An Extension of the Damage Model

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1. Introduction

Let X be an original observation subjected to a destructive process. Then, what is observed is the undamaged part of X, say Y. This is usually called the resulting random variable (r.v.). The destruction process (or the survival distribution) can be represented by the conditional distribution of Y given X $(Y \mid X)$. This model first considered by *Rao* [1963] is called a damage model.

In the simple case where the distribution of $Y \mid X$ is Binomial with parameters n, p we have

$$G_Y(t) = G_X(q + pt), \; G_{Y|X=Y}(t) = \frac{G_X(pt)}{G_X(p)}$$

$$0 < p < 1, \; q = 1 - p$$

where $G_X(t)$, $G_Y(t)$, $G_{Y|X=Y}(t)$ are the probability generating functions (p.g.f.'s) of the original r.v., the resulting r.v. and the resulting r.v. when no damage has occurred.

In section 2 of this paper we consider the problem of obtaining the p.g.f. of the resulting distribution when the parameter p of the Binomial survival is a r.v. with d.f. $F_2(p)$. Section 3 deals with the same problem when the parameter λ of the original distribution is a r.v. with d.f. $F_1(\lambda)$, and the survival distribution is Binomial. Several known distributions are derived for various forms of $F_1(\lambda)$ and $F_2(p)$. In section 4 the relation between the p.g.f.'s of the resulting distribution in general and the resulting distribution when no damage has occurred is studied. Using this relation we obtain a characterization of the Poisson distribution which gives *Rao/Rubin's* [1964] result as a special case.

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2. The Damage Model with the Survival Distribution Mixed Binomial

Let us consider the more general form of the damage model in which the parameter p of the Binomial survival distribution is not a fixed number. Instead, suppose that p is a r.v. with d.f. $F_2(p)$. In this case

$$P(Y=r \mid X=n)=\int_0^1 \binom{n}{r} p^r q^{n-r} dF_2(p), \qquad r=0,1,\ldots,n \quad n=0,1,\ldots \tag{2.1}$$

and hence

$$G_Y(t)=\int_0^1 G_X(q+pt)\,dF_2(p), \qquad q=1-p \tag{2.2}$$

and

$$G_{y\mid x=y}(t)=\frac{\int_0^1 G_X(pt)\,dF_2(p)}{\int_0^1 G_X(p)\,dF_2(p)}. \tag{2.3}$$

If we now assume that X is Poisson with parameter λ, (2.2) and (2.3) respectively become

$$G_Y(t)=M_p\,(\lambda(t-1)) \tag{2.4}$$

$$G_{Y\mid X=Y}(t)=\frac{M_p\,(\lambda t)}{M_p\,(\lambda)} \tag{2.5}$$

where $M_Z(t)$ denotes the moment generating function of the r.v. Z.

Different forms of the mixing distribution $F_2(p)$ give rise to various distributions representing the resulting distribution when the original distribution is Poisson and the survival distribution is Binomial $(n, p) \underset{p}{\wedge} F_2(p)$. Here are some examples

a) $(Y \mid X) \sim$ Binomial $(n, p) \underset{p}{\wedge}$ Beta (α, β) (Negative Hypergeometric).

$$G_Y(t)={}_1F_1\,(\alpha; \alpha+\beta; \lambda\,(t-1)), \qquad \alpha>0, \beta>0 \tag{2.6}$$

where ${}_1F_1(a; b; t)$ is the confluent hypergeometric function given by

$${}_1F_1(a; b; t)=\frac{\Gamma(b)}{\Gamma(a)\,\Gamma(b-a)}\int_0^1 e^{tu} u^{a-1}\,(1-u)^{b-a-1} du. \tag{2.7}$$

The distribution with p.g.f (2.6) was first examined by *Gurland* [*1958*].

b) $(Y \mid X) \sim$ Binomial $(n, p) \underset{p}{\wedge}$ Gamma (α, β), truncated to the right at the point 1.

$$G_Y(t) = c\ {}_1F_1\left(\alpha; \alpha+1; \lambda(t-1) - \frac{1}{\beta}\right), \quad \alpha > 0, \beta > 0 \tag{2.8}$$

where c is the normalizing constant.
The distribution with p.g.f. (2.8) has been studied by *Kemp* [1968] as a limited risk Compound Poisson Process.

