013). Yet another type of constraints takes the form of *diversity constraints*. Public schools are often required to satisfy balance on the composition of students, typically in terms of socioeconomic status (Ehlers, Hafalir, Yenmez, and Yildirim, 2014). Several mechanisms have been proposed for each of these various constraints, but previous stud-

[1]See Roth and Sotomayor (1990) for a comprehensive survey of many results in this literature.

ies have focused on tailoring mechanisms to specific settings, rather than providing a general framework.[2]

This paper develops a general framework for handling various distributional constraints, in the setting of *matching with contracts* (Hatfield and Milgrom, 2005). We begin with a simple model in which, on one side of the market, there exists just one hypothetical representative agent, *the hospitals*. Although extremely simple, this model proves useful. More specifically, we offer methods to aggregate the preferences of individual hospitals and *distributional constraints* into a preference of this representative agent and, as detailed later, use this aggregation to help analyze matching with constraints.[3]

For this model with *the hospitals*, the key of our analysis is to associate the preference of the hospitals to a mathematical concept called $M^\natural$-concavity (Murota, 2003).[4] $M^\natural$-concavity is an adaptation of concavity to functions on discrete domains, and has been studied extensively in discrete convex analysis, which is a branch of discrete mathematics. We show that if the hospitals' aggregated preference can be represented by an $M^\natural$-concave function, then the following key properties in two-sided matching hold: (i) the generalized Deferred Acceptance (DA) mechanism (Hatfield and Milgrom, 2005) is strategyproof for doctors, (ii) the resulting matching is stable (in the sense of Hatfield and Milgrom (2005)), (iii) the obtained matching is optimal for each doctor among all stable matchings, and (iv) the time complexity of the generalized DA mechanism is proportional to the square of the

[2]Examples of papers that accommodate specific constraints include Ehlers, Hafalir, Yenmez, and Yildirim (2014); Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2015); Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014); Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014); Kamada and Kojima (2015). Needless to say, we do not claim to subsume all the results in the existing studies. For instance, Kamada and Kojima (2014a) allow for general choice functions that satisfy substitutability, while our study focuses on choice functions that satisfy $M^\natural$-concavity, which is a stronger requirement. Another notable example is the study of matching with minimum quotas by Fragiadakis and Troyan (2014). Their mechanisms are different from ours, and whether there is any way to reduce their problem to our framework, or even any matching framework with substitutability, is an open question.

[3]In fact, our results readily generalize for cases with multiple, separate hospitals. The assumption of exactly one agent on the hospital side is made for simplicity only and, as stated above, that model proves sufficient for our purposes. A similar technique has been used by Kamada and Kojima (2014a).

[4]The letter M in $M^\natural$-concavity comes from "matroid," a mathematical structure that plays an important role in this paper. The symbol $\natural$ is read "natural."

number of possible contracts.

Equipped with this general result, we study conditions under which the hospitals' preference can be represented by an M$^\natural$-concave function. We start by separating the preference of the hospitals into two parts. More specifically, we divide the preference of the hospitals into hard distributional constraints for the contracts to be feasible, and soft preferences over a family of feasible contracts. Drawing upon techniques from discrete convex analysis, we first show that if the hospitals' preference is represented by an M$^\natural$-concave function, then a family of the sets of contracts that satisfy hard distributional constraints (which we call *hospital-feasible* contracts) must constitute a matroid. Next, we show that if the hard distributional constraints constitute a matroid and the soft preferences satisfy certain easy-to-verify conditions (e.g., they can be represented as a sum of values associated with individual contracts), then the hospitals' preference can be represented by an M$^\natural$-concave function.

One of the main motivations of our work is to provide an easy-to-use recipe, or a toolkit, for organizing matching mechanisms under constraints. Although our general result is stated in terms of the abstract M$^\natural$-concavity condition, market designers do not need advanced knowledge on discrete convex analysis or matching theory. On the contrary, our sufficient conditions in the preceding sections suffice for most practical applications. To use our tool, all one needs to show is that the given hard distributional constraints produce a matroid (note that requirements over soft preferences are elementary, e.g., a sum of the values of the contracts). Fortunately, there exists a vast literature on matroid theory, and what kinds of constraints produce a matroid is well-understood. Therefore, it usually suffices to show that the hard distributional constraints can be mapped into existing results in matroid theory. We confirm this fact by demonstrating that most distributional constraints can be represented using our method. The list of applications includes matching markets with regional maximum quotas (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2014a,b, 2015), individual minimum quotas (Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo, 2015), regional minimum quotas (Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo, 2014), diversity requirements in school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014), and the cadet-branch matching program (Sönmez and Switzer, 2013). We also utilize the same toolkit to obtain new results such as the existence of a stable matching under new types of constraints. As such, we believe that this study contributes to the advance of practical market design (or "economic engineering") as emphasized in the recent literature (see Roth (2002) and Milgrom (2009) for instance), by providing tools for organizing matching clearinghouses in practice.

The rest of this paper is organized as follows. First, in the rest of this section, we discuss related literature. In Section 2, we introduce our model. In Section 3, we prove that when the hospitals' preference is represented as an M$^\natural$-concave function, the above-mentioned key properties hold. In Section 4, we present several sufficient conditions under which a function becomes M$^\natural$-concave. Then, in Section 5, we examine existing works on two-sided, many-to-one matching problems and show that the sufficient conditions described in the previous section hold in these cases. In Section 6, we discuss the relations between applications presented in Section 5, as well as new applications that combine different types of distributional constraints. Finally, Section 7 concludes this paper. Proofs are deferred to Appendix unless noted otherwise.

Related literature

Although matching with constraints is a fairly new research topic, questions related to this issue have been studied in the literature in various specific contexts. In the U.K. medical match in the 20th century, some hospitals preferred to hire at most one female doctor (Roth, 1991). In school choice, schools are subject to diversity constraints in terms of socioeconomic status and academic performance (Abdulkadiroğlu, 2005; Abdulkadiroğlu and Sönmez, 2003; Ehlers, Hafalir, Yenmez, and Yildirim, 2014; Hafalir, Yenmez, and Yildirim, 2013; Kojima, 2012). Constraints placed over *sets* of agents have been studied in the context of student-project allocations (Abraham, Irving, and Manlove, 2007), college admission (Biro, Fleiner, Irving, and Manlove, 2010), and medical residency matching (Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo, 2014; Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2014a,b, 2015). Our marginal contribution over these existing studies is to present a unified framework and analyze these specific markets as well as others using a single technique: as will be seen below, our theory can be applied to a wide variety of existing models (Section 5) as well as new ones (Section 6).

Our paper is at the intersection of discrete mathematics and economics. In the former research field, there is a vast literature on discrete optimization. Their insight has been used in a broad range of applications such as scheduling, facility location, and structural analysis of engineering systems among others: see Murota (1991) or Schrijver (2003) or Korte and Vygen (2012), for instance. Recent advance in discrete convex analysis has found applications in exchange economy with indivisible goods (Murota, 2003; Murota and Tamura, 2003; Sun and Yang, 2006), systems analysis (Murota, 2003), inventory management (Huh and Janakiraman, 2010; Zipkin, 2008) and auction (Murota, Shioura, and Yang, 2013). As this long, and yet partial, list suggests, techniques from this literature can be applied to a wide variety of problems. We add matching problems to this list. As suggested by our analysis, results from discrete convex analysis may provide useful tools for studying matching in particular and economics in general.

This paper is not the first to apply discrete convex analysis to matching problems. Fujishige and Tamura (2006, 2007) and Murota and Yokoi (2013) apply discrete convex analysis to study matching problems, and some of our analysis draws upon their results.[5] Our marginal contributions are twofold. First, unlike these existing studies, we analyze incentives, i.e., we apply our technique to show that the generalized DA mechanism is strategyproof for doctors. Such a strategic question is a natural issue to study in economics, but it is rarely studied in the optimization literature. In this sense, we provide new economic questions to the discrete optimization literature. Second, we are the first to establish that various constraints found in practice can be addressed by the technique of discrete convex analysis.

This paper uses the framework of matching with contracts due to Hatfield and Milgrom (2005).[6] They identify a set of conditions for key results in matching with contracts. More specifically, if the choice function of every hospital satisfies substitutability, the law of aggregate demand, and the irrelevance of rejected contracts, then a generalized DA mechanism finds a stable allocation, and the mechanism is strategyproof for doctors.[7] Hatfield and Kojima (2009, 2010) further show that the generalized DA mechanism

[5]See also an earlier contribution by Fleiner (2001) who applies matroid theory to matching. His analysis is a special case of a more recent contribution by Fujishige and Tamura (2007).

[6]Fleiner (2003) obtains some of the results including the existence of a stable allocation in a framework that is more general than the model of Hatfield and Milgrom (2005). On the other hand, he does not show results regarding incentives, which are important for our purposes.

[7]Hatfield and Milgrom (2005) implicitly assume the irrelevance of rejected contracts in their analysis. Aygün and Sönmez (2013) point this out and show that this condition is important for the conclusions of Hatfield and Milgrom (2005).

is group strategyproof for doctors.[8] Our analysis draws upon those studies, but makes at least four marginal contributions over them. First, we show that $M^\natural$-concavity provides a single sufficient condition for these key results to hold, so mechanism designers can check just this condition instead of the existing conditions separately. Second, a large number of sufficient conditions are known for $M^\natural$-concavity, so practitioners can often implement desirable mechanisms simply off the shelf. Third, the literature has developed various ways to construct $M^\natural$-concave functions from other functions, so even if off-the-shelf solutions are not available, it is often easy to combine existing results to find a solution. Lastly, the time complexity of the generalized DA mechanism is polynomial under $M^\natural$-concavity while this property is not guaranteed under general substitutable preferences. This property is very important in practical application.

As stated above, one of the main goals of our paper is to provide a class of payoff functions that are useful and easy to use in practice. Although this research program is still in its infancy, there are notable contributions. Hatfield and Milgrom (2005) set the agenda by introducing a family of payoff functions called endowed assignment valuations. A variant of this class of functions is proposed by Milgrom (2009) and further studied by Budish, Che, Kojima, and Milgrom (2013), while Ostrovsky and Leme (2014) propose a new class of payoff functions. We contribute to this line of research in several ways. First, we identify a more general class of payoff functions, $M^\natural$-concave functions, as the key to our approach.[9] Second, in addition to various sufficient conditions, we identify a *necessary* condition for our class of payoff functions in terms of the matroid structure.

This paper is part of the literature on practical market design, both in terms of content and in terms of approach. As advocated by Roth (2002), recent market design theory has focused on solving practical problems by providing detailed and concrete solutions.[10] Real problems often share common basic features, but differ substantially in details. For instance, different

[8]Other contributions in matching with contracts include Hatfield and Kojima (2008), Hatfield and Kominers (2009, 2012), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Echenique (2012), Sönmez (2013), Sönmez and Switzer (2013), and Kominers and Sönmez (2014).

[9]Ostrovsky and Leme (2014) demonstrate that the class of endowed assignment valuations is a strict subset of the set of all $M^\natural$-concave functions.

[10]Auction market design emphasizes the importance of addressing practical problems as well (see Milgrom (2000, 2004) for instance).

school districts share some common goals such as efficiency, stability (fairness) and incentive compatibility, but can differ in some details such as diversity constraints, structure of school priorities, and authoritative power of different schools (Abdulkadiroğlu, Pathak, and Roth, 2005, 2009; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005, 2006). In this respect, our contributions are twofold. First, our framework provides mechanisms that can be applied to a variety of existing problems as discussed earlier. Second, we develop a theory of matching under constraints that could be applied to new problems that have not been found yet but may be found in the future.

Finally, this paper is part of the literature on matching and market design. The field is too large to even casually summarize here. Instead, we refer interested readers to surveys by Roth and Sotomayor (1990), Roth (2008), Sönmez and Ünver (2011), and Abdulkadiroğlu and Sönmez (2013).

2 Model

A market is a tuple $(D, H, X, (\succ_d)_{d \in D}, f)$. D is a finite set of doctors and H is a finite set of hospitals. X is a finite set of contracts. Each contract $x \in X$ is bilateral, in the sense that x is associated with exactly one doctor $x_D \in D$ and exactly one hospital $x_H \in H$. Each contract can also contain some terms of contracts such as working time and wages. Each $\succ_d$ represents the strict preference of each doctor d over acceptable contracts within $X_d = \{x \in X \mid x_D = d\}$. We assume each contract $x \in X$ is acceptable for x_H: if a hospital considers a contract unacceptable, it is not included in X. For notational simplicity, for $X' \subseteq X$ and $x \in X$, we write $X' + x$ and $X' - x$ to represent $X' \cup \{x\}$ and $X' \setminus \{x\}$, respectively. Also, when $x = \emptyset$, $X' + x$ means nothing is added to X', and $X' - y$ means nothing is removed from X'.

We assume some distributional constraints are enforced on feasible contracts. We assume such distributional constraints and hospital preferences are aggregated into a preference of a representative agent, which we call "the hospitals" (Section 5 illustrates in detail how such aggregations can be done in various applications). The preference of the hospitals is represented by a payoff function $f : 2^X \to \mathbb{R} \cup \{-\infty\}$, where $\mathbb{R}$ is the set of all real numbers. For two sets of contracts $X', X'' \subseteq X$, the hospitals strictly prefer X' over X'' if and only if $f(X') > f(X'')$ holds. If $X' \subseteq X$ violates some distributional constraint, then $f(X') = -\infty$. We assume f is normalized[11] by $f(\emptyset) = 0$. Also, we assume f is unique-selecting, i.e., for all $X' \subseteq X$, $|\arg\max_{X'' \subseteq X'} f(X'')| = 1$ holds.[12]

Now, we introduce several concepts used in this paper.

Definition 1 (feasibility)**.** For a subset of contracts $X' \subseteq X$, we say X' is **hospital-feasible** if $f(X') \neq -\infty$. We say X' is **doctor-feasible** if for all $d \in D$, either (i) $X'_d = \{x\}$ and x is acceptable for d, or (ii) $X'_d = \emptyset$ holds, where $X'_d = \{x \in X' \mid x_D = d\}$. We say X' is **feasible** if it is doctor- and hospital-feasible. We call a feasible set of contracts **matching**.

With a slight abuse of notation, for two sets of contracts X' and X'', we denote $X'_d \succ_d X''_d$ if either (i) $X'_d = \{x'\}$, $X''_d = \{x''\}$, and $x' \succ_d x''$ for some $x', x'' \in X_d$ that are acceptable for d, or (ii) $X'_d = \{x'\}$ for some $x' \in X_d$ that is acceptable for d and $X''_d = \emptyset$. Furthermore, we denote $X'_d \succeq_d X''_d$ if either $X'_d \succ_d X''_d$ or $X'_d = X''_d$. Also, we use notations like $x \succ_d X'_d$ or $X'_d \succ_d x$, where x is a contract and X' is a matching. Furthermore, for $X'_d \subseteq X_d$, we say X'_d is acceptable for d if either (i) $X'_d = \{x\}$ and x is acceptable for d, or (ii) $X'_d = \emptyset$ holds.

For each doctor d, its **choice function** Ch_d specifies her most preferred contract within $X' \subseteq X$, i.e., $Ch_d(X') = \{x\}$, where x is the most preferred acceptable contract in X'_d if one exists, and $Ch_d(X') = \emptyset$ if no such contract exists. Then, the choice function of all doctors Ch_D is defined as $Ch_D(X') := \bigcup_{d \in D} Ch_d(X')$.

For the hospitals, their choice function Ch_H is defined by $Ch_H(X') = \arg\max_{X'' \subseteq X'} f(X'')$ for each $X' \subseteq X$. Since we assume payoff function f is unique-selecting, Ch_H is uniquely determined by f.

Definition 2 (stability (Hatfield and Milgrom, 2005))**.** We say a matching X' is **stable** if $X' = Ch_H(X') = Ch_D(X')$ and there exists no $x \in X \setminus X'$

[11] As described later, this assumption is slightly stronger than mere normalization, because it implies that $\emptyset$ is hospital-feasible.

[12] Observe that strict preference of the hospitals, a standard assumption in matching, imply f is unique-selecting, but the converse does not hold. Also we note that if f is not unique-selecting, we can obtain a unique-selecting function by modifying f very slightly. Let us define a total order relation on X, and for each $x \in X$, let $\mathrm{rank}(x)$ represent the position of x within X according to this relation, i.e., $\mathrm{rank}(x) = i$ if x is ranked i-th. Also, let $v(x)$ denote $\epsilon \cdot 2^{-\mathrm{rank}(x)}$, where ϵ is a sufficiently small positive number. Then, $f(X') + \sum_{x \in X'} v(x)$ is unique-selecting and $\arg\max[f(X') + \sum_{x \in X'} v(x)] \subseteq \arg\max f(X')$.

such that $x \in Ch_H(X' + x)$ and $x \in Ch_D(X' + x)$.[13]

We sometimes refer to the stability concept in Definition 2 as Hatfield-Milgrom (HM)-stability when we discuss the relation with other stability concepts.

Let $\mathcal{X}$ be the set of all stable matchings. We say $X' \in \mathcal{X}$ is the **doctor-optimal stable matching** if $X'_d \succeq_d X''_d$ for all $X'' \in \mathcal{X}$ and $d \in D$.[14]

A mechanism φ is a function that takes a profile of preferences of doctors $\succ_D$ as an input and returns a matching $X' \subseteq X$. Let $\succ_{D\setminus\{d\}}$ denote a profile of preferences of doctors except d, and $(\succ_d, \succ_{D\setminus\{d\}})$ denote a profile of preferences of all doctors, where d's preference is $\succ_d$ and the profile of preferences of other doctors is $\succ_{D\setminus\{d\}}$. We say φ is **strategyproof for doctors** if $\varphi_d((\succ_d, \succ_{D\setminus\{d\}})) \succeq_d \varphi_d((\succ'_d, \succ_{D\setminus\{d\}}))$ holds for all d, $\succ_d$, $\succ'_d$, and $\succ_{D\setminus\{d\}}$.

3 M$^\natural$-concavity and the generalized DA mechanism

This section introduces the concept of M$^\natural$-concavity, which imposes a restriction on the way that the hospitals evaluate sets of contracts. Then we show that if the preference of the hospitals is represented as an M$^\natural$-concave function, then a number of key conclusions in matching theory hold.

Definition 3 (M$^\natural$-concavity)**.** We say that f is **M$^\natural$-concave** if for all $Y, Z \subseteq X$ and $y \in Y \setminus Z$, there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $f(Y) + f(Z) \leq f(Y - y + z) + f(Z - z + y)$ holds.

