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1982

Sankhyā : The Indian Journal of Statistics
1982, Volume **44**, Series A, Pt. 3 pp. 415–422.

ON CHARACTERIZING SOME DISCRETE DISTRIBUTIONS USING AN EXTENSION OF THE RAO-RUBIN THEOREM

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1. Introduction

Let X, Y be non-negative integer-valued r.v.'s such that $P\{X = n\} = P_n$ and

$$P(Y = r \mid X = n) = s(r, n), \; r = 0, 1, \ldots, n.$$

Using Bernstein's theorem on completely monotone functions, Rao and Rubin (1964) have shown that if

$$s(r, n) = \binom{n}{r} p^r q^{n-r}, \; p \text{ in } (0, 1), \; q = 1-p \qquad \ldots \quad (1.1)$$

then $\{P_n, n = 0, 1, \ldots\}$ is Poisson if and only if (iff)

$$P(Y = r) = P(Y = r \mid X = Y) = P(Y = r \mid X > Y), \; r = 0, 1, \quad \ldots \quad (1.2)$$

(the so called R-R condition).

Shanbhag (1974) gave an elementary proof of this result. Srivastava and Srivastava (1970) proved that if $\{P_n\}$ is Poisson, then the R-R condition holds iff $s(r, n)$, $r = 0, 1, \ldots, n$ is binomial. Rao and Rubin (1964) have also shown that if the non-negative integer-valued r.v. X takes the values $k, k+1, \ldots$; $k \geqslant 0$ and $s(r, n)$ is binomial as in (1.1), then $\{P_n, n = k, k+1, \ldots\}$ is truncated Poisson iff

$$P(Y = r \mid Y \geqslant k) = P(Y = r \mid X = Y) = P(Y = r \mid X > Y, \; Y \geqslant k),$$
$$r = k, k+1, \ldots, \qquad \ldots \quad (1.3)$$

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2. Shanbhag's result and a variant

Shanbhag (1977) gives the following extension of the R-R characterization.

Theorem 1 (Shanbhag's extension) : *Let $\{(a_n, b_n), n = 0, 1, \ldots\}$ be a sequence of real vectors with $b_0, b_1 > 0$, $a_n > 0$ for all $n \geqslant 0$ and $b_n \geqslant 0$ for $n \geqslant 2$ and let $\{c_n\}$ denote the convolution of $\{a_n\}$ and $\{b_n\}$. (Observe that $c_n > 0$, $n \geqslant 0$). Let (X, Y) be a random vector, as defined in the introduction, such that $P_0 = P[X = 0] < 1$ and whenever $P_n > 0$*

$$P(Y = r \mid X = n) = \frac{a_r b_{n-r}}{c_n} \quad r = 0, 1, \ldots, n. \qquad \ldots \quad (2.1)$$

Then

$$P(Y = r) = P(Y = r \mid X = Y) \quad r = 0, 1, \ldots \qquad \ldots \quad (2.2)$$

iff

$$\frac{P_n}{c_n} = \frac{P_0}{c_0} \theta^n \quad n = 1, 2, \ldots \quad \text{for some } \theta > 0. \qquad \ldots \quad (2.3)$$

The result of Theorem 1 will now be used to prove the following theorem.

Theorem 2 (Extension to the truncated case) : *Let $\{(a_n, b_n) : n = 0, 1, \ldots\}$ be a sequence of vectors of non-negative real numbers such that $b_0, b_1 > 0$ and $a_n > 0$ for $n \geqslant k$, $k \geqslant 0$. Let $\{c_n\}$ be the convolution of $\{a_n\}$ and $\{b_n\}$. (Observe that $c_n > 0$, $n \geqslant k$). Let (X, Y) be a vector of non-negative integer-valued r.v.'s such that $P(X = n) = P_n$ with $P_k < 1$ and X taking values $\geqslant k$ only, and whenever $P_n > 0$*

$$P(Y = r \mid X = n) = \frac{a_r b_{n-r}}{c_n} \quad \begin{array}{l} r = 0, 1, \ldots, n \\ n = k, k+1, \ldots \end{array} \qquad \ldots \quad (2.4)$$

Then

$$P(Y = r \mid Y \geqslant k) = P(Y = r \mid X = Y),\ r = k, k+1, \ldots \qquad \ldots \quad (2.5)$$

iff

$$\frac{P_n}{c_n} = \frac{P_k}{c_k} \theta^{n-k} \quad n = k, k+1, \ldots \text{ for some } \theta > 0, \qquad \ldots \quad (2.6)$$

Proof: We have been given that $X-k$ is a non-negative integer-valued random variable. Further, it follows that conditional on $Y-k \geqslant 0$, the random variable $Y-k$ is non-negative and integer valued.