3. The Damage Model with the Original Distribution Mixed Poisson

Let us now turn to the situation where the parameter λ of the original distribution is a r.v. with d.f. $F_1(\lambda)$ ($\lambda > 0$). Denote by $G_{X|\lambda}(t)$ the p.g.f. of the conditional distribution of $X \mid \lambda$, i.e. of X for given λ. Then on the assumption that the conditional distribution $Y \mid X$ (i.e. the survival distribution) is Binomial with parameters n, p we have

$$G_Y^*(t) = \int_0^\infty G_{X|\lambda}(q+pt)\, dF_1(\lambda) \tag{3.1}$$

and

$$G_{Y|X=Y}^*(t) = \frac{\int_0^\infty G_{X|\lambda}(pt)\, dF_1(\lambda)}{\int_0^\infty G_{X|\lambda}(p)\, dF_1(\lambda)}. \tag{3.2}$$

(We use the notation $G_Y^*(t)$ to indicate that this time, the mixing is taking place in the original distribution.) If, in particular $X \mid \lambda$ is Poisson (λ) (3.1), (3.2) become respectively

$$G_Y^*(t) = M_\lambda(p(t-1)) \tag{3.3}$$

and

$$G_{Y|X=Y}^*(t) = \frac{M_\lambda(pt-1)}{M_\lambda(p-1)}. \tag{3.4}$$

By making use of (3.3) one can obtain the form of the p.g.f of the resulting distribution for different forms of $F_1(\lambda)$. Here are two interesting examples.

a) $X \sim \text{Poisson}(\lambda) \underset{\lambda}{\wedge} \text{Beta}(\alpha, \beta)$.

$$G_Y^*(t) = {}_1F_1(\alpha; \alpha+\beta; p(t-1)) \tag{3.5}$$

b) $X \sim \text{Poisson}(\lambda) \underset{\lambda}{\wedge} \text{Gamma}(\alpha, \beta)$ (Negative Binomial)

$$G_Y^*(t) = \left(\frac{1}{1+p\beta}\right)^\alpha \left(1 - \frac{p\beta t}{1+p\beta}\right)^{-\alpha}, \quad \alpha, \beta > 0. \tag{3.6}$$

Clearly (3.6) is again the p.g.f. of a Negative Binomial distribution.

Remark 1. It is obvious that one can obtain the p.g.f of the resulting distribution when no damage has occurred for the examples given in sections 2 and 3 by using formulae (2.5) and (3.4) respectively.

Remark 2. Results similar to those obtained in sections 2 and 3 can be derived for discrete forms of $F_1(\lambda)$ and $F_2(p)$.

4. Relations Between $G_Y(t)$ and $G_{Y|X=Y}(t)$ in the Extended Damage Model

As *Rao* [1963] pointed out, in the simple damage model where the original distribution is Poisson and the survival distribution is Binomial the following relation holds.

$$P(Y=r)=P(Y=r\mid X=Y), \quad r=0,1,\ldots$$

which, in terms of p.g.f.'s can be written as

$$G_Y(t)=G_{Y|X=Y}(t). \tag{4.1}$$

(This condition has come to be known as the Rao-Rubin condition.)

For our extended form of the damage model the following two theorems can be established.

Theorem 1. If X is Poisson with parameter λ and $Y\mid X$ is Mixed Binomial then

$$G_Y(t+1)=c\,G_{Y|X=Y}(t) \tag{4.2}$$

where $c^{-1}=G_{Y|X=Y}(0)$ is a constant.