M$^\natural$-concavity is a discrete analogue of concavity of continuous-variable functions. To help develop intuition on M$^\natural$-concavity, consider a continuous-variable function $g : \mathbb{R} \to \mathbb{R}$. We say g is concave if for all $y, z \in \mathbb{R}$ and λ such that $0 \leq \lambda \leq 1$, the following condition holds:

$$g(y) + g(z) \leq g(y + d) + g(z - d),$$

[13] Hatfield and Milgrom (2005) as well as many others define stability in such a way that a block by a coalition that includes multiple doctors is allowed. Such a concept is identical to our definition if the hospitals have substitutable preferences.

[14] As in the case of the term stability, when we explicitly consider a set of matchings that satisfy a particular stability concept, we abuse terminology slightly by, for instance, writing "the doctor-optimal HM-stable matching."

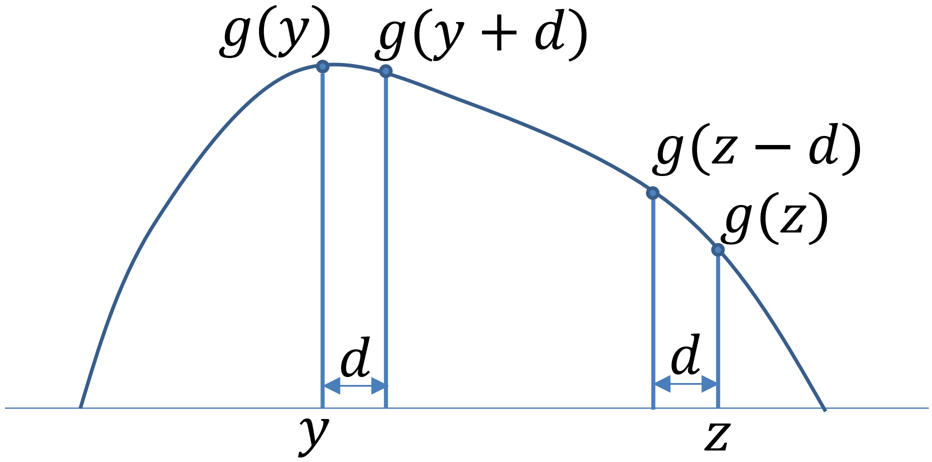

Figure 1: Concavity of a continuous-variable function

where $d = \lambda(z - y)$.[15] Assume $y < z$. Then, $y + d$ is a point reached from y by moving d to the right, and $z - d$ is a point reached from z by moving d to the left (Figure 1). In a discrete domain, we can interpret $Y - y + z$ is a point reached from Y by moving one-step closer to Z, and $Z - z + y$ is a point reached from Z by moving one-step closer to Y. Thus, M$^\natural$-concavity is a counterpart of concavity, adapted to make sense in the discrete domain.

In our context, M$^\natural$-concavity is a requirement that contracts are substitutes in a particular manner.[16] To be more precise, we can immediately derive the following proposition.

Proposition 1. *Assume f is M$^\natural$-concave. For any $Y \subset X$ and $z \in X \setminus Y$, (i) $Ch_H(Y + z) = Ch_H(Y)$, or (ii) $Ch_H(Y + z) = Ch_H(Y) + z$, or (iii) $Ch_H(Y + z) = Ch_H(Y) - y + z$ for some $y \in Ch_H(Y) \setminus Ch_H(Y + z)$.*

Proof. Let $Y^* = Ch_H(Y)$ and $Z^* = Ch_H(Y + z)$. If $z \notin Z^*$, since $Y^* \subseteq Y$ and $Z^* \subseteq Y \subset Z = Y + z$ hold and f is unique-selecting, $Y^* =$

[15] This definition is equivalent to the most common definition that $g(\lambda y + (1 - \lambda)z) \geq \lambda g(y) + (1 - \lambda)g(z)$ for all $y, z \in \mathbb{R}$ and λ with $0 \leq \lambda \leq 1$.

[16] Farooq and Tamura (2004) and Farooq and Shioura (2005) characterize M$^\natural$-concavity by substitutability. More precisely, a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with a bounded effective domain is M$^\natural$-concave if and only if $f + p$ satisfies substitutability for all linear functions $p : \mathbb{Z}^n \to \mathbb{R}$, where the effective domain $\text{dom}\, f$ of f is defined by $\{x \in \mathbb{Z}^n \mid f(x) \neq -\infty\}$. Furthermore, Fujishige and Yang (2003) show that M$^\natural$-concavity is equivalent to the gross substitutes condition due to Kelso and Crawford (1982), as well as the single improvement property due to Gul and Stacchetti (1999).

$\arg\max_{X'' \subseteq Y} f(X'') = Z^*$ holds. Thus, let us assume $z \in Z^*$. Since f is M$^\natural$-concave, either (a) there exists $y \in Y^* \setminus Z^*$ such that $f(Z^*) + f(Y^*) \leq f(Z^* - z + y) + f(Y^* - y + z)$ holds, or (b) $f(Z^*) + f(Y^*) \leq f(Z^* - z) + f(Y^* + z)$ holds. Assume (a) and $Z^* \neq Y^* - y + z$ hold. Since $Z^* = \arg\max_{Z' \subseteq Y+z} f(Z')$, $Y^* - y + z \subseteq Y + z$, and f is unique-selecting, $f(Z^*) > f(Y^* - y + z)$ holds. Also, since $Y^* = \arg\max_{Y' \subseteq Y} f(Y')$ and $Z^* - z + y \subseteq Y$, $f(Y^*) > f(Z^* - z + y)$ holds. Thus, $f(Z^*) + f(Y^*) > f(Z^* - z + y) + f(Y^* - y + z)$ holds. This is a contradiction. Thus, if (a) holds, $Z^* = Y^* - y + z$, i.e., $Ch_H(Y + z) = Ch_H(Y) - y + z$, holds. Assume (b) and $Z^* \neq Y^* + z$ hold. Since $Z^* = \arg\max_{Z' \subseteq Y+z} f(Z')$ and $Y^* + z \subseteq Y + z$, $f(Z^*) > f(Y^* + z)$ holds. Also, since $Y^* = \arg\max_{Y' \subseteq Y} f(Y')$ and $Z^* - z \subseteq Y$, $f(Y^*) > f(Z^* - z)$ holds. Thus, $f(Z^*) + f(Y^*) > f(Z^* - z) + f(Y^* + z)$ holds. This is a contradiction. Thus, if (b) holds, $Z^* = Y^* + z$, i.e., $Ch_H(Y + z) = Ch_H(Y) + z$, holds. □

This proposition provides a specific sense in which contracts are viewed as substitutes when the payoff function is M$^\natural$-concave. When a new contract z becomes available, the new chosen set of contracts $Ch_H(Y + z)$ is (i) unchanged from the original chosen set $Ch_H(Y)$ or (ii) obtained by adding z to $Ch_H(Y)$ or (iii) obtained by replacing exactly one contract y in $Ch_H(Y)$ with z. In particular, no contract that is not chosen from the original set is chosen, that is, contracts are substitutes. Note also that at most one contract becomes newly rejected, and that happens only when the new contract is accepted, so the law of aggregate demand is also satisfied (see Appendix B for formal definitions of these concepts and detailed arguments).

The generalized DA mechanism (Hatfield and Milgrom, 2005) is a generalized version of the well-known deferred acceptance algorithm (Gale and Shapley, 1962), which is adapted for the 'matching with contracts' model.[17]

Mechanism 1 (Generalized Deferred Acceptance (DA) mechanism)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor offers her most preferred contract which has not been rejected before Stage k. If no remaining contract is acceptable for d, d does not make any offer. Let X' be the set of contracts that are offered in this Step.

[17]In Hatfield and Milgrom (2005), this mechanism is called *generalized Gale-Shapley* algorithm.

Step 2 The hospitals tentatively accept $Ch_H(X')$ and reject all other contracts in X'.

Step 3 If all the contracts are tentatively accepted in Step 2, then let X' be the final matching and terminate the mechanism. Otherwise, go to Stage $k+1$.

Now we are ready to present the main result of this section. Drawing upon results in discrete convex analysis, it demonstrates that key results in matching theory hold if the preference of the hospitals can be represented by an M$^\natural$-concave function.

Theorem 1. *Suppose that the preference of the hospitals can be represented by an M$^\natural$-concave function f. Then:*

1. *The generalized DA mechanism is strategyproof for doctors. Also, it always produces a stable matching, and the obtained matching is the doctor-optimal stable matching.*

2. *The time complexity of the generalized DA mechanism is $O(T(f) \cdot |X|^2)$, assuming f can be calculated in $T(f)$ time.*

Item (1) of this result shows that the generalized DA mechanism produces a desirable matching, and incentive compatibility for doctors are guaranteed. These are the key properties emphasized in the theoretical matching literature (see Hatfield and Milgrom (2005) for example) as well as in the literature on practical market design (see, for instance, Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu, Pathak, and Roth (2009)). Item (2) further shows that the desired outcome can be easily computed by the algorithm. This latter property is not guaranteed for the general substitutable preference case. While sometimes de-emphasized in the literature, we emphasize that efficient computability is crucial for the actual implementation of the mechanism in practical market design.

Overall, this theorem suggests that the generalized DA mechanism is a compelling mechanism if preferences and constraints can be aggregated into an M$^\natural$-concave function. The remainder of this paper demonstrates that such an aggregation is indeed possible in various applied environments.

4 Conditions for M$^\natural$-concavity

In this section, we investigate conditions under which payoff function f becomes M$^\natural$-concave. Without loss of generality, we can assume payoff function f is represented by the summation of two parts, i.e., $f(X') = \widehat{f}(X') + \widetilde{f}(X')$, where $\widehat{f}$ represents hard distributional constraints for hospital-feasibility and $\widetilde{f}$ represents soft preferences over hospital-feasible contracts. More specifically, $\widehat{f}(X')$ returns 0 if X' is hospital-feasible and $-\infty$ otherwise, while $\widetilde{f}(X')$ returns a bounded non-negative value.

Let $\mathrm{dom}\, f = \mathrm{dom}\, \widehat{f} = \{X' \mid X' \subseteq X, \widehat{f}(X') \neq -\infty\}$ be the effective domain of f (or equivalently $\widehat{f}$). In the present context, $\mathrm{dom}\, f$ represents the family of hospital-feasible sets of contracts. In this section, we first show a necessary condition on $\widehat{f}$, namely, the effective domain of $\widehat{f}$ (or equivalently f) must constitute a mathematical structure called matroid. Next, we identify three sufficient conditions so that f becomes M$^\natural$-concave, assuming the effective domain of $\widehat{f}$ constitutes a matroid.

Let us first introduce the concept of matroid (Oxley, 2011).

Definition 4 (matroid)**.** Let X be a finite set, and $\mathcal{F}$ be a family of subsets of X. We say a pair $(X, \mathcal{F})$ is a **matroid** if it satisfies the following conditions.

1. $\emptyset \in \mathcal{F}$.

2. If $X' \in \mathcal{F}$ and $X'' \subset X'$, then $X'' \in \mathcal{F}$ holds.

3. If $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$, then there exists $x \in X' \setminus X''$ such that $X'' + x \in \mathcal{F}$.

The term “matroid” is created from “matrix” and “-oid”, i.e., a matroid is something similar to a matrix, and the concept of a matroid is an abstraction of some properties of matrices. To get an idea, suppose that A is a matrix and X is the set of column vectors of A. Let us assume $\mathcal{F}$ is a family of subsets of X, such that for each $X' \in \mathcal{F}$, all column vectors in X' are linearly independent. It is clear that conditions 1 and 2 of the above definition hold. Also, if X' has more elements than X'', we can always choose $x \in X' \setminus X''$ such that $X'' + x$ becomes linearly independent. Therefore condition 3 is also satisfied, which shows that $(X, \mathcal{F})$ in this example is a matroid.

The concept of matroid has been utilized in matching theory. For example, Roth, Sönmez, and Ünver (2005) show that the sets of simultaneously matchable patients induces a matroid. As we will see in this paper, matroids play an essential role in our analysis of matching under constraints. Before we present the results, however, we begin by introducing a simple matroid and methods for creating a new matroid from given matroids.

Definition 5 (uniform matroid)**.** $(X, \mathcal{F})$ is said to be a **uniform matroid** if $\mathcal{F} = \{X' \mid X' \subseteq X, |X'| \leq k\}$ for some non-negative integer k.

Definition 6 (direct sum)**.** For a set of matroids $(X_1, \mathcal{F}_1), \ldots, (X_k, \mathcal{F}_k)$, where each X_i is disjoint, their **directed sum** is defined as $(X, \mathcal{F})$, where $X = \bigcup_{1 \leq i \leq k} X_i$, $\mathcal{F} = \{X' \mid X' = \bigcup_{1 \leq i \leq k} X'_i, \text{ where } X'_i \in \mathcal{F}_i\}$.

Definition 7 (truncation)**.** Assume $(X, \mathcal{F})$ is a matroid and k is a non-negative integer. Then, its k-**truncation** is defined as $(X, \tilde{\mathcal{F}})$, where $\tilde{\mathcal{F}} = \{X' \in \mathcal{F} \mid |X'| \leq k\}$.

It is obvious that conditions 1 and 2 of a matroid hold in the above three cases. For a uniform matroid, if $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$ hold, then for any $x \in X' \setminus X''$, it follows that $|X'' + x| \leq |X'| \leq k$, so $X'' + x \in \mathcal{F}$ holds. Thus, a uniform matroid is a matroid. For a direct sum, if $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$, then for some i, $|X' \cap X_i| > |X'' \cap X_i|$ holds. By this and the assumption that $(X_i, \mathcal{F}_i)$ is a matroid, there exists $x \in (X' \cap X_i) \setminus (X'' \cap X_i)$ such that $(X'' \cap X_i) + x \in \mathcal{F}_i$ holds. Therefore $X'' + x \in \mathcal{F}$ holds, showing that a directed sum of matroids is a matroid. Finally, for k-truncation, if $X', X'' \in \tilde{\mathcal{F}}$ and $|X'| > |X''|$, then there always exists $x \in X' \setminus X''$ such that $X'' + x \in \mathcal{F}$, since $X', X'' \in \mathcal{F}$ holds. Since we also have $|X'' + x| \leq |X'| \leq k$, it follows that $X'' + x \in \tilde{\mathcal{F}}$. Thus, a k-truncation of a matroid is a matroid.

Now we are ready to present one of the connections between matroids and our theory of matching with constraints. The following theorem holds.

Theorem 2. *Under the assumption* $\emptyset \in \operatorname{dom} \widehat{f}$, $\widehat{f}$ *is* M$^\natural$*-concave if and only if* $(X, \operatorname{dom} \widehat{f})$ *is a matroid.*[18]

This theorem suggests that the matroid structure plays an important role in our analysis. In particular, the "only if" part means that, in order to utilize the theory of M$^\natural$-concavity in our analysis of matching with constraints, it is necessary for the sets of hospital-feasible contracts to constitute a matroid.

[18]Murota and Shioura (1999) show that the effective domain $\operatorname{dom} f$ of an M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ forms a generalized polymatroid. The "only if" part of Theorem 2 is a special case of this result.

Now that we have found the necessity of a matroid structure, let us turn to sufficient conditions. More specifically, we assume that $(X, \mathrm{dom}\, f) = (X, \mathrm{dom}\, \widehat{f})$ is a matroid and examine conditions on $\widetilde{f}$, i.e., the soft preference of the hospitals, for guaranteeing that $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave. This task would have been easy if the sum of two M$^\natural$-concave functions were always M$^\natural$-concave, since the hard constraint part $\widehat{f}$ is M$^\natural$-concave if $\mathrm{dom}\, \widehat{f}$ is a matroid (Theorem 2). However, the following example demonstrates that the sum of two M$^\natural$-concave functions is not guaranteed to be M$^\natural$-concave.

Example 1. Assume $X = \{x_1, x_2, x_3\}$. $f_1(X')$ is 0 if either $X' \subseteq \{x_1, x_2\}$ or $X' \subseteq \{x_1, x_3\}$, and otherwise, $-\infty$. $f_2(X')$ is 0 if either $X' \subseteq \{x_2, x_3\}$ or $X' \subseteq \{x_1, x_3\}$, and otherwise, $-\infty$. Since $(X, \mathrm{dom}\, f_1)$ and $(X, \mathrm{dom}\, f_2)$ are matroids, both f_1 and f_2 are M$^\natural$-concave from Theorem 2. However, $f = f_1 + f_2$ is not M$^\natural$-concave, since $(X, \mathrm{dom}\, f)$, where $\mathrm{dom}\, f = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}\}$, is not a matroid. To see this, observe that when $X' = \{x_1, x_3\}, X'' = \{x_2\}$, we have $X', X'' \in \mathrm{dom}\, f$ and $|X'| > |X''|$, but there exists no $x \in X' \setminus X''$ such that $X'' + x \in \mathrm{dom}\, f$ holds.

The above example shows the mere fact that both $\widehat{f}$ and $\widetilde{f}$ are M$^\natural$-concave is not sufficient for guaranteeing M$^\natural$-concavity of $f = \widehat{f} + \widetilde{f}$. Nevertheless, we demonstrate that a number of simple sufficient conditions exist when the hard constraint part induces a matroid. More specifically, we assume $(X, \mathrm{dom}\, \widehat{f})$ constitutes a matroid, and introduce three sufficient conditions to guarantee that f becomes M$^\natural$-concave: (i) $\widetilde{f}$ is a sum of contract values, (ii) $\widehat{f}$ is symmetric for groups G and $\widetilde{f}$ is order-respecting for G, and (iii) $(X, \mathrm{dom}\, \widehat{f})$ is a structure called a laminar matroid on a laminar family and $\widetilde{f}$ is a laminar concave function on it. As will be seen, most stability concepts in existing works can be understood as stability with respect to one of these M$^\natural$-concave functions.

Let us introduce the first condition, which provides a simple but very general method for obtaining an M$^\natural$-concave function.

Definition 8 (sum of contract values)**.** We say $\widetilde{f}$ is a **sum of contract values** if $\widetilde{f}(X') = \sum_{x \in X'} v(x)$, where $v : X \to (0, \infty)$ is a function such that $x \neq x'$ implies $v(x) \neq v(x')$.

As indicated by the name, function $\widetilde{f}$ in this definition is written as a sum of values of individual contracts, where $v(x)$ is interpreted as the value of contract x. Note that we assume each value is positive and different contracts are assigned different values.

Now we are ready to present the following theorem, which provides the first sufficient condition for M$^\natural$-concavity.