If we define

$$c_n^{(k)} = \sum_{r=k}^{n} a_r b_{n-r} \quad \ldots \quad (2.7)$$

we will obviously have

$$c_{n+k}^{(k)} = \sum_{r=k}^{n+k} a_r b_{n+k-r} = \sum_{r=0}^{n} a_{r+k} b_{n-r}. \quad \ldots \quad (2.8)$$

It then follows that

$$P(X-k = n \mid Y-k \geqslant 0) = \frac{1}{P(Y \geqslant k)} \frac{c_{n+k}^{(k)}}{c_{n+k}} P_{n+k} \qquad n = 0, 1, \ldots \quad \ldots \quad (2.9)$$

and

$$P(Y-k = r \mid X-k = n,\ Y-k \geqslant 0) = \frac{a_{r+k} b_{n-r}}{c_{n+k}^{(k)}} \qquad \begin{matrix} r = 0, 1, \ldots, n \\ n = 0, 1, \ldots \end{matrix} \quad \ldots \quad (2.10)$$

It also follows that (2.5) is equivalent to

$$P(Y-k = r \mid Y-k \geqslant 0) = P(Y-k = r \mid X-k = Y-k) \qquad r = 0, 1, \ldots \quad \ldots \quad (2.11)$$

Then, clearly the random vector $(X-k, Y-k)$ conditional on $Y-k \geqslant 0$ possesses all the properties required of (X, Y) in Theorem 1. Hence, applying this theorem to the random vector $(X-k, Y-k)$ conditional on $Y-k \geqslant 0$ we have that (2.11) holds iff

$$\frac{P(X-k = n \mid Y-k \geqslant 0)}{c_{n+k}^{(k)}} = \frac{P(X-k = 0 \mid Y-k \geqslant 0)}{c_k^{(k)}} \theta^n \qquad \begin{matrix} \text{for some } \theta > 0 \\ n = 0, 1, \ldots \end{matrix}$$

which in view of (2.9), is equivalent to

$$\frac{P_{n+k}}{c_{n+k}} = \frac{P_k}{c_k} \theta^n \qquad n = 0, 1, \ldots \quad \ldots \quad (2.12)$$

Since (2.11) is equivalent to (2.5), the result is now obvious.

Note 1 : Theorem 2 can also be obtained directly by appealing to the basic lemma of Shanbhag (1977).

Note 2 : It is interesting to observe that Theorems 1 and 2 remain valid if the right hand side or the left hand side of (2.2) and (2.5) is replaced by $P(Y = r \mid X > Y)$ and $P(Y = r \mid X > Y,\ Y \geqslant k)$ respectively.

In part 3, Theorem 1 is used to show that if $s(r, n)$ is negative hyper geometric (n, m, ρ) then the R-R condition holds iff $\{P_n\}$ is negative binomial $(m+\rho, p)$ without requiring additional conditions on the existence of the derivatives $G^{(r)}(t)$ of the p.g.f. G of X, as in Patil and Ratnaparkhi (1975).

Then Theorem 2 is used to show that provided again that $s(r, n)$ is negative hypergeometric, the modified R-R condition (2.5) is necessary and sufficient for P_n, $n = k, k+1, \ldots$ to be truncated negative binomial.

In part 4 we state two other characterizations based on truncated forms of the distribution of Y given X.

3. Characterizations of the Negative Binomial and Truncated Negative Binomial Distributions

Corollary 3.1 (Characterization of the Negative Binomial Distribution): *Let X and Y be non-negative integer-valued r.v.'s as defined in Theorem* 1.

Assume that

$$P(Y = r \mid X = n) = \frac{\binom{-m}{r}\binom{-\rho}{n-r}}{\binom{-m-\rho}{n}} \quad \begin{array}{l} r = 0, 1, \ldots, n \\ m > 0, \rho > 0 \end{array} \qquad \ldots \quad (3.1)$$

i.e., negative hypergeometric (n, m, ρ).