Theorem 2. If X is mixed Poisson and $Y\mid X$ is Binomial then

$$G_Y^*(t)=c^*G_{Y|X=Y}^*\left(t+\frac{q}{p}\right) \tag{4.3}$$

where $(c^*)^{-1}=G^*_{Y|X=Y}(1/p)$ is a constant.

The proof of these theorems can be easily obtained using relations (2.4), (2.5) for theorem 1 and (3.3), (3.4) for theorem 2.

Rao/Rubin [1964] used a Binomial survival distribution to show that (4.1) holds if and only if (iff) the distribution of X is Poisson.

In the sequel we extend the Rao-Rubin characterization of the Poisson distribution to the case where the survival distribution is mixed Binomial.

Theorem 3. Let us consider the random vector (X, Y) with non-negative real components such that $P(X=n)=P_n$, $n=0,1,\ldots$, with $P_0\neq 0$ and

$$Y\mid X\sim \text{Binomial}\,(n,p)\underset{p}{\wedge}F_2(p),\ p\in(0,1),\ r=0,1,\ldots,n. \tag{4.4}$$

Then condition (4.2) holds iff P_n is Poisson.

Proof. Necessity follows by theorem 1. To prove sufficiency we first observe that (2.2) can be written as

$$G_Y(t+1) = \int_0^1 G_X(pt+1)\, dF_2(p). \tag{4.5}$$

We also have

$$G_{Y|X=Y}(t) = \frac{\int_0^1 G_X(pt)\, dF_2(p)}{\int_0^1 G_X(p)\, dF_2(p)}. \tag{4.6}$$

Substituting (4.5), (4.6) in (4.2) gives

$$\int_0^1 G_X(pt+1)\, dF_2(p) = c_0 \int_0^1 G_X(pt)\, dF_2(p), \; (c_0^{-1} = \int_0^1 G_X(p)\, dF_2(p)).$$

Hence

$$\int_0^1 \sum_{n=0}^{\infty} P_n (pt+1)^n\, dF_2(p) = c_0 \int_0^1 \sum_{n=0}^{\infty} P_n (pt)^n dF_2(p) \Rightarrow$$

$$\sum_{n=0}^{\infty} \{\sum_{r=0}^{n} P_n \int_0^1 \binom{n}{r} p^r\, dF_2(p)\, t^r\} = c_0 \sum_{n=0}^{\infty} P_n \{\int_0^1 p^n\, dF_2(p)\}\, t^n \Rightarrow$$

$$\sum_{r=0}^{\infty} \{\sum_{n=r}^{\infty} P_n \int_0^1 \binom{n}{r} p^r\, dF_2(p)\}\, t^r = c_0 \sum_{r=0}^{\infty} P_r \{\int_0^1 p^r\, dF_2(p)\}\, t^r.$$

Consequently

$$\sum_{n=r}^{\infty} \binom{n}{r} P_n \int_0^1 p^r\, dF_2(p) = c_0\, P_r \int_0^1 p^r\, dF_2(p) \text{ i.e.}$$

$$\sum_{n=r}^{\infty} P_n \binom{n}{r} = c_0 P_r. \tag{4.7}$$

Taking the p.g.f.'s for both sides of (4.7) we find that

$$G_X(t+1) = c_0 G_X(t) \qquad 0 \leqslant t \leqslant 1. \tag{4.8}$$

But *Shanbhag* [1974] using an elementary approach showed that the unique solution of the functional equation

$$G(q+pt) = \frac{G(pt)}{G(p)} \qquad |t| \leqslant 1 \tag{4.9}$$

where $G(t)$ is a p.g.f., is

$$G(t) = e^{\lambda(t-1)} \qquad \text{for some } \lambda > 0.$$

Since our functional equation (4.8) is a particular case of (4.9) the result is established.

Remark 3. Clearly if the survival distribution is Binomial, i.e. if $F_2(p)$ is degenerate, then condition (4.2) reduces to (4.1) and hence theorem 3 reduces to the Rao-Rubin characterization of the Poisson distribution.

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