Theorem 3. *If $\widehat{f}$ is M$^\natural$-concave (or equivalently, $(X, \mathrm{dom}\,\widehat{f})$ is a matroid) and $\widetilde{f}(X')$ is a sum of contract values, then $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave.*

Proof. Since $\widehat{f}$ is M$^\natural$-concave, there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $\widehat{f}(Y) + \widehat{f}(Z) \leq \widehat{f}(Y - y + z) + \widehat{f}(Z - z + y)$ holds.

On the other hand, we have $\widetilde{f}(Y - y + z) + \widetilde{f}(Z - z + y) = \widetilde{f}(Y) - \widetilde{f}(\{y\}) + \widetilde{f}(\{z\}) + \widetilde{f}(Z) - \widetilde{f}(\{z\}) + \widetilde{f}(\{y\}) = \widetilde{f}(Y) + \widetilde{f}(Z)$. Thus, $f(Y) + f(Z) \leq f(Y - y + z) + f(Z - z + y)$ holds. □

We note that although $\widetilde{f}(X')$ is defined as the sum of values of X', $Ch_H(X')$ is determined only by the relative ordering of these values. Thus, the specific cardinal choice of these values is not important.

As an illustration, suppose that there exists a total ordering $\succ_H$ over X, e.g., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$, and the hospitals prefer to choose as many contracts as possible following this order, subject to a given hospital-feasibility constraint. This preference can be represented by $f = \widehat{f} + \widetilde{f}$, where $\widetilde{f}$ is a sum of contract values with respect to $v(\cdot)$ such that $v(x) > v(x')$ holds if $x \succ_H x'$.

The second sufficient condition for M$^\natural$-concavity is based on an idea of grouping contracts. We begin by formally introducing the concept of a group of contracts.

Definition 9 (group of contracts)**.** Let $G = \{g_1, \ldots, g_n\}$ be a partition of X, i.e., $g \cap g' = \emptyset$ for any $g \neq g'$ and $\bigcup_{g \in G} g = X$. We refer to each element g of G as a **group of contracts** (or simply a group) in G, and G as **groups**.

One division of contracts into groups that we use in this paper is based on hospitals, that is, we let each g_i represent the set of contracts related to hospital h_i.

Kamada and Kojima (2014a) introduce a concept called an order-respecting preference, which can model a wide variety of preferences of hospitals. Let us assume a finite sequence of groups is defined, in which a group can appear repeatedly, e.g., $g_1, g_1, g_2, g_3, g_1, g_1, g_2, g_3, \ldots$. The sequence determines a choice over the numbers of accepted contracts of each group. For example, the above sequence means that g_1 accepts two contracts, then g_2 accepts one, and g_3 accepts one, and so on, as long as there exists a contract related to each group. If a sequence is defined based on a round-robin ordering, e.g., it is given as $g_1, g_2, \ldots, g_n, g_1, g_2, \ldots, g_n, g_1, g_2, \ldots$, then it roughly means that hospitals prefer to make the numbers of accepted contracts for each group as close to one another as possible. Once the number of contracts that are accepted to each hospital h in X' is given by an order-respecting preference, each hospital chooses contracts from X'_h according to $\succ_h$.

Let us examine how to represent such an order-respecting preference by a payoff function. For a sequence of groups, let $v_g(i)$ denote the value associated with the i-th appearance of group g in this sequence. We assume that $v_g(i) > v_g(i+1)$ holds for all i, g. Also, if the i-th appearance of group g is earlier than the j-th appearance of group g' in the sequence, we assume $v_g(i) > v_{g'}(j)$ holds. In this sense, these values represent the sequence. Let us define $V_g(k) := \sum_{i=1}^{k} v_g(i)$. Also, we assume that, for each contract $x \in X$, a unique value $v(x)$ is defined.

An order-respecting payoff function is defined based on these values.

Definition 10 (order-respecting payoff function)**.** For groups G, an **order-respecting payoff function** $\widetilde{f}$ is given as follows:

$$\widetilde{f}(X') = \sum_{g \in G} V_g(|X' \cap g|) + \sum_{x \in X'} v(x),$$

where $v : X \to (0, \infty)$ is a function such that $x \neq x'$ implies $v(x) \neq v(x')$, and $|v_g(i) - v_{g'}(j)| \gg v(x)$ holds for any g, g', i, j such that $(i, g) \neq (j, g')$, and x.

Now, we introduce a further condition on a matroid so that $\widehat{f} + \widetilde{f}$ becomes M$^\natural$-concave, when $\widetilde{f}$ is an order-respecting payoff function.

Definition 11 (symmetry of groups)**.** Let $(X, \mathcal{F})$ be a matroid and G be a partition of X. We say that G is **symmetric** in $(X, \mathcal{F})$ if for all $g \in G$, for all $x, x' \in g$, and for all $X' \subset X$ such that $\{x, x'\} \cap X' = \emptyset$, $X' + x \in \mathcal{F}$ if and only if $X' + x' \in \mathcal{F}$ holds.

With the concepts of an order-respecting payoff function and symmetry of groups at hand, we present the second sufficient condition for M$^\natural$-concavity.

Theorem 4. *If $(X, \mathrm{dom}\,\widehat{f})$ is a matroid, G is symmetric in $(X, \mathrm{dom}\,\widehat{f})$, and $\widetilde{f}$ is an order-respecting payoff function for G, then $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave.*

Finally, we introduce the third sufficient condition for M$^\natural$-concavity. The crucial concept we use is the *laminar family* defined below.

Definition 12 (laminar family)**.** $\mathcal{T}$ is a **laminar family** of subsets of X if for any $Y, Z \in \mathcal{T}$, one of the following conditions holds:

1. $Y \cap Z = \emptyset$,

2. $Y \subset Z$, or

3. $Z \subset Y$.

We say $f(X') = \sum_{T \in \mathcal{T}} f_T(|X' \cap T|)$ is a **laminar concave function** if $\mathcal{T}$ is a laminar family and each f_T is a univariate concave function.

In words, a family of sets $\mathcal{T}$ is said to be a laminar family if it has a structure that can be described as layers or a hierarchy. More specifically, for any pair of sets in this family, either they are disjoint from each other or one of them is a subset of the other. Laminar families have been used for mechanism design in two-sided matching (Biro, Fleiner, Irving, and Manlove, 2010; Kamada and Kojima, 2014a), indivisible object allocation (Budish, Che, Kojima, and Milgrom, 2013), and auction (Milgrom, 2009).

A laminar family of subsets of contracts naturally induces a matroid, a structure we call a laminar matroid, as defined below.

Definition 13 (laminar matroid)**.** We say $(X, \mathcal{F})$ is a **laminar matroid** on a laminar family $\mathcal{T}$ if it is constructed as follows:

- For each $T \in \mathcal{T}$, a positive integer q_T is given.

- $\mathcal{F}$ is defined as $\{X' \subseteq X \mid |X' \cap T| \leq q_T \ (\forall T \in \mathcal{T})\}$.

A laminar matroid is a matroid, since it is obtained from uniform matroids by repeatedly taking a truncation and a directed sum (recall Definitions 5–7). With this concept at hand, we are ready to state the last sufficient condition for M$^\natural$-concavity.

Theorem 5. *If $(X, \operatorname{dom} \widehat{f})$ is a laminar matroid on a laminar family $\mathcal{T}$, and $\widetilde{f}$ is a laminar concave function on $\mathcal{T}$, then $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave.*

5 Applications

In this section, we examine existing works on matching and show that the sufficient conditions described in Section 4 hold in these cases. Through this connection, we reproduce key results in the existing literature and, for some applications, show stronger results. This enables us to provide mechanisms that are strategyproof for doctors and produce stable matchings.

As mentioned in Section 1, one of the main motivations of our work is to provide an easy-to-use recipe, or a toolkit, for organizing matching mechanisms under constraints. Consider a mechanism designer who is faced with a matching problem with constraints, and imagine that she has some initial ideas on what hard distributional constraints exist and what kind of stability properties are desired. Our recipe can be summarized as follows.

1. Check whether $(X, \mathcal{F})$, where $\mathcal{F}$ is the family of hospital-feasible sets of contracts, is a matroid. If not, modify distributional constraints so that $(X, \mathcal{F})$ becomes a matroid.

2. Compose $\widetilde{f}$, which reflects stability, such that it satisfies one of the sufficient conditions described in this paper. Modify the stability definition as necessary, by adding more desirable properties, relaxing excessively demanding requirements, or simply introducing tie-breaking.

If these two steps are successful, the job of the mechanism designer is done, because she can use an off-the-shelf mechanism, i.e., the generalized DA mechanism. More specifically, our analysis from the preceding sections guarantees that the generalized DA mechanism satisfies desirable properties. The cases we discuss in this section illustrate how the above recipe works.

5.1 Standard model (Gale and Shapley, 1962)

5.1.1 Model

A market is a tuple $(D, H, X, (\succ_d)_{d \in D}, (\succ_h)_{h \in H}, (q_h)_{h \in H})$. D is a finite set of doctors and H is a finite set of hospitals. X is a finite set of contracts. A contract $x \in X$ is a pair (d, h), which represents a matching between doctor d and hospital h. $(\succ_d)_{d \in D}$ is a profile of doctors' preferences, i.e., each $\succ_d$ represents the strict preference of each doctor d over acceptable contracts in $X_d = \{(d, h) \in X \mid h \in H\}$. $(\succ_h)_{h \in H}$ is a profile of hospitals' preferences, i.e., each $\succ_h$ represents the preference of each hospital h over the contracts that are related to it. $(q_h)_{h\in H}$ is a profile of hospitals' maximum quotas, i.e., each q_h represents the maximum quota of hospital h.

5.1.2 Feasibility

We say $X' \subseteq X$ is hospital-feasible if $|X'_h| \leq q_h$ for all h, where $X'_h = \{(d,h) \in X' \mid d \in D\}$. We say $X' \subseteq X$ is doctor-feasible if X'_d is acceptable for all d (we say X'_d is acceptable for d if either (i) $X'_d = \{x\}$ and x is acceptable for d, or (ii) $X'_d = \emptyset$ holds). We say X' is feasible if it is doctor- and hospital-feasible.

5.1.3 Stability

First, let us define the concept of a blocking pair. This definition is used throughout Section 5.

Definition 14. For a matching X', we say $(d,h) \in X \setminus X'$ is a **blocking pair** if (i) (d,h) is acceptable for d and $(d,h) \succ_d X'_d$, and (ii) either $|X'_h| < q_h$ or there exists $(d',h) \in X'$ such that $(d,h) \succ_h (d',h)$.[19]

We say a matching X' is **Gale-Shapley (GS)-stable** if there exists no blocking pair (Gale and Shapley, 1962).

5.1.4 Mechanism

The standard Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962) is defined as follows.

Mechanism 2 (standard DA)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$: Each doctor d applies to her most preferred hospital by which she has not been rejected before Stage k (if no remaining hospital is acceptable for d, d does not apply to any hospital). Each hospital h tentatively accepts doctors applying to h up to q_h according to $\succ_h$. The rest of doctors are rejected. If no doctor is rejected by any hospital,

[19] Note that we denote $X'_d \succ_d X''_d$ if either (i) $X'_d = \{x'\}$, $X''_d = \{x''\}$, and $x' \succ_d x''$ for some $x', x'' \in X_d$ that are acceptable for d, or (ii) $X'_d = \{x'\}$ for some $x' \in X_d$ that is acceptable for d and $X''_d = \emptyset$.

terminate the mechanism and return the current tentatively accepted pairs as the final matching. Otherwise, go to Stage $k+1$.

The standard DA mechanism is strategyproof (Dubins and Freedman, 1981; Roth, 1982) and obtains a GS-stable matching (Gale and Shapley, 1962).

5.1.5 Representation in our model

Let us define $\widehat{f}(X')$ as 0 if X' is hospital-feasible, i.e., $|X'_h| \leq q_h$ for all h, and otherwise as $-\infty$. Then, $(X, \mathrm{dom}\, \widehat{f})$ is a laminar matroid, since $\{X_h \mid h \in H\}$ is a laminar family and we require $|X' \cap X_h| \leq q_h$ for each h.

Let us assume a positive value $v(x)$ is defined for each $x = (d,h)$ with the property that $v((d,h)) > v((d',h))$ when $(d,h) \succ_h (d',h)$ holds.[20] With $\widetilde{f}(X') = \sum_{x \in X'} v(x)$, $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave by Theorem 3. The standard DA mechanism is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of function f defined above.

The following proposition holds.

Proposition 2. *HM-stability is equivalent to GS-stability.*

Proof. To show that HM-stability implies GS-stability, assume for contradiction that a feasible matching X' is not GS-stable. Then there exists a blocking pair (d,h). Because (d,h) is acceptable for d and $(d,h) \succ_d X'_d$, it immediately follows that $(d,h) \in Ch_D(X' + (d,h))$ by definition of Ch_D. Because either $|X'_h| < q_h$ or there exists $(d',h) \in X'$ such that $(d,h) \succ_h (d',h)$, by the definition of f, in either case it follows that $(d,h) \in \arg\max_{X'' \subseteq X' + (d,h)} f(X'') = Ch_H(X' + (d,h))$. Therefore X' is not HM-stable.

To show that GS-stability implies HM-stability, assume for contradiction that a feasible matching X' is not HM-stable. $X' = Ch_D(X')$ because X' is a matching and hence doctor-feasible, and $X' = Ch_H(X')$ because X' is a matching and hence hospital-feasible, that is, $|X'_h| \leq q_h$ for all h in the current model, and $v((d,h)) > 0$ for all $(d,h) \in X'$ by the definition of

[20]We note that although we assume a value $v(x)$ is given for each contract and f is defined by the sum of these values, $Ch_H(X')$ is determined only by the relative ordering of the values among the contracts that belong to the same hospital. Thus, the specific cardinal choice of these values, or the relative ordering among contracts for different hospitals, is not important.

$v(\cdot)$. These facts and the assumption that X' is not HM-stable imply there exists a doctor-hospital pair (d, h) such that $(d, h) \in Ch_H(X' + (d, h))$ and $(d, h) \in Ch_D(X' + (d, h))$ hold. Then $(d, h) \succ_d X'_d$ by the definition of Ch_D and, by the definition of f, either $|X'_h| < q_h$ or there exists $(d', h) \in X'$ such that $(d, h) \succ_h (d', h)$. Therefore (d, h) is a blocking pair, showing that X' is not GS-stable. Thus HM-stability is equivalent to GS-stability. □

From Proposition 2, we can guarantee that the generalized DA mechanism obtains the doctor-optimal GS-stable matching, so the generalized DA mechanism and the standard DA mechanism obtain the same outcome. Note that this fact can be derived without checking whether the two mechanisms behave exactly in the same way.

5.2 Regional maximum quotas (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014; Kamada and Kojima, 2015)

5.2.1 Model

A market is a tuple $(D, H, X, R, (\succ_d)_{d \in D}, (\succ_h)_{h \in H}, (q_h)_{h \in H}, (q_r)_{r \in R})$. The definitions of D, H, X, $\succ_d, \succ_h$, and q_h are identical to the standard model. The difference between this model and the standard model is that we assume hospitals are grouped into regions $R = \{r_1, \ldots, r_n\}$, where each region r is a subset of hospitals. $(q_r)_{r \in R}$ is a profile of regional maximum quotas, i.e., each q_r represents the regional maximum quota of r. We assume each hospital h is included in exactly one region, that is, regions partition H.[21]

5.2.2 Feasibility

For each $r \in R$, let X'_r denote $\bigcup_{h \in r} X'_h$. We say X' is hospital-feasible if $|X'_h| \leq q_h$ for all $h \in H$, and $|X'_r| \leq q_r$ for all $r \in R$. We say $X' \subseteq X$ is doctor-feasible if X'_d is acceptable for all d. Then, we say X' is feasible if it is doctor- and hospital-feasible.

[21] Kamada and Kojima (2014a) consider a more general case where regions are hierarchical. We can generalize our results to such a case by utilizing the fact that contracts related to each region form a laminar family.

5.2.3 Stability

We say a matching X' is strongly stable (Kamada and Kojima, 2014b) if the following condition holds: if (d, h), where $h \in r$, is a blocking pair (Definition 14) then (i) $|X'_r| = q_r$, (ii) $(d', h) \succ_h (d, h)$ for all $(d', h) \in X'_h$, and (iii) if $(d, h') \in X'$, then $h' \notin r$. In words, a matching is strongly stable if satisfying the desire of a blocking pair by matching them results in a violation of a regional maximum quota.

A strongly stable matching does not necessarily exist (Kamada and Kojima, 2014b). Thus, we need to consider a weaker definition of stability in order to guarantee the existence. Kamada and Kojima (2015) introduce a weaker stability concept, which we call Kamada-Kojima (KK)-stability. We say a matching X' is **KK-stable** if the following condition holds: if (d, h), where $h \in r$, is a blocking pair then (i) $|X'_r| = q_r$, (ii) $(d', h) \succ_h (d, h)$ for all $(d', h) \in X'_h$, and (iii) if $(d, h') \in X'$, then either $h' \notin r$ or $|X'_{h'}| - |X'_h| \leq 1$. The second part of condition (iii) accounts for the difference between KK-stability and strong stability; (d, h) is not regarded as a legitimate blocking pair if h and h' are in the same region and moving d from h' to h does not strictly decrease the imbalance of doctors between these hospitals.

Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) assume there exists a total preference ordering $\succ_H$ over X, i.e., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$. Here, we assume $\succ_H$ respects each $\succ_h$, i.e., if $(d, h) \succ_h (d', h)$, then $(d, h) \succ_H (d', h)$ holds. Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) introduce a weaker stability concept than strong stability based on this ordering, which we call contract-order-stability.[22] We say a matching X' is **contract-order-stable** (Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo, 2014) if the following condition holds: if (d, h), where $h \in r$, is a blocking pair then (i) $|X'_r| = q_r$ and (ii) $(d', h') \succ_H (d, h)$ for all $h' \in r$ and $(d', h') \in X'_{h'}$. We note that the condition (ii) includes the cases where $d' = d$ or $h' = h$. When $d = d'$, the condition (ii) means that (d, h) is not regarded as a legitimate blocking pair when the hospitals prefer (d, h') to (d, h).

5.2.4 Mechanism

Fix a round-robin ordering among hospitals; without loss of generality, denote it as $h_1, h_2, \ldots h_{|H|}$. Kamada and Kojima (2015) present a mechanism

[22] A contract-order stable matching is identical to a regionally fair and regionally non-wasteful matching defined in Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014).

called the Flexible Deferred Acceptance (FDA) mechanism, which utilizes this ordering. Roughly speaking, the FDA mechanism allows each hospital to sequentially accept one contract at a time according to the given round-robin ordering, subject to regional maximum quotas. Formally, the FDA mechanism is defined as follows.[23]

Mechanism 3 (FDA)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. If no remaining hospital is acceptable for d, d does not apply to any hospital. Reset X' as $\emptyset$.