Then the condition (2.2) holds iff

$$P_n(= P(X = n)) = \binom{-N}{n} p^N(-q)^n \quad N = m+\rho \quad \text{for } n = 0, 1, \ldots \qquad \ldots \quad (3.2)$$

Proof: It is obvious that Theorem 1 applies with

$$a_n = \binom{-m}{n}(-1)^n, \; b_n = \binom{-\rho}{n}(-1)^n. \qquad \ldots \quad (3.3)$$

whence

$$c_n = \binom{-m-\rho}{n}(-1)^n. \qquad \ldots \quad (3.4)$$

Corollary 3.2 (Characterization of the truncated negative binomial distribution): *Let us now assume that X, Y are non-negative integer-valued r.v.'s as defined in Theorem 2. Suppose that the conditional distribution of*

Y given X is negative hypergeometric as in (3.1). Then, the modified R-R condition (2.5) holds iff

$$P_n = \frac{\binom{-N}{n} p^N(-q)^n}{\sum_{i=k}^{\infty} \binom{-N}{i} p^N(-q)^i} \qquad n = k, k+1, \ldots \qquad \ldots \quad (3.5)$$

i.e., negative binomial truncated at $k-1$.

Proof: Define the sequences $\{a_n\}$, $\{b_n\}$, $n = 0, 1, \ldots$ as in (3.3). Consequently $P(Y = r \mid X = n)$ is again of the form $a_r b_{n-r}/c_n$, $r = 0, 1, \ldots, n$.

Hence, since all the requirements of Theorem 2, are met, (2.5) holds iff $P_n = P_k \frac{c_n}{c_k} \theta^{n-k}$ for some $\theta > 0$, and $n = k, k+1, \ldots$ i.e., iff (3.5) is valid.

Note: It is interesting to note here that if $\{P_n\}$ is negative binomial and $P(Y = r \mid X = n)$ is of the form $a_r b_{n-r}/c_n$, then the negative hypergeometric is not the only distribution of the form (2.1) that $Y \mid X$ should follow in order that the R-R condition (2.2) holds.

This is so, because there exist two independent non-negative random variables which are not negative binomial, but their sum is negative binomial. However it is interesting to observe that if we also require $\{b_n\}$ to be the s-fold convolution of $\{a_n\}$, then we get a characterization for the negative hypergeometric distribution.

4. Characterizations of the convolution of distributions with one of the members as truncated

Corollary 4.1 (Characterization of the convolution of a Poisson (μ), with a truncated poisson (λ) Distribution): *Consider the r.v.'s X and Y as in Theorem 2. Suppose that the conditional distribution of $Y \mid X$ is binomial truncated at $k-1$, i.e.,*

$$P(Y = r \mid X = n) = \frac{\binom{n}{r} p^r q^{n-r}}{\sum_{i=k}^{n} \binom{n}{i} p^i q^{n-i}} \qquad \begin{array}{l} r = k, k+1, \ldots, n \\ n = k, k+1, \ldots \\ 0 < p < 1, q = 1-p \end{array} \qquad \ldots \quad (4.1)$$

Then, the modified R-R condition (2.5) holds iff

$$P(X = n) = \frac{e^{-\mu} \sum_{r=k}^{n} \binom{n}{r} \lambda^r \mu^{n-r}}{n! \sum_{i=k}^{\infty} \frac{\lambda^i}{i!}} \qquad \begin{array}{l} \text{for some } \lambda, \mu > 0 \\ n = k, k+1, \ldots \end{array} \qquad \ldots \quad (4.2)$$

i.e., convolution of a Poisson (μ) with a Poisson (λ) truncated at $k-1$.

Proof: Consider the following sequences,

$$a_n = \begin{cases} \dfrac{\lambda^n/n!}{\sum\limits_{i=k}^{\infty} \lambda^i/i!} & n = k, k+1, \ldots \\ \\ 0 & n = 0, 1, \ldots, k-1 \end{cases} \quad \ldots (4.3)$$

$$b_n = e^{-\mu}\frac{\mu^n}{n!} \qquad n = 0, 1, \ldots \quad \ldots (4.4)$$

The convolution of (4.3) and (4.4) for $n \geqslant k$ is given by

$$c_n = \sum_{r=k}^{n} a_r b_{n-r} = \frac{e^{-\mu}\sum\limits_{r=k}^{n}\binom{n}{r}\lambda^r\mu^{n-r}}{n!\sum\limits_{i=k}^{\infty}\lambda^i/i!} \quad \ldots (4.5)$$