Step 2 For each r, iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose the hospital with the smallest index in the region first, the hospital with the second-smallest index second, and so forth and, after the last hospital, go back to the first hospital.
2. Choose doctor d who is applying to h and is not tentatively accepted or rejected yet, and is the most preferred according to $\succ_h$ among the current applicants. If there exists no such doctor, then go to the procedure for the next hospital.
3. If $|X'_h| < q_h$ and $|X'_r| < q_r$, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next hospital.
4. Otherwise, d is rejected by h. Then go to the procedure for the next hospital.

Step 3 If all the doctors are tentatively accepted in Step 2, then let X' be the final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

[23] To be more precise, Kamada and Kojima (2015) allow *a target capacity* for each hospital such that each hospital gets priority in accepting doctors up to its target capacity. For simplicity, here we consider a case where these target capacities are identical for all hospitals that belong to the same region, but allowing for more general target capacities is straightforward.

Kamada and Kojima (2015) show that the FDA mechanism is strategyproof for doctors and obtains a KK-stable matching.

Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) introduce a different mechanism, called Priority-List based Deferred Acceptance (PLDA) mechanism, which utilizes the total preference ordering $\succ_H$. Formally, the PLDA mechanism is defined as follows.

Mechanism 4 (PLDA)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. If no remaining hospital is acceptable for d, d does not apply to any hospital. Reset X' as $\emptyset$.

Step 2 For each r, iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose (d, h), where d is applying to h, d is not tentatively accepted or rejected yet, and (d, h) has the highest priority according to $\succ_H$ among the current applicants to hospitals in r.
2. If $|X'_h| < q_h$ and $|X'_r| < q_r$, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next pair.
3. Otherwise, d is rejected by h. Then go to the procedure for the next pair.

Step 3 If all the doctors who make applications in this stage are tentatively accepted in Step 2, then let X' be the final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) show that the PLDA mechanism is strategyproof for doctors and obtains a contract-order-stable matching.

5.2.5 Representation in our model

Let us define $\widehat{f}(X')$ as 0 if X' is hospital-feasible, i.e., $|X'_h| \leq q_h$ for all h and $|X'_r| \leq q_r$ for all r, and otherwise, $-\infty$. Then, $(X, \mathrm{dom}\,\widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{X_{r_1}, X_{r_2}, \ldots, X_{r_n}, X_{h_1}, X_{h_2}, \ldots, X_{h_{|H|}}\}$ is a laminar family of X.

First, we study KK-stability. As in Kamada and Kojima (2015), fix a round-robin ordering over hospitals, $h_1, h_2, \ldots, h_{|H|}$. Let $v_{h_i}(j)$ denote the value associated with the j-th choice of hospital h_i. Then, define $v_{h_i}(j)$ as $C(C-|H|\cdot j-i)$ where C is a large positive constant. Let $V_h(k) := \sum_{j=1}^{k} v_h(j)$. Then, define $\widetilde{f}(X')$ as follows:

$$\widetilde{f}(X') = \sum_{h\in H} V_h(|X'_h|) + \sum_{x\in X'} v(x), \tag{1}$$

where $C \gg v(x)$ for all $x \in X$. By choosing G as $\{X_{h_1}, X_{h_2}, \ldots, X_{h_{|H|}}\}$, it is clear that G is symmetric in $(X, \mathrm{dom}\,\widehat{f})$ and $\widetilde{f}$ defined by equation (1) is an order-respecting payoff function for G. Thus, from Theorem 4, $f(X') = \widehat{f}(X') + \widetilde{f}(X')$ is M$^\natural$-concave.

The FDA mechanism is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of function f defined above.

The following proposition holds.

Proposition 3. *HM-stability (based on $\widetilde{f}$ in equation* (1)*) implies KK-stability.*

Proof. To show that HM-stability implies KK-stability, assume X' is not KK-stable, i.e., there exists a blocking pair (d, h), where $h \in r$ and (i) $|X'_r| < q_r$, or (ii) there exists $(d', h) \in X'_h$ such that $(d, h) \succ_h (d', h)$ holds, or (iii) there exists $(d, h') \in X'$, where $h' \in r$ and $|X'_{h'}| - |X'_h| > 1$ holds. In any of these cases, clearly $(d, h) \in Ch_D(X' + (d, h))$. In case (i), obviously $(d, h) \in \arg\max_{X''\subseteq X'+(d,h)} f(X'')$ because adding (d, h) to X' does not violate the regional maximum quota for f. In case (ii), $(d, h) \in \arg\max_{X''\subseteq X'+(d,h)} f(X'')$ because adding (d, h) and subtracting $(d', h) \in X'_h$ such that $(d, h) \succ_h (d', h)$ from X', the resulting matching does not violate the regional maximum quota for r. In case (iii), by the construction of f, $(d, h) \in \arg\max_{X''\subseteq X'+(d,h)} f(X'')$. Thus, for each of the cases (i)–(iii), it follows that $(d, h) \in Ch_H(X' + (d, h))$, which implies X' is not HM-stable. Thus, HM-stability implies KK-stability. □

We note that KK-stability does not imply HM-stability. To see this, let us consider the following case. There are two hospitals h_1 and h_2, both of them belong to region r, and $q_r = 1$. There are two doctors d_1 and d_2. We assume $h_1 \succ_{d_1} h_2$, $h_2 \succ_{d_2} h_1$, $d_1 \succ_{h_1} d_2$, and $d_2 \succ_{h_2} d_1$ hold. The round-robin ordering over hospitals is defined as h_1, h_2. $X' = \{(d_2, h_2)\}$ is clearly

KK-stable, but it is not HM-stable since $(d_1, h_1) \in Ch_D(X' + (d_1, h_1))$ and $(d_1, h_1) \in Ch_H(X' + (d_1, h_1))$ hold.

The FDA (and the generalized DA) mechanism does not guarantee to obtain the doctor-optimal KK-stable matching. In fact, there is a case where the doctor-optimal KK-stable matching does not even exist. Note that the main focus of Kamada and Kojima (2015) is two-sided matching such as labor market matching, so optimality for one side of the market is not the main requirement. Note also that, despite the fact that the FDA (and hence the generalized DA) mechanism does not lead to doctor-optimality, the mechanism is still strategyproof for doctors.

Alternatively, we can define $\widetilde{f}(X')$ as $C_1 \cdot |X'| - \sum_{h \in H} C_2 \cdot |X'_h|^2 + \sum_{h \in H} C_h \cdot |X'_h| + \sum_{x \in X'} v(x)$, where $C_1 \gg C_2 \gg C_h \gg v(x)$ for all h and x, and $C_{h_1} \gg C_{h_2} \gg \ldots$. This is a laminar concave function on $X \cup \{X_{r_1}, X_{r_2}, \ldots, X_{r_n}, X_{h_1}, X_{h_2}, \ldots, X_{h_{|H|}}\}$. Thus, from Theorem 5, $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave. It is clear that Proposition 3 still holds when $\widetilde{f}$ is defined in this way.

Next, we study contract-order-stability. As in Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014), let us assume there exists a total preference ordering $\succ_H$ over X, i.e., $x_1 \succ_H x_2 \succ_H x_3 \succ_H \ldots$. Then assume a positive value $v(x)$ for each x is defined with the property that $v(x) > v(x')$ when $x \succ_H x'$. Let us assume $\widetilde{f}(X')$ is given as follows:

$$\widetilde{f}(X') = \sum_{x \in X'} v(x). \tag{2}$$

Then, f is M$^\natural$-concave by Theorem 3. The PLDA mechanism is identical to the generalized DA mechanism where Ch_H is defined by the maximizer of this function f.

The following proposition holds.

Proposition 4. *HM-stability (based on $\widetilde{f}$ in equation (2)) is equivalent to contract-order-stability.*

Proof. Assume X' is not contract-order-stable, i.e., there exists a blocking pair (d, h), where $h \in r$ and either (i) $|X'_r| < q_r$ or (ii) there exists $(d', h') \in X'$ such that $h' \in r$ and $(d, h) \succ_H (d', h')$ hold. In either case, $(d, h) \in Ch_D(X' + (d, h))$ and $(d, h) \in Ch_H(X' + (d, h))$ hold.

If X' is contract-order-stable, then the first condition for HM-stability, namely $X' = Ch_H(X') = Ch_D(X')$, is obvious. Assume there exists $(d, h) \in X \setminus X'$ such that $(d, h) \in Ch_D(X' + (d, h))$ and $(d, h) \in Ch_H(X' + (d, h))$

hold. Then, it is clear that (d,h) is a blocking pair, and either (i) $|X'_r| < q_r$ or (ii) there exists $(d',h') \in X'$ such that $h' \in r$ and $(d,h) \succ_H (d',h')$ hold. Thus, X' is not contract-order-stable. □

As in the standard model, from Proposition 4, we can guarantee that the generalized DA mechanism obtains the doctor-optimal contract-order-stable matching, so the generalized DA mechanism and the PLDA mechanism obtain the same outcome. Note that this fact can be derived without checking whether these two mechanisms behave exactly in the same way.

5.3 Regional minimum quotas (Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo, 2014)

5.3.1 Model

A market is a tuple $(D, H, X, R, (\succ_d)_{d \in D}, (\succ_h)_{h \in H}, (q_h)_{h \in H}, (p_h)_{h \in H}, (p_r)_{r \in R})$. As in the standard model, D is a finite set of doctors and H is a finite set of hospitals. $(q_h)_{h \in H}$ is a profile of maximum quotas of hospitals. $X := D \times H$ is the set of contracts. Here, we assume every doctor is acceptable to every hospital and vice versa; without this assumption, satisfying all minimum quotas can be impossible.[24]

We assume hospitals are grouped into regions $R = \{r_1, \ldots, r_n\}$, where each region r is a subset of hospitals. Here, we allow these regions to overlap. To be more precise, we assume R is a laminar family of H, i.e., these regions have a hierarchical structure. Without loss of generality, we assume $H \in R$ holds. We assume each region, as well as each individual hospital, has its minimum quota. More specifically, for each $h \in H$, p_h represents the minimum quota of hospital h, and for each $r \in R$, p_r represents the regional minimum quota of region r.

Since R is a laminar family of subsets of H, R has a tree structure, where H is the root node, and each $h \in H$ is a leaf node (as shown in

[24]This assumption is motivated by some real-life applications. For example, in many universities in Japan, an undergraduate student who majors in engineering must be assigned to a laboratory to conduct a project, and the project is required for graduation. In this setting, every student can be assumed to be acceptable to every laboratory and vice versa. In particular, since 2011, the third author has been applying a mechanism based on Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2015) to assign students to laboratories in Department of Electrical Engineering and Computer Science, School of Engineering, Kyushu University, where every student is acceptable to every laboratory and vice versa.

Figure 2 (a)). In a tree, we say region r_p is the **parent** of another region r if $r_p = \arg\min_{r' \supsetneq r} |r'|$. Similarly, we say region r_p is the parent of hospital h if $r_p = \arg\min_{r' \ni h} |r'|$. We say region r (or hospital h) is a **child** of region r_p if r_p is a parent of r (or h). For each node r, let children(r) denote the set of all children of r. Without loss of generality, we assume for each r, $p_r \geq \sum_{r' \in \text{children}(r)} p_{r'}$ holds (if this inequality does not hold, then one can redefine $p_r := \sum_{r' \in \text{children}(r)} p_{r'}$ and the constraints are unchanged). We also assume $p_H = |D|$, i.e., the minimum quota of the root is equal to the number of doctors.

The model presented in Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2015) is a special case of this model in which minimal quotas are imposed only on individual hospitals.

5.3.2 Feasibility

We say $X' \subseteq X$ is hospital-feasible if $p_h \leq |X'_h| \leq q_h$ for all h, and $p_r \leq |X'_r|$ for all r. We say X' is doctor-feasible if $|X'_d| = 1$ for all d. Then, we say X' is feasible if it is doctor- and hospital-feasible. Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) show that if $p_r \leq \sum_{h \in r} q_h$ and $p_h \leq q_h$ hold for all r and h, then a feasible matching always exists. In the rest of this section, we assume the above conditions hold.

X' is hospital-feasible only if $|X'| = |D|$ since $p_H = |D|$. Thus, feasibility of X' requires that all doctors be matched to some hospital. Let us introduce a weaker condition than hospital-feasibility. We say X' is **semi-hospital-feasible** if it is a subset of (or equal to) a hospital-feasible matching.

5.3.3 Stability

Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) introduce several concepts that are related to stability. First, in a matching X', a doctor d where $(d,h) \in X'$ has a **justified envy** towards another doctor d' where $(d',h') \in X'$, if $(d,h') \succ_d (d,h)$, and $(d,h') \succ_{h'} (d',h')$ hold.

Second, in a matching X', a doctor d where $(d,h) \in X'$ **claims an empty seat** of h' if the following conditions hold: (i) $(d,h') \succ_d (d,h)$, and (ii) $X'' = X' - (d,h) + (d,h')$ is feasible.

Third, in a matching X', a doctor d where $(d,h) \in X'$ **strongly claims an empty seat** of h' if the following conditions hold: (i) $(d,h') \succ_d (d,h)$, (ii) $X'' = X' - (d,h) + (d,h')$ is feasible, and (iii) $|X'_h| - |X'_{h'}| \geq 2$. The intuitive meaning of condition (iii) is similar to KK-stability; the claim of doctor d for moving her from h to h' is justified if such a movement strictly decreases the imbalance of doctors between these hospitals, but not otherwise.

We say a matching is **fair** if no doctor has justified envy. We say a matching is nonwasteful if no doctor claims an empty seat, and **weakly nonwasteful** if no doctor strongly claims an empty seat. In general, fairness and nonwastefulness are incompatible, i.e., there exists a case where no matching is fair and nonwasteful (Ehlers, Hafalir, Yenmez, and Yildirim, 2014). On the other hand, Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) show that a fair and weakly nonwasteful matching always exists.

5.3.4 Mechanism

Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) present a mechanism based on the FDA mechanism called Round-robin Selection Deferred Acceptance mechanism for Regional Minimum Quotas (RSDA-RQ), which is defined as follows.

Mechanism 5 (RSDA-RQ)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. Reset X' as $\emptyset$.

Step 2 For each r, iterate the following procedure until all doctors applying to hospitals in r are either tentatively accepted or rejected:

1. Choose hospital h based on the round-robin ordering.
2. Choose doctor d who is applying to h and is not tentatively accepted or rejected yet, and has the highest priority according to $\succ_h$ among the current applicants to h. If there exists no such doctor, then go to the procedure for the next hospital.
3. If $X' + (d, h)$ is semi-hospital-feasible, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next hospital.

4. Otherwise, d is rejected by h. Then go to the procedure for the next hospital.

Step 3 If all the doctors are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k+1$.

This procedure is almost identical to the FDA mechanism. The only difference is that in the RSDA-RQ mechanism, when making a decision whether to tentatively accept (d, h) or not, the RSDA-RQ mechanism checks whether $X'+(d, h)$ is semi-hospital-feasible or not. Goto, Hashimoto, Iwasaki, Kawasaki, Ueda, Yasuda, and Yokoo (2014) introduce a computationally efficient method to check semi-hospital-feasibility. The matching obtained by the RSDA-RQ mechanism is fair and weakly nonwasteful.

Fragiadakis, Iwasaki, Troyan, Ueda, and Yokoo (2015) present a mechanism based on the deferred acceptance mechanism called Extended Seat Deferred Acceptance (ESDA) mechanism. The ESDA mechanism is a special case of the RSDA-RQ mechanism for an environment in which minimal quotas are imposed only on individual hospitals. Thus, the ESDA mechanism is fair and weakly nonwasteful in that setting.

5.3.5 Representation in our model

In the face of nontrivial minimum quotas, the family of hospital-feasible sets of contracts cannot be a matroid since $\emptyset$ is not hospital-feasible. Here, we consider the family of semi-hospital-feasible sets of contracts.

We create a network flow problem (Cormen, Leiserson, Rivest, and Stein, 2009) that represents these regional constraints as follows. For notational simplicity, let q_r denote $\sum_{h \in r} q_h$.

- We set the set of start vertexes as X.
- There exists a unique terminal vertex t.
- For each h, we create an intermediate vertex v-h. There exists a directed edge from each (d, h) to this vertex, whose capacity is 1. Also, from v-h, there exists a directed edge to t, whose capacity is p_h.

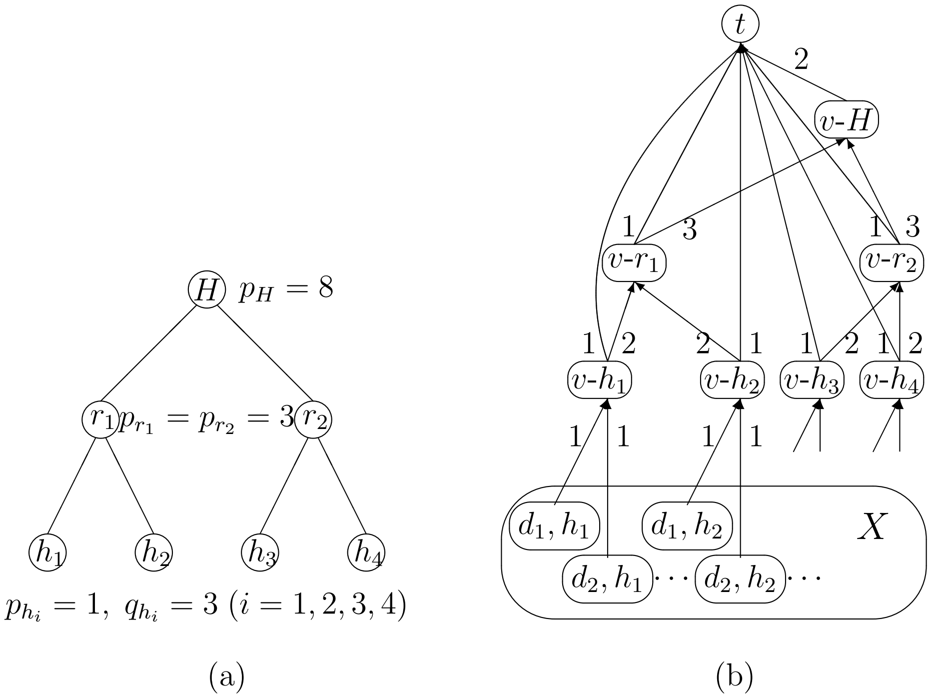

Figure 2: Example of regional minimum quotas (a) and an associated network flow problem (b)

- For each r, we create one intermediate vertex v-r. There exists a directed edge from each v-r', where $r' \in \text{children}(r)$, to v-r, whose capacity is $q_{r'} - p_{r'}$. From v-r, there exists a directed edge to t, whose capacity is $p_r - \sum_{r' \in \text{children}(r)} p_{r'}$.

We assume $\widehat{f}(X') = 0$ if there exists a **valid** network flow from X', i.e., a flow from X' to the terminal vertex t that satisfies capacity constraints of edges, and otherwise, $-\infty$.

To illustrate our construction, consider the following example. Assume there are four hospitals $h_1, \ldots, h_4$. Their maximum and minimum quotas are 3 and 1, respectively. They are divided into two regions r_1, r_2. Their minimum quotas are 3. Thus, we require at least one doctor is assigned to both h_1 and h_2, and one additional doctor is assigned to either h_1 or h_2. There are 8 doctors. Thus, p_H is set at 8 (Figure 2 (a)).

Now we are ready to illustrate our construction of the associated network flow problem. For h_1, we create one intermediate vertex v-h_1. There exists a directed edge from each contract related with h_1 to v-h_1. Also, from v-h_1, there exists a directed edge to the terminal node t, whose capacity is $p_{h_1} = 1$. Similarly, for h_2, we create one intermediate vertex v-h_2. There exists a directed edge from each contract related to h_2 to v-h_2. Also, from v-h_2, there exists a directed edge to the terminal node t, whose capacity is $p_{h_2} = 1$. The construction related to h_3 and h_4 is symmetric.

Furthermore, for r_1, we create one intermediate vertex v-r_1. There exist edges from nodes representing hospitals in children(r_1), i.e., v-h_1 and v-h_2, whose capacities are $q_{h_1} - p_{h_1} = q_{h_2} - p_{h_2} = 2$. Also, from v-r_1, there exists a directed edge to the terminal node t, whose capacity is $p_{r_1} - \sum_{r' \in \text{children}(r_1)} p_{r'} = 3 - (1 + 1) = 1$. The construction related to r_2 is symmetric. Also, for H, we create one intermediate node v-H. There exists a directed edge from v-r_1 (which is in children(H)) to v-H, whose capacity is $q_{r_1} - p_{r_1} = 3 + 3 - 3 = 3$. Similarly, there exists a directed edge from v-r_2 to v-H, whose capacity is 3. There exists a directed edge from v-H to t, whose capacity is $|D| - \sum_{r \in \text{children}(H)} p_r = 8 - (3 + 3) = 2$.

Figure 2 (b) shows the network flow problem of this example. Here, the number associated to a directed edge represents its capacity.

The following proposition holds.

Proposition 5. *X' is feasible if and only if $\widehat{f}(X') = 0$ and $|X'| = |D|$. X' is semi-hospital-feasible if and only if $\widehat{f}(X') = 0$.*

Define $\widetilde{f}(X')$ in the same way as in equation (1) in Section 5.2.5. Then, f is M$^\natural$-concave from Theorem 4 and Property 3 in Appendix A. With the help of these results, we are ready to show the following claim.

Proposition 6. *HM-stability implies fairness and weak nonwastefulness.*

Proof. Suppose X' is HM-stable. If doctor d prefers (d, h) to X'_d, then no (d', h) with $(d, h) \succ_h (d', h)$ is in X' by the definition of HM-stability and the definition of the payoff function f. Thus, HM-stability implies fairness. Also, if doctor d prefers (d, h') to (d, h) and $(d, h) \in X'$ while $X' - (d, h) + (d, h')$ is feasible, then $|X'_h| - |X'_{h'}| \leq 1$ must hold, i.e., moving d from h to h' does not strictly decrease the imbalance of doctors between h and h', by the definition of HM-stability and the construction of f. Thus, HM-stability implies weak nonwastefulness. □

We note that fairness and weak nonwastefulness do not imply HM-stability. To see this, let us consider the following case. There are three hospitals h_1, h_2

and h_3 and two doctors d_1 and d_2. The minimum quota of h_1 is 1 and the minimum quotas of the other hospitals are 0. No regional minimum quota is imposed. We assume $h_1 \succ_{d_1} h_2 \succ_{d_1} h_3$ and $h_2 \succ_{d_2} h_3 \succ_{d_2} h_1$ hold. The round-robin ordering over hospitals is defined as h_1, h_2, h_3. All hospitals prefer d_1 over d_2. We assume the individual maximum quota of each hospital is large enough. $X' = \{(d_1, h_1), (d_2, h_3)\}$ is fair and weakly nonwasteful. In particular, (d_2, h_2) satisfies conditions (i) and (ii) for strongly claiming an empty seat, but condition (iii) does not hold since $|X'_{h_3}| = 1$ and $|X'_{h_2}| = 0$. However, X' is not HM-stable since $(d_2, h_2) \in Ch_D(X' + (d_2, h_2))$ and $(d_2, h_2) \in Ch_H(X' + (d_2, h_2))$ hold.

The RSDA-RQ mechanism is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of f described above. The ESDA is identical to the generalized DA mechanism where Ch_H is defined as the maximizer of f described above, when minimal quotas are imposed only on individual hospitals.

5.4 Controlled school choice (Ehlers, Hafalir, Yenmez, and Yildirim, 2014)

5.4.1 Model

This section studies a model of matching with diversity constraints. Although the original contribution by Ehlers, Hafalir, Yenmez, and Yildirim (2014) frames the model in the context of student placement in schools, we stick to our terminology of doctors and hospitals.

A market is a tuple $(D, H, X, (\succ_d)_{d\in D}, (\succ_h)_{h\in H}, (q_h)_{h\in H}, T, \tau, (\underline{q}_h^T)_{h\in H}, (\overline{q}_h^T)_{h\in H})$. The definitions of $D, H, X, \succ_d, \succ_h$, and q_h are identical to the standard model.

One major difference between this model and the standard ones is that we assume each doctor d has her type $\tau(d) \in T = \{t_1, \ldots, t_k\}$. A type of a doctor may represent race, income, gender, or any socioeconomic status. Furthermore, each hospital has soft minimum and maximum bounds for each type t, i.e., $(\underline{q}_h^T)_{h\in H}$ and $(\overline{q}_h^T)_{h\in H}$, where $\underline{q}_h^T = (\underline{q}_h^t)_{t\in T}$ and $\overline{q}_h^T = (\overline{q}_h^t)_{t\in T}$. Each $\underline{q}_h^t$ and $\overline{q}_h^t$ represent minimum and maximum bounds for type t at hospital h. We assume $\sum_{t\in T} \underline{q}_h^t \le q_h$ holds, i.e., the minimum bounds for all types in h can be satisfied without violating the maximum quota of the hospital. For $X' \subseteq X$, let $X'_{h,t}$ denote $\{(d, h) \in X' \mid d \in D, \tau(d) = t\}$.

5.4.2 Feasibility

The bounds $\underline{q}_h^t$ and $\overline{q}_h^t$ are *soft* bounds and do not affect feasibility. We say $X' \subseteq X$ is hospital-feasible if $|X'_h| \leq q_h$ for all h. We say $X' \subseteq X$ is doctor-feasible if X'_d is acceptable for all d. Then, we say X' is feasible if it is doctor- and hospital-feasible.

5.4.3 Stability

Ehlers, Hafalir, Yenmez, and Yildirim (2014) introduce a stability concept, which we call Ehlers-Hafalir-Yenmez-Yildirim (EHYY)-stability. For a matching X', we say $(d, h) \in X \setminus X'$ is an EHYY-blocking pair, where $\tau(d) = t$, if (d, h) is acceptable for d and $(d, h) \succ_d X'_d$, and any one of the following conditions holds:

(i) $|X'_h| < q_h$,

(ii) $|X'_{h,t}| < \underline{q}_h^t$, or

(iii) there exists another doctor d', such that $(d', h) \in X'$ and $\tau(d') = t'$ hold, and any one of the following conditions holds:

   (a) $t = t'$ and $(d, h) \succ_h (d', h)$,

   (b) $t \neq t'$, $\underline{q}_h^t \leq |X'_{h,t}| < \overline{q}_h^t$, $\underline{q}_h^{t'} < |X'_{h,t'}| \leq \overline{q}_h^{t'}$, and $(d, h) \succ_h (d', h)$,

   (c) $t \neq t'$, $\underline{q}_h^t \leq |X'_{h,t}| < \overline{q}_h^t$, and $|X'_{h,t'}| > \overline{q}_h^{t'}$, or

   (d) $t \neq t'$, $|X'_{h,t}| \geq \overline{q}_h^t$, $|X'_{h,t'}| > \overline{q}_h^{t'}$, and $(d, h) \succ_h (d', h)$.

We say a matching X' is **EHYY-stable** if there exists no EHYY-blocking pair.

Intuitively, this stability concept means that for each type t, up to $\underline{q}_h^t$ doctors of type t can be assigned to hospital h without competing against doctors of other types. Then, if more type t doctors hope to be assigned to h, they can be assigned up to $\overline{q}_h^t$ but these doctors must compete against doctors of other types. Furthermore, if more type t doctors beyond $\overline{q}_h^t$ hope to be assigned to h, they can be assigned only when q_h is not filled yet by accepting doctors of type $t' \neq t$ up to $\overline{q}_h^{t'}$.

5.4.4 Mechanism

Ehlers, Hafalir, Yenmez, and Yildirim (2014) present a mechanism called the Deferred Acceptance Algorithm with Soft Bounds (DAASB), whose outcome satisfies EHYY-stability. The DAASB mechanism is defined as follows.

Mechanism 6 (DAASB)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each doctor applies to her most preferred hospital by which she has not been rejected before Stage k. If no remaining hospital is acceptable for d, d does not apply to any hospital. Reset X' as $\emptyset$.

Step 2 For each hospital h, iterate the following procedure until all doctors applying to h are either tentatively accepted or rejected:

Phase 1:

1. Choose doctor d who is applying to h and is not tentatively accepted, rejected, or postponed to the next phase yet, and has the highest priority according to $\succ_h$ among current applicants to h. If there exists no such doctor, then go to the procedure for the next phase.
2. If $|X'_{h,\tau(d)}| < \underline{q}_h^{\tau(d)}$, then d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next doctor.
3. Otherwise, the decision on d is postponed to the next phase. Go to the procedure for the next doctor.

Phase 2:

1. Choose doctor d who is applying to h and is not tentatively accepted, rejected, or postponed to the next phase yet, and has the highest priority according to $\succ_h$ among current applicants to h. If there exists no such doctor, then go to the procedure for the next phase.
2. If $|X'_h| = q_h$ then reject all doctors applying to h who have not been tentatively accepted yet. Go to the procedure for the next hospital. If $|X'_{h,\tau(d)}| < \overline{q}_h^{\tau(d)}$, then d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next doctor.

3. Otherwise, the decision on d is postponed to the next phase. Go to the procedure for the next doctor.

Phase 3: 1. Choose doctor d who is applying to h and is not tentatively accepted, or rejected yet, and has the highest priority according to $\succ_h$ among current applicants to h. If there exists no such doctor, then go to the procedure for the next hospital.

2. If $|X'_h| = q_h$ then reject all doctors applying to h who are not tentatively accepted yet. Go to the procedure for the next hospital.

3. Otherwise, d is tentatively accepted by h and (d, h) is added to X'. Then go to the procedure for the next doctor.

Step 3 If all the doctors are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

5.4.5 Representation in our model

Let us consider an extended market $(D, H, \tilde{X}, (\tilde{\succ}_d)_{d\in D}, f)$. Here, a contract $x \in \tilde{X}$ is represented as (d, h, s), where $d \in D$, $h \in H$, and $s \in \{0, 1, 2\}$. $s = 0, 1, 2$ are interpreted to mean that doctor d is accepted at hospital h for type $\tau(d)$'s priority seat, normal seat, and extended seat, respectively. As described later, we introduce such a seat distinction to satisfy EHYY-stability. From the matching in the extended market $\tilde{X}'$, the matching in the original market X' is obtained by mapping each contract (d, h, s) to (d, h). Let $\tilde{X}'_{h,t,s}$ denote $\{(d, h, s) \in \tilde{X}' \mid d \in D, \tau(d) = t\}$.

We define the modified preference of each doctor d, denoted $\tilde{\succ}_d$ such that $(d, h, s)\tilde{\succ}_d(d, h', s')$ holds for any $h \neq h'$, s, and s' if $(d, h) \succ_d (d, h')$, and $(d, h, 0)\tilde{\succ}_d(d, h, 1)\tilde{\succ}_d(d, h, 2)$ holds for any h, i.e., for each h, doctor d prefers h's priority seat over its normal seat, and its normal seat over its extended seat.

For the extended market, let us assume for each x, its value $v(x)$ is defined. We assume $v((d, h, 0)) > v((d', h, 1))$ and $v((d, h, 1)) > v((d', h, 2))$ hold for any d, d', and h, i.e., hospitals first try to fill their priority seats, then normal seats, and finally extended seats. Also, we assume $v((d, h, s)) > v((d', h, s))$ if $(d, h) \succ_h (d', h)$, i.e., the preference of an individual hospital over doctors is respected, as long as doctors have the same type.

Let us define $\widehat{f}(\tilde{X}')$ as 0 when $|\tilde{X}'_h| \leq q_h$, $|\tilde{X}'_{h,t,0}| \leq \underline{q}^t_h$, and $|\tilde{X}'_{h,t,1}| \leq \overline{q}^t_h - \underline{q}^t_h$ hold for all $h \in H$ and $t \in T$, and otherwise, $-\infty$. Intuitively, these definitions mean that the number of priority seats of hospital h for type t doctors is $\underline{q}^t_h$, and the number of normal seats is $\overline{q}^t_h - \underline{q}^t_h$.

Also, let us define $\widetilde{f}(\tilde{X}')$ as $\sum_{x \in \tilde{X}'} v(x)$. $(\tilde{X}, \mathrm{dom}\,\widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{\tilde{X}_{h,t,s} \mid h \in H, t \in T, s \in \{0,1,2\}\} \cup \{\tilde{X}_h \mid h \in H\}$ is a laminar family of $\tilde{X}$. Thus, $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave from Theorem 3. The matching obtained by the DAASB mechanism is identical to the matching in the original market mapped from the outcome of the generalized DA mechanism where Ch_H is defined as the maximizer of function f.

The following proposition holds.

Proposition 7. *HM-stability of $\tilde{X}'$ in the extended market implies EHYY-stability of X' in the original market.*

Proof. Assume (d, h) is an EHYY-blocking pair in the original market, where $\tau(d) = t$. If condition (i) of an EHYY-blocking pair holds, by choosing $x = (d, h, 2)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. Also, if condition (ii) holds, $|\tilde{X}'_{h,t,0}| < \underline{q}^t_h$ holds. Thus, by choosing $x = (d, h, 0)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. Then, in either case, $\tilde{X}'$ is not HM-stable.

Assume condition (iii) holds, so there exists another doctor d' such that $\tau(d') = t'$ and d' is assigned to h. If condition (a) holds, $(d', h, s) \in \tilde{X}'$ and $v((d, h, s)) > v((d', h, s))$ must hold. Thus, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. If condition (b) holds, $(d', h, 1) \in \tilde{X}'$, $|\tilde{X}'_{h,t,1}| < \overline{q}^t_h - \underline{q}^t_h$, and $v((d, h, 1)) > v((d', h, 1))$ must hold. Thus, by choosing $x = (d, h, 1)$, $x \in Ch_D(\tilde{X}'+x)$ and $x \in Ch_H(\tilde{X}'+x)$ hold. If condition (c) holds, $(d', h, 2) \in \tilde{X}'$, $|\tilde{X}'_{h,t,1}| < \overline{q}^t_h - \underline{q}^t_h$, and $v((d, h, 1)) > v((d', h, 2))$ must hold. Thus, by choosing $x = (d, h, 1)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. If condition (d) holds, $(d', h, 2) \in \tilde{X}'$ and $v((d, h, 2)) > v((d', h, 2))$ must hold. Thus, by choosing $x = (d, h, 2)$, $x \in Ch_D(\tilde{X}' + x)$ and $x \in Ch_H(\tilde{X}' + x)$ hold. Thus, in any of these cases, $\tilde{X}'$ is not HM-stable. □

For a matching in the extended market, the corresponding matching in the original market is uniquely determined, but multiple matchings in the extended market can be mapped onto an identical matching in the original market. Thus, whether EHYY-stability in the original market implies HM-stability in the extended market or not depends on how to determine the mapping from the original market to the extended market. However, since the generalized DA mechanism is identical to the DAASB mechanism, it obtains the doctor-optimal EHYY-stable matching.

5.5 Student-project allocation (Abraham, Irving, and Manlove, 2007)

5.5.1 Model

In the Student-Project Allocation (SPA) problem, a market is represented as a tuple $(S, P, L, X, (\succ_s)_{s \in S}, (\succ_l)_{l \in L}, (q_l)_{l \in L}, (q_p)_{p \in P})$. S is a finite set of students, P is a finite set of projects, and L is a finite set of lecturers. Each project $p \in P$ is offered by some lecturer $l \in L$. Let P_l denote the set of projects offered by lecturer l. Each contract $x \in X$ is a pair (s, p), which represents the assignment of student s to project p. For $X' \subseteq X$, let X'_s denote $\{(s, p) \in X' \mid p \in P\}$ and X'_l denote $\{(s, p) \in X' \mid s \in S, p \in P_l\}$. For each $s \in S$, $\succ_s$ represents the preference of student s over acceptable contracts in X_s. For each $l \in L$, $\succ_l$ represents the preference of lecturer l over S, and q_l represents the maximum quota of lecturer l. For each $p \in P$, q_p represents the maximum quota of project p.

5.5.2 Feasibility

We say $X' \subseteq X$ is student-feasible if X'_s is acceptable for each $s \in S$. We say $X' \subseteq X$ is lecturer-feasible if $|X'_l| \leq q_l$ holds for all $l \in L$, and $|X'_p| \leq q_p$ holds for all $p \in P$. Then, we say X' is feasible if it is student- and lecturer-feasible.

5.5.3 Stability

For a matching X', a contract $(s, p) \in X \setminus X'$, where $p \in P_l$, is an **Abraham-Irving-Manlove (AIM) blocking pair** of X' if (i) (s, p) is acceptable for s, (ii) $(s, p) \succ_s X'_s$, and (iii) one of the following conditions holds:

(a) $|X'_p| < q_p$ and $|X'_l| < q_l$.