It is clear now that the R.H.S. of (4.1) can be written as $a_r\, b_{n-r}/c_n$ with a_n, b_n, c_n given by (4.3), (4.4), (4.5) respectively and with $p = \dfrac{\lambda}{\lambda+\mu}$, $q = 1-p$. Since a_n, b_n satisfy all the conditions of Theorem 2 we have that the R-R condition (2.5) holds iff P_n satisfies (2.6) for c_n given by (4.5). This implies that (2.5) holds iff

$$P_n = P_k \frac{\sum\limits_{r=k}^{n}\binom{n}{r}\lambda^r\mu^{n-r}/n!}{\binom{k}{k}\lambda^k/k!}\,\theta^{n-k} \quad n = k, k+1, \ldots; \text{ for some } \theta > 0.$$

Noting that $\sum\limits_{n=k}^{\infty} P_n = 1$ we have

$$P_k^{-1} = e^{-\mu\theta}\frac{\sum\limits_{r=k}^{\infty}(\lambda\theta)^r/r!}{(\lambda\theta)^k/k!}$$

So, (2.5) holds iff

$$P_n = \frac{e^{-\mu\theta}\sum\limits_{r=k}^{n}\binom{n}{r}(\lambda\theta)^r(\mu\theta)^{n-r}}{n!\sum\limits_{r=k}^{\infty}(\lambda\theta)^r/r!} \quad \lambda, \mu, \theta > 0$$

i.e. iff $\{P_n\}$ is the probability distribution of the convolution of a Poisson r.v. with a $k-1$ truncated Poisson r.v.

Note: The form of the convoluted Poisson distribution examined in Corollary 4.1 is a special case of convoluted Poisson distributions studied by Samaniego (1976).

Corollary 4.2 (Characterization of the convolution of a negative binomial with a truncated negative binomial distribution): *Let again, X, Y be as in Theorem 2. Suppose that the distribution of $Y|X$ is negative hypergeometric truncated at $k-1$, i.e.,*

$$P(Y = r \mid X = n) = \frac{\binom{-m}{r}\binom{-\rho}{n-r}}{\sum\limits_{i=k}^{n}\binom{-m}{i}\binom{-\rho}{n-i}} \quad r = k, k+1, \ldots, n \;\ldots \quad (4.6)$$

In that case, the modified R-R condition (2.5) holds iff $\{P_n\}$ is the convolution of a negative binomial (ρ, p) and a negative binomial (m, p) truncated at the point $k-1$, i.e., iff

$$P_n = \frac{\sum\limits_{r=k}^{n}\binom{-m}{r}\binom{-\rho}{n-r}(-q)^n p^\rho}{\sum\limits_{r=k}^{\infty}\binom{-m}{r}(-q)^r} \quad n = k, k+1, \ldots \quad \ldots \quad (4.7)$$

Proof: Define the sequences

$$a_n = \begin{cases} \dfrac{\binom{m+n-1}{n} q^n}{\sum\limits_{i=k}^{\infty}\binom{m+i-1}{i} q^i} & n = k, k+1, \ldots \quad \ldots \quad (4.8) \\[2ex] 0 & n = 0, 1, \ldots, k-1 \end{cases}$$

$$b_n = \binom{\rho+n-1}{n} q^n \quad n = 0, 1, \ldots \quad \ldots \quad (4.9)$$

Then, the convolution of $\{a_n\}$ and $\{b_n\}$ is given by

$$c_n = \frac{\sum\limits_{r=k}^{\infty}\binom{m+r-1}{r}\binom{\rho+n-r-1}{n-r} q^n}{\sum\limits_{r=k}^{\infty}\binom{m+r-1}{r} q^r} \quad n = k, k+1, \ldots \quad \ldots \quad (4.10)$$

Clearly, $\left(\text{since } \binom{m+n-1}{n} = \binom{-m}{n}(-1)^n\right)$

$$\frac{a_r b_{n-r}}{c_n} = \frac{\binom{-m}{r}\binom{-\rho}{n-r}}{\sum\limits_{r=k}^{n}\binom{-m}{r}\binom{-\rho}{n-r}} \quad r = k, k+1, \ldots \quad \ldots \quad (4.11)$$

Consequently, the sequences $\{a_n\}$, $\{b_n\}$ as defined in (4.8), (4.9) can be used to express (4.6) in the form required by Theorem 2. Hence the result follows readily from Theorem 2.

Note : It is clear that for $k = 0$ Corollaries 4.1 and 4.2 reduce to Corollaries 3.1 and 3.2, respectively.

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Paper received : March, 1979.
Revised : April, 1980.