(b) $|X'_p| < q_p$ and $|X'_l| = q_l$, and either $X'_s = \{(s, p')\}$ and $p' \in P_l$, or there exists $(s', p'') \in X'$, such that $p'' \in P_l$ and $s \succ_l s'$.

(c) $|X'_p| = q_p$ and there exists $(s', p) \in X'$, such that $s \succ_l s'$.

We say a matching X' is **Abraham-Irving-Manlove (AIM)-stable** if it has no AIM-blocking pair.

5.5.4 Mechanism

Abraham, Irving, and Manlove (2007) present two mechanisms based on the DA mechanism. One is called SPA-student, in which students make offers, and the other is called SPA-lecturer, in which lecturers make offers. Both mechanisms produce AIM-stable matchings. Although Abraham, Irving, and Manlove (2007) do not examine strategyproofness, the SPA-student is strategyproof for students. The SPA-student is defined as follows.

Mechanism 7 (SPA-student)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each student s applies to her most preferred project by which she has not been rejected before Stage k. If no remaining project is acceptable for s, s does not apply to any project. Reset X' as $\emptyset$.

Step 2 For each lecturer l, iterate the following procedure until all students applying to projects in P_l are either tentatively accepted or rejected:

1. Choose s, where s is applying to some project $p \in P_l$, s has not been tentatively accepted or rejected yet in any previous step of this stage, and s has the highest priority according to $\succ_l$ among the students currently applying to some project offered by lecturer l.
2. If $|X'_p| < q_p$ and $|X'_l| < q_l$, s is tentatively accepted by p and (s, p) is added to X'. Then go to the procedure for the next student.
3. Otherwise, s is rejected by p. Then go to the procedure for the next student.

Step 3 If all the students are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

5.5.5 Representation in our model

Let us define $\widehat{f}(X')$ as 0 if X' is lecturer-feasible, i.e., $|X'_p| \leq q_p$ for all $p \in P$ and $|X'_l| \leq q_l$ for all $l \in L$, and otherwise, $-\infty$. Then, $(X, \mathrm{dom}\,\widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{X_{l_1}, X_{l_2}, \ldots, X_{p_1}, X_{p_2}, \ldots\}$ is a laminar family of X.

Let us assume there exists an ordering $p_1, p_2, \ldots$ among projects within P_l. Then, let us define a positive value $v(x)$ for each $x \in X$ with the following properties: for every $p, p' \in P_l$, $v((s,p)) > v((s',p'))$ if $s \succ_l s'$, and $v((s,p)) > v((s,p'))$ if p appears earlier than p' in the above ordering over P_l.

Let us assume $\widetilde{f}(X')$ is given as $\sum_{x \in X'} v(x)$. Then, f is M$^\natural$-concave by Theorem 3. SPA-student is identical to the generalized DA mechanism where the choice function of the lecturers Ch_L is defined as the maximizer of function f.

The following proposition establishes a connection between HM-stability and AIM-stability.

Proposition 8. *AIM-stability implies HM-stability.*

Proof. If X' is AIM-stable, then the first condition for HM-stability, namely $X' = Ch_L(X') = Ch_S(X')$, is obvious. We show that if there exists $(s,p) \in X \setminus X'$ such that $(s,p) \in Ch_L(X' + (s,p))$ and $(s,p) \in Ch_S(X' + (s,p))$ hold, then (s,p) is an AIM-blocking pair.

By way of contradiction, let us assume (s,p) is not an AIM-blocking pair. From the fact that $(s,p) \in Ch_S(X' + (s,p))$, (s,p) is acceptable for s. Also, either $(s,p) \succ_s (s,p')$ holds where $(s,p') \in X'$, or $X'_s = \emptyset$.

Assume $p \in P_l$. Since (s,p) is not an AIM-blocking pair, either $|X'_p| = q_p$ or $|X'_l| = q_l$ holds. Since $(s,p) \in Ch_L(X' + (s,p))$, there exists $(s',p') \in X'$ such that $(s',p') \notin Ch_L(X' + (s,p))$ and $p' \in P_l$ hold (otherwise, $|X'_p| < q_p$ and $|X'_l| < q_l$ hold). Since $(s',p') \notin Ch_L(X' + (s,p))$ and $(s,p) \in Ch_L(X' + (s,p))$, $s \succ_l s'$ or $s = s'$ hold. In either case, (s,p) becomes an AIM-blocking pair. This is a contradiction. □

We note that there exist cases in which a matching X' is HM-stable but not AIM-stable. To see this, assume there exist one student s and two projects p_1 and p_2, and both projects are provided by the same lecturer l. The order on P_l is p_2, p_1. s prefers p_1 to p_2. In this example, $X' = \{(s, p_2)\}$ is stable, since $Ch_L(X' + (s, p_1)) = X'$, but it is not AIM-stable since (s, p_1) is an AIM-blocking pair under the definition of AIM-stability. However, the following proposition holds.

Proposition 9. *The generalized DA mechanism obtains the doctor-optimal AIM-stable matching.*

Proof. Since the generalized DA mechanism obtains the student-optimal HM-stable matching and AIM-stability implies HM-stability, it suffices to show that the student-optimal HM-stable matching satisfies AIM-stability. To show the latter, let us assume by way of contradiction that X' is the student-optimal HM-stable matching but it is not AIM-stable, i.e., there is an AIM-blocking pair $(s,p) \in X \setminus X'$. Let $p \in P_l$. Since X' is HM-stable, cases (a) and (b) in the definition of an AIM-blocking pair in Section 5.5.3 are not possible. Thus the only possibility is that $|X'_p| < q_p$, $|X'_l| = q_l$, $X'_s = \{(s,p')\}$, and $p' \in P_l$ hold, i.e., s is assigned to project p', although s prefers another project p, while p' and p are held by the same lecturer l and p is not full. Let $X'' = X' - (s,p') + (s,p)$. It is clear that X'' is HM-stable, s prefers X'' over X', and other students are indifferent between X'' and X'. This contradicts the assumption that X' is the student-optimal HM-stable matching. □

In the above example, the generalized DA mechanism returns $\{(s,p_1)\}$, which is AIM-stable.

5.6 Cadet-branch matching (Sönmez and Switzer, 2013)

5.6.1 Model

A market is a tuple $(I, B, T, X, (\succ_i)_{i\in I}, \succ_B, (q_b)_{b\in B}, (p_b)_{b\in B})$. I is a finite set of cadets and B is a finite set of branches of the military. $T = \{t_0, t_+\}$ is a pair of terms, where t_0 means that a cadet serves for a standard term, and t_+ means that a cadet serves for an extended term, which is longer than the standard term. $X := I \times B \times T$ is the set of contracts. A contract $x = (i,b,t)$ means i is matched with b with term t. For $X' \subseteq X$, let X'_i denote $\{(i,b,t) \in X' \mid t \in T, b \in B\}$ and X'_b denote $\{(i,b,t) \in X' \mid i \in I, t \in T\}$.

For each $i \in I$, $\succ_i$ represents the preference of cadet i over acceptable contracts in X_i. $\succ_B$ is the priority ordering (master-list) over the cadets, which is common to all branches. For each $b \in B$, q_b represents the maximum quota of branch b, and $p_b < q_b$ represents the reserved quota for extended-term contracts at branch b.

5.6.2 Feasibility

We say $X' \subseteq X$ is cadet-feasible if X'_i is acceptable for all i. We say $X' \subseteq X$ is branch-feasible if $|X'_b| \le q_b$ holds for all $b \in B$. Then, we say X' is feasible if it is cadet- and branch-feasible.

5.6.3 Stability

We say a matching X' is **fair** if for each pair of contracts $(i, b, t), (i', b', t') \in X'$, $(i, b', t') \succ_i (i, b, t)$ implies $i' \succ_B i$ (Sönmez and Switzer, 2013). In other words, fairness requires that a higher-priority cadet never envies the assignment of a lower-priority cadet.

5.6.4 Mechanism

Sönmez and Switzer (2013) present a mechanism called the Cadet-Optimal Stable Mechanism (COSM), which is defined as follows.[25] The COSM always produces a fair matching.

Mechanism 8 (COSM)**.**
Apply the following stages from $k = 1$.

Stage $k \geq 1$

Step 1 Each cadet i chooses her most preferred contract (i, b, t) which has not been rejected before Stage k and applies to b with term t. If no remaining contract is acceptable for i, i does not apply to any branch. Reset X' as $\emptyset$.

Step 2 For each b, iterate the following procedure until all cadets applying to b are either tentatively accepted or rejected:

Phase 1: Choose cadet i such that i has the highest priority according to $\succ_B$ among the applicants to b, who are applying to b with term t (t can be either t_0 or t_+) and are not tentatively accepted yet. If there exists no such cadet, then go to the procedure for the next branch. If $|X'_b| < q_b - p_b$, tentatively accept i to b and add (i, b, t) to X', and go to the procedure for the next cadet. Otherwise, go to the procedure for the next phase.

[25] To be more precise, the COSM presented in Sönmez and Switzer (2013) is slightly different from the one presented here. The difference would matter when branch b handles two contracts offered by the same cadet i, i.e., (i, b, t_0) and (i, b, t_+). However, this difference does not matter since b handles cadet-feasible contracts only. Allowing multiple contracts between the same pair of agents, as is done in the present paper, enables the preference to satisfy substitutability. This technique has been used in Kamada and Kojima (2014a, 2015) and Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) in the context of matching with constraints. See also Hatfield and Kominers (2009, 2014) who study this issue further.

Phase 2: Choose cadet i such that i has the highest priority according to $\succ_B$ among the applicants to b, who are applying to b with the extended term t_+ and are not tentatively accepted yet. If there exists no such cadet, then go to the next phase. If $|X'_b| < q_b$, tentatively accept i to b, add (i, b, t_+) to X', and go to the procedure for the next cadet. Otherwise, reject all cadets applying to b who are not tentatively accepted yet. Go to the procedure for the next branch.

Phase 3: Choose cadet i such that i has the highest priority according to $\succ_B$ among the applicants to b, who are applying to b with the standard term t_0 and are not tentatively accepted yet. If there exists no such cadet, then go to the procedure for the next branch. If $|X'_b| < q_b$, tentatively accept i to b, add (i, b, t_0) to X', and go to the procedure for the next cadet. Otherwise, reject all cadets applying to b who are not tentatively accepted yet. Go to the procedure for the next branch.

Step 3 If all the cadets are tentatively accepted in Step 2, then let X' be a final matching and terminate the mechanism. Otherwise, go to Stage $k + 1$.

5.6.5 Representation in our model

Let us consider an extended market $(I, B, T, \tilde{X}, \tilde{\succ}_I, f)$. For each contract (i, b, t_+) in X, we create two contracts $(i, b, t_+, 0)$ and $(i, b, t_+, 1)$ in the extended market. Here, $(i, b, t_+, 0)$ and $(i, b, t_+, 1)$ are interpreted to mean i is accepted to b for its priority seat and normal seat, respectively. For each contract (i, b, t_0), we create a single contract $(i, b, t_0, 1)$ in the extended market.

We obtain the modified preference of each cadet i, denoted $\tilde{\succ}_i$, such that $(i, b, t, s)\tilde{\succ}_i(i, b', t', s')$ holds for any b, b', s, s', t, and t', if $(i, b, t) \succ_i (i, b', t')$, and $(i, b, t_+, 1)\tilde{\succ}_i(i, b, t_+, 0)$ holds for any b (as long as (i, b, t_+) is acceptable for i).

From the matching in the extended market $\tilde{X}'$, the matching in the original market X' is obtained by mapping each contract (i, b, t, s) to (i, b, t). For $\tilde{X}' \subseteq \tilde{X}$, let $\tilde{X}'_{b,s}$ denote $\{(i, b, t, s) \in \tilde{X}' \mid i \in I, t \in T\}$.

For each $x \in \tilde{X}$, we define its value $v(x)$. We assume $v(\cdot)$ respects $\succ_B$ in the sense that if $i \succ_B i'$, $v((i, b, t, s)) > v((i', b, t, s))$ holds for all b, t, and

s. Also assume $v((i, b, t_+, 0)) > v((i', b', t, 1))$ for all i, i', b, b', and t, i.e., a contract for a priority seat has a larger value than any contract for a normal seat.

Let us define $\widehat{f}(\tilde{X}')$ as 0 when $|\tilde{X}'_b| \leq q_b$ and $|\tilde{X}'_{b,0}| \leq p_b$ hold for all $b \in B$, and $-\infty$ otherwise. Also, let us define $\widetilde{f}(\tilde{X}')$ as $\sum_{x \in \tilde{X}'} v(x)$. Then, $(\tilde{X}, \mathrm{dom}\, \widehat{f})$ is a laminar matroid, since $\mathcal{T} = \{\tilde{X}_{b,s} \mid b \in B, s \in \{0, 1\}\} \cup \{\tilde{X}_b \mid b \in B\}$ is a laminar family of $\tilde{X}$. Thus, $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave by Theorem 3. COSM is identical to the generalized DA mechanism where the choice function of branches Ch_B is defined as the maximizer of function f.

The following proposition holds.

Proposition 10. *HM-stability of $\tilde{X}'$ in the extended market implies fairness of X' in the original market.*

Proof. Assume X' is not fair, i.e., there exist $(i, b, t), (i', b', t') \in X'$ such that $(i, b', t') \succ_i (i, b, t)$ and $i \succ_B i'$ hold. Consider the case where $t' = t_+$ and $(i', b', t_+, s) \in \tilde{X}'$. Then, if we choose $x = (i, b', t_+, s)$, it is clear that $x \in Ch_I(\tilde{X}' + x)$ and $x \in Ch_B(\tilde{X}' + x)$ because $v(i, b', t_+, s) > v(i', b', t_+, s)$. Consider the case where $t' = t_0$ and $(i', b', t_0, 1) \in \tilde{X}'$. Then, if we choose $x = (i, b', t_0, 1)$, it is clear that $x \in Ch_I(\tilde{X}' + x)$ and $x \in Ch_B(\tilde{X}' + x)$ because $v(i, b', t_0, 1) > v(i', b', t_0, 1)$. □

On the other hand, HM-stability is not implied by fairness. To see this, let a cadet i hope to be assigned branch b with term t_+ but she is not accepted by b in X', while $|X'_{b,0}| < p_b$. Then X' is not HM-stable, even if each cadet assigned to b has a higher priority than i (thus X' is fair).

Fairness as defined in Section 5.6.3 is a mild requirement, and the cadet-optimal fair matching does not always exist. Thus, neither the generalized DA mechanism nor the COSM always produce the cadet-optimal fair matching.

6 Discussions

6.1 Relations between applications

The SPA problem in Section 5.5 can be represented using the regional maximum quota model in Section 5.2, by letting projects provided by the same lecturer P_l form a region and its regional maximum quota be set at q_l. In the SPA problem, individual projects in P_l do not have its own preference over students; one can interpret that all projects in P_l use a common preference $\succ_l$. As a result, AIM-stability implies strong stability defined in Section 5.2. Thus, AIM-stability is stronger than KK-stability or the contract-order-stability.[26]

The model presented in Section 4 of Biro, Fleiner, Irving, and Manlove (2010) can be regarded as a generalization of the SPA problem, where the constraints have a laminar structure and maximum quotas are imposed at each element of the laminar family. Biro, Fleiner, Irving, and Manlove (2010) show that a stable matching always exists and a modification of the standard DA mechanism obtains a stable matching. Clearly, our analysis in Section 5.5 can be generalized to this environment.

6.2 Aggregation of individual hospital preferences

Recall the model of matching with regional maximum quotas in Section 5.2. As illustrated there, there may be several alternative methods for aggregating the preferences of individual hospitals into a single preference of the hospitals. One method is to introduce an order over hospitals to determine a preference over the numbers of accepted contracts at each hospital (as used in KK-stability). Then, which contracts should be accepted at each hospital is determined by the individual preference of the hospital. Another method is to generate an ordering among contracts that respects the preferences of individual hospitals (as in contract-order stability). As seen in Section 5.2, both types of aggregated hospital preferences can be represented by $\mathrm{M}^\natural$-concave functions.

Of course, what preference aggregation employ depends on what stability concept one adopts as the solution concept. This amounts to deciding a criterion for socially desirable outcomes. Recommending one criterion over another is not the goal of the present paper, because the decision would involve value judgment by the members of the society, and it is likely to depend on specific applied contexts. Our contribution is to provide a tool for achieving a desirable outcome *given societal preferences*, and we aimed at accommodating as wide a range of constraints and societal preferences as possible.

[26] AIM-stability does not coincide with strong stability or KK-stability or contract-order stability in general. Thus our analysis of the SPA problem is not subsumed by our analysis of regional maximum quotas.

In this regard, one advantage of our methodology is that it is general and flexible enough to subsume a wide class of aggregated preferences including both those corresponding to KK-stability and contract-order stability. Recall the function $\widetilde{f}$ representing the hospitals' soft preferences as in KK-stability,

$$\widetilde{f}(X') = \sum_{h \in H} V_h(|X'_h|) + \sum_{x \in X'} v(x), \tag{3}$$

where $V_h(k) = \sum_{j=1}^{k} v_h(j)$ and $v_{h_i}(j) = C(C - |H| \cdot j - i)$ with a constant $C > 0$. Contrary to the case in KK-stability, however, let us relax the assumption $C \gg v(x)$, and allow for arbitrary relations between $v_{h_i}(j)$ and $v(x)$. Even under this relaxation, by inspecting the proof of Theorem 4 it follows that function $f = \widehat{f} + \widetilde{f}$ as defined in Section 5.2.5 is $\mathrm{M}^\natural$-concave, and thus all our results hold with respect to this function f, including the existence of an HM-stable matching and strategyproofness for doctors of the generalized DA mechanism.

As mentioned above, function $\widetilde{f}$ in equation (3) generalizes the function corresponding to KK-stability. Moreover, contract-order stability corresponds to HM-stability with respect to equation (3) for the case in which $v(x) - v(x') \gg C$ for all $x \neq x'$: with this assumption, $\widetilde{f}$ primarily values accepting a contract with a high value. Thus, this paper's methodology enables us to generalize and unify Kamada and Kojima (2015) and Goto, Iwasaki, Kawasaki, Yasuda, and Yokoo (2014) in a straightforward manner.

The additional generality of our methodology also enables us to accommodate other possible societal preferences. For instance, consider a situation in which the hospitals mostly try to equalize the numbers of assigned doctors across hospitals as in KK-stability, but some special matchings are given priority; Suppose, for example, a particular hospital h has urgent needs for pediatricians, so matching pediatricians to h takes priority. Such a case can be accommodated by equation (3) if $v(x)$ is very small for most contracts,[27] but $v(x)$ is sufficiently large for any x which represents a matching of a pediatrician to the hospital h.

Another example is a situation in which there is a target capacity for each hospital that needs to be achieved first, but beyond the target capacities, applications are accepted according to a common preference ordering $\succ_H$. This criterion can be expressed by equation (3) by setting $v_h(j) \gg v(x)$

[27] More formally, $v_{h_i}(j) = C(C - |H| \cdot j - i), C \gg v(x)$ as in the case of KK-stability.

for any h, x, and j as long as j is at most the target capacity for h, while $v_h(j) \ll v(x)$ for any j that is strictly larger than the target capacity.

Both of these cases can be expressed by equation (3), and the function $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave by the preceding argument. Therefore the generalized DA mechanism is strategyproof for doctors and produces the doctor-optimal HM-stable matching. Beyond these specific examples, the major advantage of our methodology is to provide the policy maker with a general and flexible method to set a policy goal and immediately verify if such a goal is achievable and, if achievable, provide an off-the-shelf mechanism that produces a desired outcome.

6.3 Combination of distributional constraints

As emphasized earlier, the main contribution of this paper is to provide a unified framework for handling various distributional constraints. One of the advantages of such a unified framework is that it is easy to combine several kinds of distributional constraints. To demonstrate this advantage, let us investigate the case where regional maximum quotas (Section 5.2) and minimum quotas for a type (Section 5.4) coexist.

In this model, a market is a tuple

$$(D_M, D_m, H, X, R, (\succ_d)_{d \in D_M \cup D_m}, \succ_H, (p_h)_{h \in H}, (q_h)_{h \in H}, (q_r)_{r \in R}).$$

The definitions of $H, X, R, \succ_d, q_h$, and q_r are identical to the model presented in Section 5.2.

For simplicity, we assume there are two types of doctors: D_M is the set of majority doctors and D_m is the set of minority doctors. Each hospital h has a minimum quota p_h for minority doctors. As in Section 5.4, these minimum quotas are soft bounds and do not affect feasibility. We assume $\sum_{h \in r} p_h \leq q_r$ holds for all $r \in R$, i.e., the regional maximum quota of r is large enough to satisfy the minimum quotas of minority doctors for all $h \in r$. For $X' \subseteq X$, let $X'_{h,m}$ denote $\{(d, h) \in X' \mid d \in D_m\}$. As in the case of contract-order stability, we assume there exists a total order $\succ_H$ over X.

We say X' is doctor-feasible if X'_d is acceptable for all d. We say X' is hospital-feasible if $|X'_h| \leq q_h$ and $|X'_r| \leq q_r$ hold for all $h \in H$ and $r \in R$. Then, we say X' is feasible if it is doctor- and hospital-feasible.

For a matching X', we say $(d, h) \in X \setminus X'$ (where $h \in r$) is a blocking pair, if (d, h) is acceptable for d and $(d, h) \succ_d X'_d$, and any one of the following conditions holds:

(i) $d \in D_m$ and $|X_{h,m}| < p_h$,

(ii) $|X'_h| < q_h$ and $|X'_r| < q_r$,

(iii) $|X'_h| < q_h$, $|X'_r| = q_r$, there exists $h' \in r$ such that $(d, h') \in X'$, $(d, h) \succ_H (d, h')$, and either (iii-a) $d \in D_m$ and $|X'_{h',m}| > p_{h'}$, or (iii-b) $d \in D_M$ holds, or

(iv) there exists $(d', h) \in X'$, such that $(d, h) \succ_H (d', h)$, and any one of the following conditions holds:

(iv-a) $d' \in D_M$,

(iv-b) $d, d' \in D_m$, or

(iv-c) $d \in D_M, d' \in D_m$, and $|X_{h,m}| > p_h$.

We say a matching X' is stable if there exists no blocking pair according to the above definition.

In this definition, conditions (i)-(iv) list the cases in which the formation of a matching between doctor d and hospital h is regarded as a "legitimate" blocking in the presence of constraints. More specifically, (i) is a case in which d is a minority doctor and there is an available seat reserved for minority doctors in h; (ii) is a case in which h has an empty seat and its region r has room for another doctor; (iii) is a case in which the region r is full, but d is currently matched with another hospital h' in r, H prefers (d, h) over (d, h'), and moving d away from h' will not cause the minimum quota for minority at h' to be violated; (iv) is a case in which h is currently matched with a less preferred doctor d', and moving d' away from h will not cause the minimum quota for minority at h to be violated.

To represent this problem using our framework, let us consider an extended market given by tuple

$$(D_M, D_m, H, \tilde{X}, R, (\tilde{\succ}_d)_{d \in D_M \cup D_m}, \tilde{\succ}_H, (p_h)_{h \in H}, (q_h)_{h \in H}, (q_r)_{r \in R}).$$

Each contract $x \in \tilde{X}$ is represented as a triple (d, h, s) where $d \in D_M \cup D_m$, $h \in H$, and $s \in \{0, 1\}$. We assume that contracts of the form $(d, h, 0)$ are available only to minority doctors, $d \in D_m$, i.e., the triple $(d, h, 0)$ is interpreted as a contract in which $d \in D_m$ is assigned to a priority seat of hospital h, and $(d, h, 1)$ is interpreted as a contract in which $d \in D_M \cup D_m$ is assigned to a normal seat of hospital h. For $d \in D_M$, $\tilde{\succ}_d$ is obtained from $\succ_d$ such that for any $h \neq h'$, $(d,h,1)\tilde{\succ}_d(d,h',1)$ if and only if $(d,h) \succ_d (d,h')$. For $d \in D_m$, $\tilde{\succ}_d$ is obtained from $\succ_d$ such that for any $h \neq h'$, s and s', $(d,h,s)\tilde{\succ}_d(d,h',s')$ holds if and only if $(d,h) \succ_d (d,h')$, and $(d,h,0)\tilde{\succ}_d(d,h,1)$ holds for any h. $\tilde{\succ}_H$ is obtained from $\succ_H$ such that for any h, h', d, d', and s, $(d,h,s)\tilde{\succ}_H(d',h',s)$ if and only if $(d,h) \succ_H (d',h')$ holds, and $(d,h,0)\tilde{\succ}_H(d',h',1)$ holds for any h, h', d and d'. Let $v(x)$ denote the value of contract x which respects $\tilde{\succ}_H$, i.e., $v(x) > v(x')$ holds if and only if $x\tilde{\succ}_H x'$. Let $\tilde{X'}_{h,s}$ denote $\{(d,h,s) \in X' \mid d \in D_m \cup D_M\}$.

Let us define $\widehat{f}(\tilde{X}')$ as 0 when $|\tilde{X}'_h| \leq q_h$, $|\tilde{X}'_r| \leq q_r$, and $|\tilde{X}'_{h,0}| \leq p_h$ hold for all $h \in H$ and $r \in R$, and $-\infty$ otherwise. Also, let us define $\widetilde{f}(\tilde{X}')$ as $\sum_{x \in \tilde{X}'} v(x)$.

Then, $(\tilde{X}, \mathrm{dom}\,\widehat{f})$ is a laminar matroid since $\mathcal{T} = \{\tilde{X}_{h,s} \mid h \in H, s \in \{0,1\}\} \cup \{\tilde{X}_h \mid h \in H\} \cup \{\tilde{X}_r \mid r \in R\}$ is a laminar family of $\tilde{X}$. Therefore, $f = \widehat{f} + \widetilde{f}$ is M$^\natural$-concave by Theorem 3.

From the matching in the extended market $\tilde{X}'$, the matching in the original market X' is obtained by mapping each contract (d,h,s) to (d,h). The following proposition holds.

Proposition 11. *HM-stability of $\tilde{X}'$ in the extended market implies that there exists no blocking pair in the original market.*

Proof. Assume (d,h) is a blocking pair and condition (i) holds. Then, by choosing $x = (d,h,0)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold. If condition (ii) holds, by choosing $x = (d,h,1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold since $\tilde{X}' + x$ is feasible.

Assume condition (iii) holds. If (iii-a) holds, then there exists at least one doctor $d' \in D_m$ who is assigned to a normal seat of h', i.e., $(d',h',1)$ is included in $\tilde{X}'$, and $(d,h,1)\tilde{\succ}_H(d',h',1)$ hold. By choosing $x = (d,h,1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold since $(d,h,1)\tilde{\succ}_H(d',h',1)$ holds. If (iii-b) holds, $(d,h',1)$ is included in $\tilde{X}'$. By choosing $x = (d,h,1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold since $(d,h,1)\tilde{\succ}_H(d,h',1)$ holds.

Next, assume condition (iv) holds, i.e., there exists $(d',h) \in X'$ such that $(d,h) \succ_H (d',h)$ holds. If condition (iv-a) holds, then $(d',h,1) \in \tilde{X}'$ and $(d,h,1)\tilde{\succ}_H(d',h,1)$ hold. Thus, if we choose $x = (d,h,1)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold. If condition (iv-b) holds, then $(d',h,s) \in \tilde{X}'$ and $(d,h,s)\tilde{\succ}_H(d',h,s)$ hold. Thus, if we choose $x = (d,h,s)$, $x \in Ch_H(\tilde{X}' + x)$ and $x \in Ch_D(\tilde{X}' + x)$ hold. If condition (iv-c) holds, then there exists at least one doctor $d'' \in D_m$ (d'' can be equal to d'), who is assigned to a normal seat of h in $\tilde{X}'$ and $(d,h,1)\tilde{\succ}_H(d'',h,1)$ hold. Thus, if we choose $x=(d,h,1)$, $x \in Ch_H(\tilde{X}'+x)$ and $x \in Ch_D(\tilde{X}'+x)$ hold.

Thus, if there exists a blocking pair, $\tilde{X}'$ is not HM-stable. □

Proposition 11 implies that our method enables us to guarantee the existence of a matching with desirable properties–stability in an appropriately defined sense– even in a model where two kinds of distributional constraints coexist. Also, by the analysis of this paper, such a matching can be found by the off-the-shelf mechanism, i.e., the generalized DA mechanism. As illustrated by these findings, our methodology can easily be applied to cope with combinations of multiple distributional constraints.

7 Conclusion

This paper studied two-sided matching problems in which certain distributional constraints are imposed. We demonstrated that if the preference of the hospitals can be represented by an $M^\natural$-concave function, then the generalized DA mechanism is strategyproof for doctors and finds the doctor-optimal stable matching. Then we derived sufficient conditions under which the preference can be represented by an $M^\natural$-concave function, and established that those sufficient conditions are satisfied in various applied settings. Using these results, we obtained various results in the existing literature as well as new ones as corollaries of our general theorem. Because our sufficient conditions for $M^\natural$-concavity are easy to verify in many cases, they provide a recipe for non-experts in matching theory or discrete convex analysis to develop desirable mechanisms that handle distributional constraints.

Much of our effort in this paper was devoted to showing that existing results in the literature can be derived with our methodology. However, merely providing alternative proofs of existing results was not our main motivation. On the contrary, our aim was to demonstrate that this paper's methodology is flexible enough to address various types of constraints. In fact, we exploited our methodology to establish new results in Sections 6.2 and 6.3. We envision that our approach may prove useful when the match organizer is challenged by new kinds of constraints. Investigating whether this conjecture is true and, if so, in what applications, is left for future research.

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A Properties of matroids

We introduce several properties related to matroids that are used in our proofs.

Property 1 (simultaneous exchange property)**.** *Let* $(X, \mathcal{F})$ *be a matroid. Then for all* $Y, Z \in \mathcal{F}$ *and* $y \in Y \setminus Z$*, there exists* $z \in (Z \setminus Y) \cup \{\emptyset\}$ *such that* $Y - y + z \in \mathcal{F}$ *and* $Z - z + y \in \mathcal{F}$ *hold.*

Proof. Let $\mathcal{B}$ be the set of maximal elements in $\mathcal{F}$ with respect to set inclusion. $\mathcal{B}$ is called the family of **bases**. For $\mathcal{B}$, the following property holds: For all $\hat{Y}, \hat{Z} \in \mathcal{B}$ and $y \in \hat{Y} \setminus \hat{Z}$, there exists $z \in \hat{Z} \setminus \hat{Y}$ such that $\hat{Y} - y + z \in \mathcal{B}$ and $\hat{Z} - z + y \in \mathcal{B}$ hold. This property is known as the simultaneous exchange property for bases of matroids (see Theorem 39.12 of Schrijver (2003) or Condition (B) on page 69 of Murota (2003) for its proof). For Y and Z in $\mathcal{F}$, let us choose $\hat{Y}, \hat{Z} \in \mathcal{B}$ such that $\hat{Y} \supseteq Y$ and $\hat{Z} \supseteq Z$ hold. For any $y \in Y \setminus Z$, either $y \in Y \setminus \hat{Z}$ or $y \in \hat{Z}$ holds.

1. If $y \in Y \setminus \hat{Z}$, y is also included in $\hat{Y} \setminus \hat{Z}$. Thus, from the simultaneous exchange property for bases, there exists $z \in \hat{Z} \setminus \hat{Y}$ such that $\hat{Y} - y + z \in \mathcal{B}$ and $\hat{Z} - z + y \in \mathcal{B}$ holds. If $z \in Z$, both $Y - y + z$ (which is a subset of $\hat{Y} - y + z$) and $Z - z + y$ (which is a subset of $\hat{Z} - z + y$) are elements in $\mathcal{F}$. If $z \in \hat{Z} \setminus Z$, both $Y - y$ (which is a subset of $\hat{Y} - y + z$) and $Z + y$ (which is a subset of $\hat{Z} - z + y$) are elements in $\mathcal{F}$. In either case, we have established that there exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ such that $Y - y + z \in \mathcal{F}$ and $Z - z + y \in \mathcal{F}$, as desired.

2. If $y \in \hat{Z}$, both $Y - y$ (which is a subset of $\hat{Y}$) and $Z + y$ (which is a subset of $\hat{Z}$) are elements in $\mathcal{F}$. Thus the desired property holds with respect to $z = \emptyset$.

□

Property 2 (laminar concave function)**.** *Assume* $\mathcal{T}$ *is a laminar family of subsets of* X*, and* $f(X')$ *is given by a laminar concave function* $\sum_{T \in \mathcal{T}} f_T(|X' \cap T|)$*. Then,* $f(X')$ *is* M$^\natural$*-concave.*

This property is the M$^\natural$-concave version of Note 6.11 of Murota (2003).

Finally, let us introduce a matroid based on a network flow problem. Let (V, E, c, S, t) be a directed graph, where V is a set of vertexes and E is a set of directed edges. Here, $S \subset V$ is a set of start vertexes and $t \in V \setminus S$ is a unique terminal vertex. Let $c \in \mathbb{Z}^E$ represent the capacities of each edge, i.e., $c((u, v))$ is the capacity of the directed edge from u to v, where $u, v \in V$ and $\mathbb{Z}$ is the set of all integers.

A network flow is represented by $\rho \in \mathbb{Z}^E$. $\rho((u, v))$ is the flow for edge $(u, v) \in E$, i.e., the flow from u to v. For flow ρ, $\delta\rho(v)$ represents the boundary at vertex v, which is defined as $\sum_{(u,v)\in E} \rho((u, v)) - \sum_{(v,u)\in E} \rho((v, u))$, i.e., the difference between the inflow to v and the outflow from it.

We say a flow ρ is valid if for all $e \in E, \rho(e) \leq c(e)$ holds, and there exists $S' \subseteq S$ such that for all $v \in S', \delta\rho(v) = -1$, $\delta\rho(t) = |S'|$, and for all $v' \in V \setminus (S' \cup \{t\}), \delta\rho(v') = 0$ hold. We say S' is the sources of ρ.

The following property holds.

Property 3. *Let (V, E, c, S, t) be a directed graph. Then $(S, \mathcal{F})$, where $\mathcal{F} = \{S' \mid S' \subseteq S, \exists\rho \in \mathbb{Z}^E$, such that ρ is a valid flow where S' is its sources$\}$, is a matroid.*

Proof. This matroid is a variant of gammoids; see Oxley (2011) on gammoids. This is also a special case with zero costs of the network flow problem defined in Section 9.6 in Murota (2003). Theorem 9.26 in Murota (2003) shows that a function $f(S')$, which is defined as 0 if $S' \in \mathcal{F}$, and $-\infty$ otherwise, is M$^\natural$-concave. Since $\emptyset \in \mathcal{F}$, from Theorem 2, $(S, \mathcal{F})$ is a matroid.

Here, we provide a more elementary proof. It suffices to show the last condition of matroids (the other conditions are obvious). Let $S_1, S_2 \in \mathcal{F}$ with $|S_1| < |S_2|$. Let us add an artificial source s and edges (s, v) of capacity 1 for each vertex v in $S_1 \cup S_2$ to a given directed graph. We denote the new directed graph by $\hat{N} = (V \cup \{s\}, \hat{E}, \hat{c}, s, t)$, where $\hat{E} = E \cup \{(s, v) \mid v \in S_1 \cup S_2\}$ and $\hat{c}(u, v)$ is 1 if $u = s$ and $c(u, v)$ otherwise. For $i = 1, 2$, we can construct a valid flow ρ_i in $\hat{N}$ such that $\rho_i(s, v) = 1$ for all $v \in S_i$, $\delta\rho_i(t) = |S_i| = -\delta\rho_i(s)$ and $\delta\rho_i(u) = 0$ for $u \neq s, t$. By an optimality criterion of maximum flows in networks (e.g., see Corollary 10.2a of Schrijver (2003)), a valid flow $\rho \in \mathbb{Z}^{\hat{E}}$ from s to t in $\hat{N}$ is maximum (i.e., maximizes $\delta\rho(t)$ among valid flows from s to t in $\hat{N}$) if there is no directed path from s to t in the auxiliary directed graph $\tilde{N}_\rho = (V \cup \{s\}, \tilde{E}, \tilde{c}, s, t)$, where

$$\begin{aligned}
E_f &= \{(u,v) \mid (u,v) \in \hat{E},\ c(u,v) > \rho(u,v)\}, \\
E_b &= \{(v,u) \mid (u,v) \in \hat{E},\ \rho(u,v) > 0\}, \\
\tilde{E} &= E_f \cup E_b, \\
\tilde{c}(u,v) &= c(u,v) - \rho(u,v) \qquad ((u,v) \in E_f), \\
\tilde{c}(v,u) &= \rho(u,v) \qquad ((v,u) \in E_b).
\end{aligned}$$

Since ρ_1 is not maximum by $\delta\rho_1(t) < \delta\rho_2(t)$, there is a directed path P from s to t in $\tilde{N}_{\rho_1}$. By modifying ρ_1 into ρ'_1 by

$$\rho'_1(u,v) = \begin{cases} \rho_1(u,v) + 1 & ((u,v) \in P \cap E_f) \\ \rho_1(u,v) - 1 & ((v,u) \in P \cap E_b) \\ \rho_1(u,v) & \text{(otherwise)}, \end{cases}$$

we have $\delta\rho'_1(t) = \delta\rho(t) + 1$. Furthermore, since there is no edge from s to S_1 in $\tilde{N}_{\rho_1}$, P must pass through a vertex v in $S_2 \setminus S_1$. We can construct a valid flow in the original directed graph from ρ'_1 where $S_1 + v$ is its source, that is, $S_1 + v$ is a member of $\mathcal{F}$.

□

B Proof of Theorem 1

First, we prove Item (1). By Lemma 1 below, if f is M$^\natural$-concave and unique-selecting, then Ch_H satisfies the irrelevance of rejected contracts, the substitutes condition, and the law of aggregate demand. Hatfield and Milgrom (2005) show that if Ch_H satisfies these three conditions, then the generalized DA mechanism is strategyproof for doctors, and it obtains the doctor-optimal stable matching.

Lemma 1. *Ch_H satisfies the following three properties.*

Irrelevance of rejected contracts: for any $X' \subseteq X$ and any $x \in X \setminus X'$, $Ch_H(X') = Ch_H(X' + x)$ holds whenever $x \notin Ch_H(X' + x)$.

Substitutes condition: for any $X', X'' \subseteq X$ with $X' \subseteq X''$, $Re_H(X') \subseteq Re_H(X'')$ holds, where $Re_H(X') = (X' \setminus Ch_H(X'))$.

Law of aggregate demand: for any $X', X'' \subseteq X$ with $X' \subseteq X''$, $|Ch_H(X')| \leq |Ch_H(X'')|$.

Proof. Fujishige and Tamura (2006) show that the substitutes condition holds if f is M$^\natural$-concave and unique-selecting. Furthermore, Murota and Yokoi (2013) show that the law of aggregate demand holds if f is M$^\natural$-concave and unique-selecting.

Here, we provide a more elementary proof. From Proposition 1, by adding contract x to X', $Ch_H(X'+x) = Ch_H(X')$ or $Ch_H(X'+x) = Ch_H(X') + x$ or $Ch_H(X'+x) = Ch_H(X') - x' + x$ for some $x' \in Ch_H(X')$. Thus, it is clear that irrelevance of rejected contracts holds. To prove the substitutes condition and the law of aggregate demand, it is sufficient to consider the case where $X'' = X' + x$. It is clear that $|Ch_H(X')| \leq |Ch_H(X'+x)|$ and $Re_H(X') \subseteq Re_H(X'+x)$ hold. □

Next, we prove Item (2). At Step 1 in Mechanism 1, the calculation of Ch_D is $O(|X|)$ in total, because each (rejected) doctor selects her most preferred contract which has not been rejected. Hence the time complexity of the generalized DA mechanism depends on calculations of Ch_H. At Step 2 in Mechanism 1, we calculate $Ch_H(X')$ by adding newly offered contracts one by one. More precisely, we use the next relation which is guaranteed by the substitutes condition and irrelevance of rejected contracts,

$$Ch_H(X') = Ch_H(Ch_H(\cdots Ch_H(Ch_H(X'' + y_1) + y_2) + \cdots) + y_k)$$

where X'' and $\{y_1, y_2, \ldots, y_k\}$ are the tentatively accepted contracts at the previous stage (or initially $X'' = \emptyset$) and the newly offered contracts in X' at Step 1, respectively. By Proposition 1 and the fact $X'' = Ch_H(X'')$, $Ch_H(X'' + y_1)$ is determined by calculating f exactly $|X''| + 1$ times, and hence, at most $|X|$ times.[28] In the same way as above, we can determine $Ch_H(X')$ by calculating f at most $k \cdot |X|$ times. Since each contract is selected as a newly offered contract at most once in the generalized DA mechanism, the calculation of Ch_H is $O(T(f) \cdot |X|^2)$ in total. Thus, the time complexity of the generalized DA mechanism is $O(T(f) \cdot |X|^2)$.

[28]To be more precise, $Ch_H(X'' + y_1)$ is either X'', $X'' + y_1$, or $X'' - x + y_1$, where $x \in X''$. Thus, to obtain $Ch_H(X'' + y_1)$, it is sufficient to apply f to these $|X''| + 1$ candidates and to choose the one that maximizes f, because $f(X'')$ has been calculated at the previous stage.

C Proof of Theorem 2

The "if" part is obtained immediately from Property 1. The proof of the "only if" part is given as follows. Let $N = \{1, 2, \ldots, n\}$. A function $\eta : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ is said to be M$^\natural$-concave if $\operatorname{dom} \eta = \{z \in \mathbb{Z}^N \mid \eta(z) \neq -\infty\}$ is not $\emptyset$ and for all $z, z' \in \operatorname{dom} \eta$ and $i \in N$ with $z_i > z'_i$, either (a) $\eta(z) + \eta(z') \leq \eta(z - \chi_i) + \eta(z' + \chi_i)$ or (b) there exists $j \in N$ such that $z'_j > z_j$ and $\eta(z) + \eta(z') \leq \eta(z - \chi_i + \chi_j) + \eta(z' + \chi_i - \chi_j)$, holds, where χ_i is a unit vector such that its i-th element is 1 and other elements are 0. We consider the case where $\operatorname{dom} \eta$ is bounded and has 0 as the minimum point. For each $i \in N$, let $c_i = \max\{z_i \mid z \in \operatorname{dom} \eta\}$. Let us consider a finite set X and a partition $G = \{g_1, g_2, \ldots, g_n\}$ of X with $|g_i| \geq c_i$ for all $i \in N$. Let us define $\zeta(X')$ as $(|X' \cap g_1|, \ldots, |X' \cap g_n|)$. Each element $\zeta_i(X')$, where $1 \leq i \leq n$, is a non-negative integer. We define the family $\mathcal{F}$ of subsets of X defined by

$$\mathcal{F} = \{X' \subseteq X \mid \zeta(X') \in \operatorname{dom} \eta\}.$$

Then, the following lemma holds (recall that the definition of symmetry is given in Definition 11).

Lemma 2. *$(X, \mathcal{F})$ is a matroid and G is symmetric in $(X, \mathcal{F})$.*

Proof. The fact that G is symmetric in $(X, \mathcal{F})$ is obvious by the definition of $\mathcal{F}$. Since $0 \in \operatorname{dom} \eta$, we have $\emptyset \in \mathcal{F}$.

Let $X', X'' \in \mathcal{F}$ with $|X'| > |X''|$. We denote $\zeta(X')$ and $\zeta(X'')$ by z' and z'', respectively. It follows from $|X'| > |X''|$ that there exists $i \in N$ with $z'_i > z''_i$. The M$^\natural$-concavity of η guarantees that (a) $z' - \chi_i, z'' + \chi_i \in \operatorname{dom} \eta$ or (b) there exists $j \in N$ such that $z''_j > z'_j$ and $z' - \chi_i + \chi_j, z'' + \chi_i - \chi_j \in \operatorname{dom} \eta$.

In case (a), we have $X'' + x \in \mathcal{F}$ for some $x \in g_i \cap (X' \setminus X'')$. In case (b), there exist $x \in g_i \cap (X' \setminus X'')$ and $y \in g_j \cap (X'' \setminus X')$ with $\hat{X}' := X' - x + y \in \mathcal{F}$. We note that $|\hat{X}'| = |X'|$ and $X' \cap X''$ is a proper subset of $\hat{X}' \cap X''$. We replace X' by $\hat{X}'$, and continue the above discussion. After a finite number of iterations, the above (a) must occur by $|X'| > |X''|$. Hence the third condition in Definition 4 holds.

It can be immediately seen that case (a) in the above discussion happens if $X'' = \emptyset$ and $x \in X'$. Thus, we have $X' - x \in \mathcal{F}$ which implies the second condition in Definition 4. □

We finally prove Theorem 2. Suppose that $X = \{x_1, x_2, \ldots, x_m\}$. Since f is an M$^\natural$-concave function on X and $f(\emptyset) = 0$, from Lemma 2 for the partition

$G = \{\{x_1\}, \{x_2\}, \dots, \{x_m\}\}$ and X, we have $\mathcal{F} = \{X' \mid X' \subseteq X, f(X') \neq -\infty\}$ is a matroid.

D Proof of Theorem 4

Let $\mathcal{F} = \operatorname{dom} \widehat{f}$. Since $G = \{g_1, \dots, g_n\}$ is symmetric in a matroid $(X, \mathcal{F})$, when we check whether $X' \in \mathcal{F}$, only the number of members for each group matters. Then, we can assume $\widehat{f}(X')$ is equal to $\widehat{\eta}(\zeta(X'))$, where $\zeta(X') = (|X' \cap g_1|, \dots, |X' \cap g_n|)$ as defined in Appendix C, and $\widehat{\eta}(z)$ is 0 if $\exists X' \in \mathcal{F}$ such that $z = \zeta(X')$, and otherwise, $-\infty$.

We first show that the function $\acute{\eta} : \mathbb{Z}^N \to \mathbb{R} \cup \{-\infty\}$ defined by $\acute{\eta}(z) = \widehat{\eta}(z) + \sum_{1 \leq i \leq n} V_{g_i}(z_i)$ is M$^\natural$-concave (see Appendix C for the definition of M$^\natural$-concavity of functions on $\mathbb{Z}^N$).

Lemma 3. *$\acute{\eta}$ is M$^\natural$-concave, and* $\operatorname{dom} \acute{\eta}$ *has* 0 *as the minimum point.*

Proof. Since $\widehat{f}$ gives a matroid, $\operatorname{dom} \acute{\eta}$ has 0 as the minimum point. It is known that the sum of an M$^\natural$-concave function and a separable concave function is also M$^\natural$-concave (see (4) in Theorem 6.13 and Theorem 6.15 of Murota (2003)). Since $\acute{\eta}$ is the sum of $\widehat{\eta}$ and the separable concave function $\sum_{1 \leq i \leq n} V_{g_i}(z_i)$, it is enough to show that $\widehat{\eta}$ is M$^\natural$-concave. Furthermore, since the value of $\widehat{\eta}$ in its effective domain is always 0, to show the M$^\natural$-concavity of $\widehat{\eta}$, it is sufficient to show that $\operatorname{dom} \widehat{\eta}$ is an M$^\natural$-convex set, i.e., for all $z, z' \in \operatorname{dom} \widehat{\eta}$ and $i \in N$ with $z_i > z'_i$, either (a) $z - \chi_i, z' + \chi_i \in \operatorname{dom} \widehat{\eta}$ or (b) there exists $j \in N$ such that $z'_j > z_j$ and $z - \chi_i + \chi_j, z' + \chi_i - \chi_j \in \operatorname{dom} \widehat{\eta}$, holds. Let X_1, X_2 be elements of matroid $(X, \mathcal{F})$ such that $z = \zeta(X_1)$ and $z' = \zeta(X_2)$, where $\mathcal{F} = \operatorname{dom} \widehat{f}$. By the symmetry of G in $(X, \mathcal{F})$, we can assume that either $X_1 \cap g_k \subseteq X_2 \cap g_k$ or $X_2 \cap g_k \subseteq X_1 \cap g_k$ for each $k \in N$. By $z_i > z'_i$, there exists $x \in g_i \cap (X_1 \setminus X_2)$. The simultaneous exchange property (Property 1) for X_1, X_2 and x guarantees that (a') $X_1 - x, X_2 + x \in \mathcal{F}$ or (b') there exists $y \in (X_2 \setminus X_1)$ with $X_1 - x + y, X_2 + x - y \in \mathcal{F}$. In the case (a'), we have $z - \chi_i = \zeta(X_1 - x)$ and $z' + \chi_i = \zeta(X_2 + x)$, that is, (a) holds. In the case (b'), $y \notin g_i$ must hold, and therefore, there exists $j \in N$ with $y \in g_j$. By our assumption, $|X_1 \cap g_j| < |X_2 \cap g_j|$ must be satisfied. Thus, in this case, (b) holds. □

It suffices to show that the function $\acute{f}$ defined by $\acute{f}(X') = \acute{\eta}(\zeta(X'))$ is M$^\natural$-concave (as a function of X'), because f is equal to the sum of $\acute{f}$ and a linear function $\sum_{x \in X'} v(x)$.

Lemma 4. *$\acute{f}$ is M$^\natural$-concave.*

Proof. Let $X', X'' \in \operatorname{dom} \acute{f}$ and $x \in X' \setminus X''$. We assume that $x \in g_i$. We denote $\zeta(X')$ and $\zeta(X'')$ by z' and z'', respectively. If $|X' \cap g_i| \leq |X'' \cap g_i|$ then there exists $y \in g_i \cap (X'' \setminus X')$. By the symmetry of G in $(X, \operatorname{dom} \widehat{f})$, $f(X') = f(X' - x + y)$ and $f(X'') = f(X'' + x - y)$.

In the sequel, we suppose that $|X' \cap g_i| > |X'' \cap g_i|$, i.e., $z'_i > z''_i$. The M$^\natural$-concavity of $\acute{\eta}$ guarantees that:

(i) $\acute{\eta}(z') + \acute{\eta}(z'') \leq \acute{\eta}(z' - \chi_i) + \acute{\eta}(z'' + \chi_i)$

or

(ii) there exists $j \in N$ such that $z''_j > z'_j$ and

$$\acute{\eta}(z') + \acute{\eta}(z'') \leq \acute{\eta}(z' - \chi_i + \chi_j) + \acute{\eta}(z'' + \chi_i - \chi_j).$$

In case (i), we have $\acute{f}(X') + \acute{f}(X'') \leq \acute{f}(X' - x) + \acute{f}(X'' + x)$. In case (ii), there exist $y \in g_j \cap (X'' \setminus X')$ such that

$$\acute{f}(X') + \acute{f}(X'') \leq \acute{f}(X' - x + y) + \acute{f}(X'' + x - y).$$

Hence $\acute{f}$ is M$^\natural$-concave. □

E Proof of Theorem 5

Assume $\widetilde{f}(X')$ is given as $\sum_{T \in \mathcal{T}} \widetilde{f}_T(|X' \cap T|)$. Then, $f = \widehat{f} + \widetilde{f}$ can be written as $\sum_{T \in \mathcal{T}} f_T(|X' \cap T|)$, where $f_T(k) = \widehat{f}_T(k)$ if $k \leq q_T$, and otherwise, $-\infty$. This is also a laminar concave function, since each f_T is a univariate concave function. Thus, f is M$^\natural$-concave from Property 2.

F Proof of Proposition 5

First, we show that X' is feasible if and only if $\widehat{f}(X') = 0$ and $|X'| = |D|$. Assume X' is feasible. From the fact X' is hospital-feasible, $|X'| = |D|$ holds. Let us define a flow in which X' is the source as follows. For each edge from $(d, h) \in X'$ to v-h, we set its flow as 1. Since X' is hospital feasible, $p_h \leq |X'_h| \leq q_h$ holds. Thus, the total input flow to v-h is at least p_h and at most q_h. Then, for each $h \in H$, we set the flow from v-h to t as p_h, and the flow from v-h to its parent r as $|X'_h| - p_h$. This is at most $q_h - p_h$. Thus, it is within the capacity. Also, for each $r \in R - H$, the total input flow to v-r is $|X'_r| - \sum_{r' \in \text{children}(r)} p_{r'}$, which is at least $p_r - \sum_{r' \in \text{children}(r)} p_{r'}$ and at most $q_r - \sum_{r' \in \text{children}(r)} p_{r'}$. Then, we set the flow from v-r to t as $p_r - \sum_{r' \in \text{children}(r)} p_{r'}$, and the flow from v-r to its parent region r' as $|X'_r| - p_r$. This is at most $q_r - p_r$. Thus, it is within the capacity. Finally, for H, the total input flow to v-H is $|D| - \sum_{r \in \text{children}(H)} p_r$. Then, we set the flow from v-H to t as $|D| - \sum_{r \in \text{children}(H)} p_r$. It is clear the flow defined as above if valid. Thus, $\widehat{f}(X') = 0$ holds.

Next, we show that if $\widehat{f}(X') = 0$ and $|X'| = |D|$, then X' is hospital-feasible. The total capacity of edges toward t is $(|D| - \sum_{r \in \text{children}(H)} p_r) + \sum_{r \in R-H}(p_r - \sum_{r' \in \text{children}(r)} p_{r'}) + \sum_{h \in H} p_h = |D|$. Thus, if $\widehat{f}(X') = 0$ and $|X'| = |D|$, each of these edges is saturated, i.e., its flow is equal to its capacity. Thus, for each hospital $h \in H$, $|X'_h|$ is at least p_h since the edge from v-h to t is saturated. Also, since the flow is valid, at the edge from v-h to r, where r is h's parent region, its flow is at most $q_h - p_h$. Thus, $|X'_h|$ is at most q_h. Similarly, for each region $r \in R$, we can recursively show that $|X'_r|$ is at least p_r and at most q_r. Thus, X' is feasible.

For matroid $(X, \mathcal{F})$, recall that we say $X' \in \mathcal{F}$ is a base if there exists no $X'' \in \mathcal{F}$ such that $X'' \supset X'$, i.e., X' is maximal. From Definition 4, it is clear that all bases have the same size. Also, since we assume every doctor is acceptable for all hospitals, the size of a base is $|D|$.

Now, let us show that if $\widehat{f}(X') = 0$, X' is semi-hospital-feasible. We have already shown that if $\widehat{f}(X') = 0$ and $|X'| = |D|$, then X' is hospital-feasible. If $\widehat{f}(X') = 0$ and $|X'| < |D|$, let us choose a base X'' such that $X'' \supset X'$ holds. Then, $|X''| = |D|$ and $\widehat{f}(X'') = 0$. Thus, X'' is hospital-feasible and hence X' is semi-hospital-feasible.

Finally, let us show that if X' is semi-hospital-feasible, then $\widehat{f}(X') = 0$. Since X' is semi-hospital-feasible, there exists $X'' \supseteq X'$ such that X'' is hospital-feasible. Then, $\widehat{f}(X'') = 0$ and from Definition 4, $\widehat{f}(X') = 0$ holds.